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String threshold corrections in models with spontaneously broken supersymmetry[★]

E. Kiritsis^a, C. Kounnas^{a,1}, P.M. Petropoulos^{a,b,2}, J. Rizos^{a,c}

^a Theory Division, CERN, 1211 Geneva 23, Switzerland

^b Institut de Physique Théorique, Université de Neuchâtel, 2000 Neuchâtel, Switzerland

^c Division of Theoretical Physics, Physics Department, University of Ioannina, 45110 Ioannina, Greece

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Abstract

We analyse a class of four-dimensional heterotic ground states with $N = 2$ space-time supersymmetry. From the ten-dimensional perspective, such models can be viewed as compactifications on a six-dimensional manifold with $SU(2)$ holonomy, which is locally but *not globally* $K3 \times T^2$. The maximal $N = 4$ supersymmetry is *spontaneously* broken to $N = 2$. The masses of the two massive gravitinos depend on the (T, U) moduli of T^2 . We evaluate the one-loop threshold corrections of gauge and R^2 couplings and we show that they fall in several universality classes, in contrast to what happens in usual $K3 \times T^2$ compactifications, where the $N = 4$ supersymmetry is explicitly broken to $N = 2$, and where a single universality class appears. These universality properties follow from the structure of the elliptic genus. The behaviour of the threshold corrections as functions of the moduli is analysed in detail: it is singular across several rational lines of the T^2 moduli because of the appearance of extra massless states, and suffers only from logarithmic singularities at large radii. These features differ substantially from the ordinary $K3 \times T^2$ compactifications, thereby reflecting the existence of spontaneously broken $N = 4$ supersymmetry. Although our results are valid in the general framework defined above, we also point out several properties, specific to orbifold constructions, which might be of phenomenological relevance.

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¹ On leave from Laboratoire de Physique Théorique de l'École Normale Supérieure, CNRS, 24 rue Lhomond, 75231 Paris Cedex 05, France.

² On leave from Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France.

1. Introduction

In four dimensions the maximal number of possible space-time supersymmetries is $N = 8$. This upper limit on N follows from the requirement that no massless states with spin greater than 2 exist in the theory. In a realistic world, and for energies above the electroweak scale, $E > M_Z$, we need chiral matter, and among supersymmetric theories only the $N = 1$ possess chiral representations. There is a general belief that in field theory spontaneous breaking of an $N > 1$ supersymmetric theory necessarily produces a non-chiral spectrum. This impeded attempts [1] to use $N > 1$ supersymmetric theories in order to describe physics beyond the electroweak scale. In string theory, this question does not apply. In Ref. [2] it was shown that there are perturbative heterotic ground states where supersymmetry is spontaneously broken from $N = 2$ down to $N = 1$, which possess a chiral four-dimensional spectrum. This opens more possibilities in string model-building, and obviously a more careful investigation is required when $N > 1$ is spontaneously broken to chiral $N = 1$ models.

In general, we can assume that there might be a sequence of supersymmetry-breaking transitions, $N = 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$, that occur at intermediate-energy scales, A_N . We can also assume that the final scale, corresponding to the $N = 1 \rightarrow 0$ supersymmetry breaking, is relatively low, $A_{N=1} \sim O(1)$ TeV, while $M_s \sim O(10^{17})$ TeV $> A_{N>1} > O(1)$ TeV. This scenario provides a solution to the hierarchy problem, and, depending on the value of the intermediate scales A_N , $A_{N=1}$ can be pushed higher by no more than a few orders of magnitude. In this framework, it is important to estimate the physical consequences of the existence of other supersymmetry-breaking scales, $A_{N=8}$, $A_{N=4}$ and $A_{N=2}$. To do this, we need to analyse the behaviour of the couplings in string theory, and in particular their threshold corrections as a function of the compactification moduli and the supersymmetry-breaking scales A_N .

The origin of A_N (including $A_{N=1}$) can be either perturbative or non-perturbative [2]. The recent remarkable progress in understanding the non-perturbative structure of string theories gives us the possibility to study some of the non-perturbative aspects of partial breaking at scales A_N , by performing perturbative calculations in dual string theories. We are thus led to reconsider “spontaneous” versus “explicit” supersymmetry-breaking beyond perturbation theory. Indeed, in perturbative string theory there exist two qualitatively different ways of reducing the number of supersymmetries. In the language of orbifold compactification, some of the original gravitinos are projected out from the spectrum. We would like to distinguish the freely acting orbifolds (spontaneous breaking) from the non-freely acting ones (explicit breaking). A free orbifold action is the one that has no fixed points (strictly speaking, it should not be called orbifold), whereas in a non-free action there are fixed points and some extra *twisted states* are added in the theory. Such a definition relies on a geometrical interpretation of a given ground state. It can, however, be extended to non-geometric ground states. If the orbifold action that breaks supersymmetry is free, then the associated non-invariant gravitinos are not projected out but become massive. This is a stringy generalization [3,4] of the Scherk–

Schwarz idea [5] of breaking supersymmetry³ in the context of Kaluza–Klein theories. The low-energy behaviour of the couplings of these two classes is markedly different. In the non-freely acting case (explicit breaking) [8–15,20], the low-energy theory has no memory of the original supersymmetry, while in the freely acting case (spontaneous breaking) it does [2]. This was verified by explicit calculation in several classes of ground states with spontaneously broken supersymmetry: $N = 4$ to $N = 2$ [21], $N = 8$ to $N = 6$ [22] as well as $N = 8$ to $N = 3$ [23]; furthermore the non-perturbative aspects of this problem have been studied utilizing the heterotic/type II duality [2,24].

String ground states with spontaneously broken supersymmetry have very peculiar high-energy properties that might be desirable: a logarithmically growing gauge and gravitational thresholds at large moduli despite the existence of towers of charged states below the Planck mass, and a possibly special behaviour of the vacuum energy. Obviously, this issue is of crucial importance in choosing string models that should represent the real low-energy world.

In this paper our approach is more modest. One of our goals will be to understand in more detail the generic properties of low-energy couplings in heterotic models, where supersymmetry is spontaneously broken from $N = 4$ to $N = 2$, relevant for the physics at energy scales $E \sim \Lambda_{N=4}$. In situations where supersymmetry is further reduced, it turns out that the dependence of the low-energy couplings on the volumes of the internal manifold can also be obtained from the calculations presented here. In a sense this paper is a generalization of [21] to a much wider class of heterotic ground states with spontaneously broken $N = 4$ to $N = 2$ supersymmetry.

The heterotic ground states that will be considered in the following have 8 unbroken supercharges. These can be thought of as compactifications of the ten-dimensional heterotic string on a six-dimensional manifold with $SU(2)$ holonomy, which is locally but not globally of the $K3 \times T^2$ type. They are characterized by a set of shift vectors w that act on the two-torus. Some of these models can be constructed starting from the heterotic string on $T^6 = T^4 \times T^2$ and orbifolding by a symmetry that involves translations on T^2 and non-freely acting transformations on T^4 . Another orbifold construction that belongs to the above class is the following: orbifold a standard $K3 \times T^2$ compactification by using a symmetry that is non-freely acting on $K3$, but preserves the hyper-Kähler structure and acts as a translation on the two-torus. It is important to stress, however, that these examples do not exhaust all the possibilities: our analysis is valid beyond any orbifold construction.

We will restrict ourselves to the simplest translation on the two-torus, namely \mathbb{Z}_2 . In this case the shift is a half-lattice vector of the T^2 Narain lattice. We will be quite general, making no detailed assumptions on the structure of internal (4,0) superconformal theory. As will be clear from our discussion below, our techniques are directly applicable to a general translation group on the two-torus. In the entire class of models we will be

³ There are two other mechanisms for breaking the supersymmetry. The first is gaugino condensation [6], while the second uses internal magnetic fields [7]. None of them seems to fit in the stringy Scherk–Schwarz context.

dealing with, the original $N = 4$ supersymmetry is spontaneously broken to $N = 2$, and the two massive gravitinos have masses that depend on the two-torus moduli.

We will focus on threshold corrections to the gauge and R^2 couplings. In the presence of $N = 2$ space-time supersymmetry, it is known that the only perturbative corrections to such couplings come from one loop. Moreover, the only massive states that contribute at one loop are BPS multiplets, since the threshold is proportional to the supertrace of the helicity squared. Thus, such thresholds depend on the elliptic genus of the internal (4,0) superconformal field theory. This property is essential and it implies, along with modular invariance, that the thresholds possess certain universality properties and depend only on some low-energy data. This was already demonstrated for standard $K3 \times T^2$ compactifications in [19,20]. Here, the thresholds will be shown to depend on the lattice shift vector w , the beta-function coefficients b_i , the levels k_i of the associated current algebras, and the jumps of the beta functions $\delta_h b_i$ and $\delta_v b_i$ at special submanifolds of the moduli space of the two-torus, where extra massless states appear (but where gauge symmetry is not necessarily enhanced). We will show in particular that the usual decomposition for the gauge threshold corrections, which holds, for instance, in the standard $K3 \times T^2$ compactification, is not valid any longer and must be replaced by

$$\Delta_i^w = b_i \Delta^w(T, U) + \delta_h b_i H^w(T, U) + \delta_v b_i V^w(T, U) + k_i Y^w(T, U).$$

The moduli-dependent functions Δ^w , H^w and V^w are *universal*: they only depend on the shift vector w . On the other hand, Y^w depends also on the gravitational anomaly of the model: $Y^w = Y_1^w + b_{\text{grav}} Y_2^w$, with $Y_{1,2}^w$ universal.

The models under consideration in this paper have type II duals. The easiest way to see this is to employ the construction of the heterotic ones as freely acting orbifolds of the heterotic string on T^6 . Since the heterotic string on T^6 is dual (via $S \leftrightarrow T$ interchange) to the type II string on $K3 \times T^2$, freely orbifolding both sides will produce a new dual pair. In [2] heterotic/type II duality was utilized to study some aspects of this problem. In particular, it was shown that heterotic ground states with $N = 4$ supersymmetry spontaneously broken to $N = 2$ and massive gravitinos in the perturbative spectrum are sometimes dual to type II models without massive gravitinos in the perturbative spectrum. Thus, at the perturbative level, supersymmetry in the type II context seems explicitly broken. The massive gravitinos are BPS multiplets of the unbroken $N = 2$ supersymmetry. They can therefore be identified in the type II description to monopoles whose mass is of order $1/g_{\text{II}}^2$, where g_{II} is the type II string coupling. In particular they become very light at strong type II coupling, thereby enhancing the supersymmetry. This might indicate the possibility that in string theory supersymmetry is always spontaneously broken, either perturbatively, or non-perturbatively. There is another possibility, though. In both theories of the dual pair there are potential non-perturbative corrections. It is thus possible that an analogue of the Seiberg–Witten non-restoration of gauge symmetry is happening here: non-perturbative effects do not enable supersymmetry restoration at strong coupling. A more careful study of this problem is necessary, which we leave for the future.

As we pointed out previously, the appearance of non-perturbative corrections to the gauge and R^2 couplings cannot be excluded in general. In $N = 2$ ground states, we always have the appropriate Higgs expectation value that cuts off the infra-red. Thus, all non-perturbative effects are expected to be due to instantons. Moreover, since the F^2 and R^2 couplings are of the BPS-saturated type [15,25–27] only supersymmetric instantons (that preserve one out of the two supersymmetries) are expected to contribute. Thus, in the four-dimensional heterotic ground states we expect instanton corrections due to the heterotic five-brane wrapped around the compact internal six-dimensional manifold.

More information about the perturbative and non-perturbative contributions is reached by decompactifying one of the directions of the two-torus to obtain a five-dimensional theory. Here, there are two possibilities: (i) the five-dimensional model has 16 supercharges, and in this case the five-dimensional perturbative thresholds are zero, as implied by the extended supersymmetry; (ii) the five-dimensional model has only 8 supercharges and now the perturbative thresholds are non-zero. In both cases, however, there are no non-perturbative instanton corrections: the six-dimensional world-volume of the Euclidean heterotic five-brane cannot be wrapped around the internal space and have finite action.

The structure of this paper is as follows. In Section 2 we present a general description of $N = 2$ heterotic ground states in terms of their helicity-generating partition function. The latter is expressed in terms of the elliptic genus corresponding to the internal manifold. This is very useful for the determination of threshold corrections. Moreover, it allows us to define the class of models that we will be analysing throughout the paper, by giving the generic form of their elliptic genus.

In Section 3 we briefly recall the general procedure that is used for computing gauge and gravitational threshold corrections in supersymmetric string vacua. We also present the basic properties of these corrections in $N = 2$ heterotic compactifications, where a two-torus is factorized. These models play an important role in our subsequent analysis, because they turn out to share some decompactification limits with the models where the two-torus undergoes a shift and where supersymmetry is promoted to spontaneously broken $N = 4$.

Section 4 is devoted to the description of the class of models where the two-torus is not factorized. Here we stress the role of the shift on the T^2 , which is interpreted as a stringy Sherk–Schwarz mechanism. Depending on the kind of shift vector, several decompactification scenarios appear. In models where the norm λ of the shift vector vanishes, two possible decompactification limits exist in the (T, U) plane: with and without restoration of $N = 4$ supersymmetry. When $\lambda = 1$ (the only relevant alternative), $N = 4$ supersymmetry is always restored. This is in agreement with the partial breaking of the target-space duality group, which makes several directions in the moduli space inequivalent.

In Section 5 we proceed to the computation of threshold corrections. This is achieved by advocating general holomorphicity and modular-covariance properties. Most of the model and moduli dependence is lost at the level of the thresholds, which turn out to depend only on the two-torus moduli (T, U) as well as on several rational parameters

(discrete Wilson lines) related to some low-energy data of the model. These are b_i and k_i but also δb_i , the discontinuities of the beta-function coefficients along some specific lines in the two-torus moduli space, where additional vector multiplets and/or hypermultiplets become massless. Across these lines, the thresholds diverge logarithmically. We also observe that in the class of models under consideration, the above low-energy parameters are in fact related in a very specific way. This leaves some arbitrariness in the splitting of the gauge threshold corrections into gauge-factor-dependent and gauge-factor-independent pieces, even though we demand the latter contribution to be regular in the (T, U) space. Moreover, some model dependence survives in the group-factor-independent term $Y^w(T, U)$, which is not fully universal. A similar model dependence appears in the gravitational thresholds. These features are to be contrasted to what happens in models in which a two-torus is factorized: the gauge threshold is uniquely defined as a sum of two terms, one being universal and the other group-factor dependent, and both regular; the gravitational threshold corrections are model independent. Finally, the behaviour of the thresholds at various decompactification limits is analysed, and turns out to agree with what is expected on general grounds based on the restoration of $N = 4$ supersymmetry. Again the results strongly depend on whether the norm λ of the shift vector equals 0 or 1. The existence of a common decompactification limit in these models and in models with a factorized two-torus is also observed in the behaviour of the thresholds.

As an application, we examine in Section 6 the subclass of \mathbb{Z}_2 orbifolds. In this case, more can be said about the nature of the extra massless states appearing along the rational lines of the T^2 moduli space. In fact, a priori, these can be either vector multiplets or hypermultiplets depending on the specific model at hand and on the shift vector acting on the two-torus. In the case of orbifolds, *only extra hypermultiplets* become massless, except for the lines $T = U$ and $T = -1/U$, present systematically, where either hypermultiplets or vector multiplets may appear, depending on the shift vector. This information might be of some phenomenological relevance. The last part of Section 6 is devoted to some specific orbifold examples for both the situations $\lambda = 0$ and $\lambda = 1$. We construct in particular four-dimensional ground states whose gauge group contains factors such as $E_8 \times E_8$, $SO(40)$ or even E_8 realized at level 2.

Most of the technicalities are presented in appendices. In Appendix A we give an overview of \mathbb{Z}_2 -shifted $(2, 2)$ lattice sums. Rational lines and asymptotic behaviours of the latter are also analysed there. Appendices B and C contain the machinery used for the determination of gauge and gravitational corrections. Finally, in Appendix D, we perform explicitly the general integrals over the fundamental domain, which are involved in our expressions for the thresholds. We also analyse their singularities and asymptotic behaviours.

2. General description of $N = 2$ heterotic ground states

In this section we will give a brief description of the heterotic ground states that we will be studying in the following. They will be best described by writing their (four-dimensional) helicity-generating partition functions. Indeed, our motivation is eventually to compute couplings associated with interactions such as $F^n R^m$. Therefore, we need in general to evaluate amplitudes involving operators like $i(x^\mu \overleftrightarrow{\partial} x^\nu + 2\psi^\mu \psi^\nu) \bar{J}^k$, where \bar{J}^k is an appropriate right-moving current and the left-moving factor corresponds to the left-helicity operator. We will not expand here on the various procedures that have been used in order to calculate exactly (i.e. to all orders in α' by properly taking into account corrections due to the gravitational back-reaction, and without infra-red ambiguities) these correlation functions; details on the determination of the amplitude-generating functions relevant for gauge and gravitational couplings can be found in [16–18,20]. We will restrict ourselves to the helicity-generating partition functions (which are also very useful in the analysis of S -duality issues), defined as

$$Z(v, \bar{v}) = \text{Tr}' q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} e^{2\pi i(v\lambda - \bar{v}\bar{\lambda})}, \tag{2.1}$$

where the prime over the trace excludes the zero-modes related to the space-time coordinates (consequently $Z(v, \bar{v})|_{v=\bar{v}=0} = \tau_2 Z$, where Z is the vacuum amplitude), and $\lambda, \bar{\lambda}$ stand for the left- and right-helicity contributions to the four-dimensional physical helicity. Various helicity supertraces are finally obtained by taking appropriate derivatives of (2.1). More on these issues can be found in Appendix G of [28].

For four-dimensional heterotic $N = 4$ solutions with maximal-rank gauge group ($r = 22$) (2.1) reads⁴

$$\begin{aligned} Z_{N=4}(v, \bar{v}) &= \frac{1}{|\eta|^4} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v)}{\eta} \left(\frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \right)^3 \xi(v) \bar{\xi}(\bar{v}) Z_{6,22} \\ &= \frac{1}{|\eta|^4} \left(\frac{\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\frac{v}{2})}{\eta} \right)^4 \xi(v) \bar{\xi}(\bar{v}) Z_{6,22}, \end{aligned} \tag{2.2}$$

where

$$\xi(v) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^{2\pi i v})(1 - q^n e^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\vartheta_1'(0)}{\vartheta_1(v)}$$

counts the helicity contributions of the space-time bosonic oscillators, and $Z_{6,22} \equiv \Gamma_{6,22}/\eta^6 \bar{\eta}^{22}$ denotes the partition function of six compactified coordinates as well as of sixteen right-moving currents; it depends generically on 132 moduli, namely 36 internal background metric and antisymmetric tensor fields, and 96 internal background gauge fields (Wilson lines). It is possible to continuously connect several extended-symmetry points such as $U(1)^6 \times E_8 \times E_8$, $U(1)^6 \times SO(32)$ or $SO(44)$. The (4, 0)

⁴ We use the short-hand notation $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v)$ for $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v|\tau)$, and $\vartheta \begin{bmatrix} a \\ b \end{bmatrix}$ for $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0|\tau)$.

supersymmetry is read off automatically from expression (2.2), which has a fourth-order zero at $v = 0$. Theories with lower-rank gauge group and the same supersymmetry can be easily constructed by modding out discrete symmetries, which correspond to outer automorphisms and act without fixed points on the lattice [24,29].

Supersymmetry can be reduced to $N = 2$ in various ways. Generically, the helicity-generating function reads

$$Z_{N=2}^{\text{generic}}(v, \bar{v}) = \frac{1}{|\eta|^4} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v)}{\eta} \frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (\bar{v})}{\eta} \xi(v) \bar{\xi}(\bar{v}) C_{6,22} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v), \tag{2.3}$$

where $C_{6,22} \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right] (v)$ are traces in the internal superconformal field theory. We have kept explicit the ϑ -function contribution of the left-moving fermions of the two transverse space-time coordinates as well as those of the internal two-dimensional free theory. The (6, 22) internal field theory has $c_R = 22$ on the right sector, while on the left sector $c_L = 8$. This sector has a two plus four split: the two-dimensional part of it is free, with central charge $c = 2$, while the four-dimensional one is $N = 4$ superconformal with $\hat{c} = 4$ [30]. Space-time supersymmetry can be used again to write (2.3) as

$$Z_{N=2}^{\text{generic}}(v, \bar{v}) = \frac{1}{|\eta|^4} \left(\frac{\vartheta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \left(\frac{v}{2} \right)}{\eta} \right)^2 \xi(v) \bar{\xi}(\bar{v}) C_{6,22} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \left(\frac{v}{2} \right), \tag{2.4}$$

where

$$C_{6,22} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \left(\frac{v}{2} \right) = \text{Tr}_R^{\text{int}} (-1)^{F^{\text{int}}} q^{L_0^{\text{int}} - \frac{1}{3}} \bar{q}^{\bar{L}_0^{\text{int}} - \frac{11}{12}} e^{2\pi i v J_0}$$

is the (generalized) elliptic genus of the internal conformal field theory. The standard elliptic genus [31,32], relevant for the gravitational threshold corrections is obtained at $v = 0$. The charge J_0 is the sum of the internal $U(1)$ -current zero-modes of the $N = 2$ and $N = 4$ internal superconformal algebras.

An interesting class of models is provided when the ten-dimensional theory is generically compactified to six dimensions in a way that preserves 8 supercharges out of 16, and is toroidally compactified further down to four dimensions. In that case, the T^2 contribution factorizes completely. We obtain

$$C_{6,22} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \left(\frac{v}{2} \right) = C_{4,20} \left(\frac{v}{2} \right) Z_{2,2}, \tag{2.5}$$

where $C_{4,20}(v/2)$ is a (left-helicity-generating) conformal block with two left and no right world-sheet supersymmetries, and central charges $\hat{c}_L = 4$, $c_R = 20$. It accounts for the compactification from ten to six dimensions, and actually defines the generalized elliptic genus for the four-dimensional compact manifold plus a gauge bundle on it with instanton number 24. This conformal block depends in particular on several moduli (other than the two-torus ones). On the other hand, $Z_{2,2} \equiv \Gamma_{2,2}/|\eta|^4$ is the partition function of the two-torus (A.1), whose complex moduli are T and U . It is invariant

under the full target-space duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{\Gamma \leftrightarrow U}$. More details about lattice sums can be found in Appendix A.

The above $N = 2$ construction (2.5) captures many compactifications such as $K3$ or orbifold models. For example, (symmetric or asymmetric) \mathbb{Z}_2 twists acting at the level of the $N = 4$ model (2.2) leave invariant a single complex plane corresponding to a T^2 compactification from six to four dimensions. They therefore belong to the above class of $N = 2$ ground states. Their helicity-generating function is given by Eqs. (2.4) and (2.5) with

$$C_{4,20}^{\text{orb}} \left(\frac{\nu}{2} \right) = \frac{1}{2} \sum_{h,g=0}^1 \frac{\vartheta \left[\begin{smallmatrix} 1+h \\ 1+g \end{smallmatrix} \right] \left(\frac{\nu}{2} \right)}{\eta} \frac{\vartheta \left[\begin{smallmatrix} 1-h \\ 1-g \end{smallmatrix} \right] \left(\frac{\nu}{2} \right)}{\eta} Z_{4,20} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right], \tag{2.6}$$

where $Z_{4,20} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] \equiv \Gamma_{4,20} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] / \eta^4 \bar{\eta}^{20}$ summarize the bosonic orbifold blocks; in particular the untwisted partition function that describes the right-moving currents and the four compactified coordinates is $Z_{4,20} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] \equiv Z_{4,20}$; it depends on 80 moduli. The \mathbb{Z}_2 -twisted contributions are moduli independent. They can be constructed in many consistent ways, provided they satisfy the periodicity and modular-invariance requirements (see Eqs. (4.2), (4.3) with $\lambda = 0$).

The best-known example of the orbifold construction is the symmetric \mathbb{Z}_2 orbifold, which turns out to belong also to the $K3$ moduli space. This model is achieved by going to a point of the $(4, 20)$ moduli space, where a four-torus is factorized.⁵ The gauge group is $E_8 \times E_7 \times SU(2) \times U(1)^2$. Thus $N_V = 386$, while $N_H = 628$ (N_V and N_H are the number of vector multiplets and hypermultiplets, respectively).

The gravitational and gauge couplings of the $N = 2$ heterotic ground states with a factorized two-torus have been studied extensively. In the present paper, our goal is to analyse more general situations, where Eq. (2.5) does not hold any longer and is replaced by

$$C_{6,22} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] \left(\frac{\nu}{2} \right) = \frac{1}{2} \sum_{h,g=0}^1 C_{4,20}^\lambda \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] \left(\frac{\nu}{2} \right) Z_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right], \tag{2.7}$$

where $Z_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ stands for the shifted partition function of the two-torus. Such a structure appears, for instance, in freely acting orbifolds that reduce $N = 4$ supersymmetry to $N = 2$, and act with a lattice shift on the two-torus. Eqs. (2.4) and (2.7) describe more general $N = 2$ ground states, which have always a $U(1)^2$ right-moving gauge group coming from the two-torus. They are obviously not of the factorized form $K3 \times T^2$, but they correspond to compactifications on six-dimensional manifolds of $SU(2)$ holonomy. In Section 4, where these models are described in detail, we will argue that $N = 2$ supersymmetry is promoted to *spontaneously broken* $N = 4$. The corresponding threshold corrections will be computed in Section 5.

⁵ Its $(4, 20)$ lattice sum is actually given in (6.9), which provides also a relevant example in the framework of heterotic ground states where the T^2 is not factorized.

3. Thresholds in general and toroidal compactifications from six dimensions

3.1. Computation in generic heterotic supersymmetric string vacua

Threshold corrections appear in the relation between the running gauge coupling $g_i(\mu)$ of the low-energy effective field theory and the string coupling constant g_s , which is associated with the expectation value of the dilaton field. For supersymmetric ground states and non-anomalous $U(1)$'s, one can introduce, at the string level, an infra-red regularization prescription such that it becomes possible to match unambiguously string theory and low-energy effective-field-theory amplitudes [16–18,20], thus leading to the relation

$$\frac{16 \pi^2}{g_i^2(\mu)} = k_i \frac{16 \pi^2}{g_s^2} + b_i \log \frac{M_s^2}{\mu^2} + \Delta_i, \tag{3.1}$$

which is actually expected if one assumes the decoupling of massive modes [8–11]. In this expression, μ is the infra-red scale, while $M_s = 1/\sqrt{\alpha'}$ is the string scale. String unification relates the latter to the Planck scale $M_P = 1/\sqrt{32 \pi G_N}$ and to the string coupling constant. At the tree level this relation reads

$$M_s = g_s M_P; \tag{3.2}$$

notice that for supersymmetric vacua (3.2) does not receive any perturbative correction [19].

In the $\overline{\text{DR}}$ scheme for the effective field theory, the thresholds read [19,20]

$$\Delta_i = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} (F_i - b_i) + b_i \log \frac{2 e^{1-\gamma}}{\pi \sqrt{27}}, \tag{3.3}$$

where, in the presence of supersymmetry, the function F_i is defined by the following genus-one string amplitude:

$$F_i = \left\langle -\lambda^2 \left(\overline{P}_i^2 - \frac{k_i}{4\pi\tau_2} \right) \right\rangle_{\text{genus-one}}. \tag{3.4}$$

Here λ is the left-helicity operator introduced above, \overline{P}_i is the charge operator of the gauge group G_i (for conventions see Refs. [16,17]), k_i is the level of the i th gauge group factor, and b_i are the full beta-function coefficients,

$$b_i = \lim_{\tau_2 \rightarrow \infty} F_i. \tag{3.5}$$

Generically, we can express any $N = 1$ heterotic vacuum amplitude in the canonical form

$$Z = \frac{1}{\tau_2 |\eta|^4} \frac{1}{2} \sum_{a,b=0}^1 \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} C \begin{bmatrix} a \\ b \end{bmatrix}, \tag{3.6}$$

where $C \begin{bmatrix} a \\ b \end{bmatrix}$ are related to the various sectors of the internal six-dimensional partition function. By using this form, we can recast Eq. (3.4) as follows:

$$F_i = \frac{i}{2\pi} \frac{1}{|\eta|^4} \sum_{a,b=0}^1 \frac{\partial_\tau \vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \left(\overline{P}_i^2 - \frac{k_i}{4\pi\tau_2} \right) C \begin{bmatrix} a \\ b \end{bmatrix}. \quad (3.7)$$

One can similarly introduce the function

$$F_{\text{grav}} = \frac{i}{24\pi} \frac{1}{|\eta|^4} \sum_{a,b=0}^1 \frac{\partial_\tau \vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} \left(\overline{E}_2 - \frac{3}{\pi\tau_2} \right) C \begin{bmatrix} a \\ b \end{bmatrix}, \quad (3.8)$$

where E_{2n} is the n th Eisenstein series (see Appendix B). This function plays a similar role in the determination of the gravitational threshold corrections, which appear in the renormalization of the R^2 term [19]

$$A_{\text{grav}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} (F_{\text{grav}} - b_{\text{grav}}) \quad (3.9)$$

(up to a constant term), where

$$b_{\text{grav}} = \lim_{\tau_2 \rightarrow \infty} \left(F_{\text{grav}} + \frac{1}{12\bar{q}} \right) \quad (3.10)$$

is the gravitational anomaly in units where a hypermultiplet contributes $1/12$ [11].

3.2. The case of the $N = 2$ ground states with a factorized T^2

Let us now concentrate on $N = 2$ ground states that come from toroidal compactification of generic six-dimensional $N = 1$ string theories. We recall here the determination of the gravitational and gauge couplings for these models [20], because it plays a significant role in the analysis of the more general constructions presented in Section 4: those two classes of ground states turn out to share large-moduli limits (see Section 4.2). We will focus in particular on the couplings of group factors corresponding to the rank-20 part⁶ of the gauge group, which were already present in the six-dimensional theory and not on the corrections to the couplings of the $U(1)$'s originated from the two-torus (or the $SU(2)$'s or $SU(3)$ appearing at extended-symmetry points of the T, U moduli). In other words, the charge operator \overline{P}_i^2 will not act on the lattice sum $\Gamma_{2,2}$. For the models at hand the helicity-generating function is given by (2.4) with (2.5).

By comparing the latter (at $v = 0$) to (3.6), and using (3.7) and (3.8), we obtain

$$F_i = -\frac{\Gamma_{2,2}}{\bar{\eta}^{24}} \left(\overline{P}_i^2 - \frac{k_i}{4\pi\tau_2} \right) \overline{\Omega}, \quad F_{\text{grav}} = -\frac{\Gamma_{2,2}}{\bar{\eta}^{24}} \left(\frac{\overline{E}_2}{12} - \frac{1}{4\pi\tau_2} \right) \overline{\Omega}, \quad (3.11)$$

⁶ Actually, this part of the gauge group is at most of rank 20. For convenience, we will, however, keep on referring to it as the “rank-20 component”, in order to distinguish it from the two-torus contribution.

where

$$\bar{\Omega} \equiv -\frac{1}{2} \bar{\eta}^{20} C_{4,20}|_{v=0} . \tag{3.12}$$

A few remarks are in order here. The first one concerns the antiholomorphicity and universality properties of the function $\bar{\Omega}$ defined in Eq. (3.12). Indeed, by analysing the relevant two-point amplitudes in six dimensions, it was shown in [33] that, for vanishing v , $C_{4,20}$ is a purely antiholomorphic function, the elliptic genus. Put another way, $C_{4,20}|_{v=0}$ is essentially the supertrace of the left-helicity squared (see Eq. (3.4)), and thus it receives contributions from massless and massive BPS states only. Such states are necessarily of the form left-moving vacuum times right-moving excitations.

Six-dimensional anomaly cancellation⁷ forces the function $C_{4,20}|_{v=0}$ to be independent of the kind of compactification that has been used to go from ten to six dimensions. Following [19,20], we therefore conclude that for the models under consideration⁸

$$\bar{\Omega} = \bar{E}_4 \bar{E}_6 . \tag{3.13}$$

It is important to stress here that the above universality property applies exclusively to the elliptic genus and could not be promoted at the level of the full model. In other words, the data $C_{4,20}|_{v=0} = -2\bar{E}_4\bar{E}_6/\bar{\eta}^{20}$ do not enable us to reconstruct the full function $C_{4,20}(v/2)$, which is in general model (and moduli) dependent.

By using the above result (3.13), we can go further: if we advocate again holomorphicity properties and demand (3.5) as well as the absence of tachyon contribution in F_i , we determine the action of the charge operator \bar{P}_i^2 with the result [15,17,19,20]

$$\begin{aligned} F_i &= k_i \left(F_{\text{grav}} + \Gamma_{2,2} \left(\frac{\bar{j}}{12} - 84 \right) \right) + b_i \Gamma_{2,2} \\ &= -\frac{k_i}{12} \Gamma_{2,2} \left(\frac{\widehat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right) + b_i \Gamma_{2,2} , \end{aligned} \tag{3.14}$$

where \widehat{E}_2 stands for the modular covariant combination $\bar{E}_2 - 3/\pi\tau_2$ and $j(\tau) = \frac{1}{q} + 744 + O(q)$, $q = \exp 2\pi i\tau$, is the standard j -function. Therefore one can write

$$\Delta_i = b_i \Delta - k_i Y , \tag{3.15}$$

with

$$\begin{aligned} \Delta &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\Gamma_{2,2}(T, U, \bar{T}, \bar{U}) - 1 \right) + \log \frac{2e^{1-\gamma}}{\pi\sqrt{27}} \\ &= -\log \left(4\pi^2 |\eta(T)|^4 |\eta(U)|^4 T_2 U_2 \right) \end{aligned} \tag{3.16}$$

⁷ At the level of the four-dimensional spectrum, this constraint reads $N_H - N_V = 242$, at generic points of the two-torus moduli space. It translates into $b_{\text{grav}} = 22$, since in our normalizations $b_{\text{grav}} = (22 - N_V + N_H)/12$. Along the enhanced-symmetry line $T = U$, $N_H - N_V = 240$; it can even reach the values 238 or 236 when $T = U = i$ or $T = U = \rho$ respectively.

⁸ Notice that in the case of orbifolds, Eqs. (2.6) and (3.12) lead to the result (3.13) by direct calculation.

and

$$Y = \frac{1}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma_{2,2}(T, U, \bar{T}, \bar{U}) \left(\frac{\widehat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right). \tag{3.17}$$

A further analysis of this gauge-factor-independent threshold can be found in [20].

Our second comment is related to the absence of $\frac{1}{q}$ -pole in expression (3.14). If such a pole were present, combined with the lattice sum $\Gamma_{2,2}$, it would generate an extra constant term in F_i , at some special line of the two-torus moduli space (such as $T = U$, $T = U = i$ or $T = U = \rho \equiv \exp \frac{2\pi i}{3}$). This would lead to a jump in the beta-function coefficients b_i , proportional to k_i and due to extra massless states charged under the i th gauge-group factor. It is clear from the above analysis that this phenomenon does not occur. In other words the extra massless states that do appear at extended-symmetry points carry no charge with respect to the rank-20 component of the gauge group that is considered here. A straightforward consequence of this situation is the regularity of universal contributions Y (see Eq. (3.17)) all over the T^2 moduli space. As we will see in the following, this picture will change drastically in ground states where $N = 2$ supersymmetry is realized as a spontaneous breaking of $N = 4$. Notice that, already in the case at hand, the gravitational anomaly receives an extra contribution

$$\delta_i b_{\text{grav}} = -\frac{\delta}{6}, \quad \delta = 1, 2 \text{ or } 3, \tag{3.18}$$

when $T = U$, $T = U = i$, or $T = U = \rho$, respectively. This is due to the appearance of $\delta_i N_V = 2\delta$ extra vector multiplets, while $\delta_i N_H = 0$.

The gravitational thresholds are given by (see (3.9), (3.11) and (3.13))

$$4_{\text{grav}}^{\text{gen}} = -\frac{1}{12} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\Gamma_{2,2} \frac{\widehat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} + 264 \right) \tag{3.19}$$

at generic points of the T^2 moduli space and have a singular behaviour along the line $T = U$. For these values of the moduli, it is necessary to properly subtract the full $b_{\text{grav}} = 22 + \delta_i b_{\text{grav}}$ so as to avoid logarithmic divergences in the integral (3.19). Notice finally that the gravitational thresholds are identical for all $N = 2$ models with a factorized two-torus (as is the gauge-factor-independent term of the gauge thresholds).

The last observation we would like to make here concerns the determination of $\bar{\mathcal{O}}$ defined in (3.12). Although the solution given in (3.13) is the one that satisfies all the requirements (antiholomorphicity, regularity in the τ -plane, ...), there is another possibility that one should not disregard, namely $\bar{\mathcal{O}} = 0$. In that case $F_i = F_{\text{grav}} = 0$ and, as a consequence, all beta-function coefficients and the gravitational anomaly vanish: the ground state at hand actually possesses $N = 4$ supersymmetry. In that case, the two extra space-time supersymmetries appear as a conspiracy of left-moving zero-modes originated from the ($\hat{c}_L = 4, c_R = 20$) conformal block (this can happen, e.g. in orbifolds, since in some cases two extra massless gravitinos can appear from the twisted sector). Such

models will appear in Section 5, sharing decompactification limits with $N = 2$ models where $N = 4$ supersymmetry is broken spontaneously.

4. Models with spontaneously broken $N = 4$ to $N = 2$ supersymmetry

4.1. General models and helicity-generating function

As was announced at the end of Section 2, we will now construct different $N = 2$ models in four dimensions, which can be represented as ground states where $N = 4$ supersymmetry is *spontaneously broken* to $N = 2$.

We will describe a representative orbifold construction [21], which we will then generalize beyond orbifolds. Orbifolding consists in performing a \mathbb{Z}_2 rotation in the $(4, 20)$ part of the original $N = 4$ model (and which would project out two of the four gravitinos) together with a \mathbb{Z}_2 lattice shift on the T^2 . Here the orbifold group acts without fixed points. The two gravitinos that would have been projected out combine with a state carrying T^2 momentum (or winding depending on the lattice shift) and survive the orbifold projections. They are massive, however, and their mass is an easily computable function of the T^2 moduli. In these orbifolds, $N = 4$ supersymmetry is spontaneously broken to $N = 2$.

The partition function for the orbifold models under consideration can be written in the following way:

$$Z_{\text{sp br}}^{\text{orb}} = \frac{1}{\tau_2 |\eta|^4} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \left(\frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right)^2 \times \frac{1}{2} \sum_{h,g=0}^1 \frac{\vartheta \left[\begin{smallmatrix} a+h \\ b+g \end{smallmatrix} \right]}{\eta} \frac{\vartheta \left[\begin{smallmatrix} a-h \\ b-g \end{smallmatrix} \right]}{\eta} Z_{4,20}^\lambda \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] Z_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]. \quad (4.1)$$

The shifted lattice sum $\Gamma_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ appearing in (A.7)

$$Z_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] \equiv \frac{\Gamma_{2,2}^w \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]}{|\eta|^4}$$

is given in (A.3) and (A.4). It depends on two integer-valued two-vectors (see Appendix A) \mathbf{a} and \mathbf{b} , whose components are defined modulo 2, and we use the short-hand notation $w \equiv (\mathbf{a}, \mathbf{b})$. The modular properties are captured in a single $O(2, 2, \mathbb{Z})$ -invariant parameter $\lambda \equiv \mathbf{ab}$, which allows us to distinguish two cases of interest: $\lambda = 0$ and $\lambda = 1$.⁹

Modular invariance of the full partition function can be advocated for determining how the \mathbb{Z}_2 -twisted contributions $Z_{4,20}^\lambda \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ should transform. By using Eqs. (A.8) and (A.9) we find

⁹ It can be shown (see Appendix A) that other values of λ are related to the above by lattice periodicity.

$$\tau \rightarrow \tau + 1, \quad Z_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{i\pi(\frac{4}{3} + (1-\lambda)\frac{h^2}{2})} Z_{4,20}^\lambda \begin{bmatrix} h \\ h + g \end{bmatrix}, \quad (4.2)$$

$$\tau \rightarrow -\frac{1}{\tau}, \quad Z_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{-i\pi(1-\lambda)hg} Z_{4,20}^\lambda \begin{bmatrix} g \\ -h \end{bmatrix}. \quad (4.3)$$

In the case $\lambda = 0$, modular invariance allows us to use the same twisted partition functions $Z_{4,20} \begin{bmatrix} h \\ g \end{bmatrix}$ as those appearing in $N = 2$ ground states with a factorized two-torus (Eqs. (2.4) and (2.5)). The case $\lambda = 1$, however, necessitates the introduction of slightly different twists, since $Z_{4,20}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix}$ must now transform with different phases. Examples will be worked out in Section 6, Eqs. (6.9)–(6.16).

Here we would like to pause and examine the possibility of generalizing the construction presented so far to models where a \mathbb{Z}_2 acts freely on the two-torus whereas *the compactification from ten to six dimensions is not necessarily an orbifold*. This can be achieved by looking first at the helicity-generating function of the orbifold models with spontaneously broken $N = 4$ supersymmetry (4.1). This function is actually given by (2.4) with

$$C_{6,22}^{\text{orb}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{v}{2} \right) = \frac{1}{2} \sum_{h,g=0}^1 C_{4,20}^{\text{orb},\lambda} \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{v}{2} \right) Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}, \quad (4.4)$$

where

$$C_{4,20}^{\text{orb},\lambda} \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{v}{2} \right) = \frac{\vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \left(\frac{v}{2} \right)}{\eta} \frac{\vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix} \left(\frac{v}{2} \right)}{\eta} Z_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}. \quad (4.5)$$

Expression (4.4) is actually the most adequate for further generalization. Indeed, instead of (4.5), we can use more general blocks $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$ such that the internal (4, 20) theory has $N = 4$ left-moving superconformal symmetry. We can thereby construct the most general heterotic four-dimensional ground states with $N = 2$ supersymmetry, which can be enhanced to $N = 4$. For these, the helicity-generating function is (2.4) with (2.7), as advertised in Section 2. Modular covariance demands that

$$\tau \rightarrow \tau + 1, \quad v \rightarrow v, \quad C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{v}{2} \right) \rightarrow e^{i\pi(\frac{5}{3} - \lambda\frac{h^2}{2})} C_{4,20}^\lambda \begin{bmatrix} h \\ h + g \end{bmatrix} \left(\frac{v}{2} \right), \quad (4.6)$$

$$\tau \rightarrow -\frac{1}{\tau}, \quad v \rightarrow \frac{v}{\tau}, \quad C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{v}{2} \right) \rightarrow -e^{i\pi(\frac{v^2}{27} + \lambda hg)} C_{4,20}^\lambda \begin{bmatrix} g \\ -h \end{bmatrix} \left(\frac{v}{2} \right). \quad (4.7)$$

In general, the model-dependent functions $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$ depend on several (continuous or discrete) moduli. The $(N = 4)$ -sector contribution $C_{4,20}^{\lambda=0} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (v/2)$ is in fact the one given in (4.5) for $(h, g) = (0, 0)$ [30], namely

$$C_{4,20}^\lambda \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\frac{v}{2} \right) = \left(\frac{\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{v}{2} \right)}{\eta} \right)^2 Z_{4,20}, \quad (4.8)$$

and does not depend on the choice of $\lambda = 0, 1$. The other sectors, however, might or might not be connected to some orbifold realization captured in (4.5). At $v = 0$,

$C_{4,20}^{\lambda} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ vanishes because of the fermionic zero-modes and, as we will see later, for $(h, g) \neq (0, 0)$, $C_{4,20}^{\lambda} \begin{bmatrix} h \\ g \end{bmatrix} |_{v=0}$ are purely antiholomorphic functions. This last property puts severe constraints on $C_{4,20}^{\lambda} \begin{bmatrix} h \\ g \end{bmatrix} |_{v=0}$ and is responsible for the absence of continuous-moduli dependence inside the threshold corrections (see Section 5).

4.2. Decompactification limits and restoration of $N = 4$ supersymmetry

As we mentioned above, there are two possible values for the parameter λ , which lead to fundamentally different ground states. In the case $\lambda = 0$, any model with spontaneously broken $N = 4$ supersymmetry of the type (2.4) with (2.7) can be mapped onto a ground state with $N = 2$ supersymmetry of the type $K3 \times T^2$ studied in Section 2. This mapping is achieved by defining the function $C_{4,20}(v/2)$, which appears in (2.5) in terms of the blocks $C_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$ appearing in (2.7):

$$C_{4,20} \left(\frac{v}{2} \right) = \frac{1}{2} \sum_{h,g=0}^1 C_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} \left(\frac{v}{2} \right), \tag{4.9}$$

in agreement with all properties (modular transformations, ...) that these functions must satisfy. For $\lambda = 1$ it is not possible to establish such a kind of mapping.

This manipulation is actually deeper than a formal construction, the two models being closely related in their six-dimensional decompactification limit. In fact, as we point out in Appendix A (see Eq. (A.16) for the $\lambda = 0$ shifted lattice sum I) any $\lambda = 0$ shifted lattice sum possesses a decompactification limit in which $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ are equal for all (h, g) (and in particular equal to the limit of $\Gamma_{2,2}$). By comparing (2.5) and (2.7), we thus conclude that, in this six-dimensional limit,¹⁰ the $N = 2$ ground state with spontaneously broken supersymmetry (i.e. with the shifted (2, 2) lattice) and the ordinary $N = 2$ ground state (i.e. with the unshifted (2, 2) lattice), which is mapped on the former through (4.9), are in fact identical. It can also be argued that these two ground states, related through (4.9), possess actually the same gauge group. Their matter content is, however, different.

On the other hand, any $\lambda = 0$ or $\lambda = 1$ model with spontaneously broken space-time supersymmetry of the type (2.7) can be mapped onto an $N = 4$ model by keeping the $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ sector only. Indeed, as we mentioned above, this sector is the $N = 4$ sector of the original model whose conformal block $C_{4,20}^{\lambda} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (v/2)$ is given in (4.8). The corresponding $N = 4$ heterotic model (see (2.2)) is therefore defined by a (6, 22) lattice factorized as

$$Z_{6,22} \rightarrow Z_{4,20} Z_{2,2}. \tag{4.10}$$

Again, this formal connection between an $N = 2$ model with shifted lattice and an $N = 4$ model can be made more concrete by observing that they do have a common six-dimensional limit. Indeed, either in $\lambda = 0$ or in $\lambda = 1$ shifted sums, there is a

¹⁰For this lattice sum I, this limit is $T_2 \rightarrow 0, U_2 = 1$.

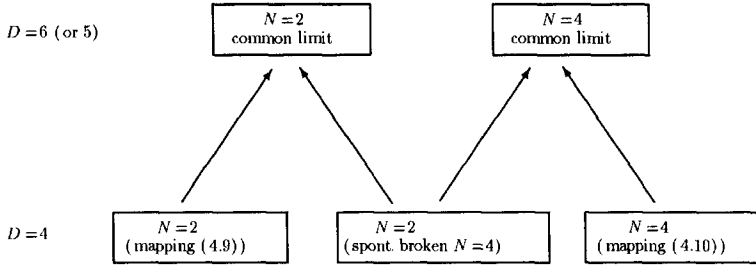


Fig. 1. Decompactification scheme of generic models with $\lambda = 0$ shifted $(2, 2)$ lattice.

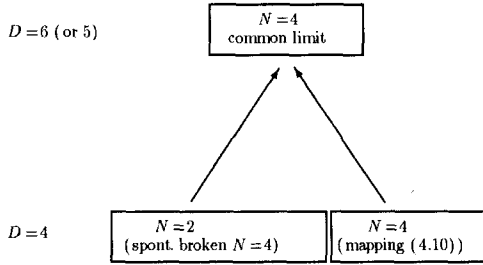


Fig. 2. Decompactification scheme of models with $\lambda = 1$ shifted $(2, 2)$ lattice.

decompactification limit¹¹ where only $\Gamma_{2,2}^{w, [0]} \equiv \Gamma_{2,2}$ survives, thereby selecting the $N = 4$ sector of the model (2.7). Thus the original $N = 2$ ground state and the $N = 4$ ground state obtained by using the above mapping are identical in that limit, provided the (T, U) moduli are appropriately rescaled in order to re-absorb the factor $1/2$ present in (2.7) (for the lattices I and X, for example, we must perform $T \rightarrow 2T$ in the $N = 2$ model).

The previous observations show that $N = 4$ supersymmetry is restored in some appropriate six-dimensional decompactification limit of the $N = 2$ model built up with shifted lattices. This is a manifestation of the underlying Scherk–Schwarz mechanism responsible for the spontaneous breaking of the $N = 4$ supersymmetry. The same conclusion can be reached by analysing the behaviour of the $(T$ - and U -dependent) mass of the two gravitinos (see Refs. [2,21]). For $\lambda = 0$ ground states, there are two inequivalent limits in the (T, U) moduli where the masses of both gravitinos either vanish or become infinite. These two limits, when $N = 4$ supersymmetry is and is not restored, coincide respectively with the limits of some ordinary (i.e. with factorized two-torus) $N = 4$ and $N = 2$ models. When the shift vector of the $(2, 2)$ lattice is of the type $\lambda = 1$, the mass gap of two gravitinos always vanishes at the decompactification limit, and the $N = 4$ supersymmetry is always restored in six dimensions.

We have summarized the above results in Figs. 1 and 2. As a final remark, we would like to mention another possibility that can appear in $N = 2$ ground states constructed

¹¹ These limits are $T_2 \rightarrow \infty$ and $U_2 = 1$ for both models I ($\lambda = 0$) and X ($\lambda = 1$) (see Eqs. (A.15), (A.19) and (A.20)).

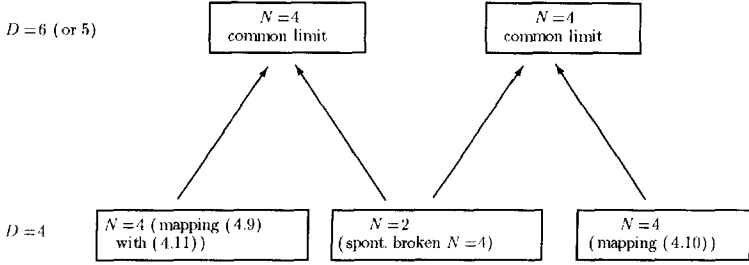


Fig. 3. Decompactification scheme for models with $\lambda = 0$ shifted $(2, 2)$ lattice of class (ii).

with $(2, 2)$ lattices such that the shift vector satisfies $\lambda = 0$. As we will see in Section 5, it can happen that the ordinary $N = 2$ model obtained through the mapping (4.9) possesses the following property:

$$\sum_{h,g=0}^1 C_{4,20}^{\lambda=0} \left[\begin{matrix} h \\ g \end{matrix} \right] \left(\frac{U}{2} \right) \xrightarrow{v \rightarrow 0} 0. \tag{4.11}$$

The $N = 2$ supersymmetry of the latter model is actually promoted to $N = 4$ (see discussion at the end of Section 3.2), therefore leading to the picture summarized in Fig. 3. Ground states that possess the property (4.11) will be referred to as belonging to class (ii), whereas generic $\lambda = 0$ models (Fig. 1) will be of class (i).

5. Thresholds in models with spontaneously broken $N = 4$ supersymmetry

Our starting point is now (2.4), (2.7). In that case (3.7) and (3.8) read respectively¹²

$$F_i^w = \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] F_i^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \tag{5.1}$$

and

$$F_{\text{grav}}^w = \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] F_{\text{grav}}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right], \tag{5.2}$$

where we focused, as previously, on the corrections to gauge couplings corresponding to the rank-20 factors of the gauge group. We have also introduced

$$F_i^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] = -\frac{1}{\bar{\eta}^{24}} \left(\bar{P}_i^2 - \frac{k_i}{4\pi\tau_2} \right) \bar{\Omega}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \tag{5.3}$$

and

$$F_{\text{grav}}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] = -\frac{1}{\bar{\eta}^{24}} \left(\frac{\bar{E}_2}{12} - \frac{1}{4\pi\tau_2} \right) \bar{\Omega}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right], \tag{5.4}$$

¹² The prime summation over (h, g) stands for $(h, g) = \{(0, 1), (1, 0), (1, 1)\}$.

where

$$\overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} = -\frac{1}{4} \bar{\eta}^{20} C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \Big|_{v=0} \tag{5.5}$$

(notice that $\overline{\Omega}^\lambda \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ vanishes).¹³ By using (5.4), one can recast (5.3) in the form

$$F_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix} = k_i F_{\text{grav}}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} + \overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix}, \tag{5.6}$$

which involves the functions

$$\overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix} = -\frac{1}{\bar{\eta}^{24}} \left(\overline{P}_i^2 - k_i \frac{\overline{E}_2}{12} \right) \overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}. \tag{5.7}$$

The functions $\overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ are antiholomorphic for the same reason that $\overline{\Omega}$ in Eq. (3.12) is antiholomorphic in the case of $N = 2$ models that are toroidal compactifications of six-dimensional $N = 1$ theories; the same holds therefore for $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \Big|_{v=0}$, as we advertised in Section 4, as well as for $\overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$. The modular-transformation properties of these functions are (see (4.6) and (4.7)):

$$\tau \rightarrow \tau + 1, \quad \overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{-i\pi\lambda \frac{h^2}{2}} \overline{\Omega}^\lambda \begin{bmatrix} h \\ h+g \end{bmatrix}, \quad \overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{-i\pi\lambda \frac{h^2}{2}} \overline{A}_i^\lambda \begin{bmatrix} h \\ h+g \end{bmatrix}, \tag{5.8}$$

$$\tau \rightarrow -\frac{1}{\tau}, \quad \overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow \bar{\tau}^{10} e^{i\pi\lambda hg} \overline{\Omega}^\lambda \begin{bmatrix} g \\ -h \end{bmatrix}, \quad \overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{i\pi\lambda hg} \overline{A}_i^\lambda \begin{bmatrix} g \\ -h \end{bmatrix}. \tag{5.9}$$

These transformation properties together with the singularity structure inside the τ -plane allow us to determine the most general functions $\overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ that could be obtained starting from any consistent ground state of the type (2.4), (2.7). They turn out to depend on several discrete Wilson lines that appear in the functions $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \Big|_{v=0}$, but most of the model and moduli dependence present in $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$ is lost.¹⁴ In the following, we will present this analysis for the relevant cases ($\lambda = 0$ and $\lambda = 1$). We will show how these functions $\overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ can indeed be realized, and eventually evaluate $\overline{A}_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$. We will therefore determine completely F_i^v in terms of several physical parameters of the model, among which the beta-function coefficients b_i , much as we have reached (3.14) for $N = 2$ ground states with a factorized two-torus.

In order to make the subsequent analysis more transparent, we introduce the functions

$$F_i^{\lambda(\pm)} = F_i^\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm F_i^\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

¹³ In the case of orbifold models, the functions $\overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ are given in (B.1) in terms of the twisted lattices $I_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$.

¹⁴ Similarly to Section 3.2, the knowledge of $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ at $v = 0$ does not enable us to reconstruct the full functions $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$.

and similarly for $F_{\text{grav}}^{\lambda(\pm)}$. The gauge and gravitational functions (5.1) and (5.2) now read

$$F_i^w = \Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_i^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \Gamma_{2,2}^{w(+)} F_i^{\lambda(+)} + \Gamma_{2,2}^{w(-)} F_i^{\lambda(-)} \quad (5.10)$$

and

$$F_{\text{grav}}^w = \Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} F_{\text{grav}}^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \Gamma_{2,2}^{w(+)} F_{\text{grav}}^{\lambda(+)} + \Gamma_{2,2}^{w(-)} F_{\text{grav}}^{\lambda(-)}, \quad (5.11)$$

respectively ($\Gamma_{2,2}^{w(\pm)}$ is given in (A.10)).

5.1. The case $\lambda = 0$

The threshold corrections in this case have been calculated in [21] for a specific model that corresponds to the Scherk–Schwarz version of the symmetric \mathbb{Z}_2 orbifold. Here we will present the results for the general case (see (2.7)), when $\lambda = 0$.

The simplest way to derive the most general Ω^λ 's for a given value of λ is to extend the results of a particular model. In the case $\lambda = 0$, one could choose the symmetric \mathbb{Z}_2 orbifold of [21]. However, for simplicity, we shall consider the $E_8 \times E_8 \times SO(8) \times U(1)^2$ model presented in Appendix B, which leads to the functions $\Omega_{(0)}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ given in Eq. (B.6). A natural generalization of these functions is given by

$$\begin{aligned} \Omega^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= g(1-x) \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \Omega^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= g(x) \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \Omega^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= g\left(\frac{x}{x-1}\right) \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned} \quad (5.12)$$

where $x \equiv (\vartheta_2/\vartheta_3)^4$. The function $g(x)$ must satisfy the constraints

$$g(x) = g\left(\frac{1}{x}\right),$$

required for modular covariance (see Eqs. (5.8) and (5.9)). Unitarity now demands that $\Omega^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ have a regular expansion without poles inside the fundamental domain, except at $\tau = i\infty$. It follows that $g(x)$ can have poles at $x = 0, 1$ as well as at the roots of $\Omega^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. More details on the geometry and singularities on the three-punctured sphere can be found in [34]. Putting everything together, we obtain

$$g(x) = \xi_1 \frac{x^2}{(x^2 - x + 1)^2} + \xi_2 \frac{x}{x^2 - x + 1} + \xi_3. \quad (5.13)$$

In this representation, the $E_8 \times E_8 \times SO(8) \times U(1)^2$ ground state of Appendix B corresponds therefore to $\xi_1 = \xi_2 = 0$ and $\xi_3 = 1$. It is clear from Eqs. (5.12) and (5.13) that

$$\begin{aligned}
 \Omega^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{2} (\vartheta_3^4 + \vartheta_4^4) (\xi_1 \vartheta_3^8 \vartheta_4^8 + \xi_2 \vartheta_3^4 \vartheta_4^4 E_4 + \xi_3 E_4^2) , \\
 \Omega^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\xi_1 \vartheta_2^8 \vartheta_3^8 + \xi_2 \vartheta_2^4 \vartheta_3^4 E_4 + \xi_3 E_4^2) , \\
 \Omega^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{2} (\vartheta_2^4 - \vartheta_4^4) (\xi_1 \vartheta_2^8 \vartheta_4^8 - \xi_2 \vartheta_2^4 \vartheta_4^4 E_4 + \xi_3 E_4^2) ,
 \end{aligned} \tag{5.14}$$

which satisfy

$$\sum'_{(h,g)} \Omega^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = (\xi_1 + \xi_2) E_4 E_6 . \tag{5.15}$$

The above result deserves a discussion. The functions $\Omega^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ (the same actually holds in the case $\lambda = 1$) depend on three parameters only: $(\xi_1, \xi_2, \xi_3) \equiv \xi$, which, as we will see very soon, are subject to several constraints and can take only some discrete values. These parameters exhaust all moduli dependence of $\Omega^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ and define a kind of universality classes. Consequently, despite the model dependence of $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} (v/2)$, which is a priori a function of a large number of moduli, $C_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} |_{v=0}$ (see Eq. (5.5)) is almost universal.

Our goal is to compute threshold corrections for the gauge couplings. We must therefore determine the full gauge function F_i^w (Eqs. (5.1) and (5.6)), which implies the computation of the functions $A_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$, following (5.7). In the general case, however, it is difficult to proceed in this way, because the gauge group is unknown and so is the action of the covariant derivative. Thus we shall follow a different method. It consists in writing down the most general functions $A_i^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ compatible with modular covariance, unitarity, etc., as we did for the Ω 's, and in determining the various free parameters that appear in those functions in terms of some low-energy physical quantities (i.e. related to the massless spectrum) such as the beta-function coefficients and the affine-Lie-algebra levels. This is exactly the method that was used in order to reach (3.14) in the case of $N = 2$ models that are toroidal compactifications from six to four dimensions. As a corollary of our analysis, the universality-class vector ξ will also be expressed in terms of physical parameters of the ground state.

We can find the most general functions $A_i^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ by following the same lines of thought as for the determination of the general Ω 's given in (5.12) and (5.13):

$$\begin{aligned}
 A_i^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= f_i^{\lambda=0} (1 - x) A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \\
 A_i^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= f_i^{\lambda=0} (x) A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \\
 A_i^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= f_i^{\lambda=0} \left(\frac{x}{x-1} \right) A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} ;
 \end{aligned} \tag{5.16}$$

here $f_i^\lambda(x)$ is the ratio of two polynomials of x , satisfying $f_i^\lambda(x) = f_i^\lambda(1/x)$, and $A_{(0)E_8}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ are given in Eq. (B.7). Following arguments similar to the ones advocated for

$g(x)$, we obtain

$$f_i^{\Lambda=0}(x) = \frac{A_i(x^6 + 1) + B_i x^2(x^2 + 1) + C_i x(x^4 + 1) + D_i x^3}{(x^2 - x + 1)(x + 1)^2(x - 2)(2x - 1)}. \quad (5.17)$$

The determination of the constants A_i, B_i, C_i, D_i necessitates the introduction of several constraints, which involve various physical parameters. The relevant quantities for this analysis are $F_{\text{grav}}^{\Lambda=0} [g]$ and $F_i^{\Lambda=0} [g]$ (Eqs. (5.4) and (5.6)). By using Eqs. (5.12), (5.13), (5.16) and (5.17) in Eqs. (5.4) and (5.6), we obtain explicit expressions for $F_{\text{grav}}^{\Lambda=0} [g]$ and $F_i^{\Lambda=0} [g]$, which we can further expand in powers of \bar{q} . The results for these expansions are summarized in Eqs. (C.1)–(C.6). In order to determine the various parameters (namely A_i, B_i, C_i, D_i and ξ) appearing in these expressions in terms of low-energy quantities related to the model, we proceed as follows.

We first observe that the tachyon, being the lowest-lying state, is not charged and, therefore, cannot contribute to the gauge function (5.10). Taking into account the structure of the shifted lattice, namely the fact that the lattice sum $\Gamma_{2,2}^w [1]^{[0]}$ is always of the form $\Gamma_{2,2}^w [1]^{[0]} = 1 + \dots$, whereas $\Gamma_{2,2}^{w(\pm)}$ never contain the unity (see Appendix A, Eqs. (A.11)–(A.14)), we conclude that the coefficient of the $1/\bar{q}$ term in $F_i^{\Lambda=0} [1]^{[0]}$ of (C.1) must be zero. Moreover, as we notice in Appendix A, for any w , there is always a corner in the (T, U) moduli space where $\Gamma_{2,2}^w [g]^{[h]}$ are equal for all (h, g) ¹⁵ (up to exponentially suppressed terms). In that limit, the $\frac{1}{\bar{q}}$ -pole present in $F_i^{\Lambda=0(+)}$ of (C.2) contributes as does the $\frac{1}{\bar{q}}$ -pole of $F_i^{\Lambda=0} [1]^{[0]}$ of (C.1), and its coefficient must then be set equal to zero.

We now turn to constraints originated from the identification of the beta-function coefficients, according to (3.5). For *generic values of the two-torus moduli*, the only contribution comes from the constant term of $F_i^{\Lambda=0} [1]^{[0]}$, which has therefore to be identified with b_i . However, it is important to observe that there are regions of the (T, U) moduli space where *extra charged massless states* (vector multiplets and/or hypermultiplets) appear and contribute to the beta-function coefficients, which therefore become $b_i \rightarrow b_i + \delta b_i$; those must in particular be considered in expressions such as (3.3) in order to properly determine the thresholds. This enlargement of the massless spectrum can occur at the decompactification limit, where $\Gamma_{2,2}^w [g]^{[h]}$ become equal for all (h, g) (see Eq. (A.16) for the case $\Gamma_{2,2}^l [g]^{[h]}$). In this case only hypermultiplets might become massless (the gauge symmetry remains unchanged). The extra contribution to the beta-function coefficients will be denoted $\delta_i b_i$, and will be identified with the constant term of $F_i^{\Lambda=0(+)}$ of (C.2). On the other hand, along the line $T = f_h^w(U)$, extra vector multiplets and/or hypermultiplets become massless. This enhancement of the massless spectrum is originated from the $(2, 2)$ lattice, as is clear from Eqs. (A.14) and (C.3); we will have $\delta_h b_i$, which has to be identified with twice the coefficient of the $1/\sqrt{\bar{q}}$

¹⁵ For example, for a shift vector w corresponding to model I in Table A.1, this happens indeed in the limit $T_2 \rightarrow 0, U_2 = 1$, as is clear from (A.16).

term of $F_i^{\lambda=0(-)}$ in (C.3).¹⁶

Finally, there are two constraints that are obtained by inspecting the gravitational function (5.11). The latter receives a tachyon contribution. At generic values of the moduli (T, U) , this contribution is given by the coefficient of the $1/\bar{q}$ term in $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ of (C.4), which must then be equal to $-1/12$ in our normalizations. Furthermore, *the gravitational anomaly at generic values of the two-torus moduli*, b_{grav} , has to be identified with the constant term of $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (see Eq. (3.10) and the structure of the shifted lattice sums). We will come back later to the discontinuities $b_{\text{grav}} \rightarrow b_{\text{grav}} + \delta b_{\text{grav}}$ that occur along special lines.

The seven constraints obtained so far are summarized in Appendix C, where we solve them in order to express the parameters A_i, B_i, C_i, D_i and ξ in terms of $b_i, \delta_v b_i, \delta_h b_i$ and b_{grav} . By inspecting expressions (C.12), (C.13) and (C.14), which give ξ , we can draw a straightforward conclusion: the parameters $b_i, \delta_v b_i$ and $\delta_h b_i$ are related in the sense that the combination $2b_i - 12\delta_h b_i - \delta_v b_i$ is necessarily proportional to k_i and the latter captures the whole group-factor dependence. As we will see in the following, in the framework of $\lambda = 0$ models, the constant of proportionality can be determined in terms of b_{grav} only.

Before we turn to the determination of the threshold corrections, several comments related to the above analysis are in order.

We first observe that in the models under consideration, where the two-torus undergoes a shift leading to a spontaneous breaking of $N = 4$ supersymmetry down to $N = 2$, there is room for discontinuities in the beta-function coefficients. This phenomenon, as we pointed out in Section 3, does not occur in ground states where a two-torus is factorized. Here it occurs along the line $T = f_h^w(U)$, where extra massless states appear, charged under the rank-20 component of the gauge group.

The beta-function coefficients also suffer from discontinuity at the decompactification limit where all $F_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ become equal (see (A.16) and left-hand part of Fig. 1).¹⁷ This is specific to $\lambda = 0$ lattices and, as explained in Section 4.2, in this limit, the model at hand becomes identical to an $N = 2$ model with factorized two-torus obtained through the mapping (4.9) (see Fig. 1); therefore $b_i + \delta_v b_i$ is to be identified with the beta-function coefficient of the latter model (called \tilde{b}_i in Ref. [21]).

More information about this $N = 2$ ground state with factorized two-torus, sharing a common limit with our $N = 4$ $\lambda = 0$ model, can be obtained by analysing the corresponding functions F_{grav} and F_i . According to the mapping (4.9) these read

¹⁶ Note that at some isolated points of this line, as explained in Appendix A, the lattice multiplicity doubles and the beta-function coefficients become $b_i + 2\delta_h b_i$ instead of $b_i + \delta_h b_i$ which is their value at a generic point along $T = f_h^w(U)$.

¹⁷ We also have a trivial discontinuity in the other decompactification limit, namely (A.15) depicted in the right-hand part of Fig. 1. In that limit $N = 4$ supersymmetry is restored and the full functions F_i^w and F_{grav}^w vanish as a consequence of the exponential suppression of the lattice sums (see Eqs. (5.1) and (5.2)). So do the beta functions and the gravitational anomaly. This discontinuity, however, does not introduce any further physical parameter. The same phenomenon actually appears in the unique decompactification limit (A.19) of $\lambda = 1$ models (see Fig. 2).

$$F_{\text{grav}} = \Gamma_{2,2} \sum'_{(h,g)} F_{\text{grav}}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix},$$

$$F_i = k_i F_{\text{grav}} + \Gamma_{2,2} \sum'_{(h,g)} \bar{A}_i^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}.$$

By using the solutions (C.8)–(C.13) of Appendix C, we can compute $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ as well as $\bar{A}_i^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$, and, thanks to (5.15), perform the summation with the result (we keep here the explicit dependence with respect to ξ_3)

$$F_{\text{grav}} = -\frac{(1-\xi_3)}{12} \Gamma_{2,2} \frac{\widehat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} \quad (5.18)$$

and

$$F_i = -\frac{k_i(1-\xi_3)}{12} \Gamma_{2,2} \left(\frac{\widehat{E}_2 \bar{E}_4 \bar{E}_6}{\bar{\eta}^{24}} - \bar{j} + 1008 \right) + (b_i + \delta_i b_i) \Gamma_{2,2}, \quad (5.19)$$

which can be compared to the results for $N = 2$ models with factorized T^2 (Eq. (3.14)). Eq. (5.18) shows that in order for those limiting models to possess the correct tachyon contribution (normalization of the $\frac{1}{q}$ -pole in F_{grav}), the original $\lambda = 0$ models have either

$$\xi_3 = \begin{cases} 0 & \text{class (i),} \\ 1 & \text{class (ii);} \end{cases} \quad (5.20)$$

we refer to the classes of $\lambda = 0$ models introduced in Section 4.2.

The models in the first class remain genuine $N = 2$ in the limit under consideration, whereas those in class (ii) actually become $N = 4$ models: F_{grav} vanishes and F_i must also vanish, implying $\delta_i b_i = -b_i$. In this class the mapping (4.9) actually satisfies (4.11), and we are in the situation represented in Fig. 3. By using Eq. (C.14), we can recast (5.20) in terms of physical parameters, which therefore satisfy

$$\begin{cases} 4b_i - 24\delta_h b_i - 2\delta_i b_i = 9k_i (b_{\text{grav}} - 6), & \text{class (i),} \\ \delta_i b_i = -b_i, \quad 2b_i - 8\delta_h b_i = 3k_i (b_{\text{grav}} + 2), & \text{class (ii).} \end{cases} \quad (5.21)$$

These relations show that there always exists in the string ground states considered here, a combination of physical, gauge-group-dependent parameters (such as b_i , $\delta_i b_i$ and $\delta_h b_i$), which depends only on the level of the affine Lie algebras. As we will see in the following, this implies that there is no unambiguous way of defining a group-factor-independent threshold correction Y as was the case in $N = 2$ models with a factorized two-torus (see Eq. (3.15)).

Discontinuities like those discussed above also occur in the gravitational anomaly. The expansions (C.4), (C.5) and (C.6) of $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$, together with the enhancement properties of the massless spectrum and the large- or small-radius behaviours of the lattice sums $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ (see Eqs. (A.11)–(A.16)) show that there are several possibilities.

In the limit that we have just analysed, where all $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ become equal (limit (A.16) for shifted lattice I), the gravitational anomaly acquires an extra piece $\delta_i b_{\text{grav}}$, which is

the constant term of (C.5). Eqs. (C.12), (C.13), (C.14) and (5.21) then allow us to recast this discontinuity as

$$\delta_t b_{\text{grav}} = \begin{cases} 22 - b_{\text{grav}}, & \text{class (i),} \\ -b_{\text{grav}}, & \text{class (ii),} \end{cases} \quad (5.22)$$

where it appears, as expected, that the limiting ground states of class (i) are $N = 2$ ground states with gravitational anomaly 22, whereas for class (ii) we reach $N = 4$ models with vanishing gravitational anomaly.

Along the line $T = f_h^w(U)$, as can be seen from Eqs. (A.14) and (C.6), another discontinuity appears, $\delta_h b_{\text{grav}}$, which has to be identified with twice the coefficient of the $1/\sqrt{q}$ term of $F_{\text{grav}}^{\lambda=0(-)}$; it can be expressed as

$$\delta_h b_{\text{grav}} = \frac{1}{24} \begin{cases} 2 - 3b_{\text{grav}}, & \text{class (i),} \\ -30 - 3b_{\text{grav}}, & \text{class (ii).} \end{cases} \quad (5.23)$$

Furthermore, in the case of the gravitational anomaly, extra discontinuities appear along the rational lines $T = U$ and $T = -1/U$ (see Eqs. (A.11) and (A.12)), which play no role in the beta-function coefficients of the rank-20 component of the gauge group ($\delta_t b_i = \delta'_t b_i = 0$), because of the absence of tachyonic contribution in F_i . The same phenomenon also occurs in the ordinary $N = 2$ models, as discussed at the end of Section 3.2, although in that case the lines $T = U$ and $T = -1/U$ are equivalent as a consequence of the $SL(2, \mathbb{Z})_T$. Here the $\frac{1}{q}$ -pole of $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (C.4) leads to

$$\delta_t b_{\text{grav}} = -\frac{\delta_t^w}{6}, \quad \text{at } T = U \quad (5.24)$$

and

$$\delta'_t b_{\text{grav}} = -\frac{\delta_t^w}{6}, \quad \text{at } T = -\frac{1}{U}, \quad (5.25)$$

where

$$\delta_t^w = (-)^{a_1 - b_1}, \quad (5.26)$$

which becomes $(-)^{a_1 - b_1} + (-)^{a_2 - b_2}$ when $T = U = i$, or

$$(-)^{a_1 - b_1} + (-)^{a_2 - b_2} ((-)^{a_1} + (-)^{b_1})$$

if $T = U = \rho$ or $-1/\rho$, and

$$\delta_t^w = (-)^{a_2 - b_2}, \quad (5.27)$$

which becomes $(-)^{a_2 - b_2} + (-)^{a_1 - b_1} ((-)^{a_2} + (-)^{b_2})$ if $T = -1/U = \rho$ or $-1/\rho$. As we will see in Section 6, both vector multiplets and hypermultiplets (uncharged under the rank-20 component of the gauge group) can in general become massless along these lines, whereas in $N = 2$ models with a factorized two-torus, only vectors appear (see Eq. (3.18)). Therefore, *symmetry is not necessarily enhanced*.

Finally, in ground states belonging to class (ii), i.e. when $\xi_3 = 1$, we also have

$$\delta'_v b_{\text{grav}} = \frac{1}{6} \quad (5.28)$$

at the line $T = f'_v(U)$, as is clear from Eqs. (C.5) and (A.13). Notice, however, that along this line $\delta'_v b_i = 0$ because of the absence of $\frac{1}{q}$ -pole in $F_i^{\lambda=0(+)}$ of (C.2). This was one of the physical constraints imposed above, namely the absence of charged tachyon anywhere in moduli space.

We now come to the computation of the threshold corrections. Collecting the results (5.12)–(5.17) and (C.8)–(C.14) in Eqs. (5.1)–(5.6), we can recast Eq. (3.3), for generic values of T and U , as was advertised in the introduction:

$$\Delta_i^w = b_i \Delta^w(T, U) + \delta_h b_i H^w(T, U) + \delta_v b_i V^w(T, U) + k_i Y^w(T, U), \quad (5.29)$$

where

$$\begin{aligned} \Delta^w(T, U) = & \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\mathcal{O}}_{(0)}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{\delta}_g^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] + \overline{\Lambda}_{(0)i}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{\delta}_f^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \right) - 1 \right) \\ & + \log \frac{2}{\pi} \frac{e^{1-\gamma}}{\sqrt{27}}, \end{aligned} \quad (5.30)$$

$$H^w(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\mathcal{O}}_{(0)}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{h}_g^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] + \overline{\Lambda}_{(0)i}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{h}_f^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \right), \quad (5.31)$$

$$V^w(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\mathcal{O}}_{(0)}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{v}_g^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] + \overline{\Lambda}_{(0)i}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{v}_f^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \right), \quad (5.32)$$

$$Y^w(T, U) = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\mathcal{O}}_{(0)}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{y}_g^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] + \overline{\Lambda}_{(0)i}^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \overline{y}_f^\lambda \left[\begin{matrix} h \\ g \end{matrix} \right] \right). \quad (5.33)$$

The functions $\delta_{g,f}^{\lambda=0}$, $h_{g,f}^{\lambda=0}$, $v_{g,f}^{\lambda=0}$ and $y_{g,f}^{\lambda=0}$ appearing in the above integrals are given in Eqs. (C.15). The integrals themselves can be evaluated by unfolding the fundamental domain and reducing the summations over the modular group orbits (the modular group now being reduced as explained in Appendix A) in the spirit of Refs. [9,15]. Some preliminary results were given in [12,21]. More complete formulas are presented in Appendix D.

The functions $\Delta^w(T, U)$ and $Y^w(T, U)$ defined in (5.30) and (5.33) are finite all over the two-torus moduli space. However, the function $H^w(T, U)$ becomes singular along the line $T = f'_h(U)$, because of the extra constant contribution of the integrand in (5.31), which originates from extra massless states that lead to a logarithmic divergence (see Eqs. (D.31)). Along this line, we must therefore substitute in Eq. (5.29)

$$\begin{aligned}
 H^w(T, U) &\rightarrow H^w(f_h^w(U), U) \\
 &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (f_h^w(U), U) \right. \\
 &\quad \times \left. \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \bar{h}_g^\lambda \begin{bmatrix} h \\ g \end{bmatrix} + \overline{A}_{(0)i}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \bar{h}_f^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \right) - 1 \right) \\
 &\quad + \log \frac{2}{\pi} \frac{e^{1-\gamma}}{\sqrt{27}}, \tag{5.34}
 \end{aligned}$$

which accounts for the extra massless states by subtracting the contribution $\delta_h b_i$. Eq. (5.29) now leads to the correct thresholds.

The function $V^w(T, U)$ remains finite in the bulk of moduli space. On the other hand, it develops an extra logarithmic singularity in the decompactification limit (A.16), where all $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ become equal. Then, by formally substituting

$$\begin{aligned}
 V^w(T, U) &\rightarrow V^w(T_{\text{lim}}, U_{\text{lim}}) \\
 &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (T_{\text{lim}}, U_{\text{lim}}) \right. \\
 &\quad \times \left. \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\Omega}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} \bar{v}_g^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} + \overline{A}_{(0)i}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} \bar{v}_f^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} \right) - 1 \right) \\
 &\quad + \log \frac{2}{\pi} \frac{e^{1-\gamma}}{\sqrt{27}}, \tag{5.35}
 \end{aligned}$$

which again regularizes the extra massless contributions, the threshold (5.29) matches in this limit (see Eqs. (5.18) and (5.19)) the ordinary $N = 2$ thresholds

$$\Delta_i = (b_i + \delta_v b_i) \Delta - k_i (1 - \xi_3) Y, \tag{5.36}$$

where Δ and Y are given respectively in (3.16) and (3.17). For $T_2 \rightarrow 0$ and $U_2 = 1$ the latter behaves as (see Refs. [17,20])

$$\Delta_i \rightarrow (b_i + \delta_v b_i) \left(\frac{\pi}{3} \frac{1}{T_2} + \log T_2 - \log 4\pi^2 |\eta(i)|^4 \right) - k_i (1 - \xi_3) \left(\frac{4\pi}{T_2} + 20 \kappa T_2 \right), \tag{5.37}$$

up to exponentially suppressed terms (κ is given in (D.30)). For class-(i) models, the dominant behaviour is linear with respect to the volume of the decompactifying manifold. If the model under consideration belongs instead to class (ii), the matching (5.36) shows that Δ_i^w vanish, which reflects the restoration of $N = 4$ supersymmetry.

Actually, substitution of (5.35) is formal in the sense that the function $V^w(T_{\text{lim}}, U_{\text{lim}})$ defined in (5.35) is divergent everywhere except for the limiting value $T_{\text{lim}}, U_{\text{lim}}$ (i.e. $T_2 \rightarrow 0, U_2 = 1$ for the lattice I). Without this substitution, Δ_i^w given in (5.29) with

V^w given in (5.32) possesses, for the models of class (ii), a logarithmic divergence in the limit considered here: $\Delta_i^l \sim -\delta_i b_i \log T_2 = b_i \log T_2$ (more details about those limits, including subleading terms, can be found in Appendix D). Although the $N = 4$ supersymmetry is restored, the accumulation of extra massless states not properly regularized in the infra-red (one subtracts b_i in the integrals and not $b_i + \delta_i b_i$ as one would to follow the formal prescription (5.35)) leads to that logarithmic behaviour. A proper treatment of this infra-red divergence necessitates the introduction of Wilson lines as explained in [21].

Finally, as we have already pointed out several times, in the $\lambda = 0$ models, there is another decompactification limit ($T_2 \rightarrow \infty, U_2 = 1$ in models with shifted lattice I) where $N = 4$ supersymmetry is always restored. In that limit, Δ_i^w given in (5.29) diverges logarithmically (e.g. $\Delta_i^l \sim -b_i \log T_2$), which is the same infra-red phenomenon as appeared in the previous case.

Our last comment about the gauge corrections Δ_i^w concerns the group-factor-independent thresholds $Y^w(T, U)$ appearing in the decomposition (5.29). In $N = 2$ models with a factorized T^2 (see Section 3.2), the decomposition (3.15) is unique because there is no a priori relation between b_i and k_i . Moreover, Y (see (3.17)) is absolutely model independent, and this is also a consequence of anomaly cancellations in six dimensions. However, in ground states with spontaneously broken $N = 4$ supersymmetry under consideration, the various physical parameters appearing in the decomposition (5.29) are not independent. They are related through (5.21). We have therefore the freedom of adding to $Y^w(T, U)$, defined in (5.33), any function that is regular everywhere in the moduli space, invariant under the relevant duality group, and properly behaved in the decompactification limits; then, by using (5.21), we compensate the other functions Δ^w, H^w and V^w without disturbing the decomposition in terms of $b_i, \delta_h b_i, \delta_v b_i$ and k_i . This arbitrariness cannot be reduced unless $Y^w(T, U)$ is related to some other physical quantities such as the one-loop correction to the Kähler potential in the spirit of Ref. [20], where this was done for ordinary $N = 2$ models. Furthermore, by using Eqs. (5.33) and (C.15), $Y^w(T, U)$ is recast as follows:

$$Y^w = Y_1^w + b_{\text{grav}} Y_2^w, \quad (5.38)$$

which shows that some *model dependence*, captured in the parameter b_{grav} and the shift vector w , is now left in the gauge-factor-independent threshold. Notice that the freedom in the decomposition (5.29) makes it possible to discard either Y_1^w or Y_2^w , which are both regular everywhere in the moduli space.

By repeating the above steps, we can proceed to the determination of the gravitational corrections (3.9). At *generic points* of the two-torus moduli space, they read

$$\Delta_{\text{grav}}^w = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(-\frac{1}{12} \sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\Omega}_{(0)}^\Lambda \begin{bmatrix} h \\ g \end{bmatrix} \bar{g} \begin{bmatrix} h \\ g \end{bmatrix} - b_{\text{grav}} \right), \quad (5.39)$$

where $g \begin{bmatrix} h \\ g \end{bmatrix}$ are given in (5.13). This threshold is not model independent as it was in ground states with a factorized two-torus (see (3.19)), where anomaly cancellation in

six dimensions was advocated. Its model dependence is captured in the parameters ξ , appearing in $g^{[h]}_g$, that can be expressed in terms of b_{grav} by using (C.12)–(C.14), together with the result (5.21):

$$\begin{cases} \xi_1 = \frac{31}{32} + \frac{3}{64} b_{\text{grav}}, & \xi_2 = \frac{1}{32} - \frac{3}{64} b_{\text{grav}}, & \xi_3 = 0, & \text{class (i),} \\ \xi_1 = \frac{63}{32} + \frac{3}{64} b_{\text{grav}}, & \xi_2 = -\frac{63}{32} - \frac{3}{64} b_{\text{grav}}, & \xi_3 = 1, & \text{class (ii).} \end{cases} \quad (5.40)$$

Since the gravitational anomaly (see (6.2)) is related to the number of massless vector multiplets and hypermultiplets, it is clear from the above expressions that the parameters ξ can only take discrete rational values, as advertised previously. This defines several universality classes for the gravitational thresholds, which eventually read

$$A_{\text{grav}}^w = A_{\text{grav},1}^w + b_{\text{grav}} A_{\text{grav},2}^w. \quad (5.41)$$

The actual expressions for $A_{\text{grav},1}^w$ and $A_{\text{grav},2}^w$ depend on whether the model belongs to the class (i) or (ii), but are universal within each of the two classes where they turn out to depend only on the shift vector w .

The thresholds (5.39) diverge logarithmically along the lines $T = U$, $T = -1/U$, $T = f_h^w(U)$ and $T = f_v^w(U)$ (see Eqs. (D.31)), where the subtraction of b_{grav} does not account for all massless contributions. The exact thresholds are obtained by replacing b_{grav} with the actual value of the gravitational anomaly at the considered line (see Eqs. (5.22)–(5.25)). In the decompactification limit (A.16), where an ordinary $N = 2$ model is matched, the gravitational thresholds diverge linearly for the class (i) and logarithmically for the class (ii) since, then, $N = 4$ is actually restored. In the decompactification limit (A.15), where $N = 4$ supersymmetry is systematically restored, the divergence is always logarithmic. All these behaviours are summarized at the end of Appendix D.

5.2. The case $\lambda = 1$

We now turn to the determination of threshold corrections in models where the shifted lattice $\Gamma_{2,2}^w [g]$ is of the type $\lambda = 1$. We must therefore determine the functions $\Omega^{\lambda=1} [g]$ and $A_i^{\lambda=1} [g]$ appearing in $F_{\text{grav}}^{\lambda=1} [g]$ and $F_i^{\lambda=1} [g]$ (Eqs. (5.4) and (5.6)). Our starting point will now be the model $E_8 \times E_8 \times U(1)^2$ of Appendix B. We construct the generalized $\Omega^{\lambda=1} [g]$ and $A_i^{\lambda=1} [g]$ in a way similar to what was done in the previous section, using the $\Omega_{(0)}^{\lambda=1} [g]$ and $A_{(0)i}^{\lambda=1} [g]$ of (B.9) and (B.10), and Eqs. (5.12) and (5.16) with $\lambda = 1$ instead of $\lambda = 0$. Repeating the steps of the $\lambda = 0$ calculation we find that $g(x)$ is again given by (5.13), which translates into

$$\begin{aligned} \Omega^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \vartheta_3^2 \vartheta_4^2 (\xi_1 \vartheta_3^8 \vartheta_4^8 + \xi_2 \vartheta_3^4 \vartheta_4^4 E_4 + \xi_3 E_4^2), \\ \Omega^{\lambda=1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\vartheta_2^2 \vartheta_3^2 (\xi_1 \vartheta_2^8 \vartheta_3^8 + \xi_2 \vartheta_2^4 \vartheta_3^4 E_4 + \xi_3 E_4^2), \end{aligned}$$

$$\Omega^{\lambda=1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\vartheta_2^2 \vartheta_4^2 (\xi_1 \vartheta_2^8 \vartheta_4^8 - \xi_2 \vartheta_2^4 \vartheta_4^4 E_4 + \xi_3 E_4^2); \quad (5.42)$$

we also find that

$$f_i^{\lambda=1}(x) = \frac{A_i(x^4 - x^3 + x^2 - x + 1) + B_i x(x^2 - x + 1) + C_i x^2}{(x^2 - x + 1)(x - 2)(2x - 1)}. \quad (5.43)$$

The determination of the constants A_i, B_i, C_i as well as of the parameters ξ of (5.13) in terms of physical parameters can be carried out as in the previous case. We first expand $F_i^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix}$ and $F_{\text{grav}}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix}$ (see Appendix C). By taking into account the structure of the $\lambda = 1$ shifted lattice as explained in Appendix A (see Eqs. (A.11), (A.12), (A.17) and (A.18)), we can identify the coefficients of the various negative or zero powers of \bar{q} in expressions (C.16)–(C.21) with the physical parameters.

The absence of charged tachyon requires the vanishing of the $1/\bar{q}$ term in $F_i^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The constant term of $F_i^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ must be identified with the beta-function coefficient at generic moduli b_i . The $1/\bar{q}^{3/4}$ term in $F_i^{\lambda=1(+)}$ plays a role along the line $T = f_v^w(U)$ where extra states become massless. Twice its coefficient will be therefore identified with $\delta_v b_i$. Similarly, twice the coefficient of the $1/\bar{q}^{1/4}$ term in $F_i^{\lambda=1(-)}$ will be called $\delta_h b_i$, which represents the discontinuity of the beta-function coefficient along the line $T = f_h^w(U)$.¹⁸ In contrast to what happens in $\lambda = 0$ models, the two discontinuities of the beta-function coefficients arise at finite values of the moduli, where in general either vector multiplets and/or hypermultiplets appear. Remember that for $\lambda = 0$, this happens only for $\delta_h b_i$, since $\delta_v b_i$ was the discontinuity at the $N = 2$ limit of the moduli space. Here $N = 4$ supersymmetry is restored in both limits (see Fig. 2), with vanishing of all beta-function coefficients and gravitational anomalies, and without any new physical parameter.

Two more constraints are needed in order to determine all the above parameters. These are obtained by inspecting the expansion of $F_{\text{grav}}^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (Eq. (C.19)). Normalization of the tachyon contribution in the gravitational corrections imposes the residue of the pole to be $-1/12$. Furthermore, the constant term is the gravitational anomaly.

The above constraints lead to six Eqs. (C.22), the solution of which is given in (C.23)–(C.28). As we already mentioned in the $\lambda = 0$ case, we observe that the combination $b_i - 2\delta_h b_i - 8\delta_v b_i$ is necessarily proportional to k_i . In contrast to the $\lambda = 0$ case (see Eq. (5.21)), however, the proportionality constant cannot be expressed in terms of b_{grav} only. It necessitates the introduction of a new parameter, although not independent, such as the discontinuity of the gravitational anomaly along the line $T = f_v^w(U)$, $\delta_v b_{\text{grav}}$, which is twice the coefficient of the $1/\bar{q}^{3/4}$ term in $F_{\text{grav}}^{\lambda=1(+)}$ (C.20). We therefore find the relation

$$b_i - 2\delta_h b_i - 8\delta_v b_i = 3k_i(16\delta_v b_{\text{grav}} + b_{\text{grav}} - 2). \quad (5.44)$$

¹⁸Note again that at some isolated points of the lines $T = f_v^w(U)$ or $T = f_h^w(U)$, the multiplicity of the relevant terms in the lattice sums can double and the beta-function coefficients become $b_i + 2\delta_v b_i$ or $b_i + 2\delta_h b_i$.

Note that the gravitational anomaly possesses another discontinuity $\delta_h b_{\text{grav}}$ along the $T = f_h^w(U)$, which can be read off from $F_{\text{grav}}^{\lambda=1(-)}$ (C.21) and expressed in terms of other physical parameters by using (C.26)–(C.28) and (5.44):

$$\delta_h b_{\text{grav}} = -28 \delta_v b_{\text{grav}} - b_{\text{grav}} + \frac{10}{3}. \tag{5.45}$$

Further discontinuities $\delta_t b_{\text{grav}}$ and $\delta'_t b_{\text{grav}}$ arise along $T = U$ and $T = -1/U$, respectively, due to the tachyon pole of $F_{\text{grav}}^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ combined with the $\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ lattice sum. These are the same as those appearing in the $\lambda = 0$ case, and are given in (5.24) and (5.25).

The computation of the threshold corrections goes on as in the $\lambda = 0$ situation. The gauge corrections can be decomposed as in (5.29) with all (T, U) -dependent functions given in (5.30)–(5.33) and $\delta_{g,f}^{\lambda=1}, \dots, y_{g,f}^{\lambda=1}$ displayed in (C.29). Again due to the relation (5.44), the definition of the group-factor-independent contribution $Y^w(T, U)$ is not unique, although it is taken to be regular everywhere; it is also model dependent through b_{grav} , and can be decomposed as in Eq. (5.38) (see (C.29)).

Singularities appear at $T = f_h^w(U)$, where $H^w(T, U)$ exhibits a logarithmic behaviour (see Eqs. (D.32)). This can be cured on the line $T = f_h^w(U)$ by properly subtracting the full contribution of the massless states, i.e. by performing the substitution (5.34). The same phenomenon occurs across $T = f_v^w(U)$, where $V^w(T, U)$ diverges and where the substitution

$$\begin{aligned} V^w(T, U) &\rightarrow V^w(f_v^w(U), U) \\ &= \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (f_v^w(U), U) \right. \\ &\quad \times \left(-\frac{1}{12} \frac{\widehat{E}_2}{\widehat{\eta}^{24}} \overline{\Omega}_{(0)}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} \overline{v}_g^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} + \overline{A}_{(0)i}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} \overline{v}_f^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} \right) - 1 \Big) \\ &\quad + \log \frac{2 e^{1-\gamma}}{\pi \sqrt{27}} \end{aligned} \tag{5.46}$$

is compulsory in order for the thresholds Δ_i^w to make sense. Finally, at the limits (A.19) (see Fig. 2), where the $N = 4$ supersymmetry is restored and $F_i^{\lambda=1}$ vanishes, the thresholds (5.29) diverge logarithmically (e.g. $\Delta_i^X \sim \pm b_i \log T_2$, the minus sign corresponding to the large- T_2 limit, see Appendix D); this is, as described previously, the consequence of an incomplete infra-red regularization.

For the models at hand, gravitational corrections are still given in (5.39), where the parameters ξ appearing in $g \begin{bmatrix} h \\ g \end{bmatrix}$ (see Eq. (5.13)) are now taken in (C.26)–(C.28), which, thanks to (5.44), are expressed in the more convenient way

$$\xi_1 = \frac{3}{4} \delta_v b_{\text{grav}} + \frac{3}{64} b_{\text{grav}} + \frac{27}{32}, \quad \xi_2 = -\frac{3}{2} \delta_v b_{\text{grav}} - \frac{3}{64} b_{\text{grav}} + \frac{5}{32}, \quad \xi_3 = \frac{3}{4} \delta_v b_{\text{grav}}. \tag{5.47}$$

Here also we observe that only rational values are allowed for these parameters. However, the decomposition (5.41) must now be replaced with

$$\Delta_{\text{grav}}^w = \Delta_{\text{grav},1}^w + b_{\text{grav}} \Delta_{\text{grav},2}^w + \delta_v b_{\text{grav}} \Delta_{\text{grav},3}^w, \quad (5.48)$$

where $\Delta_{\text{grav},i}^w$ $i = 1, 2, 3$ are universal (only shift-vector-dependent) functions. As far as the singularity of the corrections is concerned, the same comments as before apply here; details can be found in Appendix D.

6. The case of orbifolds

6.1. Some general results

In Sections 4 and 5 we described a general class of heterotic constructions with spontaneously broken $N = 4$ supersymmetry, for which we computed the gravitational and gauge threshold corrections. Those turn out to depend on moduli (T, U) as well as on several low-energy parameters of the model, such as beta-function coefficients and the gravitational anomaly and their discontinuities across rational lines or on the border of the moduli space. The gravitational thresholds, in particular, depend on a very specific combination of these low-energy data, namely on ξ (see (5.40) or (5.47)). The latter are discrete Wilson lines and are the only parameters entering the elliptic genus (see (5.5), (5.14) and (5.42)), which therefore exhibits the universality properties that we already discussed in Section 5.

The above parameters (or equivalently the elliptic genus) do not contain enough information to reconstruct all properties of the massless spectrum, such as the number of vector multiplets and hypermultiplets. Only the differences $N_V - N_H$ or $\delta N_V - \delta N_H$ can be determined through b_{grav} or δb_{grav} . However, if we restrict ourselves to the subclass of the \mathbb{Z}_2 orbifolds, more information can be reached. Indeed, for these models the function $C_{4,20}^{\lambda, [h]}(v/2)$ is given in (4.5), and it is possible to explicitly compute the helicity supertrace B_4 . We can then extract the massless part of the latter, and identify it with the low-energy formula, which reads

$$B_4^0 = \frac{62 + 7N_V - N_H}{4}, \quad (6.1)$$

for heterotic ground states. By using the low-energy expression for the gravitational anomaly,

$$b_{\text{grav}} = \frac{22 - N_V + N_H}{12}, \quad (6.2)$$

we can determine N_V and N_H as well as the discontinuities of these numbers all over the moduli space.

The helicity supertrace $B_4 = \langle (\lambda + \bar{\lambda})^4 \rangle$ is obtained at one loop by acting on $Z(v, \bar{v})$ (Eqs. (2.4), (2.7) and (4.5)) with $\frac{1}{16\pi^4} (\partial_v - \partial_{\bar{v}})^4$ at $v = \bar{v} = 0$. After some algebra (details can be found in Ref. [24]), we find

$$B_4 = \frac{3}{4} \frac{\Gamma_{2,2} \Gamma_{4,20}}{\bar{\eta}^{24}} + \frac{1}{2} \sum'_{(h,g)} \Gamma_{2,2}^w \left[\begin{matrix} h \\ g \end{matrix} \right] \left(\frac{H_{[g]}^{[h]}}{2} + 2 - \bar{E}_2 \right) \frac{\bar{\Omega}^{\lambda} [g]}{\bar{\eta}^{24}}, \quad (6.3)$$

where

$$H \begin{bmatrix} h \\ g \end{bmatrix} = \frac{12}{\pi i} \partial_\tau \log \frac{\vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}}{\eta} = \begin{cases} \vartheta_3^4 + \vartheta_4^4 & (h, g) = (0, 1), \\ -\vartheta_2^4 - \vartheta_3^4 & (h, g) = (1, 0), \\ \vartheta_2^4 - \vartheta_4^4 & (h, g) = (1, 1). \end{cases}$$

Expression (6.3) is valid for \mathbb{Z}_2 -orbifold constructions with spontaneously broken $N = 4$ to $N = 2$ supersymmetry. The second term of (6.3) results from $N = 2$ sectors and possesses the same universality properties as those described previously for the gravitational threshold corrections: the model and moduli dependence has shrunk to (T, U) and ξ ; put differently, this term depends on the elliptic genus only. However, the first term of B_4 , originated from the $N = 4$ sector, spoils this universality: as expected from general considerations, it introduces a full dependence on the various moduli of $\Gamma_{4,20}$.

Let us now concentrate on the massless contributions to B_4 . By using the full machinery introduced so far, we can compute B_4^0 in terms of ξ , and use (5.40) or (5.47) to trade the latter for the parameter b_{grav} . We find

$$B_4^0 = \frac{3}{2} N_I + 54 - 12 b_{\text{grav}}. \tag{6.4}$$

This formula is valid for any shift vector w ($\lambda = 0$ or 1), at any *generic* point of the (T, U) moduli space. We also assumed that, at generic values of the other moduli,¹⁹ we have the behaviour: $\Gamma_{4,20} = 1 + 2N_I \bar{q} + \dots$. This introduces a new parameter of the orbifold, which captures all the extra moduli dependence at the level of the massless spectrum. By using (6.1) and (6.2) we can recast (6.4) as $N_V + N_H = 22 + 2N_I$, where N_V and N_H are the number of vector multiplets and hypermultiplets *at generic* (T, U) moduli. In turn, these are given by

$$N_V = 22 + N_I - 6b_{\text{grav}}, \tag{6.5}$$

$$N_H = N_I + 6b_{\text{grav}}. \tag{6.6}$$

We can go further and describe the behaviour of B_4^0 across rational lines in the (T, U) moduli space. This can be achieved thanks to the various expansions and limits introduced in Appendices A and C for lattices and the gravitational functions $F_{\text{grav}}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$. Notice that the actual value of N_I plays no role in this analysis.

(a) *The case $\lambda = 0$*

- The decompactification limit where $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ become equal for all (h, g) .

In this limit, present for any shift vector w , the massless spectrum is enhanced and discontinuities $\delta_i b_i$ and $\delta_i b_{\text{grav}}$ appear. In the situation (i) (see Section 4.2), namely when $\xi_3 = 0$, this limit is also shared by an $N = 2$ model with factorized two-torus

¹⁹ For truly generic values of the 80 moduli of $\Gamma_{4,20}$, we would have $\Gamma_{4,20} = 1 +$ non-integer values of q and \bar{q} . However, orbifold constructions often necessitate some of the moduli to be fixed at specific values.

(mapping (4.9), see Fig. 1), which has beta-function coefficients $\tilde{b}_i = b_i + \delta_v b_i$, and gravitational anomaly $\tilde{b}_{\text{grav}} = b_{\text{grav}} + \delta_v b_{\text{grav}}$. We find

$$\delta_v B_4^0 = -3\delta_v b_{\text{grav}} .$$

This result can be recast in the form (see (5.22), (6.1) and (6.2))

$$\delta_v N_V = 0, \quad \delta_v N_H = 264 - 2b_{\text{grav}} .$$

The number of vector multiplets remains constant, as was already stressed in the general case (see Section 5); only extra hypermultiplets appear.

When $\xi_3 = 1$ (class (ii)), in the limit under consideration, the model matches an $N = 4$ orbifold with a factorized two-torus obtained through the mapping (4.9) with (4.11) (see Fig. 3). We obtain now

$$\delta_v B_4^0 = -18 - 3\delta_v b_{\text{grav}} .$$

In the limit at hand, $N = 4$ supersymmetry is restored and the number of $N = 2$ vector multiplets and hypermultiplets has no longer any meaning. Instead the number of $N = 4$ vectors makes sense and turns out to be equal to the number of $N = 2$ vector multiplets present before reaching the $N = 4$ limit, as can be seen from the result

$$\tilde{B}_4^0 = B_4^0 + \delta_v B_4^0 = 3 + \frac{3}{2}N_V ,$$

N_V given in (6.5). In other words, the massless spectrum is reshuffled in such a way that the gauge group remains unchanged, as it should, when $N = 4$ supersymmetry is restored.

- The line $T = f_h^w(U)$.

Here we find for both situations (i) and (ii)

$$\delta_h B_4^0 = -3\delta_h b_{\text{grav}} ,$$

which leads to (see (5.23), (6.1) and (6.2))

$$\delta_h N_V = 0, \quad \delta_h N_H = \begin{cases} -\frac{3}{2}b_{\text{grav}} + 1, & \text{class (i),} \\ -\frac{3}{2}b_{\text{grav}} - 15, & \text{class (ii).} \end{cases}$$

Now the absence of extra vectors is specific to orbifolds.

- The line $T = f_v^w(U)$.

For models of class (i) no extra massless states appear along this line. In the case (ii), however, although the beta-function coefficients (of the rank-20 factor of the gauge group) remain unchanged and the gauge thresholds are regular, extra massless states appear since the gravitational anomaly has a discontinuity (see (5.28)). Similarly

$$\delta_v' B_4^0 = -\frac{1}{2} ,$$

and consequently

$$\delta_v' N_V = 0, \quad \delta_v' N_H = 2 .$$

The extra hypermultiplets are singlets under the rank-20 component of the gauge group.

(b) *The case $\lambda = 1$*

In this case $N = 4$ supersymmetry is restored in all decompactification limits, and there is not much to say about them. The interesting phenomena occur along the lines $T = f_h^w(U)$ and $T = f_v^w(U)$, where we find

$$\delta_{h,v} B_4^0 = -3\delta_{h,v} b_{\text{grav}} ,$$

leading to

$$\delta_{h,v} N_V = 0 , \quad \delta_{h,v} N_H = 12 \delta_{h,v} b_{\text{grav}}$$

(remember that in the situation $\lambda = 1$, $\delta_v b_{\text{grav}}$ is considered as an input parameter together with b_i , $\delta_v b_i$, $\delta_h b_i$, and b_{grav} , whereas $\delta_h b_{\text{grav}}$ is related to the others through (5.45)). Again, the absence of extra vectors is not to be considered as a generic feature of the class of models analysed in this paper, but instead as a property of the orbifold constructions.

(c) *The lines $T = U$ and $T = -U^{-1}$*

These deserve a special treatment, because they appear both in models with spontaneously broken $N = 4$ supersymmetry and in ordinary $N = 2$ models with a factorized two-torus (although in the latter they are equivalent). In all cases the beta-function coefficients (of the rank-20 factor of the gauge group) remain unchanged, $\delta_t b_i = \delta'_t b_i = 0$, and the corresponding gauge threshold corrections are regular. This is a consequence of the absence of any charged tachyon. The gravitational anomaly, however, receives extra contributions. For the models with a factorized two-torus, this is given in (3.18), and accounts for the appearance of 2, 4 or 6 extra vector multiplets: the $U(1)^2$ factor of the two-torus becomes $U(1) \times SU(2)$, $SU(2) \times SU(2)$ or $SU(3)$ at $T = U$, $T = U = i$ or $T = U = \rho$.

In the case of models with spontaneously broken $N = 4$ supersymmetry we are considering here, the discontinuities of the gravitational anomaly are given in (5.24) and (5.25). Moreover, for orbifold constructions, using (6.3) we find

$$\delta_t B_4^0 = \frac{3}{2} \delta - 12 \delta_t b_{\text{grav}} , \tag{6.7}$$

which leads to

$$\delta_t N_V = \delta + \delta_t^w , \quad \delta_t N_H = \delta - \delta_t^w \tag{6.8}$$

for the line $T = U$, and similarly for the line $T = -1/U$ with δ_t^w replaced with $\delta_t'^w$. (δ , δ_t^w and $\delta_t'^w$ are defined in Eqs. (3.18), (5.26) and (5.27), respectively). We observe that not only extra vector multiplets, but also extra hypermultiplets, singlets under the rank-20 component of the gauge group, appear when $T = U$ or $T = -1/U$ are reached. Since $\delta_t^w \neq \delta_t'^w$, the spectrum of extra massless states is different along the two lines under consideration, which translates the breaking of the duality group. Moreover, it is

determined exclusively by the shift vector w and does not depend on any low-energy parameter of the model (as, for instance, b_{grav}).

6.2. Examples

(a) The case $\lambda = 0$, class (i)

These models fit to the scheme presented in Fig. 1. They can be seen as deformations of the known six-dimensional \mathbb{Z}_2 orbifolds. This perspective has been adopted in [21], where a model was constructed as a deformation of the symmetric \mathbb{Z}_2 orbifold. This will be our first example. Following this procedure, more orbifold models can be constructed. The number of possible models in this class is equal to the number of ordinary \mathbb{Z}_2 $N = 2$ orbifolds. They are related by the mapping (4.9).

- $E_8 \times E_7 \times SU(2) \times U(1)^2$.

In the notation used here this model can be recovered with the lattice

$$\Gamma_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \bar{\vartheta}^6 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a} - h \\ \bar{b} - g \end{bmatrix} \bar{E}_4, \tag{6.9}$$

where

$$Z_{4,4} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \equiv Z_{4,4} \equiv \frac{\Gamma_{4,4}}{|\eta|^8} \tag{6.10}$$

is the partition function of four compactified bosons, which depends on 16 moduli while, for $(h, g) \neq (0, 0)$,

$$Z_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \equiv \frac{\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^8} = \frac{16 |\eta|^4}{\left| \vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix} \right|^2} \tag{6.11}$$

are the ordinary \mathbb{Z}_2 -twisted contributions. Here $N_I = 240$. One can use Eqs. (B.1) to determine the corresponding functions $\Omega_{[g]}^{[h]}$; after comparison with (5.14), the latter give $\xi = (0, 1, 0)$. This determines the universality class of the model, and in particular $b_{\text{grav}} = -62/3$, $\delta_v b_{\text{grav}} = 128/3$ and $\delta_h b_{\text{grav}} = 8/3$. We find $N_V = 386$ and $N_H = 116$, in agreement with the gauge group and the matter massless spectrum, which is $(\mathbf{1}, \mathbf{56}, \mathbf{2}) + 4 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})$, with $b_{E_8} = -60$, $b_{E_7} = -12$ and $b_{SU(2)} = 52$. We also find $\delta_v N_H = 512$ extra hypermultiplets in the $N = 2$ decompactification limit, originated from the twisted sector, and falling in $8 \times (\mathbf{1}, \mathbf{56}, \mathbf{1}) + 32 \times (\mathbf{1}, \mathbf{1}, \mathbf{2})$. They lead to $\delta_v b_{E_8} = 0$, $\delta_v b_{E_7} = 96$ and $\delta_v b_{SU(2)} = 32$. Finally, we find $\delta_h N_H = 32$ extra hypermultiplets originating from the twisted sector in $16 \times (\mathbf{1}, \mathbf{1}, \mathbf{2})$, with $\delta_h b_{E_8} = 0$, $\delta_h b_{E_7} = 0$ and $\delta_h b_{SU(2)} = 16$. Eq. (5.21) is verified by the above low-energy parameters.

- $SO(12) \times SO(20) \times U(1)^2$.

In this case the $(4, 20)$ lattice sum is

$$\Gamma_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \bar{\vartheta}^6 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \bar{\vartheta}^5 \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix} \bar{\vartheta}^5 \begin{bmatrix} \bar{a} - h \\ \bar{b} - g \end{bmatrix}. \quad (6.12)$$

Now $N_\Gamma = 240$, $\xi = (1, 0, 0)$; $b_{\text{grav}} = 2/3$, $\delta_\nu b_{\text{grav}} = 64/3$ and $\delta_h b_{\text{grav}} = 0$. We find $N_V = 258$ and $N_H = 244$, in agreement with the gauge group and the matter massless spectrum, which is here $(\mathbf{12}, \mathbf{20}) + 4 \times (\mathbf{1}, \mathbf{1})$, with $b_{SO(12)} = 20$ and $b_{SO(20)} = -12$. We also find $\delta_\nu N_H = 256$ extra hypermultiplets falling in $8 \times (\mathbf{32}, \mathbf{1})$. They lead to $\delta_\nu b_{SO(12)} = 64$ and $\delta_\nu b_{SO(20)} = 0$. Finally, we find $\delta_h N_H = 0$, so that nothing happens along $T = f_h^w(U)$ ($\delta_h N_V$ vanishes for all orbifold models). All low-energy parameters are consistent with Eq. (5.21).

(b) The case $\lambda = 0$, class (ii)

Let us now proceed to present two models where $N = 4$ supersymmetry is restored in both six-dimensional limits (see Fig. 3). This is related to the fact that they are constructed by a pure shift in the $\Gamma_{2,2}$ lattice and no twist action in the $\Gamma_{4,4}$ lattice. Since in both limits the shift action is effectively removed, and no twist action is present, supersymmetry is always restored to $N = 4$.

- $E_8 \times E_8 \times SO(8) \times U(1)^2$.

The simplest model is the one presented in Appendix B. It corresponds to the lattice (B.5). Notice that it is quite remarkable to find an $E_8 \times E_8$ factor together with $N = 2$ supersymmetry in a four-dimensional construction. It has $N_\Gamma = 252$, $\xi = (0, 0, 1)$; $b_{\text{grav}} = -42$, $\delta_\nu b_{\text{grav}} = 42$ and $\delta_h b_{\text{grav}} = 4$. We find $N_V = 526$ and $N_H = 0$. There is no matter here and $b_{E_8} = -60$, $b_{SO(8)} = -12$. We have $\delta_\nu b_{E_8} = 60$, $\delta_\nu b_{SO(8)} = 12$ because of the $N = 4$ restoration. We also find $\delta_h N_H = 48$ extra hypermultiplets along the line $T = f_h^w(U)$ falling in $6 \times (\mathbf{1}, \mathbf{1}, \mathbf{8})$. They lead to $\delta_h b_{E_8} = 0$ and $\delta_h b_{SO(8)} = 12$, in agreement with Eq. (5.21).

- $SO(40) \times U(1)^2$.

This model is obtained with

$$\Gamma_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \vartheta^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \vartheta \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix} \vartheta \begin{bmatrix} \bar{a} - h \\ \bar{b} - g \end{bmatrix} \bar{\vartheta}^{20} \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix}. \quad (6.13)$$

It has the largest single group factor that can be obtained in this construction. We find $N_\Gamma = 380$, $\xi = (-1, 1, 1)$; $b_{\text{grav}} = -190/3$, $\delta_\nu b_{\text{grav}} = 190/3$ and $\delta_h b_{\text{grav}} = 20/3$. There is no matter here, $N_V = 782$ and $b_{SO(40)} = -76$, $\delta_\nu b_{SO(40)} = 76$. Now $\delta_h N_H = 80$ extra hypermultiplets appear along the line $T = f_h^w(U)$ falling in $2 \times (\mathbf{40})$. They lead to $\delta_h b_{SO(40)} = 4$, in agreement with Eq. (5.21).

The enlargement of the gauge group in the last two models is also a result of the absence of twist, which allows the $\Gamma_{4,4}$ right-moving fermions to get gauged.

(c) *The case $\lambda = 1$*

As explained in Section 5, $\lambda = 1$ models fall in the situation depicted in Fig. 2. The two six-dimensional limits are equivalent (which was not true in the previous case) and therefore restore the $N = 4$ supersymmetry since the orbifold twist is removed. We will give three examples, one of these being the celebrated E_8 level 2 construction.

- $E_8 \times E_8 \times U(1)^2$.

This is the simplest model of this category and is the one presented in Appendix B. The $(4, 20)$ lattice is given in (B.8). It has $N_I = 240$, $\xi = (0, 0, 1)$; $b_{\text{grav}} = -118/3$, $\delta_v b_{\text{grav}} = 4/3$ and $\delta_h b_{\text{grav}} = 16/3$. We find $N_V = 498$, and $N_H = 4$ hypermultiplets, which are singlets of the non-Abelian gauge group factor. The beta-function coefficient is $b_{E_8} = -60$. Moreover, along the rational lines $T = f_v^w(U)$ and $T = f_h^w(U)$, we find $\delta_v N_H = 16$ and $\delta_h N_H = 64$ extra hypermultiplets, singlets of the non-Abelian gauge group factor, leading to $\delta_v b_{E_8} = 0$ and $\delta_h b_{E_8} = 0$. This is in agreement with Eq. (5.44).

- $SO(16) \times SO(16) \times U(1)^2$.

Another model can be obtained by using the lattice

$$\Gamma_{4,20}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \bar{\vartheta}^8 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \bar{\vartheta}^8 \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix}. \tag{6.14}$$

We now have $N_I = 240$, $\xi = (1, 0, 0)$, $b_{\text{grav}} = 10/3$, $\delta_v b_{\text{grav}} = 0$ and $\delta_h b_{\text{grav}} = 0$. We find $N_V = 242$ and $N_H = 260$, in agreement with the gauge group and the matter massless spectrum, which is here $(\mathbf{16}, \mathbf{16}) + 4 \times (\mathbf{1}, \mathbf{1})$, with $b_{SO(16)} = 4$. Finally $\delta_{v,h} N_H = 0$ and $\delta_{v,h} b_{SO(16)} = 0$, in agreement with Eq. (5.44).

- $E_8 \times U(1)^2$.

This model has recently attracted much attention in the framework of heterotic/type II dual-pair construction [35]. On the $\Gamma_{0,16}$ lattice, the \mathbb{Z}_2 permutes the two E_8 's. A single E_8 current algebra survives, which is realized at level 2. Eventually, the $(4, 20)$ lattice sum reads

$$\Gamma_{4,20}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{E_{8|2}} \begin{bmatrix} h \\ g \end{bmatrix}, \tag{6.15}$$

where

$$\Gamma_{E_{8|2}} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \bar{E}_4(\bar{\tau}), \quad \Gamma_{E_{8|2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \bar{E}_4(2\bar{\tau}) \frac{\bar{\eta}^{16}(\bar{\tau})}{\bar{\eta}^8(2\bar{\tau})}, \tag{6.16}$$

and $\Gamma_{E_{8|2}} \begin{bmatrix} 1 \\ 0 \text{ or } 1 \end{bmatrix}$ are obtained by the modular transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\tau^{-1}$.

In the model at hand, $N_I = 240$, $\xi = (15/16, 1/16, 0)$; $b_{\text{grav}} = 2$, $\delta_v b_{\text{grav}} = 0$ and $\delta_h b_{\text{grav}} = 4/3$. We find $N_V = 250$, and $N_H = 252$ hypermultiplets, which are in $(\mathbf{248}) + 4 \times (\mathbf{1})$. The beta-function coefficient is $b_{E_8} = 0$. Moreover, along the rational line $T = f_v^w(U)$, $\delta_v N_H = 0$ and consequently $\delta_v b_{E_8} = 0$. On the other hand, $\delta_h N_H = 16$ hypermultiplets appear at $T = f_h^w(U)$ and, being singlets of E_8 , give $\delta_h b_{E_8} = 0$. Again, beta-function coefficients fulfill Eq. (5.44).

7. Conclusions

In this paper we have analysed threshold corrections to gauge and gravitational couplings in four-dimensional heterotic models where $N = 4$ space-time supersymmetry is spontaneously broken to $N = 2$.

Such ground states can be viewed as obtained by compactifying the ten-dimensional heterotic string on a six-dimensional compact manifold of $SU(2)$ holonomy. This manifold is locally but not globally of the product form $K3 \times T^2$. In these models there are two massive gravitinos, whose masses are calculable functions of the torus moduli. These masses become vanishing, and thus supersymmetry is restored to $N = 4$, in an appropriate decompactification limit. The analysis of the decompactification limits exhibits three subclasses of models ($\lambda = 0$ (i) and (ii), and $\lambda = 1$).

The properties mentioned above are expected to significantly affect the high-energy running of effective coupling constants; this was shown to be true in some sample ground states in [21].

Here we have derived explicit expressions for generic models of the above type without knowledge of their detailed structure. The important ingredients that appear in the expressions for the one-loop gauge and gravitational thresholds are properties of the massless and BPS spectrum; more precisely, beta-function coefficients and affine-Lie-algebra levels, as well as jumps of the beta-functions along submanifolds of the torus moduli space where extra BPS multiplets become massless. In fact, in contrast to what happens in models with a factorized two-torus, several rational lines appear, where the gauge threshold corrections are singular (singularities of the gravitational thresholds appear independently of the factorization of the two-torus). However, these lines do not necessarily correspond to an enhancement of gauge symmetry: δN_V and δN_H are not a priori determined.

We have thus found that the universality properties, observed in $K3 \times T^2$ -like compactifications [19,20] as a consequence of six-dimensional anomaly cancellations, are slightly modified here, although they can still be traced to modular invariance and unitarity, and to the fact that the couplings studied are of the BPS-saturated type [25]. For the gravitational thresholds, the explicit expression exhibits a model dependence, which is captured in the shift vector w and the rational parameters ξ namely b_{grav} (and $\delta_i b_{\text{grav}}$ for $\lambda = 1$). The latter can be interpreted as discrete Wilson lines (or instanton numbers of the \mathbb{Z}_2 -shift embedding), and define the various universality classes where all models under consideration fall. These are genuine classes in the sense that they contain more than a single representative. As far as the gauge threshold corrections are concerned, the usual decomposition in two terms no longer holds. A gauge-factor-independent term can still be defined. However, there is some arbitrariness in its definition due to a relation between the various low-energy parameters involved. Moreover, this term depends explicitly on the value of the gravitational anomaly of the ground state.

By using our expressions for the threshold corrections (for which we have also explicitly performed the integrals over the fundamental domain), we have analysed the behaviour at large radii of compactification. In agreement with the expected supersym-

metry-restoration properties, the thresholds are linearly or logarithmically divergent. In the second case, the $N = 4$ supersymmetry is restored, and the logarithmic divergence is actually an infra-red artefact due to an accumulation of massless states, which can be lifted by switching on appropriate Wilson lines. Indeed, the thresholds should vanish as expected when supersymmetry is extended to $N = 4$.

For generic orbifold constructions falling in our general class of heterotic ground states, the enhancement of the massless spectrum along specific submanifolds of the moduli space can be unambiguously determined. Except for the lines $T = U$ and $T = -1/U$, only hypermultiplets become massless. In the framework of orbifolds, we have also presented several specific constructions, where the gauge group contains factors such as $E_8 \times E_8$, $SO(40)$ or even $E_{8|2}$ (in four dimensions).

The results presented here are a priori applicable to $N = 2$ supersymmetric theories. In fact, they can serve for realistic $N = 1$ models that are orbifolds of the ground states studied in this paper. The internal moduli dependence of the couplings would be coming from $N = 2$ sectors and will thus be given by the expressions we have derived above.

The formalism we developed so far can also be useful for analysing the issue of non-perturbative phenomena in $N = 2$ type II dual models. The extra δN_V and δN_H massless states that appear on the rational lines will then correspond to monopoles or dyons *à la* Seiberg–Witten. Work in this direction will appear soon [36].

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Appendix A. Two-torus lattice sums

In this appendix we give our notation and conventions for the usual and \mathbb{Z}_2 -shifted $(2, 2)$ lattice sums. We also analyse the behaviours of those sums all over the moduli space as well as in various decompactification limits.

A.1. \mathbb{Z}_2 -shifted lattice sums

The $(2, 2)$ lattice sum is given by

$$\Gamma_{2,2}(T, U, \bar{T}, \bar{U}) = \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}} \exp\left(2\pi i \bar{\tau} \mathbf{m} \mathbf{n} - \frac{\pi \tau_2}{T_2 U_2} |T n_1 + T U n_2 + U m_1 - m_2|^2\right). \tag{A.1}$$

It is invariant under the full target-space duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$.

The \mathbb{Z}_2 -shifted lattice sum of the two-torus $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ depends on two integer-valued two-vectors $(\mathbf{a}, \mathbf{b}) \equiv w$. Independently of the shift vector w ,

$$\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 0 \end{bmatrix} \equiv \Gamma_{2,2}, \tag{A.2}$$

given in (A.1); for $(h, g) \neq (0, 0)$, $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ is obtained from $\Gamma_{2,2}$ by inserting $(-1)^{g(na+mb)}$ and shifting $\mathbf{m} \rightarrow \mathbf{m} + \mathbf{a}h/2$ and $\mathbf{n} \rightarrow \mathbf{n} + \mathbf{b}h/2$. There are many choices for the \mathbb{Z}_2 translation on the T^2 . The choice of the vectors \mathbf{a} and \mathbf{b} determines the kind of states (winding and/or momentum) that are projected out by the orbifold. We find

$$\begin{aligned} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} &= \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}} (-1)^{g(na+mb)} \exp\left(2\pi i \bar{\tau} \left(\mathbf{m} + \mathbf{a} \frac{h}{2}\right) \left(\mathbf{n} + \mathbf{b} \frac{h}{2}\right)\right. \\ &\quad \left. - \frac{\pi \tau_2}{T_2 U_2} \left|T \left(n_1 + b_1 \frac{h}{2}\right) + TU \left(n_2 + b_2 \frac{h}{2}\right)\right.\right. \\ &\quad \left.\left. + U \left(m_1 + a_1 \frac{h}{2}\right) - \left(m_2 + a_2 \frac{h}{2}\right)\right|^2\right) \end{aligned} \tag{A.3}$$

in the Hamiltonian representation, or

$$\begin{aligned} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} &= \frac{T_2}{\tau_2} \sum_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}} e^{i\pi \mathbf{a} (n\mathbf{g} - m\mathbf{h} - \mathbf{b} \frac{gh}{2})} \exp\left[-\frac{\pi}{\tau_2} \sum_{i,j} \left[m_i + b_i \frac{g}{2} + \left(n_i + b_i \frac{h}{2}\right) \tau\right]\right. \\ &\quad \left. \times (G_{ij} + B_{ij}) \left[m_j + b_j \frac{g}{2} + \left(n_j + b_j \frac{h}{2}\right) \bar{\tau}\right]\right) \end{aligned} \tag{A.4}$$

in the Lagrangian representation, where as usual

$$G = \frac{T_2}{U_2} \begin{pmatrix} 1 & U_2 \\ U_2 & |U|^2 \end{pmatrix}, \quad B = T_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.5}$$

It is easy to check the periodicity properties (h, g integers)

$$\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{2,2}^w \begin{bmatrix} h+2 \\ g \end{bmatrix} = \Gamma_{2,2}^w \begin{bmatrix} h \\ g+2 \end{bmatrix} = \Gamma_{2,2}^w \begin{bmatrix} -h \\ -g \end{bmatrix}, \tag{A.6}$$

as well as the modular transformations that expression

$$\mathbb{Z}_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \equiv \frac{\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^4} \tag{A.7}$$

obeys

Table A.1
The nine models with $\lambda = 0$

Case	a	b
I	(0, 0)	(1, 0)
II	(0, 0)	(0, 1)
III	(0, 0)	(1, 1)
IV	(1, 0)	(0, 0)
V	(0, 1)	(0, 0)
VI	(1, 1)	(0, 0)
VII	(1, 0)	(0, 1)
VIII	(0, 1)	(1, 0)
IX	(1, -1)	(1, 1)

Table A.2
The six models with $\lambda = 1$

Case	a	b
X	(1, 0)	(1, 0)
XI	(1, 0)	(1, 1)
XII	(1, 1)	(1, 0)
XIII	(0, 1)	(0, 1)
XIV	(0, 1)	(1, 1)
XV	(1, 1)	(0, 1)

$$\tau \rightarrow \tau + 1, \quad Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{i\pi ab \frac{h^2}{2}} Z_{2,2}^w \begin{bmatrix} h \\ h + g \end{bmatrix}, \tag{A.8}$$

$$\tau \rightarrow -\frac{1}{\tau}, \quad Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{-i\pi abhg} Z_{2,2}^w \begin{bmatrix} g \\ -h \end{bmatrix}. \tag{A.9}$$

The relevant parameter for these transformations is $\lambda \equiv ab$.

We would now like to give a few properties of the shifted lattice sums. It is clear from expression (A.3) or (A.4) that the integers a_i and b_i are defined modulo 2, in the sense that adding 2 to anyone of them amounts at most to a change of sign in $F_{2,2}^w \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Such a modification is necessarily compensated by an appropriate one in $C_{4,20}^\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (see Eq. (2.7)) in order to ensure modular invariance, and thus we are left with the same string ground state. On the other hand, adding 2 to a_i or b_i translates into adding a multiple of 2 to λ . Therefore, although λ can be any integer, *only* $\lambda = 0$ and $\lambda = 1$ correspond to truly different situations.

In Tables A.1 and A.2, we list all physically distinct models with $\lambda = 0$ and $\lambda = 1$, respectively. In each of these classes, all the models are related to one another by transformations that belong to $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$.

Another issue that we would like to discuss here is that of target-space duality in the presence of a \mathbb{Z}_2 translation. The moduli dependence of the two-torus shifted sectors

(see Eq. (A.3) or (A.4)) reduces in general the duality group to some subgroup²⁰ of $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$. Transformations that do not belong to this subgroup map a model $w = (\mathbf{a}, \mathbf{b})$ to some other model $w' = (\mathbf{a}', \mathbf{b}')$, leaving however $\lambda = \mathbf{a}\mathbf{b} = \mathbf{a}'\mathbf{b}'$ invariant. This plays an important role in string constructions such as those described in (2.4) with (2.7), where a \mathbb{Z}_2 translation appears, giving a moduli-dependent mass to half of the gravitinos. Indeed, for such a model, decompactification limits that are related by transformations that do not belong to the actual duality group are no longer equivalent. Therefore, the spontaneously broken $N = 4$ supersymmetry might or might not be restored (see Section 4).

To be more specific, by using expression (A.3), we can determine the transformation properties of $\Gamma_{2,2}^{w, [h]} [g]$ under the full group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$:

$$SL(2, \mathbb{Z})_T : \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \rightarrow \begin{pmatrix} d & 0 & 0 & b \\ 0 & d & -b & 0 \\ 0 & -c & a & 0 \\ c & 0 & 0 & a \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}, \quad ad - bc = 1,$$

$$SL(2, \mathbb{Z})_U : \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \rightarrow \begin{pmatrix} a' & -c' & 0 & 0 \\ -b' & d' & 0 & 0 \\ 0 & 0 & d' & b' \\ 0 & 0 & c' & a' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}, \quad d'd' - b'c' = 1$$

and

$$\mathbb{Z}_2^{T \leftrightarrow U} : \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}.$$

Thus, we can determine the duality group for a given model by demanding that the components of the vectors \mathbf{a} and \mathbf{b} remain invariant modulo 2. For example, in the situation I ($\lambda = 0$) defined by $\mathbf{a} = (0, 0)$ and $\mathbf{b} = (1, 0)$, the target-space duality group turns out to be $\Gamma^+(2)_T \times \Gamma^-(2)_U$, whereas for the case X with $\lambda = 1$ and $\mathbf{a} = (1, 0)$, $\mathbf{b} = (1, 0)$, we find $\Gamma(2)_T \times \Gamma(2)_U \times \mathbb{Z}_2^{T \leftrightarrow U}$.

A.2. Rational lines and asymptotic behaviours

Finally, we would like to analyse the behaviour of the shifted lattices over the moduli space. This includes the identification of special lines in the (T, U) -plane, where extra massless states can appear in the spectrum, as well as some large-radius properties. Notice that these special lines are not necessarily lines of enhanced symmetry, since

²⁰ The subgroups of $SL(2, \mathbb{Z})$ that will actually appear in the following are $\Gamma^\pm(2)$ and $\Gamma(2)$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents an element of the modular group, $\Gamma^+(2)$ is defined by a, d odd and b even, while for $\Gamma^-(2)$ we have a, d odd and c even. Their intersection is $\Gamma(2)$.

in some situations only extra hypermultiplets appear. For this analysis we introduce the combinations

$$\Gamma_{2,2}^{w(\pm)} = \frac{1}{2} \left(\Gamma_{2,2}^w \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \Gamma_{2,2}^w \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \quad (\text{A.10})$$

which turn out to be convenient in the computation of the threshold corrections (see Section 5).

(a) *The case* $\lambda = 0$

We focus here on the appearance of $O(\bar{q})$ terms in $\Gamma_{2,2}^{w \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ or $\Gamma_{2,2}^{w(+)}$, and $O(\sqrt{\bar{q}})$ terms in $\Gamma_{2,2}^{w(-)}$. These situations are indeed possible along some specific lines in the moduli space, although they are not simultaneously realized. The results are summarized as follows:

$$\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + (-)^{a_1-b_1} 2\bar{q} + \dots, \quad \text{for } T = U \quad (\text{A.11})$$

$$= 1 + (-)^{a_2-b_2} 2\bar{q} + \dots, \quad \text{for } T = -\frac{1}{U}, \quad (\text{A.12})$$

$$\Gamma_{2,2}^{w(+)} = \dots + 2\bar{q} + \dots, \quad \text{for } T = f_v^w(U), \quad (\text{A.13})$$

$$\Gamma_{2,2}^{w(-)} = \dots + 2\sqrt{\bar{q}} + \dots, \quad \text{for } T = f_h^w(U). \quad (\text{A.14})$$

The lines $T = U$ and $T = -1/U$ are no longer equivalent. In these expressions, the multiplicities are valid for generic points along the indicated lines. They can, however, be modified at some particular values of the moduli. For instance,

$$\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + \left((-)^{a_1-b_1} + (-)^{a_2-b_2} \right) 2\bar{q} + \dots \quad \text{for } T = U = i,$$

$$\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + \left((-)^{a_1-b_1} + (-)^{a_2-b_2} \left((-)^{a_1} + (-)^{b_1} \right) \right) 2\bar{q} + \dots$$

for $T = U = \rho$ or $-1/\rho$,

and

$$\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + \left((-)^{a_2-b_2} + (-)^{a_1-b_1} \left((-)^{a_2} + (-)^{b_2} \right) \right) 2\bar{q} + \dots$$

for $T = -1/U = \rho$ or $-1/\rho$.

On the other hand, the functions $f_v^w(U)$ and $f_h^w(U)$ depend on the particular shift vector w . For concreteness, we concentrate on the particular case I (see Table A.1); any other situation is obtained by duality transformation. In this case, $f_v^I = 4U$ and $f_h^I = 2U$. Moreover, for $\Gamma_{2,2}^{I(+)}$, the multiplicity is doubled at $T = 4U = 1 + i\sqrt{3}$, whereas it is doubled at $T = 2U = 1 + i$ for $\Gamma_{2,2}^{I(-)}$.

Whatever the value of λ , the existence of the above lines $T = f_{v,h}^w(U)$ translates an underlying \mathbb{Z}_2 symmetry of the shifted lattice or of a sublattice of the latter. For example, $\Gamma_{2,2}^{I(-)}$ is invariant under $T \leftrightarrow 2U$, whereas only a sublattice of $\Gamma_{2,2}^{I(+)}$ is invariant under $T \leftrightarrow 4U$; similarly, a sublattice of $\Gamma_{2,2}^{w[0]}$ is invariant under $T \leftrightarrow U$ or $T \leftrightarrow -1/U$.

In contrast to what happens in the case of ordinary lattice sums, the behaviour of the $\lambda = 0$ shifted lattice sums in the decompactification limit depends on whether one considers large or small moduli. This is due to the partial breaking of the duality group. For definiteness, let us focus on model I and consider two six-dimensional limits: $T_2 \rightarrow \infty, U_2 = 1$ (i.e. $R_1 \rightarrow \infty, R_2 \rightarrow \infty^{21}$) on the one hand, and $T_2 \rightarrow 0, U_2 = 1$ (i.e. $R_1 \rightarrow 0, R_2 \rightarrow 0$) on the other. These two limits are mapped onto each other under the combined transformation $T \rightarrow -1/T$ and $U \rightarrow -1/U$, which does not leave model I invariant (it actually gives model IV). Therefore, they are not expected to be equivalent and it is easy to verify that

$$\Gamma_{2,2}^I \begin{bmatrix} h \\ g \end{bmatrix} \xrightarrow{T_2 \rightarrow \infty, U_2 = 1} \begin{cases} T_2/\tau_2 & \text{for } h = g = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{A.15}$$

whereas

$$\Gamma_{2,2}^I \begin{bmatrix} h \\ g \end{bmatrix} \xrightarrow{T_2 \rightarrow 0, U_2 = 1} \frac{1}{T_2 \tau_2} \quad \forall h, g, \tag{A.16}$$

up to exponentially suppressed terms.

Similar conclusions can be reached for other $\lambda = 0$ models by considering the relevant $SL(2, \mathbb{Z})_T \times SU(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$ transformations: *there are always two distinct decompactification limits where either all $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ survive and are equal, or only $\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ survives.*

We would like to emphasize again that the nature of the extra massless states (vector multiplets and hypermultiplets) appearing across the lines $T = U, T = -1/U, T = f_{v,h}^w(U)$ as well as in the two distinct decompactification limits, is not determined by the structure of the shifted lattice only: it depends on the full structure of the string ground state.

(b) The case $\lambda = 1$

In this case, we are interested in terms of order \bar{q} in $\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These are given in (A.11) and (A.12), with the same modifications of their multiplicity at $T = U = i$ and at other special points, as explained above. Moreover, terms of order $\bar{q}^{3/4}$ and $\bar{q}^{1/4}$ are generated in $\Gamma_{2,2}^{w(+)}$ and $\Gamma_{2,2}^{w(-)}$, respectively,

$$\Gamma_{2,2}^{w(+)} = \dots + 2 \bar{q}^{3/4} + \dots, \quad \text{for } T = f_v^w(U) \tag{A.17}$$

$$\Gamma_{2,2}^{w(-)} = \dots + 2 \bar{q}^{1/4} + \dots, \quad \text{for } T = f_h^w(U), \tag{A.18}$$

²¹ Remember that when $T_1 = U_1 = 0, T_2$ and U_2 are parametrized as follows: $U_2 = R_2/R_1$ and $T_2 = R_1 R_2$, where R_1 and R_2 are the radii of compactification.

where for the model X (see Table A.2) $f_v^X = 3U$ or $U/3$ and $f_h^X = U$. Again, the generic multiplicity is 2 and can be promoted to 4 at some particular points on the lines $T = f_{v,h}^w(U)$. Results for other models in Table A.2 are obtained by performing appropriate $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \rightarrow U}$ transformations.

We now turn to the decompactification limit of $\lambda = 1$ models. Let us consider again a specific model, namely model X, in the limits $T_2 \rightarrow \infty, U_2 = 1$ and $T_2 \rightarrow 0, U_2 = 1$. There is a major difference with respect to the $\lambda = 0$ case studied above: *the duality transformation that maps the limits at hand onto each other now leaves the model invariant*. These two limits are therefore equivalent and the $\lambda = 1$ shifted lattice under consideration possesses a unique behaviour, which is

$$\Gamma_{2,2}^X \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow 0 \quad \forall (h, g) \neq (0, 0) \tag{A.19}$$

in both $T_2 \rightarrow \infty, U_2 = 1$ and $T_2 \rightarrow 0, U_2 = 1$ limits, whereas

$$\Gamma_{2,2}^X \begin{bmatrix} 0 \\ 0 \end{bmatrix} \equiv \Gamma_{2,2} \rightarrow \begin{cases} T_2/\tau_2 & \text{for } T_2 \rightarrow \infty, U_2 = 1, \\ 1/T_2\tau_2 & \text{for } T_2 \rightarrow 0, U_2 = 1. \end{cases} \tag{A.20}$$

The same holds for more general $\lambda = 1$ models. There is essentially a *unique decompactification limit where only $\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ survives*.

Appendix B. Two four-dimensional $E_8 \times E_8$ orbifold models

We present here two typical \mathbb{Z}_2 -orbifold models with $N = 4$ supersymmetry broken to $N = 2$ and determine some quantities relevant in Section 5 to the general analysis of the threshold corrections. The partition function for the \mathbb{Z}_2 -orbifold constructions is given in (4.1), which we recall here:

$$\begin{aligned} Z_{\text{sp br}}^{\text{orb}} &= \frac{1}{\tau_2 \eta^{12} \bar{\eta}^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \\ &\quad \times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \Gamma_{4,20}^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}, \end{aligned}$$

where $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ is the shifted two-torus lattice sum (see Eq. (A.3) or (A.4)). For these constructions, we can recast the threshold functions $\bar{\Omega}^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$ defined in (5.5) by using (4.5). We find

$$\begin{aligned} \bar{\Omega}^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{\vartheta_3^2 \vartheta_4^2} \Gamma_{4,20}^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ \bar{\Omega}^\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\frac{1}{\vartheta_2^2 \vartheta_3^2} \Gamma_{4,20}^\lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \bar{\Omega}^\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= -\frac{1}{\vartheta_2^2 \vartheta_4^2} \Gamma_{4,20}^\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{B.1}$$

We also recall the Eisenstein series, which will appear in the following considerations:

$$E_2 = \frac{12}{i\pi} \partial_\tau \log \eta = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \tag{B.2}$$

$$E_4 = \frac{1}{2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \tag{B.3}$$

$$E_6 = \frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_3^4 + \vartheta_4^4) (\vartheta_4^4 - \vartheta_2^4) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \tag{B.4}$$

(a) *The case $\lambda = 0$*

We can choose the following (4, 20) twisted lattice:

$$\Gamma_{4,20}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \vartheta^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \vartheta \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix} \vartheta \begin{bmatrix} \bar{a} - h \\ \bar{b} - g \end{bmatrix} \bar{\vartheta}^4 \begin{bmatrix} \bar{a} + h \\ \bar{b} + g \end{bmatrix} \bar{E}_4^2, \tag{B.5}$$

which leads to an $N = 2$ four-dimensional model with gauge group $E_8 \times E_8 \times SO(8) \times U(1)^2$ with $N_V = 526, N_H = 0$. Using (B.1) we can explicitly determine the Ω 's, which now read

$$\begin{aligned} \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{2} E_4^2 (\vartheta_3^4 + \vartheta_4^4) = -\frac{1}{2} \vartheta_3^{20} (x^2 - x + 1)^2 (x - 2), \\ \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\frac{1}{2} E_4^2 (\vartheta_2^4 + \vartheta_3^4) = -\frac{1}{2} \vartheta_3^{20} (x^2 - x + 1)^2 (x + 1), \\ \Omega_{(0)}^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{2} E_4^2 (\vartheta_2^4 - \vartheta_4^4) = \frac{1}{2} \vartheta_3^{20} (x^2 - x + 1)^2 (2x - 1). \end{aligned} \tag{B.6}$$

We introduced, as previously, the variable $x = (\vartheta_2/\vartheta_3)^4$, which allows us in particular to recast $E_4 = \vartheta_3^8 (x^2 - x + 1)$.

The A 's corresponding to the E_8 factors of the gauge group are determined in a straightforward way, by using Eq. (5.7) as well as the identity [17]

$$-\frac{E_4}{\eta^{24}} \left(P_{E_8}^2 - \frac{E_2}{12} \right) E_4 = \frac{E_4}{E_6} \frac{j - j(i)}{12}.$$

We find

$$\begin{aligned} A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{24} \frac{E_4 E_6}{\eta^{24}} (\vartheta_3^4 + \vartheta_4^4) = -\frac{16}{3} \frac{(x^2 - x + 1)(x + 1)(x - 2)^2 (2x - 1)}{x^2 (x - 1)^2} \\ A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\frac{1}{24} \frac{E_4 E_6}{\eta^{24}} (\vartheta_2^4 + \vartheta_3^4) = -\frac{16}{3} \frac{(x^2 - x + 1)(x + 1)^2 (x - 2)(2x - 1)}{x^2 (x - 1)^2} \\ A_{(0)E_8}^{\lambda=0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \frac{1}{24} \frac{E_4 E_6}{\eta^{24}} (\vartheta_2^4 - \vartheta_4^4) = \frac{16}{3} \frac{(x^2 - x + 1)(x + 1)(x - 2)(2x - 1)^2}{x^2 (x - 1)^2}. \end{aligned} \tag{B.7}$$

(b) The case $\lambda = 1$

Similarly a $\lambda = 1$ model can be obtained with

$$\Gamma_{4,20}^{\lambda=1} \begin{bmatrix} h \\ g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \bar{E}_4^2. \quad (\text{B.8})$$

The gauge group (in a generic point of the $\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ lattice) is now $E_8 \times E_8 \times U(1)^2$ and $N_V = 498$, $N_H = 4$. Following the same procedure as in the previous case, we obtain

$$\begin{aligned} \Omega_{(0)}^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= E_4^2 \vartheta_3^2 \vartheta_4^2 = \vartheta_3^{20} (x^2 - x + 1)^2 \sqrt{1-x}, \\ \Omega_{(0)}^{\lambda=1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -E_4^2 \vartheta_2^2 \vartheta_3^2 = -\vartheta_3^{20} (x^2 - x + 1)^2 \sqrt{x}, \\ \Omega_{(0)}^{\lambda=1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= -E_4^2 \vartheta_2^2 \vartheta_4^2 = -\vartheta_3^{20} (x^2 - x + 1)^2 \sqrt{x(1-x)}, \end{aligned} \quad (\text{B.9})$$

and, for the E_8 factors,

$$\begin{aligned} A_{(0)E_8}^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \frac{1}{12} \frac{E_4 E_6}{\eta^{24}} \vartheta_3^2 \vartheta_4^2 = \frac{32}{3} \frac{(x^2 - x + 1)(x + 1)(x - 2)(2x - 1)\sqrt{1-x}}{x^2(x-1)^2}, \\ A_{(0)E_8}^{\lambda=1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= -\frac{1}{12} \frac{E_4 E_6}{\eta^{24}} \vartheta_2^2 \vartheta_3^2 = -\frac{32}{3} \frac{(x^2 - x + 1)(x + 1)(x - 2)(2x - 1)\sqrt{x}}{x^2(x-1)^2}, \\ A_{(0)E_8}^{\lambda=1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= -\frac{1}{12} \frac{E_4 E_6}{\eta^{24}} \vartheta_2^2 \vartheta_4^2 \\ &= -\frac{32}{3} \frac{(x^2 - x + 1)(x + 1)(x - 2)(2x - 1)\sqrt{x(1-x)}}{x^2(x-1)^2}. \end{aligned} \quad (\text{B.10})$$

Appendix C. Some details on the threshold calculation

In this appendix we collect technicalities that appear in the determination of the threshold corrections (see Section 5) for models with spontaneously broken $N = 4$ supersymmetry, for which the helicity-generating function is given in Eqs. (2.4) and (2.7).

C.1. Models with $\lambda = 0$ shifted lattice

In order to express the constants A_i, B_i, C_i, D_i and ξ , which appear in the functions $F_i^{\lambda=0}$ and $F_{\text{grav}}^{\lambda=0}$, in terms of the physical parameters of the model, namely $b_i, \delta_h b_i, \delta_v b_i$ and b_{grav} , we must identify the latter with the various coefficients that appear in the large- τ_2 expansions of $F_{\text{grav}}^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ and $F_i^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix}$ (Eqs. (5.4) and (5.6)). Neglecting the $\frac{1}{\tau_2}$ -suppressed contributions, which play no role in our argument, these expansions read

$$F_i^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\bar{q}} \left(-\frac{2A_i + 2B_i + 2C_i + D_i}{48} - \frac{\xi_1 + \xi_2 + \xi_3}{12} k_i \right) + \frac{1}{6} (-282A_i - 26B_i - 122C_i + 3D_i + (4\xi_1 - 124\xi_2 - 252\xi_3) k_i) + O(\bar{q}), \tag{C.1}$$

$$F_i^{\lambda=0(+)} = \frac{1}{\bar{q}} \left(-\frac{A_i}{24} + \frac{\xi_3}{12} k_i \right) + \left(-47A_i - \frac{32}{3}B_i - \frac{80}{3}C_i + \frac{1}{3}(64\xi_1 + 128\xi_2 + 126\xi_3) k_i \right) + O(\bar{q}), \tag{C.2}$$

$$F_i^{\lambda=0(-)} = \frac{1}{\sqrt{\bar{q}}} \left(-\frac{6A_i + 2C_i}{3} + \frac{4\xi_2 + 6\xi_3}{3} k_i \right) + O(\sqrt{\bar{q}}) \tag{C.3}$$

and

$$F_{\text{grav}}^{\lambda=0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\bar{q}} \left(-\frac{\xi_1 + \xi_2 + \xi_3}{12} \right) + \frac{2}{3} (\xi_1 - 31\xi_2 - 63\xi_3) + O(\bar{q}), \tag{C.4}$$

$$F_{\text{grav}}^{\lambda=0(+)} = \frac{1}{\bar{q}} \frac{\xi_3}{12} + \frac{64\xi_1 + 128\xi_2 + 126\xi_3}{3} + O(\bar{q}), \tag{C.5}$$

$$F_{\text{grav}}^{\lambda=0(-)} = \frac{1}{\sqrt{\bar{q}}} \left(\frac{4\xi_2 + 6\xi_3}{3} \right) + O(\sqrt{\bar{q}}). \tag{C.6}$$

The various constraints and identifications explained in the text lead to the following equations:

$$\begin{aligned} 2A_i + 2B_i + 2C_i + D_i + 4(\xi_1 + \xi_2 + \xi_3) k_i &= 0, \\ \frac{1}{6} (-282A_i - 26B_i - 122C_i + 3D_i + \frac{1}{6}(4\xi_1 - 124\xi_2 - 252\xi_3) k_i) &= b_i, \\ -A_i + 2\xi_3 k_i &= 0, \\ -\frac{3A_i + C_i}{3} + \frac{2\xi_2 + 3\xi_3}{3} k_i &= \frac{\delta_h b_i}{4}, \\ -47A_i - \frac{32}{3}B_i - \frac{80}{3}C_i + \frac{2}{3}(32\xi_1 + 64\xi_2 + 63\xi_3) k_i &= \delta_v b_i, \\ \xi_1 + \xi_2 + \xi_3 &= 1, \\ \frac{2}{3}(\xi_1 - 31\xi_2 - 63\xi_3) &= b_{\text{grav}}. \end{aligned} \tag{C.7}$$

The solutions read

$$A_i = \frac{1}{36} (4b_i - 24\delta_h b_i - 2\delta_v b_i + 54k_i - 9b_{\text{grav}} k_i), \tag{C.8}$$

$$B_i = \frac{1}{144} (53b_i - 48\delta_h b_i - 40\delta_v b_i + 990k_i - 99b_{\text{grav}} k_i), \tag{C.9}$$

$$C_i = \frac{1}{288} (-112b_i + 456\delta_h b_i + 56\delta_v b_i - 1494k_i + 225b_{\text{grav}} k_i), \tag{C.10}$$

$$D_i = \frac{1}{44}(-26 b_i - 168 \delta_h b_i + 40 \delta_v b_i - 1494 k_i + 45 b_{\text{grav}} k_i), \quad (\text{C.11})$$

$$\xi_1 = \frac{1}{576 k_i}(32 b_i - 192 \delta_h b_i - 16 \delta_v b_i + 990 k_i - 45 b_{\text{grav}} k_i), \quad (\text{C.12})$$

$$\xi_2 = \frac{1}{576 k_i}(-64 b_i + 384 \delta_h b_i + 32 \delta_v b_i - 846 k_i + 117 b_{\text{grav}} k_i), \quad (\text{C.13})$$

$$\xi_3 = \frac{1}{72 k_i}(4 b_i - 24 \delta_h b_i - 2 \delta_v b_i + 54 k_i - 9 b_{\text{grav}} k_i). \quad (\text{C.14})$$

Let us now introduce several “elementary” functions, which will enable us to express the quantities appearing in (5.30)–(5.33) in a compact way. As usual, $f(x) = f\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and consequently $f\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = f(1-x)$, $f\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = f(x/(x-1))$. We have

$$\begin{aligned} \sigma(x) &= -\frac{(x-1)^2}{3x}, \\ \phi(x) &= \frac{(x-1)^6}{(x^2-x+1)(x+1)^2(x-2)(2x-1)}, \\ \chi(x) &= \frac{(x-1)^4}{(x^2-x+1)(x-2)(2x-1)}, \\ \psi(x) &= \frac{(x-1)^2}{2(x^2-x+1)}. \end{aligned}$$

With these conventions

$$\begin{aligned} \delta_g^{\lambda=0} &= \frac{2}{9}\psi^2, \\ h_g^{\lambda=0} &= -\frac{4}{3}\psi^2, \\ v_g^{\lambda=0} &= -\frac{1}{9}\psi^2, \\ y_g^{\lambda=0} &= \left[3 - \frac{1}{24}\frac{1}{\sigma} + \frac{4}{9}\frac{1}{\sigma^2} - \frac{b_{\text{grav}}}{16}\left(8 - \frac{1}{\sigma}\right)\right]\psi^2, \\ \delta_f^{\lambda=0} &= \left(\frac{1}{9} - \frac{5}{48}\frac{1}{\sigma} - \frac{1}{48}\frac{1}{\sigma^2}\right)\phi, \\ h_f^{\lambda=0} &= \left(-\frac{2}{3} + \frac{29}{36}\frac{1}{\sigma}\right)\phi, \\ v_f^{\lambda=0} &= \left(-\frac{1}{18} + \frac{5}{108}\frac{1}{\sigma}\right)\phi, \\ y_f^{\lambda=0} &= \left[\left(\frac{3}{2} - \frac{61}{48}\frac{1}{\sigma} - \frac{1}{24}\frac{1}{\sigma^2} + \frac{4}{27}\frac{1}{\sigma^3}\right) - b_{\text{grav}}\left(\frac{1}{4} - \frac{23}{96}\frac{1}{\sigma} - \frac{1}{48}\frac{1}{\sigma^2}\right)\right]\phi. \end{aligned} \quad (\text{C.15})$$

C.2. Models with $\lambda = 1$ shifted lattice

$F_i^{\lambda=1}$ and $F_{\text{grav}}^{\lambda=1}$ in terms of the various physical parameters. Neglecting the $\frac{1}{r_2}$ -suppressed contributions, the expansions of $F_i^{\lambda=1} [h]$ and $F_{\text{grav}}^{\lambda=1} [h]$ are given by

$$F_i^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\bar{q}} \left(-\frac{A_i + B_i + C_i}{12} - \frac{\xi_1 + \xi_2 + \xi_3}{12} k_i \right) + \frac{2}{3} (-97 A_i - 33 B_i - C_i + (5 \xi_1 - 27 \xi_2 - 59 \xi_3) k_i) + O(\bar{q}), \quad (\text{C.16})$$

$$F_i^{\lambda=1(+)} = \frac{1}{\bar{q}^{3/4}} \left(-\frac{A_i}{3} + \frac{2 \xi_3}{3} k_i \right) + O(\bar{q}^{1/4}), \quad (\text{C.17})$$

$$F_i^{\lambda=1(-)} = \frac{1}{\bar{q}^{1/4}} \left(-\frac{44 A_i + 16 B_i}{3} + \frac{32 \xi_2 + 8 \xi_3}{3} \right) + O(\bar{q}^{3/4}), \quad (\text{C.18})$$

and

$$F_{\text{grav}}^{\lambda=1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\bar{q}} \left(-\frac{\xi_1 + \xi_2 + \xi_3}{12} \right) + \frac{2}{3} (5 \xi_1 - 27 \xi_2 - 59 \xi_3) + O(\bar{q}), \quad (\text{C.19})$$

$$F_{\text{grav}}^{\lambda=1(+)} = \frac{1}{\bar{q}^{3/4}} \frac{2 \xi_3}{3} + O(\bar{q}^{1/4}), \quad (\text{C.20})$$

$$F_{\text{grav}}^{\lambda=1(-)} = \frac{1}{\bar{q}^{1/4}} \frac{32 \xi_2 + 8 \xi_3}{3} + O(\bar{q}^{3/4}). \quad (\text{C.21})$$

The equations now are

$$\begin{aligned} A_i + B_i + C_i + (\xi_1 + \xi_2 + \xi_3) k_i &= 0, \\ \frac{2}{3} \left(-97 A_i - 33 B_i - C_i + (5 \xi_1 - 27 \xi_2 - 59 \xi_3) k_i \right) &= b_i, \\ -\frac{A_i}{6} + \frac{\xi_3}{3} k_i &= \frac{\delta_v b_i}{4}, \\ \frac{-22 A_i - 8 B_i}{3} + \frac{16 \xi_2 + 4 \xi_3}{3} k_i &= \frac{\delta_h b_i}{4}, \\ \xi_1 + \xi_2 + \xi_3 &= 1, \\ \frac{2}{3} (5 \xi_1 - 27 \xi_2 - 59 \xi_3) &= b_{\text{grav}}, \end{aligned} \quad (\text{C.22})$$

which we can solve as

$$A_i = \frac{1}{32} (b_i - 2 \delta_h b_i - 56 \delta_v b_i + 6 k_i - 3 b_{\text{grav}} k_i), \quad (\text{C.23})$$

$$B_i = \frac{1}{64} (9 b_i + 12 \delta_h b_i + 336 \delta_v b_i - 34 k_i + 21 b_{\text{grav}} k_i), \quad (\text{C.24})$$

$$C_i = \frac{1}{64} (7 b_i - 8 \delta_h b_i - 224 \delta_v b_i - 42 k_i - 15 b_{\text{grav}} k_i), \quad (\text{C.25})$$

$$\xi_1 = \frac{1}{64 k_i} (b_i - 2 \delta_h b_i - 8 \delta_v b_i + 60 k_i), \quad (\text{C.26})$$

$$\xi_2 = \frac{1}{64 k_i} (-2 b_i + 4 \delta_h b_i + 16 \delta_v b_i - 2 k_i + 3 b_{\text{grav}} k_i) , \quad (\text{C.27})$$

$$\xi_3 = \frac{1}{64 k_i} (b_i - 2 \delta_h b_i - 8 \delta_v b_i + 6 k_i - 3 b_{\text{grav}} k_i) . \quad (\text{C.28})$$

Finally, we have

$$\begin{aligned} \delta_g^{\lambda=1} &= \frac{1}{16} \psi^2 , \\ h_g^{\lambda=1} &= -\frac{1}{8} \psi^2 , \\ v_g^{\lambda=1} &= -\frac{1}{2} \psi^2 , \\ y_g^{\lambda=1} &= \left(\frac{3}{8} - \frac{3}{32} b_{\text{grav}} - \frac{5}{24} \frac{1}{\sigma} + \frac{4}{9} \frac{1}{\sigma^2} \right) \psi^2 , \\ \delta_f^{\lambda=1} &= \left(\frac{1}{32} + \frac{1}{64} \frac{1}{\sigma} \right) \chi , \\ h_f^{\lambda=1} &= -\frac{1}{16} \chi , \\ v_f^{\lambda=1} &= -\frac{7}{4} \chi , \\ y_f^{\lambda=1} &= \left[\frac{3}{16} - \frac{1}{96} \frac{1}{\sigma} - \frac{1}{9} \frac{1}{\sigma^2} - b_{\text{grav}} \left(\frac{3}{32} + \frac{1}{64} \frac{1}{\sigma^2} \right) \right] \chi . \end{aligned} \quad (\text{C.29})$$

Appendix D. Fundamental-domain integrals

D.1. General evaluation of the integrals

In this appendix we evaluate the following integrals:

$$I(T, U) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w [h] \bar{\Lambda}^\lambda [h] [g] - c_0 \right) \quad (\text{D.1})$$

and

$$\tilde{I}(T, U) = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w [h] \hat{E}_2 \bar{\Phi}^\lambda [h] [g] - \hat{c}_0 \right) , \quad (\text{D.2})$$

and present some relevant asymptotic behaviours. Integrals invariant under $\Gamma(2)$ such as (D.1) were first evaluated in [12] and later in [21] in special cases, and then more generally in [24] and the last of [13].

The functions $\Lambda^\lambda [h]_g$ and $\Phi^\lambda [h]_g$ possess the following properties: (i) they transform in a way that ensures modular invariance of the first term of the integrand in both $\lambda = 0$

and $\lambda = 1$ cases (see Eqs. (5.8) and (5.9) with $\Phi^\lambda \begin{bmatrix} h \\ g \end{bmatrix} \sim (1/\eta^{24}) \Omega^\lambda \begin{bmatrix} h \\ g \end{bmatrix}$); (ii) they are holomorphic with Fourier expansion²² in terms of q ,

$$\begin{aligned}
 A^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \sum_{n \geq -1} c_n q^n, \\
 A^{\lambda(+)} &= \sum_{n \geq -1} a_n q^{n+\frac{1}{4}}, \\
 A^{\lambda(-)} &= \sum_{n \geq -1} b_n q^{n+\frac{1}{2}+\frac{1}{4}}, \\
 \Phi^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \sum_{n \geq -1} c_n q^n, \\
 \Phi^{\lambda(+)} &= \sum_{n \geq -1} a_n q^{n+\frac{1}{4}}, \\
 \Phi^{\lambda(-)} &= \sum_{n \geq -1} b_n q^{n+\frac{1}{2}+\frac{1}{4}}, \\
 E_2 \Phi^\lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \sum_{n \geq -1} \hat{c}_n q^n, \\
 E_2 \Phi^{\lambda(+)} &= \sum_{n \geq -1} \hat{a}_n q^{n+\frac{1}{4}}, \\
 E_2 \Phi^{\lambda(-)} &= \sum_{n \geq -1} \hat{b}_n q^{n+\frac{1}{2}+\frac{1}{4}}.
 \end{aligned} \tag{D.3}$$

As a corollary of these properties, in the $\lambda = 0$ case, we have

$$\sum'_{(h,g)} A^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \alpha + \beta j,$$

which implies that the coefficients $a_n + c_n$ are in this case closely related to the Fourier coefficients of the j -function. Similarly,

$$\sum'_{(h,g)} \Phi^{\lambda=0} \begin{bmatrix} h \\ g \end{bmatrix} = \gamma \frac{E_4 E_6}{\eta^{24}} + \delta \frac{E_4}{E_6}$$

(α , β , γ and δ are constants) in general, although in our computations of gravitational corrections δ turns out to vanish systematically.

The above integrals are expected to converge in the (T, U) plane, with logarithmic singularities on the lines $T = U$, $T = -1/U$, $T = f_v^w(U)$ and $T = f_h^w(U)$ due to the presence of c_{-1} , a_{-1} and b_{-1} terms, respectively, in (D.3) (see Appendix A).

The starting point is the Hamiltonian representation of the lattice sums, which reads

²² As usual, $f(\pm) = f \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm f \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\tau_2 I_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} = \sum_A T^w[A] \begin{bmatrix} h \\ g \end{bmatrix}, \tag{D.4}$$

where, in some specific Poisson-resummed form,

$$T^w[A] \begin{bmatrix} h \\ g \end{bmatrix} = T_2 e^{-i\pi\lambda\frac{hg}{2}} \exp \left[i\pi \begin{pmatrix} a_1 & a_2 \end{pmatrix} A \begin{pmatrix} g \\ -h \end{pmatrix} \right] e^{2\pi i \bar{T} \det A} \\ \times \exp \left[-\frac{\pi T_2}{\tau_2 U_2} \left| \begin{pmatrix} 1 & U \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right] \tag{D.5}$$

and the summation is performed over a set of matrices of the form (remember that $w = (\mathbf{a}, \mathbf{b})$ with $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$)

$$A = \begin{pmatrix} n_1 + b_1 \frac{h}{2} & m_1 + b_1 \frac{g}{2} \\ n_2 + b_2 \frac{h}{2} & m_2 + b_2 \frac{g}{2} \end{pmatrix}.$$

In order to evaluate the integrals (D.1) and (D.2), we generalize the method of modular orbits, which was first introduced in [9] and later applied to various situations. The idea is to reduce the set of matrices to a fundamental one and simultaneously unfold the integration domain by performing $PSL(2, \mathbb{Z})$ transformations on the τ variable. In this way, each term of the resulting series can be integrated separately. This operation assumes the exchange of summations and integrations, which can be invalid because of tachyon-like divergences. Depending on the values of the moduli T and U , we must therefore utilize other Poisson resummations than the one presented in (D.5).

The set of fundamental matrices depends on the vector \mathbf{b} . For concreteness, we will analyse two situations only, in which the shift vectors are w_I and w_X , corresponding respectively to $\lambda = 0$ and $\lambda = 1$ lattices. Any other case in Tables A.1 and A.2 can be obtained by duality transformations.

(a) *Evaluation of I for shift vectors w_I and w_X*

In the case at hand, $\mathbf{b} = (1, 0)$ and there is no null orbit:

$$I = I_{nd} + I_{dg}, \tag{D.6}$$

where “nd” and “dg” stand for non-degenerate and degenerate orbits, respectively, and

$$I_{nd} = \sum'_{(h,g)} I_{nd} \begin{bmatrix} h \\ g \end{bmatrix}, \tag{D.7}$$

$$I_{dg} = I_{dg} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{D.8}$$

After the identification of the set of fundamental matrices, we obtain

$$I_{nd} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \int_{\mathcal{H}} \frac{d^2\tau}{\tau^2} \sum_{k>0} \sum_{k>j\geq 0} \sum_{p \neq 0} T^{I \text{ or } X} \left[A = \begin{pmatrix} k & j + \frac{1}{2} \\ 0 & p \end{pmatrix} \right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{A} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{D.9}$$

$$I_{\text{nd}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2^2} \sum_{k \geq 0} \sum_{k \geq j \geq 0} \sum_{p \neq 0} T^{\text{I or X}} \left[A = \begin{pmatrix} k + \frac{1}{2} & j \\ 0 & p \end{pmatrix} \right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{A} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (\text{D.10})$$

$$I_{\text{nd}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \int_{\mathcal{H}} \frac{d^2\tau}{\tau_2^2} \sum_{k > 0} \sum_{k > j > 0} \sum_{p \neq 0} T^{\text{I or X}} \left[A = \begin{pmatrix} k + \frac{1}{2} & j + \frac{1}{2} \\ 0 & p \end{pmatrix} \right] \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{A} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (\text{D.11})$$

$$I_{\text{dg}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lim_{N \rightarrow \infty} \left\{ \int_{\mathcal{S}} \frac{d^2\tau}{\tau_2^2} \sum_{j,p} T^{\text{I or X}} \left[A = \begin{pmatrix} 0 & j + \frac{1}{2} \\ 0 & p \end{pmatrix} \right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{A} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(1 - e^{-\frac{\pi}{2}} \right) - c_0 \left(\log N + \gamma + 1 + \log \frac{2}{3\sqrt{3}} \right) \right\}, \quad (\text{D.12})$$

where \mathcal{H} is the upper half-plane and \mathcal{S} is the strip $\{\tau \in \mathcal{H}, |\tau_1| < 1/2\}$. By using the standard machinery and the appropriate Poisson resummation of (D.5) to cover the whole moduli space, we obtain the following results:

$$\begin{aligned} I^{\text{I}}(T, U) = & -c_0 \left(\log |\vartheta_4(T)|^4 |\vartheta_2(U)|^4 T_2 U_2 - \gamma + 1 + \log \frac{\pi}{6\sqrt{3}} \right) \\ & + \left(c_0 - \frac{a_0}{2} \right) \log \left| \frac{\vartheta_4(T)}{\eta(T)} \right|^4 \\ & + \frac{\pi}{9} \left(a_0 - 2c_0 - 48(a_{-1} + c_{-1}) \right) \\ & \times \left(\frac{T_2}{2} \Theta \left(\frac{T_2}{2} - 2U_2 \right) + 2U_2 \Theta \left(2U_2 - \frac{T_2}{2} \right) \right) \\ & + \frac{\pi}{9} \left(a_0 - 2c_0 + 24(a_{-1} + c_{-1}) \right) \left(T_2 \Theta(T_2 - U_2) + U_2 \Theta(U_2 - T_2) \right) \\ & + 4 \operatorname{Re} \left\{ -c_{-1} \operatorname{Li}_1 \left(e^{2\pi i(T_1 - U_1 + i|T_2 - U_2|)} \right) \right. \\ & + a_{-1} \operatorname{Li}_1 \left(e^{2\pi i(\frac{T_1}{2} - 2U_1 + i|\frac{T_2}{2} - 2U_2|)} \right) + b_{-1} \operatorname{Li}_1 \left(e^{2\pi i(\frac{T_1}{2} - U_1 + i|\frac{T_2}{2} - U_2|)} \right) \\ & + \sum_{k, \ell > 0} \left(-c_{k\ell} \operatorname{Li}_1 \left(e^{2\pi i(Tk + U\ell)} \right) + a_{k\ell} \operatorname{Li}_1 \left(e^{2\pi i(\frac{T}{2}k + 2U\ell)} \right) \right. \\ & + (2c_{2k\ell} - a_{2k\ell}) \operatorname{Li}_1 \left(e^{2\pi i(Tk + 2U\ell)} \right) \\ & \left. \left. + b_{2k\ell - k - \ell} \operatorname{Li}_1 \left(e^{2\pi i(\frac{T}{2}(2k-1) + U(2\ell-1))} \right) \right) \right\} \end{aligned} \quad (\text{D.13})$$

and

$$\begin{aligned} I^{\text{X}}(T, U) = & -c_0 \left(\log |\vartheta_2(T)|^4 |\vartheta_2(U)|^4 T_2 U_2 - \gamma + 1 + \log \frac{\pi}{96\sqrt{3}} \right) \\ & - \pi c_0 \left(T_2 \Theta(T_2 - U_2) + U_2 \Theta(U_2 - T_2) \right) \\ & + 4 \operatorname{Re} \left\{ c_{-1} \operatorname{Li}_1 \left(e^{2\pi i(T_1 - U_1 + i|T_2 - U_2|)} \right) \right\} \end{aligned}$$

$$\begin{aligned}
 &+ a_{-1} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{3v_1}{2} - \frac{u_1}{2} + i \left| \frac{3v_2}{2} - \frac{u_2}{2} \right| \right)} \right) \\
 &+ a_{-1} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T}{2} - \frac{3u_1}{2} + i \left| \frac{T_2}{2} - \frac{3u_2}{2} \right| \right)} \right) + b_{-1} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T_1}{2} - \frac{u_1}{2} + i \left| \frac{T_2}{2} - \frac{u_2}{2} \right| \right)} \right) \\
 &+ \sum_{k, \ell > 0} \left(-c_{k\ell} \operatorname{Li}_1 \left(e^{2\pi i (Tk + U\ell)} \right) + 2c_{4k\ell} \operatorname{Li}_1 \left(e^{2\pi i (2Tk + 2U\ell)} \right) \right. \\
 &+ 2c_{4k\ell - 2k - 2\ell + 1} \operatorname{Li}_1 \left(e^{2\pi i (T(2k-1) + U(2\ell-1))} \right) \\
 &+ a_{4k\ell - 3k - 3\ell + 2} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T}{2}(4k-3) + \frac{U}{2}(4\ell-3) \right)} \right) \\
 &+ a_{4k\ell - k - \ell} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T}{2}(4k-1) + \frac{U}{2}(4\ell-1) \right)} \right) \\
 &+ b_{4k\ell - k - 3\ell} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T}{2}(4k-3) + \frac{U}{2}(4\ell-1) \right)} \right) \\
 &\left. + b_{4k\ell - 3k - \ell} \operatorname{Li}_1 \left(e^{2\pi i \left(\frac{T}{2}(4k-1) + \frac{U}{2}(4\ell-3) \right)} \right) \right) \Big\}. \tag{D.14}
 \end{aligned}$$

The polylogarithms are defined as usual,

$$\operatorname{Li}_1(x) = -\log(1-x), \quad \operatorname{Li}_2(x) = \sum_{j>0} \frac{x^j}{j^2}, \quad \operatorname{Li}_3(x) = \sum_{j>0} \frac{x^j}{j^3}.$$

Several comments are in order here. We first notice the partial breaking of the duality group $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T \leftrightarrow U}$ as explained in Appendix A (the $\mathbb{Z}_2^{T \leftrightarrow U}$ symmetry survives in the second case). We also observe the appearance of logarithmic singularities, as expected from Eqs. (A.11)–(A.14) and (A.17), (A.18). In I^1 , these take place at $T = U$, $T = -1/U$, $T = 4U$ and $T = 2U$. For the situation I^X ($\lambda = 1$), the divergences occur at $T = U$, $T = -1/U$, $T = 3U$ and $T = U/3$. The leading behaviours are

$$\begin{aligned}
 I^1 &\sim 4c_{-1} \log|T - U| \quad \text{at } T = U, \\
 I^1 &\sim -4a_{-1} \log|T - 4U| \quad \text{at } T = 4U, \\
 I^1 &\sim -4b_{-1} \log|T - 2U| \quad \text{at } T = 2U,
 \end{aligned} \tag{D.15}$$

and

$$\begin{aligned}
 I^X &\sim -4(c_{-1} + b_{-1}) \log|T - U| \quad \text{at } T = U, \\
 I^X &\sim -4a_{-1} \log|T - 3U| \quad \text{at } T = 3U, \\
 I^X &\sim -4a_{-1} \log \left| T - \frac{U}{3} \right| \quad \text{at } T = \frac{U}{3}.
 \end{aligned} \tag{D.16}$$

The residues at $T = -1/U$ can be determined in both cases I and X by performing appropriate Poisson resummations.

The most obvious example for the situation with $\lambda = 0$ is the constant function. In that case only the first term of (D.13) survives, in agreement with [21]. A somewhat less trivial situation is provided by the function $A \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right] = j, \forall (h, g) \neq (0, 0)$, for which

$$I^I[j] = -744 \left(\log |\vartheta_4(T)|^4 |\vartheta_2(U)|^4 T_2 U_2 - \gamma + 1 + \log \frac{\pi}{6\sqrt{3}} \right) - 8 \log \left| j \left(\frac{T}{2} \right) - j(2U) \right| + 4 \log |j(T) - j(U)| .$$

This is precisely what is obtained by using the results of [15] together with the identity

$$\sum'_{(h,g)} \Gamma_{2,2}^I \left[\begin{matrix} h \\ g \end{matrix} \right] = 2 \Gamma_{2,2} \left(\frac{T}{2}, 2U \right) - \Gamma_{2,2}(T, U) . \tag{D.17}$$

Finally, we would like to analyse the behaviour of the above integrals in the two limits that were considered in Appendix A, and which play a role in our analysis of the decompactification problem. Up to exponentially suppressed terms,

$$I^I(T, U) , I^X(T, U) \xrightarrow{T_2 \rightarrow \infty, U_2 = 1} -c_0 \log T_2 - c_0 \mu \tag{D.18}$$

$$I^I(T, U) \xrightarrow{T_2 \rightarrow 0, U_2 = 1} \frac{\pi}{3} \left(a_0 + c_0 - 24 (a_{-1} + c_{-1}) \right) \frac{1}{T_2} + c_0 \log T_2 - c_0 \mu - \left(3c_0 + \frac{a_0}{2} \right) \log 2 \tag{D.19}$$

and

$$I^X(T, U) \xrightarrow{T_2 \rightarrow 0, U_2 = 1} c_0 \log T_2 - c_0 \mu ; \tag{D.20}$$

we have assumed $T_1 = U_1 = 0$ and μ is a constant,

$$\mu = 4 \log |\eta(i)| - \gamma + 1 + \log \frac{\pi}{3\sqrt{3}} . \tag{D.21}$$

(b) Evaluation of \tilde{I} for shift vectors w_1 and w_X

The insertion of $\widehat{E}_2 = \overline{E}_2 - 3/\pi\tau_2$, for the cases at hand (i.e. without null orbit), leads to the following result:

$$\tilde{I} = \hat{I} - \frac{3}{\pi} I' , \tag{D.22}$$

where \hat{I} is the integral (D.6) evaluated above, with all coefficients c_n, a_n, b_n substituted with $\hat{c}_n, \hat{a}_n, \hat{b}_n$; on the other hand,

$$I' = \sum'_{(h,g)} I'_{\text{nd}} \left[\begin{matrix} h \\ g \end{matrix} \right] + I'_{\text{dg}} \left[\begin{matrix} 0 \\ 1 \end{matrix} \right] , \tag{D.23}$$

where $I'_{\text{nd}} \left[\begin{matrix} h \\ g \end{matrix} \right]$ are given in (D.9)–(D.11) with all $\overline{A} \left[\begin{matrix} h \\ g \end{matrix} \right]$ substituted with $\tau_2^{-1} \overline{\Phi} \left[\begin{matrix} h \\ g \end{matrix} \right]$ and

$$I'_{\text{dg}} \left[\begin{matrix} 0 \\ 1 \end{matrix} \right] = \int_S \frac{d^2\tau}{\tau_2^3} \sum_{j,p} T^{1 \text{ or } X} \left[A = \begin{pmatrix} 0 & j + \frac{1}{2} \\ 0 & p \end{pmatrix} \right] \left[\begin{matrix} 0 \\ 1 \end{matrix} \right] \overline{\Phi} \left[\begin{matrix} 0 \\ 1 \end{matrix} \right] \tag{D.24}$$

(this integral is infra-red-finite; the cut-off present in (D.2) plays a role in \hat{I} only). After some algebra we find

$$\begin{aligned}
 I'(T, U) = & \frac{4}{T_2 U_2} \operatorname{Re} \left\{ \sum_{k>0} \left((c_0 - a_0) \mathcal{P}(Tk) + a_0 \mathcal{P}\left(\frac{T}{2}k\right) \right) \right. \\
 & + \sum_{\ell>0} \left(-c_0 \mathcal{P}(U\ell) + 2c_0 \mathcal{P}(2U\ell) \right) + \frac{c_0}{4\pi} \zeta(3) \\
 & + \frac{\pi^2}{108} \left[\frac{1}{30} (16c_0 + a_0) \left(16U_2^3 \Theta\left(\frac{T_2}{2} - 2U_2\right) + \frac{T_2^3}{4} \Theta\left(2U_2 - \frac{T_2}{2}\right) \right) \right. \\
 & + \frac{1}{15} (7c_0 - 8a_0) (U_2^3 \Theta(T_2 - U_2) + T_2^3 \Theta(U_2 - T_2)) \\
 & + (a_0 + c_0 - 48(a_{-1} + c_{-1})) T_2 U_2 \left(\frac{T_2}{2} \Theta\left(\frac{T_2}{2} - 2U_2\right) \right. \\
 & \left. \left. + 2U_2 \Theta\left(2U_2 - \frac{T_2}{2}\right) - \frac{T_2}{2} \Theta(T_2 - U_2) - \frac{U_2}{2} \Theta(U_2 - T_2) \right) \right] \\
 & - c_{-1} \mathcal{P}(T_1 - U_1 + i|T_2 - U_2|) \\
 & + a_{-1} \mathcal{P}\left(\frac{T_1}{2} - 2U_1 + i\left|\frac{T_2}{2} - 2U_2\right|\right) \\
 & + b_{-1} \mathcal{P}\left(\frac{T_1}{2} - U_1 + i\left|\frac{T_2}{2} - U_2\right|\right) \\
 & + \sum_{k, \ell>0} \left(-c_{k\ell} \mathcal{P}(Tk + U\ell) + a_{k\ell} \mathcal{P}\left(\frac{T}{2}k + 2U\ell\right) \right. \\
 & + (2c_{2k\ell} - a_{2k\ell}) \mathcal{P}(Tk + 2U\ell) \\
 & \left. \left. + b_{2k\ell-k-\ell} \mathcal{P}\left(\frac{T}{2}(2k-1) + U(2\ell-1)\right) \right) \right\} \quad (D.25)
 \end{aligned}$$

and

$$\begin{aligned}
 I^{X'}(T, U) = & \frac{4}{T_2 U_2} \operatorname{Re} \left\{ c_0 \sum_{k>0} \left(2\mathcal{P}(2Tk) - \mathcal{P}(Tk) \right) + c_0 \sum_{\ell>0} \left(2\mathcal{P}(2U\ell) - \mathcal{P}(U\ell) \right) \right. \\
 & + \frac{c_0}{4\pi} \zeta(3) + \frac{\pi^2}{12} c_0 (U_2^3 \Theta(T_2 - U_2) + T_2^3 \Theta(U_2 - T_2)) \\
 & + c_{-1} \mathcal{P}(T_1 - U_1 + i|T_2 - U_2|) \\
 & + a_{-1} \mathcal{P}\left(\frac{3T_1}{2} - \frac{U_1}{2} + i\left|\frac{3T_2}{2} - \frac{U_2}{2}\right|\right) \\
 & + a_{-1} \mathcal{P}\left(\frac{T_1}{2} - \frac{3U_1}{2} + i\left|\frac{T_2}{2} - \frac{3U_2}{2}\right|\right) \\
 & + b_{-1} \mathcal{P}\left(\frac{T_1}{2} - \frac{U_1}{2} + i\left|\frac{T_2}{2} - \frac{U_2}{2}\right|\right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k,\ell>0} \left(-c_{k\ell} \mathcal{P}(Tk + U\ell) + 2c_{4k\ell} \mathcal{P}(2Tk + 2U\ell) \right. \\
 & + 2c_{4k\ell-2k-2\ell+1} \mathcal{P}(T(2k-1) + U(2\ell-1)) \\
 & + a_{4k\ell-3k-3\ell+2} \mathcal{P}\left(\frac{T}{2}(4k-3) + \frac{U}{2}(4\ell-3)\right) \\
 & + a_{4k\ell-k-\ell} \mathcal{P}\left(\frac{T}{2}(4k-1) + \frac{U}{2}(4\ell-1)\right) \\
 & + b_{4k\ell-k-3\ell} \mathcal{P}\left(\frac{T}{2}(4k-3) + \frac{U}{2}(4\ell-1)\right) \\
 & \left. + b_{4k\ell-3k-\ell} \mathcal{P}\left(\frac{T}{2}(4k-1) + \frac{U}{2}(4\ell-3)\right) \right), \tag{D.26}
 \end{aligned}$$

where we have introduced [15]

$$\mathcal{P}(x) = \text{Im } x \text{Li}_2(e^{2\pi ix}) + \frac{1}{2\pi} \text{Li}_3(e^{2\pi ix}).$$

Regarding the singularities and the breaking of the duality group, the same observations can be made, as in the cases without insertion of \widehat{E}_2 . In particular, Eqs. (D.15) and (D.16) hold also for the functions $\tilde{I}(T, U)$ and $\tilde{I}^X(T, U)$. When the shift vector is w_1 , the simplest situations arise with $\Phi \begin{smallmatrix} h \\ g \end{smallmatrix} = E_4 E_6 / \eta^{24}$ or E_4 / E_6 , for all $(h, g) \neq (0, 0)$. We can compute these integrals by using the identity (D.17) and the results of [15]; they turn out to be in agreement with our general formulas (D.22), (D.25).

The asymptotic behaviours read here

$$\tilde{I}(T, U), \tilde{I}^X(T, U) \xrightarrow{T_2 \rightarrow \infty, U_2=1} -(c_0 - 24 c_{-1}) \log T_2 - (c_0 - 24 c_{-1}) \mu - \frac{c_0 \rho}{T_2}, \tag{D.27}$$

$$\begin{aligned}
 \tilde{I}(T, U) \xrightarrow{T_2 \rightarrow 0, U_2=1} & -8\pi \left((a_{-1} + c_{-1}) - \frac{a_0 + c_0}{48} \right) \frac{1}{T_2} + (c_0 - 24 c_{-1}) \log T_2 \\
 & - \left(3(c_0 - 24 c_{-1}) + \frac{1}{2}(a_0 - 24 a_{-1}) \right) \log 2 \\
 & - (c_0 - 24 c_{-1}) \mu - (c_0 \kappa + a_0 \nu) T_2, \tag{D.28}
 \end{aligned}$$

$$\tilde{I}^X(T, U) \xrightarrow{T_2 \rightarrow 0, U_2=1} (c_0 - 24 c_{-1}) \log T_2 - (c_0 - 24 c_{-1}) \mu - c_0 \rho T_2, \tag{D.29}$$

up to exponentially suppressed terms. Again, we assumed $T_1 = U_1 = 0$ and introduced the constants

$$\begin{aligned}
 \rho &= \frac{12}{\pi} \sum_{j>0} \left(\frac{1}{j^2} \left(\frac{1}{2} + \frac{1}{\sinh^2 2\pi j} - \frac{1}{4 \sinh^2 \pi j} \right) + \frac{1}{j^3} \frac{1}{4\pi} \tanh \pi j \right), \\
 \kappa &= \frac{12}{\pi} \sum_{j>0} \left(\frac{1}{j^2} \left(\frac{1}{30} + \frac{1}{4 \sinh^2 \pi j} \right) + \frac{1}{j^3} \frac{1}{4\pi} \coth \pi j \right),
 \end{aligned}$$

$$\nu = \frac{12}{\pi} \sum_{j>0} \left(\frac{1}{j^2} \left(-\frac{7}{240} + \frac{1}{8 \sinh^2 \frac{\pi j}{2}} \right) + \frac{1}{j^3} \frac{1}{4\pi \sinh \pi j} \right). \tag{D.30}$$

D.2. Application to threshold corrections

One can use the results obtained so far to further investigate the threshold corrections, Eqs. (5.29) and (5.39), of the models with spontaneously broken supersymmetry described in Section 4.

For lattices with shift vectors w_1 and w_X ($\lambda = 0$ and 1 respectively) we obtain the following singularity properties (see (D.15), (D.16)):

$$\begin{aligned} \Delta_{\text{grav}}^1 &\sim -\frac{1}{3} \log |T - U|, & \Delta_i^1 &\text{finite, at } T = U, \\ \Delta_{\text{grav}}^1 &\sim \frac{1}{3} \log \left| T + \frac{1}{U} \right|, & \Delta_i^1 &\text{finite, at } T = -\frac{1}{U}, \\ \Delta_{\text{grav}}^1 &\sim \left\{ \begin{array}{l} \text{finite in class (i),} \\ -\frac{1}{3} \log |T - 4U| \text{ in class (ii)} \end{array} \right\}, & \Delta_i^1 &\text{finite, at } T = 4U, \\ \Delta_{\text{grav}}^1 &\sim -2\delta_h b_{\text{grav}} \log |T - 2U|, & \Delta_i^1 &\sim -2\delta_h b_i \log |T - 2U|, \text{ at } T = 2U, \end{aligned} \tag{D.31}$$

and

$$\begin{aligned} \Delta_{\text{grav}}^X &\sim \left(\frac{1}{3} - 2\delta_h b_{\text{grav}} \right) \log |T - U|, & \Delta_i^X &\sim -2\delta_h b_i \log |T - U|, \text{ at } T = U \\ \Delta_{\text{grav}}^X &\sim \frac{1}{3} \log \left| T + \frac{1}{U} \right|, & \Delta_i^X &\text{finite, at } T = -\frac{1}{U}, \\ \Delta_{\text{grav}}^X &\sim -2\delta_v b_{\text{grav}} \log |T - 3U|, & \Delta_i^X &\sim -2\delta_v b_i \log |T - 3U|, \text{ at } T = 3U, \\ \Delta_{\text{grav}}^X &\sim -2\delta_v b_{\text{grav}} \log |U - 3T|, & \Delta_i^X &\sim -2\delta_v b_i \log |U - 3T|, \text{ at } T = \frac{U}{3}. \end{aligned} \tag{D.32}$$

Finally, we can analyse the behaviour of the corrections in the various decompactification limits. We will give the results containing leading terms and subleading corrections, up to exponentially suppressed ones. We assume again $T_1 = U_1 = 0$, and use Eqs. (D.18)–(D.20) and (D.27)–(D.29).

(a) *The limit $T_2 \rightarrow \infty$, $U_2 = 1$*

In this limit, $N = 4$ supersymmetry is restored in both $\lambda = 0$ and $\lambda = 1$ lattices. The behaviours are

$$\Delta_{\text{grav}}^1, \quad \Delta_{\text{grav}}^X \rightarrow -b_{\text{grav}} (\log T_2 + \mu) - (b_{\text{grav}} - 2) \frac{\rho}{T_2}$$

and

$$\Delta_i^1, \quad \Delta_i^X \rightarrow -b_i \left(\log T_2 + \mu - \log \frac{2 e^{1-\gamma}}{\pi 3\sqrt{3}} \right) - k_i (b_{\text{grav}} - 2) \frac{\rho}{T_2}.$$

(b) The limit $T_2 \rightarrow 0$, $U_2 = 1$

This limit is ($N = 4$)-supersymmetric for $\lambda = 0$ models of class (ii) and models with $\lambda = 1$. For $\lambda = 0$ models belonging to class (i), the supersymmetry remains $N = 2$:

$$\begin{aligned} \Delta_{\text{grav}}^I (\text{class (i)}) &\rightarrow \frac{4\pi}{T_2} + b_{\text{grav}} \left(\log T_2 - \mu - \frac{5}{2} \log 2 \right) - 11 \log 2 \\ &\quad - (\kappa (b_{\text{grav}} - 2) - \nu (b_{\text{grav}} - 22)) T_2, \\ \Delta_i^I (\text{class (i)}) &\rightarrow (b_i + \delta_i b_i - 12 k_i) \frac{\pi}{3T_2} + b_i \left(\log T_2 - \mu + \log \frac{1}{\pi} \frac{e^{1-\gamma}}{12\sqrt{3}} \right) \\ &\quad - \frac{\delta_i b_i}{2} \log 2 - k_i (\kappa (b_{\text{grav}} - 2) - \nu (b_{\text{grav}} - 22)) T_2, \\ \Delta_{\text{grav}}^I (\text{class (ii)}) &\rightarrow b_{\text{grav}} \left(\log T_2 - \mu - \frac{5}{2} \log 2 \right) - (b_{\text{grav}} - 2) (\kappa - \nu) T_2, \\ \Delta_i^I (\text{class (ii)}) &\rightarrow b_i \left(\log T_2 - \mu + \log \frac{1}{\pi} \frac{e^{1-\gamma}}{6\sqrt{6}} \right) - k_i (b_{\text{grav}} - 2) (\kappa - \nu) T_2, \\ \Delta_{\text{grav}}^X &\rightarrow -b_{\text{grav}} (-\log T_2 + \mu) - (b_{\text{grav}} - 2) \rho T_2, \\ \Delta_i^X &\rightarrow -b_i \left(-\log T_2 + \mu - \log \frac{2}{\pi} \frac{e^{1-\gamma}}{3\sqrt{3}} \right) - k_i (b_{\text{grav}} - 2) \rho T_2. \end{aligned}$$

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