Textures for neutrino mass matrices

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We give a classification of heavy Majorana neutrino mass matrices with up to three texture zeros, assuming the Dirac masses of the neutrinos to be of the same form as the ones of the up quarks in the five texture zero solutions for the quark matrices. This is the case for many unified and partially unified models. We find that it is possible to have solutions which account for the solar and atmospheric neutrino problems as well as the COBE observations simultaneously, and we motivate the existence of such solutions from symmetries. [S0556-2821(96)04711-X]

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I. INTRODUCTION

Recently, there has been a lot of work on the origin of fermion masses and mixing angles [1–9]. The observed pattern of masses and mixings can be explained if some structure in addition to that of the standard model exists at higher scales. Within the context of supersymmetry, unification has had considerable success in determining the parameters of the standard model [10]. In addition to the successful predictions of the gauge couplings, the pattern and magnitude of spontaneous symmetry breaking at the electroweak scale [11], and b-τ unification, it was also found that the fermion mixing angles and masses have values consistent with the appearance of ''texture'' zeros in the mass matrices [12,1]. Such textures indicate the existence of additional symmetries beyond the standard model and together with the hierarchical structure observed in the quark and lepton mass matrices imply that an underlying family symmetry [e.g., a U(1) family symmetry] with breaking characterized by a small parameter λ might exist [4]. For an exact symmetry, only the third generation would be massive and all mixing angles zero. However, symmetry-breaking terms gradually fill the mass matrices in powers of λ, generating a hierarchy of mass scales and mixing angles. Thus, a broken symmetry can explain the ''texture'' zeros as well as the relative magnitude of the nonzero elements.

In a previous work [5,6], the implications of such a scheme for neutrino masses and mixing angles in the case of the minimal multiplet content of the minimal supersymmetric standard model (MSSM), extended to include right-handed neutrino components (plus the standard model Higgs singlets needed to generate their masses and to break the extended gauge family symmetry) were considered. Alternative schemes have also been proposed [7,8]. In [5,6] right-handed fields got Majorana masses from a term of the form $\nu_R\nu_R\Sigma$ where Σ is a SU(3)$\otimes$SU(2)$\otimes$U(1) invariant Higgs scalar field with $I_W=0$ and ν_R is a right-handed neutrino. In many models Σ is not an elementary field, but a combination of scalar fields $\Sigma=\tilde{\bar{\nu}}_R\tilde{\bar{\nu}}_R$, where $\tilde{\bar{\nu}}_R$ is the scalar component of a right-handed antineutrino supermultiplet $\overline{\nu_R}$ [13]. It was found that, up to a discrete ambiguity, the Majorana mass matrix is determined and that, unlike previous assumptions for this matrix [14], a large splitting between the entries exists.

The main conclusions of this work are the following.

(i) The heaviest neutrino has a mass (0.4–4) eV for a top quark of 200 GeV, thus being of the right size for structure formation [15].

(ii) The light neutrinos have masses and mixing angles of the magnitude needed to explain solar neutrino oscillations [16,17], in the small mixing angle region of the Mikheyev-Smirnov-Wolfenstein (MSW) effect.

(iii) In the simplest scheme, described in [5], it was not possible to obtain large mixing angles, without fine-tuning of the Yukawa couplings. Such a large mixing is required for a vacuum solution of the solar neutrino problem as well as the neutrino oscillation solution to the atmospheric neutrino problem [18]. In [6] it was found, however, that in order to obtain neutrino masses in a phenomenologically interesting region while retaining bottom-τ unification, in the small tanβ regime, large mixing in the μ-τ charged leptonic sector has to occur.[1] This mixing can in principle appear also in the large tanβ regime and in particular in a subclass of the textures of [5], when dropping a residual Z_2 symmetry. Still, for a single Σ field, the eigenvalues of the light Majorana mass exhibit large splittings. Therefore, although we had been able

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[1]The distortions to the bottom-τ unification would appear as an implication of the structure emerging from the U(1) symmetry, if the right-handed neutrinos have Yukawa couplings of the same order as the up quarks, thus affecting the radiative corrections in the model in the small tanβ regime [19]. An alternative solution arises in a subclass of models where the Dirac-type Yukawa coupling of the neutrino is very suppressed [20]. In the large tanβ regime due to the infrared fixed point for the bottom coupling, which is described by analytical expressions in [21], the effect of the neutrino coupling to bottom-τ unification is negligible.

to obtain two classes of solutions where we address either the solar and the atmospheric neutrino problem, or the solar neutrino problem and the Cosmic Background Explorer (COBE) data, it was not possible to solve all three problems simultaneously.

However, in principle there is no reason why this particular conclusion in the simplest extension of the standard model should apply in the case of a more complicated symmetry or with more than one pair of singlet fields $\Sigma,\overline{\Sigma}$ present in the theory. Since in such a case there are many possible patterns, instead of making a complete search based on symmetries, we follow the opposite strategy.

(i) We assume the very large class of models from underlying unified models [such as strings and grand unified theories (GUT's)] or partially unified models that fix the neutrino Dirac mass matrix to be proportional to the u-quark mass matrix. This of course is a simplified assumption [2,17,14]. In fact we know that if a Dirac-type mass is generated by a vacuum expectation value (VEV) of the $\overline{\mathbf{126}}$ of SO(10), then $m_\nu^D=-3m_u$. Thus, the m_ν^D,m_u relation is more complicated if both **10** and $\overline{\mathbf{126}}$ of SO(10) develop nonzero VEV's along the directions contributing to the Dirac-type neutrino masses. To simplify the analysis, one usually assumes that for each entry only one type of Higgs field contributes. But different Higgs fields may develop vacuum expectation values in different directions [8].

(ii) We then study all possible Majorana neutrino mass matrices with three exact and an arbitrary number of phenomenological texture zeros. It is clear that we are looking for solutions with at least one large mixing angle (to explain the atmospheric neutrino deficit) and nearly degenerate neutrino mass eigenvalues.

(iii) We then motivate these phenomenological solutions from symmetries. As a result we find only a small number of possible Majorana neutrino mass matrices. This gives a constraint on the underlying theory in terms of necessary couplings.

We will start by briefly reviewing the experimental limits on neutrino masses and mixing angles, in Sec. II. In Sec. III we will discuss the whole framework of unification and mass matrices. In Sec. IV we then give a first example of how we find solutions with exact texture zeros that allow large mixing angles and nearly degenerate mass eigenvalues. The complete results of this analysis are tabulated in the Appendix. These findings at high scales are then confronted with the low energy requirements for such textures in Sec. V. Here a classification of phenomenological texture zeros is given. Section VI addresses the derivation of such textures from underlying U(1) symmetries. The connection of the textures at high and low scales via renormalization is discussed in Sec. VII. The conclusions are given in Sec. VIII. Finally, the complete approach of finding textures is summarized in the Appendix.

II. NEUTRINO PHENOMENOLOGY

Various recent data, confirmed by many experimental groups, may be explained if the neutrinos have nonvanishing masses. In such a case, the neutral leptons produced in weak interactions are in general not stationary. The weak eigenstates are linear combinations of the neutrino mass eigenstates, implying neutrino mixing. Although the standard model theory does not give masses to the neutrinos, most extensions of the standard model predict small masses and mixing. Before discussing such extensions, we first review the experimental situation and give some indication of possible explanations.

If the neutrinos are light, neutrino oscillation experiments are the best candidates to measure the small mass differences δm^2 (from 1 eV^2 down to 10^{-10} eV^2). Furthermore, neutrino oscillations may explain the solar neutrino problem, i.e., the apparent reduction of the ν_e flux at earth compared to that predicted by the standard solar model [22] (SSM). If the neutrino mixing is small, the mechanism of neutrino oscillations is not effective. Nevertheless, under the conditions of high density encountered in the sun's interior, the oscillation can be enhanced from the MSW effect [16], since small mixing angles can be converted into large effective mixing angles, due to resonant scattering of ν_e neutrinos by electrons. The data from the solar neutrino experiments can thus be described either by assuming resonant transitions (MSW effect) or vacuum oscillations. These two possibilities require the following ranges for masses and mixing angles.

(a) The small mixing angle solution for the MSW effect requires

$$\delta m^2_{\nu_e\nu_\alpha}\approx(0.6-1.2)\times10^{-5}\ \mathrm{eV}^2, \tag{1}$$

$$\sin^2 2\theta_{\alpha e}\approx(0.6-1.4)\times10^{-2}. \tag{2}$$

(b) Vacuum oscillations can solve the solar neutrino puzzle if

$$\delta m^2_{\nu_e\nu_\alpha}\approx(0.5-1.1)\times10^{-10}\ \mathrm{eV}^2, \tag{3}$$

$$\sin^2 2\theta_{\alpha e}\geqslant0.75, \tag{4}$$

where α is μ or τ. The most natural solution in unified models is obtained through the MSW mass and mixing angle ranges. This solution in particular requires a light neutrino Majorana mass of the order

$$m_\odot\approx\sqrt{\delta m^2}\approx3.0\times10^{-3}\ \mathrm{eV}, \tag{5}$$

as already given in (1). Such ultralight masses can be generated effectively in GUT's [23] and supersymmetric (SUSY) GUT's [24] by the well known ''seesaw'' mechanism [25].

The atmospheric neutrino problem may be explained in the case that a large mixing and small mass splitting involving the muon neutrino exists [18,26]. Taking into account the bounds from accelerator and reactor disappearance experiments one finds that, for ν_e-ν_μ or ν_τ-ν_μ oscillations,

$$\delta m^2_{\nu_\alpha\nu_\mu}\leqslant10^{-2}\ \mathrm{eV}^2, \tag{6}$$

$$\sin^2 2\theta_{\mu\alpha}\geqslant0.51-0.6, \tag{7}$$

where α stands for e,τ and in (7) the larger lower limit for $\sin^2 2\theta_{\mu\alpha}$ refers to ν_μ-ν_τ oscillations. Finally we have already mentioned that neutrinos are possible candidates for structure formation provided they have a mass ~1 eV. This value is consistent (with a small margin according to some measurements) with the bounds from neutrinoless double β ($\beta\beta_{0\nu}$) decay. In terms of the neutrino masses and mixing angles, the relevant $\beta\beta_{0\nu}$ measurable quantity can be written as

$$|\langle m_{\nu_e}\rangle| = \left|\sum_i^3 (U_{ei})^2 m_{\nu_i} e^{i\lambda_i}\right| \leqslant 1 \text{ eV}, \tag{8}$$

where $e^{i\lambda_i}$ is the *CP* parity of the ith neutrino, while U_{ei} are the elements of the unitary transformation relating the weak and mass neutrino eigenstates.

What are the possible theoretical solutions that are consistent with this data? Only a partial solution of all these problems may be obtained easily in simple models with a general hierarchical pattern of neutrino masses. In most cases it is possible to obtain a solution to the solar neutrino problem with $m_{\nu_1} \ll m_{\nu_2} \approx m_\odot$, and $m_{\nu_3} \sim 1$ eV to interpret the COBE data. Then, the $\beta\beta_{0\nu}$ bound is satisfied due to the smallness of the U_{e3} mixing element predicted by the theory. Indeed, assuming the above hierarchy, the quantity $\langle m_{\nu_e}\rangle$ may be approximated by $|U_{e3}|^2 m_{\nu_3}$. Because of the assumed hierarchy, it follows easily that oscillation experiments are sensitive only to δm^2_{13}, δm^2_{23}, since oscillations related to δm^2_{12} are too rapid. Thus the formula for the oscillations $P(\nu_e \to \nu_e)$ [27] is simplified to

$$P(\nu_e \to \nu_e) = 1 - 4|U_{e3}|^2 (1 - |U_{e3}|^2) \sin^2\left(\frac{\pi x}{\ell}\right), \tag{9}$$

where several trigonometric identities and $m_1, m_2 \ll m_3$ are used. Setting $|U_{e3}| = \sin\Theta_{ee}$ this may be rewritten as

$$P(\nu_e \to \nu_e) = 1 - \sin^2\Theta_{ee} \sin^2\left(\frac{\pi x}{\ell}\right). \tag{10}$$

Taking $\sin^2\Theta_{ee} = 0.2$, we find $|U_{e3}| \approx 0.23$. This in turn would imply, i.e., $m_{\nu_3} \approx 18.9$ eV for $\langle m_{\nu_e}\rangle \approx 1$ eV from above. Obviously, the atmospheric neutrino data do not fit in the above scenario.

It thus appears that the experimental data require nearly degenerate mass eigenstates $m_{\nu_i} \approx m_0$, $i = 1,2,3$ [28], since, first of all, structure formation in the Universe and the COBE data require $\Sigma_i m_i \approx 3$ eV. This sets the scale of the masses. The data from the atmospheric and solar neutrino experiments force the involved masses to be very similar. In this case the $\beta\beta_{0\nu}$ bound may be respected due to mutual cancellations in (8) by opposite *CP* phases $e^{i\lambda_i}$. Introducing an average mass m_0 one finds that

$$\delta m^2_{12} \approx 2 m_0 |m_2 - m_1| \approx 10^{-5} \text{ eV}^2, \tag{11}$$

$$\delta m^2_{23} \approx 2 m_0 |m_2 - m_3| \leqslant 10^{-2} \text{ eV}^2, \tag{12}$$

$$\langle m_{\nu_e}\rangle \approx m_0 \sum_{i=1}^3 (U_{ei})^2 e^{i\lambda_i} \leqslant 1 \text{ eV}. \tag{13}$$

With the mentioned mutual cancellations, $m_0 \approx (1-2)$ eV can be consistent with all data. Our aim in the present paper is to use this observation and constraints from the low energy theory, in order to determine the optimal Majorana mass matrices with zero textures, for a wide class of theories. We consider the cases with (a) hierarchical light-neutrino masses (partial solutions) and with (b) nearly degenerate neutrino mass eigenstates of order ~ 1 eV (complete solutions).

The necessary mixing may occur in two ways: either purely from the neutrino sector of the theory or by the charged lepton mixing. In the second case the mixing is typically too small to have any impact on the atmospheric neutrino problem, but may still account for the solar neutrino problem. In the former case, the mixing may be such as to account for both deficits. One, of course, can consider mixing in both sectors. In this paper, we will search systematically for solutions with one large mixing angle, stemming from the need to accommodate the atmospheric neutrino data from the beginning. Such an origin of a large mixing actually seems to be the most reasonable case. We also chose the small mixing angle solution in order to address the solar neutrino puzzle. We are not discussing the possibility of a second large angle (as required for the vacuum oscillation solution for the solar neutrinos), since the analysis will be more complicated. Therefore we have two possibilities.

(1) The solar neutrino problem is resolved by ν_e-ν_μ oscillations and the atmospheric neutrino problem by ν_μ-ν_τ oscillations. For this possibility to be viable, we need a large mixing angle, in the 2-3 entries.

(2) The solar neutrino problem is resolved by ν_e-ν_τ oscillations and the atmospheric neutrino problem by ν_e-ν_μ oscillations. In this case the large angle should be in the 1-2 entries.

III. UNIFICATION AND MASS MATRICES

In this section we discuss how predictions for mass matrices arise in unified theories. We will start by resuming the discussion of [3,4] for quark and lepton mass matrices and show how this extends to include neutrino mass matrices, if we assume certain unifications. So far, there has been a lot of progress in the construction of viable string theories. Although many models seem to have fundamental problems in resembling the standard model at low energies, most of them can not be totally excluded, because of their very complex structure [30]. On the other hand, there exist many more models which have not been studied at all. Therefore one is very interested in having an additional criterion to distinguish between all these possibilities and to single out those that may lead to the standard model at low energies.

The idea in [4] was (instead of taking specific models and studying their parameters) to use additional discrete symmetries, which appear vastly in many string models, and study their implications for fermion mass patterns. Discrete symmetries lead quite naturally to hierarchies of parameters (Yukawa couplings, etc.) and therefore to predictions that are largely independent of specific numerical values. A certain model is realistic only if it possesses such a hierarchical structure, without any fine-tuning. Indeed, it is a general observation that a string model exhibits all possible couplings allowed by the discrete symmetries, which are typically[2] of

[2]Here we are referring to models close to, e.g., the conformal point and not to large moduli VEV (vacuum expectation value) cases.

order and there are hardly any *accidental* zeros for the values of the Yukawa couplings.

To see how symmetries may imply a hierarchical pattern of Yukawa couplings (and therefore masses), one can look at the mass matrices for a two-quark doublet, assuming an additional U(1) symmetry to the standard model gauge group, under which $t_{L,R}, b_{L,R}$ have charge α_1 and $c_{L,R}, s_{L,R}$ have charge α_2. The form of the actual mass matrix depends also on the transformation of the Higgs fields H_1, H_2 which give masses to the fermions. Taking these charges to be $-2\alpha_1$, only the t and b quarks acquire a mass in the electroweak breaking and the up or down quark mass matrix has only one nonzero element. However, if the U(1) symmetry is broken to a discrete symmetry Z_N by the VEV of a field θ with charge -1, there are several mechanisms that give structure to the mass matrices: (1) *higher dimensional operators* $q_L q_R H_i \theta^n / M^n$, where M is the scale where these terms are created; then in general the mass matrices are of the form

$$\begin{pmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{pmatrix}, \tag{14}$$

where $\lambda = \langle \theta \rangle / M$ and a hierarchy in terms of λ arises; (2) *mixing of the coupling Higgs field*; and (3) *mixing of the coupling matter fields.*

Thus additional symmetries, together with stages of spontaneous symmetry breaking, allow for a natural explanation of hierarchies of masses. This actually makes use of the huge discrete symmetry groups appearing in string models; therefore, one may hope that the patterns of mass matrices may help to determine the underlying discrete symmetries and in turn to give restrictions on a possible fundamental string theory.

Let us note here that there are *two types* of texture zeros: *exact* and *phenomenological*. The first type is a zero implied by a symmetry. The second type relaxes the zeros in a way that does not change the hierarchical structure of a given matrix. If we are assuming a fundamental theory that has only certain couplings at the tree level, like a pure GUT theory, it is clear from the start that we deal with exact zeros. On the other hand, zeros incorporating mechanisms 1–3 to create entries in the mass matrices seem bound to be phenomenological.

After giving the underlying idea of creating hierarchies of couplings and therefore masses, let us briefly quote the ‘‘phenomenological’’ results of Ramond, Roberts, and Ross [3], which we will need for the discussion that follows. Here, the authors looked for parametrizations of symmetric quark mass matrices in terms of possible texture zeros and a hierarchical parameter λ that is in accordance with experiment. Those structures would have then to be explained by symmetries of the underlying theory. Although a pattern of zeros in one single mass matrix does not have a meaning on its own (because of the possible redefinitions of the quark fields), it has a meaning for the up and down quark mass matrices together. Therefore, one encounters a *relative structure* (e.g., one matrix may be made diagonal by redefinitions, but the texture zero structure determines now the other matrix.) A complete study of 5 and 6 texture zeros in the two mass matrices has been carried out along these lines. Studying systematically all the possible cases and taking into account the running of the renormalization group equations between the unification scale M_X, where the texture zeros are assumed, and M_W, five realistic pairs of texture zero patterns for the quark mass matrices were found. These appear in Table I.

TABLE I. Approximate forms for the symmetric textures.

Solution	Y_u, m_ν^D	Y_d
1	$\begin{pmatrix} 0 & \sqrt{2}\lambda^6 & 0 \\ \sqrt{2}\lambda^6 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 4\lambda^3 \\ 0 & 4\lambda^3 & 1 \end{pmatrix}$
2	$\begin{pmatrix} 0 & \lambda^6 & 0 \\ \lambda^6 & 0 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 2\lambda^3 \\ 0 & 2\lambda^3 & 1 \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & \sqrt{2}\lambda^4 \\ 0 & \lambda^4 & 0 \\ \sqrt{2}\lambda^4 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 4\lambda^3 \\ 0 & 4\lambda^3 & 1 \end{pmatrix}$
4	$\begin{pmatrix} 0 & \sqrt{2}\lambda^6 & 0 \\ \sqrt{2}\lambda^6 & \sqrt{3}\lambda^4 & \lambda^2 \\ 0 & \lambda^2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 0 & \lambda^4 \\ 0 & \sqrt{2}\lambda^4 & \lambda^2/\sqrt{2} \\ \lambda^4 & \lambda^2/\sqrt{2} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 2\lambda^4 & 0 \\ 2\lambda^4 & 2\lambda^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

In general the texture zero structure is not unaffected by the running of the renormalization group. Nevertheless the hierarchy structure is preserved, indicating that the texture zeros are kept at least as *phenomenological* ones. We refer the discussion to the following sections.

Let us now turn from the quark masses to lepton and neutrino masses. In the case of the neutrino masses an additional complication, through Majorana masses, arises. The experimentally relevant light neutrino mass matrix is given by

$$m_\nu^{\rm eff} = m_\nu^D (M_{\nu_R})^{-1} m_\nu^{D\dagger}, \tag{15}$$

where m_ν^D is the Dirac neutrino mass matrix and M_{ν_R} the heavy Majorana neutrino mass matrix. The matrices m_ν^D and M_{ν_R} are completely generic, unless we assume a unification that makes the prediction $m_\nu^D \sim m_u$. There are two ways of reasoning behind such an assumption: either by a unified or by a partially unified theory. Such relations are then based on a GUT or a string theory.[3] For the GUT theories, the gauge groups E_6, SU(5), and SO(10) allow for an interesting phe-

[3]One should think that the (heterotic) string theory is the preferred scenario, since it allows the solution of more fundamental problems and also delivers a rich structure of discrete symmetries which may serve to introduce zeros and hierarchical patterns in the mass matrices.

TABLE II. The form of the up quark and Dirac mass matrices in the Harvey-Ramond-Reiss (HRR) ansatz [2].

m_u	m_ν^D
$\begin{pmatrix} 0 & P/V & 0 \\ P/V & 0 & Q/V \\ 0 & Q/V & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & P/V & 0 \\ P/V & 0 & -3Q/V \\ 0 & -3Q/V & 1 \end{pmatrix}$

nomenology. In the case of string gauge groups such as E_6 or subgroups of the same rank (after Wilson line breaking) as well as in the flipped version of SU(5), interesting relations between fermion masses also appear quite naturally. Many of these models contain multiplets[4] that allow for the same structure of the u-quark mass matrix m_u and the Dirac neutrino mass matrix m_ν^D. However, as we mentioned in the Introduction, if the unified gauge group is not of string origin, representations such as the $\overline{\mathbf{126}}$ of SO(10) may be present in the theory and the Dirac neutrino mass matrix is not directly proportional to the up quark mass matrix. Actually, it can even be complex and non-Hermitian [8]. In the model of [2] mentioned above, for example, the nonzero matrix elements of the up quark mass matrix have the form of texture 2 of Table I. Assuming that the elements (2,3) and (3,2) are generated by the $\overline{\mathbf{126}}$ representation of SO(10), while (1,2), (2,1), and (3,3) are obtained by a nonzero VEV of the 10 of SO(10), one finds [2,29] (with $P\approx\sqrt{m_u m_c}/m_t$ and $Q\approx\sqrt{m_c/m_t}$) the modified matrix that appears on the right side in Table II.

In the following analysis, we will find it convenient to demand Hermitian matrices M_{ν_R}. This will enable us to classify the general three exact and any number of phenomenological texture zero solutions and to give some insight into solutions with less exact texture zeros. For the reasons explained above, will further restrict most of the analysis in the $m_\nu^D=m_u$ case, but we will also present few examples for more complicated cases.

Before passing to specific examples we have to discuss the lepton mass matrices, since their diagonalizing matrix enters in the mixing matrix of the charged leptonic currents. In complete analogy to the quark currents the leptonic [Kobayashi-Maskawa (KM)] mixing matrix is

$$V_{\mathrm{tot}}=V_\ell V_\nu^\dagger, \tag{16}$$

where V_ℓ diagonalizes the charged lepton mass matrix, while V_ν diagonalizes the light neutrino mass matrix. Instead of making a specific assumption for the lepton mass matrix or (equivalently) the associated Yukawa couplings, we just treat them as parameters. We will only apply the observation that the charged lepton hierarchical structure usually does not lead to large mixings. E.g., the ansatz in [31], which is case 3 in Table I for the quark masses and $m_e\approx m_d$ up to a numerical factor, has been studied extensively and it was found [32] that the mixing matrix due to charged current interactions, V_ℓ, is given by

$$V_\ell=\begin{pmatrix} 1 & s_3 & -s_2 \\ -s_3 & 1 & s_1 \\ s_2 & -s_1 & 1 \end{pmatrix}, \tag{17}$$

where the parameters $s_{1,2,3}$ are determined in terms of fermion mass ratios. For this ansatz,

$$s_3=6.9\times 10^{-2},$$

$$s_1=3.95\times 10^{-2},$$

$$s_2\sim\frac{m_c}{m_t}\sim 10^{-2}. \tag{18}$$

(Here we omitted possible complex phases, since they should be irrelevant when just discussing the mixing alone.) This indicates that while the e-τ mixing is too small to have any importance for the MSW effect, the e-μ mixing is sufficiently large. The total mixing matrix for the neutrinos is given by (16). This indicates that in this ansatz, even if MSW oscillations cannot be generated only via V_ν, including the mixing coming from the charged current interactions may lead to a solution.

IV. FORM OF THE MAJORANA MASS MATRIX: A FIRST EXAMPLE

In this section we will consider a first example of a model with exact texture zeros, which potentially allows the consistent incorporation of all experimental data. Here we will study the case with a strong mixing in the 2-3 entries of the effective neutrino mass matrix (ν_μ-ν_τ mixing). This will then enable a solution of the atmospheric neutrino problem. For simplicity, we assume that MSW oscillations are generated due to the mixing that arises from the charged current interactions. Furthermore, we want to have nearly degenerate masses. To simplify the analysis, we take the 1-2 and 1-3 mixing angles to be zero in this simple example. The Dirac mass matrix is taken to be given by the Giudice ansatz.

In order to identify the possible forms of the heavy Majorana mass matrix, we start from an effective light mass matrix with a strong mixing. We then investigate which form of the heavy Majorana mass matrix is compatible with the specific form of the neutrino Dirac mass matrix. This is the procedure we will follow in the Appendix, in order to obtain a full range of viable patterns for the heavy Majorana mass matrix. There, we also discuss the issue of the complex phases involved in all the mixings.

We invert (15),

$$m_{\mathrm{eff}}^{-1}=(m_\nu^{D\dagger})^{-1}(M_{\nu_R})(m_\nu^D)^{-1}, \tag{19}$$

to get

$$M_{\nu_R}=m_\nu^{D\dagger}m_{\mathrm{eff}}^{-1}m_\nu^D. \tag{20}$$

where $m_{\mathrm{eff}}^{-1\,\mathrm{diag}}$ is given by

[4] This, however, has to be taken with a grain of salt, since in the (very interesting) case of Wilson line breaking of E_6 in a heterotic string theory this structure is not perpetuated to the broken theory [13,30].

$$m_{\rm eff}^{-1\,{\rm diag}}=\begin{pmatrix}\frac{1}{m_1} & 0 & 0\\ 0 & \frac{1}{m_2} & 0\\ 0 & 0 & \frac{1}{m_3}\end{pmatrix}. \tag{21}$$

With the mixing matrix

$$V_\nu=\begin{pmatrix}1 & 0 & 0\\ 0 & c_1 & -s_1\\ 0 & s_1 & c_1\end{pmatrix}, \tag{22}$$

$m_{\rm eff}^{-1}=V_\nu m_{\rm eff}^{-1{\rm diag}}V_\nu^T$ has the form

$$m_{\rm eff}^{-1}=\begin{pmatrix}\frac{1}{m_1} & 0 & 0\\ 0 & \frac{c_1^2}{m_2}+\frac{s_1^2}{m_3} & c_1s_1\left(\frac{1}{m_2}-\frac{1}{m_3}\right)\\ 0 & c_1s_1\left(\frac{1}{m_2}-\frac{1}{m_3}\right) & \frac{c_1^2}{m_3}+\frac{s_1^2}{m_2}\end{pmatrix}$$

$$\equiv\begin{pmatrix}a & 0 & 0\\ 0 & b & d\\ 0 & d & c\end{pmatrix}. \tag{23}$$

Identifying the entries gives

$$\sin^2 2\theta_1=\frac{4d^2}{(m_2^{-1}-m_3^{-1})^2},$$

$$m_1^{-1}=a,$$

$$m_2^{-1}=\frac{b}{2}+\frac{c}{2}+\frac{1}{2}\sqrt{b^2-2bc+c^2+4d^2},$$

$$m_3^{-1}=\frac{b}{2}+\frac{c}{2}-\frac{1}{2}\sqrt{b^2-2bc+c^2+4d^2}, \tag{24}$$

where θ_1 is the ν_μ-ν_τ mixing angle.

The case of the absolute value of the three masses equal[5] (i.e., $m_1=m_2$, $m_3=-m_2$) is equivalent to

$$b=c=0,\quad a=d. \tag{25}$$

Therefore

$$\sin^2 2\theta_1=1,\quad \theta_1=45°. \tag{26}$$

The form of the heavy Majorana mass matrix may then easily be found from (20). For the Giudice ansatz,[6] where (after rescaling)

$$m_\nu^D=\begin{pmatrix}0 & 0 & x\\ 0 & x & 0\\ x & 0 & 1\end{pmatrix}, \tag{27}$$

we find that

$$\begin{pmatrix}x^2\left(\frac{c_1^2}{m_3}+\frac{s_1^2}{m_2}\right) & x^2\frac{\sin 2\theta_1}{2}\left(\frac{1}{m_2}-\frac{1}{m_3}\right) & x\left(\frac{c_1^2}{m_3}+\frac{s_1^2}{m_2}\right)\\ x^2\frac{\sin 2\theta_1}{2}\left(\frac{1}{m_2}-\frac{1}{m_3}\right) & x^2\left(\frac{c_1^2}{m_2}+\frac{s_1^2}{m_3}\right) & x\frac{\sin 2\theta_1}{2}\left(\frac{1}{m_2}-\frac{1}{m_3}\right)\\ x^2\left(\frac{c_1^2}{m_3}+\frac{s_1^2}{m_2}\right) & x\frac{\sin 2\theta_1}{2}\left(\frac{1}{m_2}-\frac{1}{m_3}\right) & \frac{x^2}{m_1}+\frac{c_1^2}{m_3}+\frac{s_1^2}{m_2}\end{pmatrix}. \tag{28}$$

For the above values of the three light masses this becomes

$$M_{\nu_R}=\begin{pmatrix}0 & M_Nx & 0\\ M_Nx & 0 & M_N\\ 0 & M_N & M_Nx\end{pmatrix}, \tag{29}$$

where $M_N=xd\approx 10^{11}-10^{13}$ GeV. Thus, we see that in this example the degeneracy of all three masses and one large mixing angle is consistent and may be understood in terms of texture zeros of the heavy Majorana neutrino mass matrix M_{ν_R} at the scale M_X. We stress that such three texture zero solutions are maximal. More zeros normally[7] imply a vanishing determinant and less texture zeros are less predictive.

If we only have $c=0$, then the heavy Majorana mass matrix becomes

[5]The fact that these relative signs between the masses are of fundamental importance will be discussed in the Appendix.

[6]The reader should keep in mind that this ansatz differs from case 3 in Table I by two factors of $\sqrt{2}$. Therefore we take here x instead of λ to denote the difference.

[7]See also the discussion about this point in the Appendix.

$$M_{\nu_R}=\begin{pmatrix} 0 & M_N x & 0 \\ M_N x & \frac{b}{d}M_N x & M_N \\ 0 & M_N & \frac{a}{d}M_N x \end{pmatrix}, \tag{30}$$

where $M_N \approx xd$ and it possesses less texture zeros than before.

For the systematic study of the three texture zero solutions, we refer the reader to the Appendix. Here all possible cases of solutions with at least one large mixing angle are given.

Let us now consider an SO(10) covering group, and assume that the entries of the up quark and the Dirac neutrino mass matrices can arise from couplings to scalar fields which belong to the 10 and/or $\overline{\mathbf{126}}$ representations of the group. If only $\overline{\mathbf{126}}$ representations are involved, we have[8] the relation $m_\nu^D \sim m_u$. In the case that both types of representations are involved, one obtains factors of -3 in certain entries of the mass matrix and unity factors in others, resulting in matrices with a slightly different structure. This would lead to predictions for the heavy Majorana mass matrix that differ only by factors from the ones we have presented. In the simple example we gave above, for $b=c=0$, a Dirac neutrino mass matrix of the form

$$M_\nu^D \sim \begin{pmatrix} 0 & 0 & x \\ 0 & -x/3 & 0 \\ x & 0 & 1 \end{pmatrix}, \tag{31}$$

which would result (for the same phenomenological choice of $m_\nu^{\rm eff}$) in the heavy Majorana mass matrix

$$M_{\nu_R} \sim \begin{pmatrix} 0 & M_N x & 0 \\ M_N x & 0 & M_N \\ 0 & M_N & -3M_N x \end{pmatrix}. \tag{32}$$

In the case of the Harvey-Ramond-Reiss (HRR) ansatz of Table II, when the coefficients that appear on the right-hand side of Table II are included, the resulting heavy Majorana neutrino mass matrix for $b=c=0$ and large mixing in the 2-3 sector takes the form

$$M_{\nu_R}=\begin{pmatrix} 0 & -3d\lambda^8 & d\lambda^6 \\ -3d\lambda^8 & a\lambda^{12} & 9d\lambda^4 \\ d\lambda^6 & 9d\lambda^4 & -6d\lambda^2 \end{pmatrix} M_N. \tag{33}$$

The latter should be compared with the matrix obtained with our procedure (solution 2 of Table IV which appears in the following section).

[8]Here, to be accurate, we should note that an overall factor of 3 multiplies the entries of the neutrino mass matrix, since in this case the VEV is pointing parallel to the hypercharge generator with elements unities and -3 in the diagonal entries. However, this does not affect the structure of the matrices and can be absorbed in the overall scale that multiplies the matrices.

Since in our numerical analysis only the form of $m_\nu^{\rm eff}$ enters, and *not* the Dirac or the heavy Majorana mass matrices, we opted to use as few input parameters as possible. These seem to be adequate for the description of the known neutrino properties. Of course, in the most general case, one could also take into account the possibility to have nonsymmetric mass matrices, as it was done in [8]. This, however, goes beyond the scope of the analysis that we present here.

V. STUDY OF VIABLE MAJORANA MASS MATRICES

In this section we will discuss mass matrices at the low energy scale M_W. So far we studied exact texture zeros of neutrino mass matrices at the unification scale M_X. To investigate their impact on mass matrices at M_W, one has to perform a renormalization group analysis. As already mentioned, the exact texture zeros are in general not preserved. Nevertheless, the hierarchical structure is kept. Or, to say it in other words, *exact* texture zeros become *phenomenological* ones. We therefore want to study such phenomenological zeros at M_W here and confront these solutions with the preliminary solutions at M_X in Sec. VII. At this stage we want to stress that a discussion of phenomenological zeros at M_W immediately applies to M_X as well, the reason being the preservation of the hierarchical structure by the renormalization group running between the two scales.

We take the admissible Dirac mass matrices from Table I and study again solutions of (19), assuming one large angle to solve the atmospheric neutrino and drop any further mixing in $m_{\rm eff}^{-1}$. We may then imagine the small mixing (needed for the solar neutrino deficit) to be due to the phenomenological character of the zeros or to reside in the charged lepton mixing. Therefore this approach is less stringent than the one in Sec. IV and the Appendix, where the small mixing at M_X was taken to be zero or physically trivial for all three zero textures. Here we will parametrize the small mixing in the appropriate way.

We start with an atmospheric neutrino mixing residing in the 2-3 submatrix. $m_{\rm eff}^{-1}$ then takes the form (23), which we use as a convenient parametrization. The solutions (20) of (19) allowing for texture zeros[9] in M_{ν_R} are given in Tables III and IV. Textures arising from a large mixing in the 1-2 submatrix appear in Table V. Here $m_{\rm eff}^{-1}$ takes a form similar to (23), where the off-diagonal elements d appear in the 1-2 submatrix.

We now pass to a discussion of the phenomenology induced by the forms of $m_{\rm eff}^{-1}$ that have been quoted. We investigate the case of a large mixing in the 2-3 submatrix. The case of large mixing in the 1-2 submatrix is very much the same and leads to analogous conclusions.

There are two possibilities[10] for texture zero solutions: $b=0$ or $c=0$ that follow.

(i) $c=0$. Imposing this constraint onto (23) suggests a

[9]These zeros are only of phenomenological type mainly due to the effect of the renormalization group (RG) on m_ν^D.

[10]The case $b=c=0$, e.g., has been already discussed for the Dirac mass matrix pattern 3 of Table I in Sec. IV and implies $\xi=1$, and is therefore in accordance with (45).

TABLE III. The texture zero solutions of the Majorana mass matrices associated with each of the Dirac mass textures of Table I with a large mixing in the 2-3 submatrix. We present here cases where either $b=0$ or $c=0$. The nonleading powers are in brackets except for the terms containing the parameter $a=1/m_1$.

Solution	M_{ν_R}/M_N	Comments
1a	$\begin{pmatrix} 0 & 0 & \frac{d}{c}\sqrt{2}\lambda^6 \\ 0 & 2\frac{a}{c}\lambda^{12} & \frac{d}{c}\lambda^4 \\ \frac{d}{c}\sqrt{2}\lambda^6 & \frac{d}{c}\lambda^4 & 1 \end{pmatrix}$	for $b=0$
1b	$\begin{pmatrix} 2\frac{b}{d}\lambda^8 & \sqrt{2}\frac{b}{d}\lambda^6 & \sqrt{2}\lambda^2 \\ \sqrt{2}\frac{b}{d}\lambda^6 & \frac{b}{d}\lambda^4+2\frac{a}{d}\lambda^8 & 1 \\ \sqrt{2}\lambda^2 & 1 & 0 \end{pmatrix}$	for $c=0$
2	$\begin{pmatrix} 0 & \frac{d}{c}\lambda^8 & \frac{d}{c}\lambda^6 \\ \frac{d}{c}\lambda^8 & \lambda^4+\frac{a}{c}\lambda^{12} & \lambda^2\left[+\frac{d}{c}\lambda^4\right] \\ \frac{d}{c}\lambda^6 & \lambda^2\left[+\frac{d}{c}\lambda^4\right] & 1\left[+2\frac{d}{c}\lambda^2\right] \end{pmatrix}$	for $b=0$
3a	$\begin{pmatrix} 0 & \sqrt{2}\lambda^4 & 0 \\ \sqrt{2}\lambda^4 & \frac{b}{d}\lambda^4 & 1 \\ 0 & 1 & 2\frac{a}{d}\lambda^4 \end{pmatrix}$	for $c=0$
3b	$\begin{pmatrix} 2\lambda^8 & \sqrt{2}\frac{d}{c}\lambda^8 & \sqrt{2}\lambda^4 \\ \sqrt{2}\frac{d}{c}\lambda^8 & 0 & \frac{d}{c}\lambda^4 \\ \sqrt{2}\lambda^4 & \frac{d}{c}\lambda^4 & 1+2\frac{a}{c}\lambda^8 \end{pmatrix}$	for $b=0$
4	$\begin{pmatrix} 0 & \frac{d}{c}\sqrt{2}\lambda^8 & \frac{d}{c}\sqrt{2}\lambda^6 \\ \frac{d}{c}\sqrt{2}\lambda^8 & \lambda^4\left[+2\sqrt{3}\frac{d}{c}\lambda^6\right]+2\frac{a}{c}\lambda^{12} & \lambda^2\left[+(1+\sqrt{3})\frac{d}{c}\lambda^4\right] \\ \frac{d}{c}\sqrt{2}\lambda^6 & \lambda^2\left[+(1+\sqrt{3})\frac{d}{c}\lambda^4\right] & 1\left[+2\frac{d}{c}\lambda^2\right] \end{pmatrix}$	for $b=0$
5	$\begin{pmatrix} 0 & \lambda^6 & \frac{1}{2}\lambda^4 \\ \lambda^6 & \sqrt{2}\lambda^4 & \frac{1+2\sqrt{2}}{4}\lambda^2\left[+\frac{b}{d}\frac{1}{\sqrt{2}}\lambda^4\right] \\ \frac{1}{2}\lambda^4 & \frac{1+2\sqrt{2}}{4}\lambda^2\left[+\frac{b}{d}\frac{1}{\sqrt{2}}\lambda^4\right] & 1\left[+\frac{b}{d}\frac{\lambda^2}{2\sqrt{2}}\right]+\frac{a}{d}\frac{1}{\sqrt{2}}\lambda^6 \end{pmatrix}$	for $c=0$

TABLE IV. Cases as in Table II, but with $b=c=0$.

Solution	M_{ν_R}/M_N	Comments
1	$\begin{pmatrix} 0 & 0 & \sqrt{2}d\lambda^6 \\ 0 & 2a\lambda^{12} & d\lambda^4 \\ \sqrt{2}d\lambda^6 & d\lambda^4 & 0 \end{pmatrix}$	for $b=c=0$
2	$\begin{pmatrix} 0 & d\lambda^8 & d\lambda^6 \\ d\lambda^8 & a\lambda^{12} & d\lambda^4 \\ d\lambda^6 & d\lambda^4 & 2d\lambda^2 \end{pmatrix}$	for $b=c=0$
3	$\begin{pmatrix} 0 & \sqrt{2}d\lambda^8 & 0 \\ \sqrt{2}d\lambda^8 & 0 & d\lambda^4 \\ 0 & d\lambda^4 & 2a\lambda^8 \end{pmatrix}$	for $b=c=0$
4	$\begin{pmatrix} 0 & \sqrt{2}d\lambda^8 & \sqrt{2}d\lambda^6 \\ \sqrt{2}d\lambda^8 & 2\sqrt{3}d\lambda^6+2a\lambda^{12} & d\lambda^4+\sqrt{3}d\lambda^4 \\ \sqrt{2}d\lambda^6 & d\lambda^4+\sqrt{3}d\lambda^4 & 2d\lambda^2 \end{pmatrix}$	for $b=c=0$
5	$\begin{pmatrix} 0 & \sqrt{2}d\lambda^8 & \frac{d\lambda^6}{\sqrt{2}} \\ \sqrt{2}d\lambda^8 & 2d\lambda^6 & \frac{d\lambda^4}{2}+\sqrt{2}d\lambda^4 \\ \frac{d\lambda^6}{\sqrt{2}} & \frac{d\lambda^4}{2}+\sqrt{2}d\lambda^4 & \sqrt{2}d\lambda^2+a\lambda^8 \end{pmatrix}$	for $b=c=0$

rewriting in terms of the parameter $\xi=-m_2/m_3>0$. Then

$$c_1=\frac{1}{\sqrt{1+\xi}},\quad s_1=\frac{\sqrt{\xi}}{\sqrt{1+\xi}}, \tag{34}$$

$$m_{\rm eff}^{-1}=\begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1-\xi}{m_2} & \frac{\sqrt{\xi}}{m_2} \\ 0 & \frac{\sqrt{\xi}}{m_2} & 0 \end{pmatrix}, \tag{35}$$

and thus

$$\sin^2 2\theta_1=\frac{4\xi}{(1+\xi)^2}. \tag{36}$$

The neutrino oscillation probabilities are given in terms of the mixing matrix (where the origin of θ_e is undetermined, as already said)

$$V_{\rm tot}=V_e^{\dagger}V_\nu=\begin{pmatrix} c_e & -s_e & 0 \\ s_e & c_e & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{pmatrix}, \tag{37}$$

$$V_{\rm tot}=\begin{pmatrix} c_e & -s_e c_1 & -s_e s_1 \\ s_e & c_e c_1 & c_e s_1 \\ 0 & -s_1 & c_1 \end{pmatrix}, \tag{38}$$

where we take

$$s_e\approx\sqrt{\frac{m_e}{m_\mu}}\approx 0.07,\quad c_e\approx 1. \tag{39}$$

Such an ansatz for the charged leptons is most commonly used [1]. Furthermore, the *block* form (37) seems appropriate to accommodate the data, since (1), (2), (6), and (7) strongly suggest this. A more general ansatz is definitely more difficult to handle.

It is now straightforward to calculate the oscillations $P(\nu_\alpha\to\nu_\beta)$ for (38), using some identities and the general formula from [27]. We thus obtain

$$P(\nu_\mu\to\nu_\tau)=c_e^2\,\frac{4\xi}{(1+\xi)^2}\sin^2\frac{m_2^2(1/\xi^2-1)x}{4E_\nu}, \tag{40}$$

$$P(\nu_e\to\nu_\tau)=s_e^2\,\frac{4\xi}{(1+\xi)^2}\sin^2\frac{m_2^2(1/\xi^2-1)x}{4E_\nu}, \tag{41}$$

$$P(\nu_e\to\nu_\mu)=\sin^2 2\theta_e\left[\frac{1}{(1+\xi)}\sin^2\frac{(m_2^2-m_1^2)x}{4E_\nu}+\frac{\xi}{(1+\xi)}\sin^2\frac{(m_3^2-m_1^2)x}{4E_\nu}-\frac{\xi}{(1+\xi)^2}\sin^2\frac{(m_3^2-m_2^2)x}{4E_\nu}\right]. \tag{42}$$

TABLE V. The texture zero solutions of the Majorana mass matrices associated with each of the Dirac mass textures of Table I with a large mixing in the 1-2 submatrix, for the examples with $b=0$. Only cases for $b=0$ emerge and the solutions for $a=b=0$ follow immediately.

Solution	M_{ν_R}/M_N	Comments
1	$\begin{pmatrix} 0 & 2\frac{d}{c}\lambda^{12} & 0 \\ 2\frac{d}{c}\lambda^{12} & 2\sqrt{2}\frac{d}{c}\lambda^{10}+2\frac{a}{c}\lambda^{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$	for $b=0$
2	$\begin{pmatrix} 0 & \frac{d}{c}\lambda^{12} & 0 \\ \frac{d}{c}\lambda^{12} & \lambda^4+\frac{a}{c}\lambda^{12} & \lambda^2+\frac{d}{c}\lambda^8 \\ 0 & \lambda^2+\frac{d}{c}\lambda^8 & 1 \end{pmatrix}$	for $b=0$
3	$\begin{pmatrix} 2\lambda^8 & 0 & \sqrt{2}\lambda^4 \\ 0 & 0 & \sqrt{2}\frac{d}{c}\lambda^8 \\ \sqrt{2}\lambda^4 & \sqrt{2}\frac{d}{c}\lambda^8 & 1+2\frac{a}{c}\lambda^8 \end{pmatrix}$	for $b=0$
4	$\begin{pmatrix} 0 & 2\frac{d}{c}\lambda^{12} & 0 \\ 2\frac{d}{c}\lambda^{12} & 2\sqrt{6}\frac{d}{c}\lambda^{10}+2\frac{a}{c}\lambda^{12} & \lambda^2+\sqrt{2}\frac{d}{c}\lambda^8 \\ 0 & \lambda^2+\sqrt{2}\frac{d}{c}\lambda^8 & 1 \end{pmatrix}$	for $b=0$
5	$\begin{pmatrix} \lambda^8 & \frac{\lambda^6}{\sqrt{2}} & \lambda^4 \\ \frac{\lambda^6}{\sqrt{2}} & \frac{\lambda^4}{2} & \frac{\lambda^2}{\sqrt{2}}+\sqrt{2}\frac{d}{c}\lambda^8 \\ \lambda^4 & \frac{\lambda^2}{\sqrt{2}}+\sqrt{2}\frac{d}{c}\lambda^8 & 1+2\frac{d}{c}\frac{\lambda^6}{\sqrt{2}}+\frac{a}{c}\lambda^8 \end{pmatrix}$	for $b=0$

(ii) $b=0$. In this case we obtain, with the same parametrization,

$$c_1=\frac{\sqrt{\xi}}{\sqrt{1+\xi}},\quad s_1=\frac{1}{\sqrt{1+\xi}}, \tag{43}$$

$$m_{\rm eff}^{-1}=\begin{pmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\xi}}{m_2} \\ 0 & \frac{\sqrt{\xi}}{m_2} & \frac{1-\xi}{m_2} \end{pmatrix}, \tag{44}$$

and again the expression (36) for $\sin^2 2\theta_1$. The oscillation probabilities for $\nu_\mu \to \nu_\tau$ and $\nu_e \to \nu_\tau$ remain the same. For the oscillation $\nu_e \to \nu_\mu$ we have to substitute $\xi \to 1/\xi$.

Let us now compare these two possibilities for textures with the experimental data. The atmospheric neutrino data implies via (6) that ξ is in between

$$\xi_1=0.23 \quad \text{and} \quad \xi_2=4.4. \tag{45}$$

Since $\xi_1\xi_2=1$ the value of ξ selected merely determines which of the neutrino masses is heavier, as well as the magnitude of the masses. Indeed, from $m_3^2=\delta m^2/(1-\xi^2)$ and $m_2^2=m_3^2-\delta m^2$, we observe that, for a value $\delta m^2 \approx 0.01$ eV^2 as implied by the atmospheric neutrino data, only values of ξ very near unity would give neutrino masses of order ~ 1 eV. In particular, one may see that

$$m_3 \approx m_2 \approx 1 \text{ eV} \quad \text{for } \xi = 0.995. \tag{46}$$

Here we should note that this is found by using the results of [26] which are quoted in the Introduction and are stricter than those of [18]. In this last reference, δm^2 for μ-τ oscillations can be as high as 0.5 eV^2. In this case one finds, e.g.,

$$m_3 = 1.62 \text{ eV}, \quad m_2 = 1.45 \text{ eV} \quad \text{for } \xi = 0.90. \tag{47}$$

After accommodating the atmospheric neutrino data, we turn to a discussion of the solar neutrino numbers, and in this example we interpret them as $\nu_e \rightarrow \nu_\mu$ oscillations. From (42) we may obtain an effective $\sin^2 2\theta_{e\mu}$. Depending on the size of ξ, the $1/(1+\xi)$ or $\xi/(1+\xi)$ term dominates:

$$\xi \ll 1: \quad \sin^2 2\theta_{e\mu} \approx \sin^2 2\theta_e \frac{1}{1+\xi} \approx 1.6 \times 10^{-2}; \tag{48}$$

$$\xi \gg 1: \quad \sin^2 2\theta_{e\mu} \approx \sin^2 2\theta_e \frac{\xi}{1+\xi} \approx 1.6 \times 10^{-2}, \tag{49}$$

when inserting the value of θ_e from (39) and ξ from (45). This is just in agreement with the MSW solution (2). To satisfy the mass constraints, m_1 must be nearly equal to m_2. For an average mass $m_0 \approx 1$ eV, $\delta m_{12}^2 \approx 2m_0|m_2 - m_1| \approx 10^{-5}$ eV^2 indicates the need for a very big degeneracy. Such a high degree of degeneracy is extremely hard to explain from an underlying theory without fine-tuning, unless the masses are forced to such values by symmetries. In Sec. VI we are going to show why this is the case.

Finally we want to discuss neutrinoless double β decay and the COBE data. For the first one, from (8) and (38), we obtain

$$|\langle m_{\nu_e} \rangle| = \left| c_e^2 m_1 + e^{i(\lambda_2 - \lambda_1)} s_e^2 \left(c_1^2 - \frac{s_1^2}{\xi} e^{i(\lambda_3 - \lambda_2)} \right) m_2 \right|, \tag{50}$$

where $e^{i(\lambda_2 - \lambda_1)}$ is the relative *CP* eigenvalue of ν_1 and ν_2 (the masses here are positive). Taking ν_2 and ν_3 to have the same *CP* eigenvalues (as already discussed in Sec. II), we obtain

$$|\langle m_{\nu_e} \rangle| = \left| c_e^2 m_1 + e^{i(\lambda_2 - \lambda_1)} s_e^2 \left(c_1^2 - \frac{s_1^2}{\xi} \right) m_2 \right|. \tag{51}$$

Now we may again study the texture zeros. With (34) we get

$$|\langle m_{\nu_e} \rangle| = c_e^2 m_1 \approx m_1 = \sim 1 \text{ eV}, \tag{52}$$

which is consistent with the bound (8). The above predictions are consistent with the COBE data, as well, since the sum of the masses for the parameter range we indicate can be of order a few eV's, as required. Therefore we conclude that there is no problem in accommodating the experimental data for the phenomenological texture zero solutions. An identical situation occurs when the large mixing which explains the atmospheric neutrino deficit is in the 1-2 entries of the neutrino mass matrices.

VI. DERIVATION OF TEXTURES FROM U(1) SYMMETRIES

In Sec. III, we already gave the motivation for looking for texture zeros, arising due to symmetries in the underlying string or GUT theory. After obtaining a classification of three exact and the general phenomenological texture zero solutions in the Appendix (respectively Sec. V), we want to demonstrate how such patterns come about. Let us consider the possibility of obtaining the above textures from additional U(1) symmetries, following from the work of Ibáñez and Ross (IR) [4] as well as [5,6]. We stress again that such additional U(1) symmetries appear most naturally in string theories (especially at the ''conformal point''). The $\text{U}(1)_{FD}$ charges assigned to the matter fields can be found in IR. They are chosen in such a way as to make the mass matrices symmetric (respectively Hermitian). Moreover, the lighter generation charges are fixed by the need to have anomaly cancellation, which is ensured by taking the U(1) to be traceless. Then one obtains the structure

$$m_u \approx \begin{pmatrix} \epsilon^{|-4\alpha_1 - 2\alpha_2|} & \epsilon^{|-3\alpha_1|} & \epsilon^{|-\alpha_2 - 2\alpha_1|} \\ \epsilon^{|-3\alpha_1|} & \epsilon^{|2(\alpha_2 - \alpha_1)|} & \epsilon^{|\alpha_2 - \alpha_1|} \\ \epsilon^{|-\alpha_2 - 2\alpha_1|} & \epsilon^{|\alpha_2 - \alpha_1|} & 1 \end{pmatrix}, \tag{53}$$

which exhibits the relations

$$m_{11}^u \simeq \frac{(m_{13}^u)^2}{m_{33}^u}, \quad m_{22}^u \simeq \frac{(m_{23}^u)^2}{m_{33}^u}. \tag{54}$$

This structure is consistent with solutions 1, 2, and 4 of the textures shown in Table I. This is because a texture zero in the (1,3) position is correlated with a texture zero in the (1,1) position. In [5,6] a similar analysis had been done to derive the Majorana mass matrices from U(1) symmetries for the case of the up quark matrix, Eq. (53). In this work, we examined the simplest case which arises when adding only one new pair of singlet fields Σ, $\bar{\Sigma}$ with zero hypercharge, but charged under the new U(1) symmetry.

Here, we will show how one can derive by symmetries cases 1 and 3, which seem to have the optimal structure, especially for a heavy Majorana mass matrix with many texture zeros, as we can see from Tables II and III. In [4], the correct u-quark mass matrix is found by making the choice of α_2/α_1, which generates the right order for the nonzero elements of the solutions 1, 2, or 4. By demanding that the powers of the (1,2) and (2,3) matrix elements be in the ratio 3:1 (as needed for solution 2 or 4), $\alpha_2 = 2\alpha_1$ and the u-quark mass matrix has the form

$$m_u \approx \begin{pmatrix} \epsilon^8 & \epsilon^3 & \epsilon^4 \\ \epsilon^3 & \epsilon^2 & \epsilon \\ \epsilon^4 & \epsilon & 1 \end{pmatrix}. \tag{55}$$

Here one also uses the freedom to set $\alpha_1 = 1$ through a redefinition of the parameter ϵ and α_2 [i.e., $\epsilon \rightarrow (\epsilon)^{\alpha_1}, \alpha_2 \rightarrow \alpha_2/\alpha_1$]. Notice that, e.g., in solution 2 the (2,2) element is zero, but it actually does not affect the phenomenology if it is up to ϵ^2. Further study reveals that one may obtain exactly the same hierarchy structure for m_d, where one encounters only a different parameter $\overline{\epsilon}$ [4]. These two matrices closely resemble case 2 in Table I and are in agreement with the data.

For possible choices of the lepton masses, we refer the reader to [4] and the extension in [5]. We remark that $m_u \sim m_\nu^D$ is more or less the simplest choice for the Dirac neutrino masses. But what are the predictions for the Majorana neutrino mass matrix? The most obvious choice leads to the same charge pattern as for the u quarks, with an additional complication coming from the presence of a singlet field. As we have already mentioned, right-handed fields get Majorana masses from a term of the form $\nu_R \nu_R \Sigma$ where Σ is a SU(3)⊗SU(2)⊗U(1) invariant Higgs scalar field with $I_W=0$ and ν_R is a right-handed neutrino. If we assume a Σ field with charge -1, it will make the (2,3) entry of the resulting mass matrix 1. Indeed, what we obtain in terms of $\overline{\epsilon}$ is [5,6]

$$M_{\nu_R} \approx \begin{pmatrix} 0 & \overline{\epsilon^{(-3-1)}} & \overline{\epsilon^{(-4-1)}} \\ \overline{\epsilon^{(-3-1)}} & \overline{\epsilon} & 1 \\ \overline{\epsilon^{(-4-1)}} & 1 & \overline{\epsilon^{(-1)}} \end{pmatrix}, \tag{56}$$

where we have set the smaller entry to zero and have not yet taken the absolute values of the charges in the exponents, since at the next stage we are going to introduce a second singlet field, which alters the structure of the heavy Majorana mass matrix and we want to have the effect of the charge of each field manifest. The matrix in Eq. (56) has a similar structure to the one we derived in (30), for the 2-3 sector (that is the down right 2×2 submatrix). To obtain the desired structure for the complete matrix, we add a second Σ' field which develops a similar VEV and has a quantum number $+2$ under U(1) symmetry. This means that now, in the heavy Majorana neutrino mass matrix, the dominant element will be the one with the biggest absolute power in $\overline{\epsilon}$. I.e., the elements (2,2), (2,3), and (3,3) would still arise mainly due to the couplings to the Σ field with charge -1, while the (1,2) and (1,3) elements arise from the couplings to to Σ'. Then the complete matrix is

$$M_{\nu_R} \approx \begin{pmatrix} 0 & \overline{\epsilon} & \overline{\epsilon^2} \\ \overline{\epsilon} & \overline{\epsilon} & 1 \\ \overline{\epsilon^2} & 1 & \overline{\epsilon} \end{pmatrix}, \tag{57}$$

and the structure would be that of the example in Sec. IV. Actually, this is in fact the solution with only $c=0$ [where the (2,2) element is of order $\overline{\epsilon}$].

Is that generic, in the sense that we may create any mass matrix in that way? Within the simple procedure of adding only a U(1) symmetry and more singlet fields, the answer is probably negative. Nevertheless, going beyond the simple descriptions given above, while assuming more than one U(1) symmetries, the phenomenologically viable Majorana mass matrices obtained in this work may be derived naturally.

After examining how the structure of the heavy Majorana mass matrix may arise, we come back to the generation of the quark mass matrices for the preferred cases 1 and 3. We start with the u-quark mass matrix for the two cases. In case 1 we need the (2,3) element to be zero. This can be done by assuming that the total charge of this entry is half-integer and therefore gets banned by a Z_2 symmetry. In principle, we could choose a large U(1) charge, for this entry, which would make it small. However, the (2,2) entry is the square of the (2,3) entry as we see from (54). Therefore the first choice is the correct one. The (1,2) entry, which is $\epsilon^{|-3\alpha_1|}$, has to be nonzero. Thus α_1 is integer, α_2 half-integer. This implies that not is only the (2,3) entry zero, but also the (1,3). The form of the mass matrix is

$$m_u \approx \begin{pmatrix} \epsilon^{|-4\alpha_1-2\alpha_2|} & \epsilon^{|-3\alpha_1|} & 0 \\ \epsilon^{|-3\alpha_1|} & \epsilon^{|2(\alpha_2-\alpha_1)|} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{58}$$

Setting $\alpha_2 = c(w/2)\alpha_1$, w = odd integer, one may choose w so as to have the (1,1) entry in high power and thus effectively zero, while getting the hierarchical structure between the two nonzero entries (1,2) and (2,2). The relation of the above ϵ to the one in [4] and [5] should be clear from the context.

Case 3 (the Giudice ansatz) may occur in the following way. We need the (1,2) entry to be 0; thus, it is α_1 which we take to be half-integer. Taking α_2 integer, the (2,3) entry is automatically zero and the mass matrix is of the form

$$m_u \approx \begin{pmatrix} \epsilon^{|-4\alpha_1-2\alpha_2|} & 0 & \epsilon^{|-\alpha_2-2\alpha_1|} \\ 0 & \epsilon^{|2(\alpha_2-\alpha_1)|} & 0 \\ \epsilon^{|-\alpha_2-2\alpha_1|} & 0 & 1 \end{pmatrix}. \tag{59}$$

The entry (1,1) is once more effectively zero, since it appears at a high power. The (1,3) and (2,2) entries can be made the same (up to a coefficient) by setting

$$\alpha_2 = 4\alpha_1. \tag{60}$$

After the u-quark mass matrices, we have to tackle the structures for the d-quark mass matrices. Here things are more complicated, but the main idea has already been given in [3]. One has to use different mixings in the light Higgs fields H_1, H_2. In principle one may create any structure from a complicated enough mixing. Here we want to demonstrate that often already simple mixing will do. The general form is

$$H_{1,2}^{\text{light}} = H_{1,2} + \sum_r \left(H_{1,2}^r \frac{\langle\theta\rangle^r}{M_{1,2}^r} + H_{1,2}^{-r} \frac{\langle\overline{\theta}\rangle^{-r}}{M_{1,2}^r} \right), \tag{61}$$

where we denote by $H_{1,2}^r$ a Higgs field[11] carrying a U(1) charge r. Which elements of a specific mass matrix are actually created depends entirely on the terms in the sum on the right-hand side (RHS).

In this way, one can reproduce the down quark mass matrix for case 1. The (1,3) entry then can be almost zero, because it can be related to a higher charge, as both the terms that contain $\alpha_{1,2}$ have the same sign. However, now, the (2,3) entry is no longer zero. There seems to be a small

[11] Here we assumed for simplicity one pair of fields $\theta, \overline{\theta}$ with charge ± 1. One might also have several pairs with different charges and couplings.

problem here. This is that the (2,2) and (2,3) entries are related, and while now we get $(2{,}3)^2 \approx (2{,}2)$, in the textures of Ramond, Roberts, and Ross, they are of the same order. However, note that this can be fixed by a choice of coefficients.

Similarly, we can get case 3 by a suitable mixing. The d-quark mass matrix here is of the form

$$m_d \approx \begin{pmatrix} 0 & \overline{\epsilon^3} & \overline{\epsilon^4} \\ \overline{\epsilon^3} & \overline{\epsilon^2} & \overline{\epsilon} \\ \overline{\epsilon^4} & \overline{\epsilon} & 1 \end{pmatrix}. \tag{62}$$

This structure for the down mass matrix is viable. But is the up? We saw before how we can get it in general (59). We observe that the (1,2) and (2,3) entries, which we want to vanish, are odd numbers, while the others that we have to retain are even. We therefore need a Z_2 symmetry to ban the odd charges in (62). But this can be done easily if there are only fields with even charges in the light Higgs fields (61) that couples to m_u. Then we get

$$m_u \approx \begin{pmatrix} 0 & 0 & \epsilon^4 \\ 0 & \epsilon^2 & 0 \\ \epsilon^4 & 0 & 1 \end{pmatrix}, \tag{63}$$

which just gives the (2,2) entry larger than what we would like. However, the basic structure is the same and coefficients can make it even better.

VII. RENORMALIZATION EFFECTS

Up to now, in Sec. IV (respectively, the Appendix) we found all exact three texture zero solutions at the scale M_X, while in Sec. V we discussed phenomenological texture zero solutions at M_W and found that there is hardly any difficulty in accommodating the experimental data. From the range of solutions for ξ in (45) it is clear that there is no problem in reconciling the solutions at both scales. A natural solution to all the neutrino puzzles may be obtained when the light effective Majorana mass scale is ~ 1 eV. In the context of a grand unified theory such a small scale is obtained by the implementation of the *seesaw* mechanism, resulting in the effective light (Majorana) mass matrix $m_\nu^{\rm eff} = m_\nu^{D^2}/M_{\nu_R}$. As already discussed, in GUT's the scale m_ν^D is usually fixed by the quark mass matrix $m_\nu^D \sim m_Q$; therefore, the right-handed neutrino scale should be around $M_N \sim 10^{12} - 10^{13}$ GeV, i.e., at least three orders smaller than the gauge unification scale $M_X \sim 10^{16}$ GeV. Then, the running of the couplings from the unification scale M_X down to the scale of M_{ν_R} must include possible radiative corrections from ν_R neutrinos. After that scale, ν_R's decouple from the spectrum, and the effective *seesaw* mechanism discussed above is operative. This running, even if of order 1, will not be able to spoil the neutrino hierarchy of the mass matrices. For the phenomenological zeros at M_X this is even more obvious.

It has already been observed that the main result of the presence of ν_R is a 10% effect in the b-τ unification, and that only for small $\tan\beta$ ($h_t \gg h_b$) [19]. It is well known that, in most of the unified gauge groups, the b-τ equality is a standard successful prediction. Indeed, after taking into account renormalization group effects from M_X down to M_W, the correct m_b/m_τ ratio at low energies is obtained naturally if the Yukawa couplings h_b, h_τ are equal at the GUT scale. In the presence of the right-handed neutrino, however, the renormalization group equations (RGE's) get modified for small $\tan\beta$.

Since at this stage we already make a distinction between small and large $\tan\beta$, we should note the following concerning mass matrices: In the simplest scheme (IR) where one tries to derive the known fermion masses from U(1) symmetries, the model is forced to be in the large $\tan\beta$ regime. This is because at the tree level the U(1) quantum numbers of the light Higgs fields H_1, H_2 allow them to couple to the third generation and an effective $SU(2)_l \otimes SU(2)_R$ symmetry of the couplings ensures equal Yukawa couplings $h_b \approx h_t$. Nevertheless, the model is easily modified if there is an additional heavy state $H_i, \overline{H_i}$, $i=1$ or 2, with the same U(1) quantum number. Then mixing effects can generate different h_b and h_t couplings, allowing for any value of $\tan\beta$.

The RGE's for small $\tan\beta$ and for the third generation Yukawa coupling can be approximated as

$$16\pi^2 \frac{d}{dt} h_t = (6h_t^2 + h_N^2 - G_U) h_t, \tag{64}$$

$$16\pi^2 \frac{d}{dt} h_N = (4h_N^2 + 3h_t^2 - G_N) h_N, \tag{65}$$

$$16\pi^2 \frac{d}{dt} h_b = (h_t^2 - G_D) h_b, \tag{66}$$

$$16\pi^2 \frac{d}{dt} h_\tau = (h_N^2 - G_E) h_\tau, \tag{67}$$

where h_N is the largest Yukawa coupling of the right-handed neutrinos. The $G_\alpha = \Sigma_{i=1}^3 c_\alpha^i g_i(t)^2$ are functions that depend on the gauge couplings and the coefficients c_α^i. Below M_N, the right-handed neutrino decouples from the massless spectrum and we are left with the standard spectrum of the MSSM. Thus for scales t beyond M_N the gauge and Yukawa couplings evolve according to the standard renormalization group equations. We may see clearly the effect of the ν_R threshold on the b-τ unification if we write the relation between the Yukawa couplings at the M_{ν_R} scale:

$$h_b(t_N) = \rho \xi_t \frac{\gamma_D}{\gamma_E} h_\tau(t_N), \tag{68}$$

with $\rho = h_{b_0}/h_{\tau_0} \xi_N$ and

$$\gamma_\alpha(t) = \exp\left(\frac{1}{16\pi^2} \int_{t_0}^{t} G_\alpha(t) dt \right), \tag{69}$$

$$\xi_i = \exp\left(\frac{1}{16\pi^2} \int_{t_0}^{t} h_i^2 dt \right), \tag{70}$$

where t_0 is at the high scale M_X. Here $\xi_i \leqslant 1$. In the case of b-τ unification we have $h_{\tau_0} = h_{b_0}$. Thus in the absence of the right-handed neutrino $\xi_N \equiv 1$, which implies $\rho = 1$, and the m_b mass has the phenomenologically reasonable prediction at low energies. However, in the presence of ν_R, if $h_{\tau_0} = h_{b_0}$ at the GUT scale, the parameter ρ is no longer equal to unity since $\xi_N < 1$. In fact the parameter ξ_N becomes smaller for lower M_N scales. Therefore in order to restore the correct m_b/m_τ prediction at low energies we need $\rho = 1$, which corresponds to

$$h_{b_0} = h_{\tau_0} \xi_N. \tag{71}$$

This would seem to alter the relative structure between the mass matrices; however, there exists a natural way to retain the successful b-τ unification, as is predicted by GUT's, with the simultaneous presence of the desired neutrino mass scale M_N to resolve the neutrino puzzles. Such a solution has been proposed in [6] in the context of fermion mass textures predicted by U(1) symmetries. It was found that it is possible to retain the $m_b^0 = m_\tau^0$ GUT prediction of the (3,3) elements of the corresponding mass matrices, provided there is sufficient mixing in the charged lepton mass matrix between the two heavier generations. But this mixing is also what one needs in order to solve the atmospheric neutrino problem.

All this is true for the small $\tan\beta$ regime. In the case of a large $\tan\beta$ the first thing to note is that there are important corrections to the bottom mass from one-loop graphs involving SUSY scalar masses and the μ parameter, which can be of the order of $(30-50)\%$. In addition to this, the effect of the heavy neutrino scale is much smaller, since now the bottom Yukawa coupling also runs to a fixed point; therefore, its initial value does not play an important role. To compare things, we look at the maximal possible effect on the b-τ unification, which would occur for a scale $M_N = 10^{12}$ GeV, and an upper limit for the running bottom mass $m_b = 4.35$. In this case, for the parameter space where $h_t = 2.0$ and $h_b = 0.0125$ lead to a factor $\xi_N = 0.86$, when we set $h_b = 2.0$, $\xi_N = 0.96$. Moreover, for the same example, if we allow for a running bottom mass $m_b = 4.4$, $\xi_N = 0.99$ (remember that the effect of the neutrino is to increase the bottom/τ mass ratio). For higher heavy neutrino scales, the relevant effect is even smaller. However, even for large $\tan\beta$, a strong mixing is also desired in order to solve the atmospheric neutrino problem.

Finally, an additional effect of renormalization effects is that, for large lepton couplings, they amplify the neutrino mixing angle at the GUT scale when going to low energies [33]. This is in the correct direction for a solution to the atmospheric neutrino problem.

VIII. CONCLUSIONS

We have explored the possibility of deriving simple Majorana mass matrices of right-handed neutrinos, which may explain simultaneously all the neutrino experimental data (atmospheric neutrino oscillations, solar neutrino oscillations in the MSW approach, neutrinoless double β decay, and the COBE data). This can be accomplished by assuming the existence of a right-handed neutrino Majorana mass matrix M_{ν_R} with a scale $(10^{12}-10^{13})$ GeV. The solution of the atmospheric neutrino puzzle resides in a large mixing stemming from the neutrino mass matrix. Some type of unification or partial unification implying $m_\nu^D \sim m_u$ was adopted. This is a common relation in successful GUT's, since it minimizes the number of arbitrary parameters and increases the predictability of the theory.

Along these lines, we gave a complete classification of *exact* three zero texture solutions at large scales M_X. These solutions allow just one large mixing.[12] On the other hand, we studied *phenomenological* texture zero solutions at any scale. It was found that there is no problem to reconcile both types of zero solutions with the experimental data.[13] As we see from Table III, in the large $\tan\beta$ case, a natural derivation of the right-handed neutrino mass matrix M_{ν_R} in terms of the low energy constraints is obtained for cases 1 and 3.

The inclusion of renormalization group effects due to the right-handed neutrino threshold does not spoil these observations. The main effect of including a neutrino running coupling is that, retaining the successful $m_b^0 = m_\tau^0$ prediction at the GUT scale, in the simplest schemes, it is now possible only in the large $\tan\beta$ case. In the small $\tan\beta$ scenario, the restoration of b-τ equality at $M_{\rm GUT}$ requires a large mixing in the charged lepton sector between the two heavier families, which is sufficient to solve the atmospheric neutrino puzzle [6]. Interestingly enough, some of the phenomenologically derived mass textures that are presented can be obtained using additional simple U(1) symmetries along the lines of [4], assuming proper U(1) charges for the standard matter fields and additional singlets acquiring vacuum expectation values.

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APPENDIX

In this appendix we carry out the systematic study of the three texture zero solutions that allow one large mixing angle. In particular, we are interested in three zeros, since this is in general the maximally allowed number for three nonzero eigenvalues. Indeed, although there exist three cases of matrices with four texture zeros and three nonvanishing eigenvalues, applying the discussion of Sec. IV to this case leads to an overdetermined set of constraints. Only if there are additional structures (like the block structure of m_ν^D in case 1 of Table I) do solutions exist at all but even in this case no large mixing may be obtained.

The inverse light neutrino Majorana matrix is

$$m_{\rm eff}^{-1} = R m_{\rm eff}^{-1\,{\rm diag}} R^T, \tag{A1}$$

where R are appropriate *rotations*. What type of rotations do we have to study? Since $m_\nu^D \sim m_u$ and m_u is real and

[12] We remark that all such solutions are given in the Appendix, even if they do not allow the required mass degeneracies.

[13] See also previous footnote.

TABLE VI. Possible textures for large ν_e-ν_μ mixing.

M_N matrices for textures of Dirac mass matrix 1				
$\begin{pmatrix} 0 & 0 & c \\ 0 & d & e \\ c & e & 0 \end{pmatrix}$	$c_1^2=\frac{1}{2}$	$e_2=0$	$e_3=0$	$-m_2=m_3$
M_N matrices for textures of Dirac mass matrix 2				
$\begin{pmatrix} 0 & b & c \\ b & 0 & e \\ c & e & 0 \end{pmatrix}$	$c_1^2=\frac{1}{2}$	$e_2\neq 0$	$e_3=0$	$\pm m_1 << -m_2=m_3$
M_N matrices for textures of Dirac mass matrix 3				
$\begin{pmatrix} 0 & b & 0 \\ b & 0 & e \\ 0 & e & f \end{pmatrix}$	$c_1^2=\frac{1}{2}$	$e_2=0$	$e_3=0$	$-m_2=m_3$

symmetric,[14] m_ν^D is real symmetric as well. M_{ν_R} and $m_{\rm eff}$ are complex symmetric, and the complex phases are in general relevant. Trying to absorb them by redefinitions would let them reappear in m_ν^D. However, instead of taking the general R required for the diagonalization of a complex symmetric matrix (Schur rotation), we restrict R to those diagonalizing a real symmetric matrix *and* allow $m_{\rm eff}^{-1\,{\rm diag}}$ to possess negative entries. One may convince oneself that this resumes in general all possible cases. Taking into account also complex phases will only lead to further constraints on solutions. Nevertheless, we have to stress that only including negative eigenvalues for $m_{\rm eff}^{-1}$ allows nontrivial solutions. Thus we only consider

$$R=R_iR_jR_k, \quad i,j,k\in 1,2,3, \text{ not equal}, \tag{A2}$$

where R_i denotes a rotation in the j-k plane, with $i\neq j,k$.

We are now looking for all possible three texture zero solutions that allow at least one large mixing angle.[15] There are 20 possibilities for three texture zeros in M_{ν_R}. Because of the experiment, there has to be one large mixing either in the 1-2 or 2-3 submatrix to explain the atmospheric neutrino data. We restrict ourselves here to the case of one large mixing and take the others to be small. (In the actual numerical study, nevertheless, the precise formulas have been taken.) We therefore to solve

$$M_{\nu_R}=m_\nu^{D\dagger}(Rm_{\rm eff}^{-1\ {\rm diag}}R^T)m_\nu^D, \tag{A4}$$

where

$$R=R_1R_2R_3, \tag{A5}$$

or permutations with

$$R_1=\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{pmatrix},$$

$$R_2=\begin{pmatrix} 1 & 0 & -e_2 \\ 0 & 1 & 0 \\ e_2 & 0 & 1 \end{pmatrix},$$

$$R_3=\begin{pmatrix} 1 & -e_3 & 0 \\ e_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A6}$$

for the large mixing in the 2-3 submatrix and its analogue for the other case. The results are given in Tables VI and VII.

Here, one has to note that the results are dependent of the order in which we multiply the R_i in (76). In general we find that $R=R_1R_2R_3$ exhausts all possibilities and that any permutation is a subclass. But we will now also see why this dependence is somehow trivial. We denote with an asterisk the angles that are associated with two degenerate eigenvalues. As explained in [27], one is able to redefine the physical states in those cases and a mixing has no physical meaning (as is, e.g., the case for all neutrinos massless). Solutions with the mixing being undetermined in this way have been dropped from the tables. The reason for this is that all solutions with degenerate eigenvalues only make sense when the texture zeros are not exact (otherwise the experimental data

[14]In the framework of [3], m_u can always be chosen to be real symmetric. Possible phases reside in m_d and are suppressed in Table I.

[15]Since

$$m_{\rm eff}=Rm_{\rm eff}^{\rm diag}R^T, \tag{A3}$$

we have the same R in (A1).

TABLE VII. Possible heavy Majorana textures for large ν_μ-ν_τ mixing. An asterisk in the table denotes arbitrary angles that are trivial, since they are associated with a mixing of degenerate eigenvalues. If there are only two masses given on the RHS, this implies that the third one is arbitrary. The sign $<>$ means that solutions with $<$ and $>$ are found.

Heavy Majorana textures of Dirac mass matrix 1				
	$c_3^2=\frac{1}{2}$	*	$e_2=0$	$-m_1=m_2=m_3$
		$e_1=0$	*	$m_1=m_3=-m_2$
		$e_1=0$	$e_2=0$	$-m_1=m_2$
$\begin{pmatrix} 0 & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{pmatrix}$	$0<c_3^2<1$	*	$e_2=0$	$-m_1<>m_2=m_3$
		$e_1=0$	*	$m_1=m_3<>\pm m_2$
		$e_1=0$	$e_2=0$	$-m_1<>m_2$
	$c_3^2=\frac{1}{2}$	*	$e_2=0$	$-m_1\approx m_2=m_3$
		$e_1=0$	*	$m_1=m_3\approx -m_2$
		$e_1=0$	$e_2=0$	$-m_1\approx m_2$
$\begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix}$	$0<c_3^2<1$	*	$e_2=0$	$-m_1<>m_2=m_3$
		$e_1=0$	*	$m_1=m_3<>\pm m_2$
		$e_1=0$	$e_2=0$	$\pm m_1<>m_2$
	$c_3^2=\frac{1}{2}$	*	$e_2=0$	$-m_1\approx m_2=m_3$
		$e_1=0$	*	$m_1=m_3\approx -m_2$
		$e_1=0$	$e_2=0$	$-m_1\approx m_2$
$\begin{pmatrix} a & b & 0 \\ b & 0 & 0 \\ 0 & 0 & f \end{pmatrix}$	$0<c_3^2<1$	*	$e_2=0$	$-m_1<>m_2=m_3$
		$e_1=0$	*	$m_1=m_3<>-m_2$
		$e_1=0$	$e_2=0$	$-m_1<>m_2$
Heavy Majorana textures of Dirac mass matrix 2				
$\begin{pmatrix} 0 & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{pmatrix}$	$0<c_3^2<1$	$e_1=0$	$e_2=0$	$(\pm m_1<>\mp m_2)<<m_3$
Heavy Majorana textures of Dirac mass matrix 3				
$\begin{pmatrix} a & 0 & c \\ 0 & 0 & e \\ c & e & 0 \end{pmatrix}$	$0<c_3^2<1$	$e_1=0$	$e_2=0$	$\begin{cases} m_1<<(-m_2<>m_3) \\ (m_1<>-m_2)<<m_3 \\ (m_1<>m_3)>>-m_2 \end{cases}$
Heavy Majorana textures of Dirac mass matrix 4				
$\begin{pmatrix} 0 & b & 0 \\ b & 0 & e \\ 0 & e & f \end{pmatrix}$	$0<c_3^2<1$	$e_1=0$	$e_2=0$	$\begin{cases} (\pm m_1<>\mp m_2)<<m_3 \\ (\pm m_1<>m_3)>>\mp m_2 \\ \pm m_1<<(\mp m_2<>m_3) \end{cases}$
$\begin{pmatrix} 0 & b & 0 \\ b & d & 0 \\ 0 & 0 & f \end{pmatrix}$	$0<c_3^2<1$	$e_1=0$	$e_2=0$	$\begin{cases} (\pm m_1<>\mp m_2)<<m_3 \\ (\pm m_1<>m_3)>>\mp m_2 \\ \pm m_1<<(\mp m_2<>m_3) \end{cases}$

cannot be explained), implying by the smallness of these entries that such an undetermined mixing angle will give a negligible effect.

Let us now summarize and discuss the results of this classification. As can be seen from the tables there are several solutions for different Dirac neutrino mass matrices. It is clear that the three texture zero solutions in the exact form (*exact zeros*) allow *only* one large mixing. All the small mixings are either zero or trivial. Therefore it is necessary that the mixing for the solar neutrino reside in V_{ℓ} in (16). Nevertheless, if the texture zeros are assumed to be only of *phenomenological* nature, additional small mixings might be created. Finally, we point out that one may easily rewrite the found solutions for M_{ν_R} in Tables VI and VII in a form that is an analogue of the solutions of [3]. This had been done, e.g., for the example in (29) in Sec. IV.

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