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A Cosmic String Specific Signature on the Cosmic Microwave Background

R. Moessner[1] L. Perivolaropoulos[2 1] R. Brandenberger[1]

Abstract

Using an analytical model for the string network we show that the kurtosis of cosmic microwave background (CMB) temperature gradient maps is a good statistic to distinguish between the cosmic string model and inflationary models of structure formation. The difference between the stringy and inflationary value for the kurtosis is inversely proportional to the angular resolution and to the number of strings per Hubble volume of the strings' scaling solution. If strings are indeed responsible for CMB anisotropies then experiments with resolutions of a couple of arcminutes or smaller could determine it using this statistic.

[1] Department of Physics, Brown University, Providence, R.I. 02912.

[2]Division of Theoretical Astrophysics, Harvard-Smithsonian Center for Astrophysics, 60 Garden St., Cambridge, Mass. 02138.

1 Introduction

Presently there are two main models for the formation of large-scale structure. In the one, quantum fluctuations originating in an inflationary epoch become classical density perturbations (Guth & Pi 1982; Hawking 1982; Starobinsky 1982; Bardeen, Steinhardt & Turner 1983), which grow by gravitational instability into the structures seen today. In the other, topological defects arising in a phase transition in the early Universe act as seeds for galaxy formation (Kibble 1976; Vilenkin 1985). Galaxy redshift surveys, measurements of peculiar velocities of galaxies and of temperature anisotropies in the cosmic microwave background (CMB) are being used to test these theories. But so far experimental data have not been sufficient to rule in favour of one model or the other, being in more or less good agreement with both (see e.g. Brandenberger 1992).

Our goal is to find, using an analytical approach, a signature of cosmic strings on the CMB which is unique to them and not predicted in inflationary models. Cosmic strings are one-dimensional topological defects which form when a symmetry is spontaneously broken to a smaller subgroup such that the first homotopy group of the vacuum manifold is nontrivial. They are very thin lines of trapped energy with mass per unit length μ; the spacetime around them is flat, but with a deficit angle of $\alpha = 8\pi \mathrm{G}\mu$ (Vilenkin 1981).

Inflationary models predict a gaussian distribution for the CMB temperature anisotropies measured at each angular scale (Efstathiou 1989). Temperature anisotropies from cosmic strings have been shown numerically by Gott et al. (1990) and analytically by Perivolaropoulos (1993b) to be also nearly normally distributed, in spite of the inherently non-gaussian nature of the effect of a single string: it causes linear steplike discontinuities in the microwave sky (Kaiser & Stebbins 1984) (for a study of the statistics of density perturbations due to point-like seeds see Scherrer & Bertschinger (1991)). By the central limit theorem the combined effect of all strings results in a gaussian signal for a large number of strings . However, the probability distribution for temperature *gradients* at small enough angular scales does show a departure from normality (Gott et al. 1990). We use a model for the string network to calculate the gradient probability distribution (actually the moment generating function which is equivalent to it) and show that its kurtosis is a good statistic to discriminate between strings and inflation in CMB measurements at angular scales below a few arcminutes. The model is also implemented as a Monte-Carlo simulation, which is used to obtain actual CMB maps from cosmic strings and check the analytical results.

The reason why the distribution function for temperature gradients is more nongaussian than that for the anisotropies is that a superposition of δ-function perturbations, the gradients of step-functions, approaches the gaussian distribution much more slowly than the superposition of the step-like temperature perturbations themselves.

We are making use of an analytical model (see e.g. Perivolaropoulos (1993a,b)) for the temperature anisotropies caused by strings which the photons encounter between the time of last scattering and today. This model was applied to the study of peculiar velocities by Vachaspati (1992) and Perivolaropoulos & Vachaspati (1993)). It has been used to calculate the power spectrum and the amplitude of CMB anisotropies, as well as their probability distribution and corresponding moments. The temperature fluctuations were found (Perivolaropoulos 1993a) (see also Hara, Mähönen and Miyoshi (1993)) to be well approximated by a scale-invariant Harrison-Zel'dovich spectrum normalized to a value consistent with observations of large-scale structure (Perivolaropoulos, Brandenberger & Stebbins 1990) (for other studies of large scale structure formation by cosmic strings see Hara & Miyoshi (1993); Stebbins et. al. (1987); Vollick (1992)). The distribution function of temperature anisotropies was found (Perivolaropoulos 1993b) to be approximated by a gaussian to an accuracy of better than one percent. These studies were useful tests of the consistency of the cosmic string model with existing CMB data, but did not provide a way to distinguish it from other consistent theories based on inflation.

An alternative to our approach is a full numerical simulation of the evolution of the string network, and a subsequent numerical evolution of the photons through it (Bennett, Stebbins & Bouchet 1992). An advantage of our analytical approach is that the explicit dependence on string parameters and angular scale can be shown, and the statistical fluctuations around the predicted values for actual experiments, due to having only a finite number of measurements, can be calculated.

2 Theoretical Model and Experimental Setup

According to the scaling solution for cosmic strings (Bennett & Bouchet 1988; Allen & Shellard 1990), there is a fixed number M$\approx$ 10 of strings with curvature radius of the order of the horizon per Hubble volume at any given time. Their orientations and velocities are uncorrelated over distances larger than the horizon, and they move with an rms velocity of $v_s = 0.15c$. In our model the strings are approximated as straight, so that the Kaiser-Stebbins formula for the temperature anisotropy due to the deficit angle of spacetime around the string can be used (Stebbins 1988):

$$\delta T/T = \pm 4\pi G\mu \mid \hat{k} \cdot (\gamma_s \vec{v_s} \times \hat{e_s}) \mid \tag{1}$$

where $\hat{k}$ is the direction of observation, $\vec{v_s}$ is the velocity of the string, $\hat{e_s}$ its orientation, and $\gamma_s = (1 - v_s/c)^{-1/2}$. Photons passing in front of the string are redshifted, while those passing behind it are blueshifted. The deficit angle extends only out to a distance of one Hubble radius (H^{-1}) from the string; this is due to compensation (Traschen, Turok & Brandenberger 1986; Veeraraghavan & Stebbins 1990; Magueijo 1992): background matter redistributes itself when strings are formed so that the monopole and dipole moments of the energy density prturbations are zero. Therefore photons passing a string at a distance larger than the horizon scale are not affected by it. This cutoff is quite sharp because of gravitational shock waves caused by the compensating underdensities (Magueijo 1992).

The time between now and last scattering is divided into N Hubble times t_i with $t_{i+1} = 2t_i$, and $N = log_2(t_0/t_{ls}) \approx 15$ for a redshift of last scattering $z_{ls} = 1000$. The apparent angular size of a Hubble volume at time t_i is given by $\theta_{H_i} \sim z_i^{-1/2} \sim t_i^{1/3}$ for large redshifts in the matter dominated era and assuming $\Omega_0 = 1$. Therefore, $\theta_{H_{i+1}} = 2^{1/3}\theta_{H_i}$, and $\theta_{H_{ls}} = z_{ls}^{-1/2} rad = 1.8^\circ$. The strings encountered by the photon at subsequent Hubble times are assumed to be uncorrelated. This is justified because after a time t_i a photon has traversed a large part of its original Hubble volume, so that at $2t_i$ its new Hubble volume is mostly new space. Moreover a string moves a distance $v_s t_i = 0.15ct_i$ which is approximately equal to the strings' mean separation, so that it is likely to encounter another string and intercommute with it. At each t_i we assume the M string segments (each of length $2H_i^{-1}$ since the strings' orientations and velocities are uncorrelated over larger distances) per Hubble volume to have random positions, orientations and velocities.

The effect of all strings together is taken to be a superposition of the effect of the individual long strings. Finally, the effect of loops is considered unimportant compared to that of long strings, and the initial temperature inhomogeneities at the surface of last scattering are assumed to be negligible compared to those induced by the string network at later times.

Neglecting the effect of loops can be justified as follows: The typical loop size is about 10^{-4} of the Hubble radius (Bennett & Bouchet 1988 ; Allen & Shellard 1990). At recombination this subtends an angle of $0.7''$. Only at a redshift $z \sim 2$ the angle becomes equal to $18''$, the smallest scale which we consider in this paper. Thus, the loops are too small for the Kaiser-Stebbins effect to apply. Instead, loops contribute to $\delta T/T$ via the usual Sachs-Wolfe integral (Sachs & Wolfe 1967), and the result of this integral is given by $\delta T/T = \frac{1}{3}\Phi$, where Φ denotes the relativistic potential at $t_{\rm ls}$. Since on scales corresponding to a beam width of b$\geq 18''$ many small loops contribute to Φ, the combined effect can (by the central limit theorem) be expressed as gaussian noise. We will estimate the magnitude of this noise in Section 6 and demonstrate that it is indeed negligible, in agreement with the conclusions of Stebbins (1993).

Individual cosmic strings produce linelike discontinuities and plateaux in the microwave background. These are features which are most easily seen in temperature gradient maps. Experiments could either make line surveys or map out the temperature over a certain region of the sky, say a square of size $\theta^\circ \times \theta^\circ$. Small angular resolutions b provide a better chance of seeing these features. We will quantify this statement in the following sections by calculating the moment generating function for the temperature difference between neighbouring areas that can be resolved by the experiment.

Fig.1 shows the result of a simulation for the temperature anisotropies expected in an experiment with $b = 80''$ and $\theta = 2.2^\circ$ (i.e.100×100 pixels). At each Hubble time t_i n_i string segments of length θ_{H_i} were placed at random over an area of size $(\theta + \theta_{H_i})^2$ (see Fig.2). By the scaling solution

$$n_i = M(\theta + \theta_{H_i})^2/\theta_{H_i}^2 \tag{2}$$

Note that the discontinuity lines seen in the map are not all at the position of the strings themselves. Each string at each of the 15 Hubble times gives rise to five such lines, one at its position, two at a distance H^{-1} parallel to it, and another two because it has finite length. Since strings are one-dimensional objects, a square region with sides equal to one quarter of the horizon is crossed by about $M/4$ strings. By equation (1) the string shown changes the temperature of region A by βr , and that of region B by $-\beta r$, where

$$\beta \equiv 4\pi G\mu\gamma_s v_s \tag{3}$$

and $r = |\hat{k} \cdot (\hat{v_s} \times \hat{e_s})|$, $\hat{v_s} = \vec{v_s}/v_s$. The direction of observation $\hat{k}$ is approximately constant over the whole observed area, and $\hat{e} = \hat{e_s} \times \hat{v_s}$ is a random unit vector since the strings' orientations and velocities are random, so that $r = |\hat{k} \cdot \hat{e}|$ is uniformly distributed over the interval [0,1]. Each beam is assigned the temperature averaged over the area covered by it.

3 Moment Generating Function for Temperature Gradients

Let X be the random variable (RV) whose realizations are the possible values for the temperature differences between two neighbouring beams (pixels). Our goal is to derive its distribution function. We do so by calculating the moment generating function $M_X(t)$ of X. The moment generating function of a RV is an important concept in statistics, it is defined by $M_X(t) =< e^{tX} >$, where <> denotes the expectation value. The jth moment $\mu_j =< X^j >$ of X is obtained from $M_X(t)$ by differentiation:

$$\mu_j = \left(\frac{d^j}{dt^j}\right)_{t=0} M_X(t) \tag{4}$$

An inversion formula gives the (cumulative) distribution function $F_X(x) \equiv P(X \leq x)$, the probability that $X \leq x$ (Lukacs 1960):

$$F_X(x+\delta) - F_X(x-\delta) = \lim_{T\to\infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin t\delta}{t} e^{-itx} \phi_X(t) dt \tag{5}$$

provided that $x + \delta$ and $x - \delta$ are continuity points of F. $\phi_X(t) = M_X(it)$ is called the characteristic function of X. Equation (5) is the general formula for a RV that need neither be absolutely continuous nor purely discrete. If it is continuous, it can be shown from (5) that a probability density $p(x) = F'(x)$ exists and that it is the Fourier transform of $\phi_X(t)$.

In the remaining part of this section we will derive $M_X(t)$. The reader who is not interested in the details is advised to jump to equation (21) and continue reading from there.

Since the temperature difference results from the superposition of the effects of the individual strings, we can decompose X into $X = \sum_{i=1}^{15} X^i$, where X^i is the RV for the temperature difference due to all the strings at Hubble time t_i. In turn we can write X^i as $X^i = \sum_{j=1}^{n_i} X^i_j$, where X^i_j is the RV for the temperature difference due to the jth string at time t_i.

Now the moment generating function has the nice property that $M_{X+Y}(t) = M_X(t)M_Y(t)$ if X and Y are independent RV. Since the X^i_j are independent in our model, $M_X(t)$ can be written as

$$M_X(t) = \prod_{i=1}^{15} M_{X^i}(t) \quad , \qquad M_{X^i}(t) = \prod_{j=1}^{n_i} M_{X^i_j}(t) \tag{6}$$

Therefore only $M_{X^i_j}(t) = < e^{tX^i_j} >$ has to be calculated, which is done below by determining the possible values for X^i_j together with their probabilities.

The two beams have the same temperature if they both lie outside the string's region of influence or both in region A or B (see Fig. 2). This is true because the observation direction $\hat{k}$ is almost exactly the same for both beams for $b \leq \theta_{H_{ls}}$. We are only interested in such small angular resolutions because the simulations have shown that for larger ones the sought-after non-gaussian signature disappears. Moreover we will find from the analytical result that the non-gaussian signature goes to zero as b tends to $\theta_{H_{ls}}$. The beams have unequal temperatures if either the string itself or one of the lines of discontinuity at Hubble volume cutoff (see dotted lines in Fig.2) pass through either or both of the beams.

First consider the string itself. The probability for it to cross the right beam (see Fig.3 and Fig.2) is

$$p^i \approx \frac{\theta_{H_i}/b}{((\theta_{H_i} + \theta)/b)^2} \tag{7}$$

for $b \ll \theta_{H_i}$. If it crosses the right beam it has uniform probability to be a distance $n\frac{b}{2}, n \in [0,1]$, away from its centre, i.e. to be tangent to a circle C_n of radius $n\frac{b}{2}$ around the centre. If it is tangent to C_n to the right of the beam's centre and such that it does not cross the left beam (i.e. somewhere along the arc between points A and B), then the possible values for X^i_j are (for r=1)

$$X^i_j = \pm\beta(n-1) \tag{8}$$

each with probability $p^i \frac{p_1}{2}$, where p_1 is the probability for a string which is tangent to C_n to be so between A and B. To obtain Eq.(8) we have used that $b \ll \theta_{H_i}$ and approximated the circular beams by squares:

$$X^i_j = \pm[\frac{1}{b^2}(b(\frac{b}{2} - n\frac{b}{2})\beta - b(\frac{b}{2} + n\frac{b}{2})\beta) - (-\beta)] = \pm\beta(1-n) \tag{9}$$

From Fig.3 it can be seen that

$$p_1 = \frac{\alpha_1}{\pi} = \frac{1}{\pi}\arccos(\frac{1-n}{2}) \tag{10}$$

Similarly one obtains

$$X^i_j = \pm\beta(n+1) \tag{11}$$

if the string is tangent to C_n to the left of the beam's centre such that it does not cross the left beam (i.e. between C and D), each with probability $p^i \frac{p_2}{2}$, where

$$p_2 = \frac{1}{\pi}\arccos(\frac{n+1}{2}) \tag{12}$$

Finally, if the string is tangent to any other point of the circle, it passes through the left beam, and it has uniform probability to do so at distances $m\frac{b}{2}$ away from that beam's centre, $m \in [0,1]$. For a given m the possible values are

$$X^i_j = \pm\beta(n+m) \quad \text{and} \pm\beta(n-m) \tag{13}$$

each with probability $p^i \frac{p_3}{4}$, and $p_3 = 1 - (p_1 + p_2)$. Since we could have started with the other beam, all the probabilities given above for the values of X^i_j have to be multiplied by 2.

For each of the other four lines of discontinuity the same results apply, but with β replaced by $\beta/2$. The probability for $X^i_j = 0$ is $1 - 2p_i(1+4) = 1 - 10p^i$. Finally, r is uniformly distributed over the interval [0,1], so $X^i_j \to rX^i_j$.

Putting all of this together (and setting $\beta = 1$) we arrive at

$$M_{X^i_j}(t) =$$

$$1 - 10p^i + 2p^i[\int_0^1 dr \int_0^1 dm \int_0^1 dn \frac{p_3}{4}(e^{(n+m)tr} +$$

$$
\begin{aligned}
&e^{-(n+m)tr} + e^{(n-m)tr} + e^{-(n-m)tr} + \\
&4(e^{(n+m)tr/2} + e^{-(n+m)tr/2} + e^{(n-m)tr/2} + e^{-(n-m)tr/2})) + \\
&\int_0^1 dr \int_0^1 dn\{\frac{p_1}{2}(e^{(n-1)tr} + e^{(-(n-1)tr} + \\
&4(e^{(n-1)tr/2} + e^{-(n-1)tr/2})) + \\
&\frac{p_2}{2}(e^{(n+1)tr} + e^{-(n+1)tr} + 4(e^{(n+1)tr/2} + e^{-(n+1)tr/2}))\}]
\end{aligned} \tag{14}
$$

We approximate p_1,p_2 and p_3 as constant over the interval $0 \leq n \leq 1$. For $p_1 = p_3 = 0.4$ and $p_2 = 0.2$ the integrals can be done with the result

$$
\begin{aligned}
M_{X_j^i}(t) &= 2p^i\{0.4\frac{1}{t^2}(-\sinh^2(t) - 16\sinh^2(\frac{t}{2})) + \\
&\frac{1}{t}(0.6\mathrm{shi}(2t) + 5\mathrm{shi}(t) + 1.6\mathrm{shi}(\frac{t}{2}))\} + 1 - 10p^i
\end{aligned} \tag{15}
$$

where $\mathrm{shi}(t) = \int_0^t dx \frac{\sinh x}{x}$. The next step is to calculate

$$
M_{X^i}(t) = \prod_{j=1}^{n_i} M_{X_j^i}(t) = (M_{X_j^i}(t))^{n_i} \tag{16}
$$

since the X_j^i are identically distributed. Define

$$
\lambda^i \equiv p^i n_i \approx M\frac{b}{\theta_{H_i}} \tag{17}
$$

Then

$$
M_{X^i}(t) = \left(1 + \frac{2\lambda^i(f(t) - 5)}{n_i}\right)^{n_i} \longrightarrow e^{2\lambda^i(f(t)-5)} \tag{18}
$$

for n^i large, which is the case here (f(t) is the function inside the curly brackets of Eq.(15)). This approximation makes it possible to do the final sum over Hubble times:

$$
M_X(t) = \prod_{i=1}^{15} M_{X^i}(t) = e^{2\sum_{i=1}^{15}\lambda^i(f(t)-5)} \tag{19}
$$

Since

$$
\sum_{i=1}^{15}\lambda^i = Mb\sum_{i=1}^{15}\frac{1}{\theta_{H_{ls}}2^{(i-1)/3}} = 4.7\frac{Mb}{\theta_{H_{ls}}} \equiv 4.7\lambda \tag{20}
$$

the final expression for the moment generating function is

$$M_X(t) = \\ e^{9.4\lambda(0.4\frac{1}{t^2}(-\sinh^2(t) - 16\sinh^2(\frac{t}{2})))} \\ \times e^{9.4\lambda(\frac{1}{t}(0.6\mathrm{shi}(2t) + 5\mathrm{shi}(t) + 1.6\mathrm{shi}(\frac{t}{2})) - 5)} \tag{21}$$

4 Probability Distribution and Kurtosis

Fig.4 shows a plot of frequencies of temperature differences for $b = 18''$ obtained from the moment generating function $M_X(t)$ via the inversion formula Eq.(5) and from a simulation. The two curves are in good agreement. A gaussian distribution is shown for comparison.

The kurtosis of the distribution of temperature gradients is a good statistic to distinguish between the string model and inflationary models. The kurtosis, defined as

$$k_4 \equiv \frac{<(X- <X>)^4}{<(X- <X>)^2>^2} \tag{22}$$

measures the peakedness of a distribution and how far out its tails extend. But this is how a gradient distribution caused by strings differs from a gaussian one (which is predicted by inflationary models): The sharp discontinuities along the location of the string produce a higher number of large gradients, while the plateaux give a higher number of small gradients than would be expected in the gaussian case.

For a RV with zero mean

$$k_4 = \frac{\mu_4}{\mu_2^2} \tag{23}$$

where μ_j is the jth moment given by Eq.(4). A Taylor expansion of $M_X(t)$ around $t = 0$ gives

$$k_4 = 3 + \frac{0.14}{\lambda} \tag{24}$$

(see Fig.5). The gaussian value for the kurtosis is 3. Recall that $\lambda = M\frac{b}{\theta_{H_{ls}}}$. So the departure from the gaussian value is inversely proportional to angular resolution and to the parameter M describing the string network. This is the main result of our paper. For $z_{ls} = 1000$ and $M = 10$ we have

$$k_4 = 3 + \frac{1.5}{\text{b (in arcminutes)}} \tag{25}$$

This shows that for beam sizes of a few arcminutes or smaller the string model predicts a non-gaussian kurtosis, whereas for larger beam sizes values very close to the gaussian one are predicted; consequently the latter experiments cannot be used to discriminate between strings and inflation using this statistic. Experiments with small angular resolutions have not yet detected CMB anisotropies. In the next section we discuss some of them to see whether they should in future measure a nongaussian value for k_4 if the anisotropies are caused by strings.

5 Confrontation with Observation

Experiments only make a finite number of measurements and therefore do not measure the true moments. In order to distinguish between different models the distribution of the measured quantities around the exact ones has to be determined in each model.

An experiment as described in Section 2 makes $n = (\frac{\theta}{b})^2$ measurements of temperature differences between beams lying next to each other along a particular direction. We can interpret the ith measurement as one realization of the RV $X^{(i)}$, where $X^{(i)}$ is the RV X of the previous sections for the ith pair of beams. If we define

$$Y^i = \frac{(X^i - < X^i >)^4}{< (X^i - < X^i >)^2 >^2} \qquad i = 1, \cdots, n \tag{26}$$

then the $Y^{(i)}$ are identically distributed, $\frac{1}{n}\sum_{i=1}^n Y^{(i)}$ is the measured kurtosis k_4^m, $< Y^{(i)} >$ is the true kurtosis k_4 of the underlying distribution, and

$$Var(Y^{(1)}) \equiv < (Y^{(1)})^2 > - < Y^{(1)} >^2 = k_8 - k_4^2 \tag{27}$$

where $k_8 = \frac{\mu_8}{\mu_2^4}$ is the normalized eighth moment. By the central limit theorem the RV

$$\frac{\sum_{i=1}^n (Y^{(i)} - < Y^{(i)} >)}{\sqrt{n}\sqrt{Var(Y^{(1)})}} \tag{28}$$

is normally distributed with unit variance for large n (Feller 1971), i.e.

$$P\left(\frac{\sum_{i=1}^n (Y^{(i)} - < Y^{(i)} >)}{\sqrt{n}\sqrt{Var(Y^{(1)})}} < \epsilon\right) = \Phi(\epsilon) \quad , \quad \Phi(\epsilon) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\epsilon} dx e^{-x^2/2}$$

$$\Leftrightarrow \quad P\left(\frac{1}{n}\sum_{i=1}^{n} Y^{(i)} - < Y^{(1)} > \quad < \epsilon\sqrt{\frac{Var(Y^{(1)})}{n}}\right) = \Phi(\epsilon)$$

$$\Leftrightarrow \quad P(k_4^m - k_4 \quad < \epsilon \ ) = \Phi\left(\epsilon\sqrt{\frac{n}{Var(Y^{(1)})}}\right) \tag{29}$$

From Eqs.(27) and (29) we can see that the measured kurtosis is normally distributed around the true value with variance $\sigma^2 = \frac{k_8 - k_4^2}{n}$. For a gaussian distribution $k_8^g = 105$, while for a stringy distribution the Taylor expansion of $M_X(t)$ to eighth order gives

$$k_8 = 105 + \frac{29}{\lambda} + \frac{1.8}{\lambda^2} + \frac{0.017}{\lambda^3} \tag{30}$$

Fig.6 shows the distribution of the measured kurtosis about the predicted values in the two models for an experiment by Fomalont et al. (1990), who map out two areas each of size $7' \times 7'$ with angular resolutions ranging from $10''$ to $80''$. For a resolution of $18''$ a clear nongaussian signal is predicted, while for $b = 80''$ the number of measurements is not sufficient to discriminate between the two models. If n were larger however, it would be possible to do so at $80''$.

A second experiment we want to discuss is one proposed by the Cambridge group which is to start construction in the second half of 1993 (Lasenby 1992). It intends to map out an area of size $10° \times 10°$ at resolutions ranging from $10'$ to $2°$, and it was planned, among other things, to look for possible signatures from topological defects. Fig.7 shows that for its highest resolution of $10'$ and $M = 10$ it would not be able to tell whether the underlying distribution is gaussian or due to strings. It would be nice if this new experiment could be designed also to make measurements at the scale of 1 arcminute, in which case it would be able to rule in favour of or against strings. Fig.7 also shows a curve for a different value of the string scaling solution parameter M. If M should indeed be as low as 3 then strings would give a nongaussian kurtosis even at $10'$.

Based on our statistic, the Owens Valley experiment (Readhead et al. 1989) with a beam width of $108''$ could detect a signal for cosmic strings if a map of contiguous patches were produced. Also, Melchiorri et al. (1993) are planning a 3 m telescope balloon experiment with a beam width of $2'$ which will have the potential of detecting non-gaussian signatures from cosmic strings.

At $10'$ the nongaussian signature has all but disappeared, so that for experiments done at degree scale or at several degrees (e.g. COBE where $b \approx 7^\circ$), strings do *not* predict a nongaussian distribution of either temperature gradients or temperature values.

6 Effect of Noise

Gaussian noise diminishes the nongaussian signature expected from strings alone. The moment generating function for the temperature difference between two neighbouring beams due to gaussian noise is

$$M_Y(t) = e^{\frac{s^2 t^2}{2}} \tag{31}$$

where s^2 has to be determined for each kind of noise. The stringy moment generating function with $\beta = 4\pi G\mu\gamma_s v_s$ not set equal to 1 is

$$\begin{aligned} M_X(t) = & \\ & e^{9.4\lambda(0.4\frac{1}{(t\beta)^2}(-\sinh^2(\beta t) - 16\sinh^2(\frac{\beta t}{2})))} \\ & \times e^{9.4\lambda(\frac{1}{\beta t}(0.6\mathrm{shi}(2\beta t) + 5\mathrm{shi}(\beta t) + 1.6\mathrm{shi}(\frac{\beta t}{2})) - 5)} \end{aligned} \tag{32}$$

The measured temperature difference is given by X+Y, and since X and Y are independent, $M_{X+Y}(t) = M_X(t)M_Y(t)$. Without noise the second moment of the probability distribution for temperature differences is $\mu_2 = 5.43\lambda\beta^2$ and the kurtosis is $k_4 = 3 + \frac{0.14}{\lambda}$. With gaussian noise the second moment is

$$\mu_2^{\mathrm{noise}} = 5.43\lambda\beta^2 + s^2 \tag{33}$$

and the kurtosis is

$$k_4^{\mathrm{noise}} = 3 + \frac{0.14}{\lambda}(1 + \frac{s^2}{5.43\lambda\beta^2})^{-2} \tag{34}$$

If $\mu_2 = s^2$ the nongaussian part of the kurtosis is reduced by a factor of four. This noise can either be instrumental or come from density perturbations at last scattering. Instrumental noise is uncorrelated, so that $s^2 = 2\sigma^2$ if the noise has variance σ^2 at each pixel. Gaussian noise from last scattering is correlated, and some cold dark matter models predict the correlation function (Gott et al. 1990)

$$C(\theta) = C(0)\frac{1}{1+\frac{\theta^2}{2\alpha^2}} \tag{35}$$

for the temperature anisotropies (RV T) for scales $\theta < 2^\circ$, where $\alpha = 8'$ is the coherence angle, and $C(0) = \sigma^2$ is the variance of T. We can approximate the temperature difference between two neighbouring beams by $Y = bT'$ and use the fact that if a gaussian RV T has the correlation function $C(\theta)$, then its derivative T' has a gaussian distribution with variance $-C''(0)$ (Vanmarcke 1983). Since $M_{bX}(t) = M_X(bt)$, $s^2 = -b^2C''(0) = \frac{b^2}{\alpha^2}\sigma^2$. A more careful analysis shows that this approximation is good for $b < \alpha$. Fig.8 shows the dependence of the kurtosis on noise from density perturbations at last scattering, and Fig.9 for instrumental noise. For resolutions of a few arcminutes or below, noise from density perturbations at last scattering does not pose much of a problem because it is correlated, but instrumental noise can wash out the stringy signal if it is too large.

As an example, we can use the above results to estimate the effect of cosmic string loops on the kurtosis of the CMB spatial gradient map. As discussed in section 2, loops contribute as gaussian noise via the Sachs-Wolfe integral.

The Sachs-Wolfe integral due to cosmic string loops has been estimated in Brandenberger and Turok (1986) (see also Traschen, Turok and Brandenberger (1986)). In that work it was shown that the largest loops at $t_{\rm ls}$ dominate the Sachs-Wolfe integral. The result for $\Delta T(\theta)$, the temperature difference between two points separated by an angle θ, scales as $\alpha^{1/4}$, where αt is the radius of the largest loops at time t. Taking $\alpha = 1$ and using $G\mu = 2 \cdot 10^{-6}$ gave the value

$$\left\langle \left(\frac{\Delta T}{T}\right)^2 \right\rangle^{1/2} \sim 5 \cdot 10^{-5} \sin^{1/2}\frac{\theta}{2} \qquad\qquad \theta < 0.5^\circ \tag{36}$$

For $\theta = 18''$, corresponding to the beam width of the VLA experiment, and with $\alpha = 10^{-2}$ we obtain a value of $1.5 \cdot 10^{-7}$. So the gaussian RV Y for the temperature gradients from loops has variance

$$s^2 = \left\langle \left(\frac{\Delta T}{T}\right)^2 \right\rangle (\theta = b) \tag{37}$$

From Fig.9 it follows that the effect of loops on the kurtosis is negligible.

7 Discussion

We have looked at a statistic which can distinguish strings from inflation and which can also be calculated analytically in a model for the string network, showing the dependence on string parameters. This is useful as a complementary approach to numerical simulations which have to deal with singularities in the evolution equations of the strings leading to some uncertainty about the results. The statistic investigated here might not be the most sensitive one, we intend to look at others in the future. The genus curve (Gott et al. 1990) for example, a topological statistic, makes use of the two-dimensional pattern of the anisotropies. A multifractal analysis (Pompilio 1993) is sensitive to higher moments of the temperature gradients.

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9 Figure Captions

Figure 1 : Simulated map of CMB temperature anisotropies from strings for a resolution of $80''$. The map has size $2.2° \times 2.2°$ (100×100 pixels). There are M=10 strings per Hubble volume, and units are such that $\beta = 4\pi G\mu\gamma_s v_s = 1$. The colour scale ranges from $\delta T/T$ = -15 to +15.

Figure 2 : Model for the effect of a single string on the CMB temperature of a patch of sky of size $\theta \times \theta$ mapped out by circular beams. Strings located within the big square of size $(\theta + \theta_H)^2$ can affect the temperature of the small patch.

Figure 3 : Calculation of temperature difference of two adjacent beams due to one string.

Figure 4 : Frequencies of temperature differences X (in units where $\beta = 1$) for $b = 18''$ and a binsize of 0.2 for strings. The gaussian fit is chosen to have the same variance as the result for strings.

Figure 5 : Kurtosis of the distribution of temperature differences between adjacent beams expected for strings (with $M = 10$) as a function of beam size b.

Figure 6 : Probability density functions of kurtosis of temperature gradients measured in the experiment by Fomalont et al. for an underlying stringy (for M = 10) and gaussian distribution.

Figure 7 : Probability density of kurtosis measured in the experiment by Lasenby at $10'$ for strings (for $M = 10$ and $M = 3$) and an underlying gaussian distribution.

Figure 8 : Dependence of kurtosis of temperature gradients on noise from density perturbations at last scattering giving rise to temperature anisotropies with variance σ^2 at each pixel. Here and in the next figure the values $M = 10$, $G\mu = 2 \times 10^{-6}$ and $\gamma_s v_s = 0.15$ are used.

Figure 9 : Dependence of kurtosis of temperature gradients on instrumental noise (with variance σ^2 at each pixel).

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