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Anomalous $U(1)$'s in type I string vacua

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Abstract

We perform a systematic string computation of the masses of anomalous $U(1)$ gauge bosons in four-dimensional orientifold vacua, and we study their localization properties in the internal (compactified) space. We find that $N = 1$ supersymmetric sectors yield four-dimensional contributions, localized in the whole six-dimensional internal space, while $N = 2$ sectors give contributions localized in four internal dimensions. As a result, the $U(1)$ gauge fields can be much lighter than the string scale, so that when the latter is at the TeV, they can mediate new non-universal repulsive forces at submillimeter distances much stronger than gravity. We also point out that even $U(1)$'s which are free of four-dimensional anomalies may acquire non-zero masses as a consequence of six-dimensional anomalies.

1. Introduction

Anomalous $U(1)$ gauge symmetries appear generically in string vacua. The massless charge spectrum is anomalous in the sense that the traditional triangle (or polygon in dimensions higher than four) diagrams are non-zero. However, the anomaly is cancelled via a generalization of the Green–Schwarz mechanism [1,2]. In four dimensions, a scalar axion (zero-form, or its dual two-form) is responsible for the anomaly cancellation. In six dimensions, both zero-forms (or their duals four-forms) and two-forms can participate in anomaly cancellation [3]. However, only zero-forms (or their duals) can give mass to

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the $U(1)$ gauge boson and break the gauge symmetry. Thus, a necessary consequence in four dimensions is that the (quasi)anomalous gauge symmetry is broken. Moreover, in the presence of supersymmetry (at least), an anomalous $U(1)$ is accompanied by a D-term potential that involves the charged scalars, shifted by a term proportional to the CP-even partner of the respective axion [2].

In perturbative heterotic vacua, at most one anomalous $U(1)$ can appear in four-dimensional $N = 1$ compactifications. The relevant axion is the four-dimensional dual of the Neveu–Schwarz (NS) two-form, which was shown to develop the appropriate couplings and transformation properties, needed to cancel all relevant anomalies [2]. Moreover, the scalar modulus appearing in the D-term potential is the dilaton, and for non-trivial vacua, vanishing of the D-term implies generically that charged scalars get a non-trivial vacuum expectation value (VEV) breaking the associated global $U(1)$ symmetry.

The situation is richer and more interesting in perturbative orientifold vacua. Here, there are in general several anomalous $U(1)$'s and the cancellation of anomalies is achieved via the coupling of twisted Ramond–Ramond (RR) axions [4]. The D-term potentials involve the twisted NS–NS moduli. However, at the orientifold point their expectation values vanish, and this allows to have a spontaneously broken gauge symmetry with the global $U(1)$ unbroken in perturbation theory [5]. The global symmetry may be broken non-perturbatively due to instanton effects, which however are small at weak coupling.

Orientifold vacua are prime candidates for realizing the Standard Model as a brane-world, in the context of perturbative string theory with low string scale and large internal dimensions [6] (for earlier attempts see [7]). As pointed out in [8], any minimal realization of the Standard Model in this context contains at least 2 anomalous $U(1)$'s that are expected to obtain a mass. Abelian gauge symmetries have been also used on the world-brane or in the bulk, in order to impose approximate global symmetries, such as baryon number or Peccei–Quinn symmetries [9,10]. It is therefore important to compute their masses and study their localization properties in the internal compact space.

It turns out that the mass of anomalous $U(1)$'s in orientifold vacua can be unambiguously calculated by a direct one-loop string computation (although a disk calculation may also give the mass modulo normalization ambiguities). In this work, we perform such a computation and we derive a formula for the mass matrix of $U(1)$ gauge bosons. We also study some explicit examples of Z_N and $Z_N \times Z_M$ orientifold vacua.

We find the following general features.

(1) The gauge boson masses are given by an ultraviolet contact term of the one-loop annulus diagram with the gauge bosons inserted one at each boundary. There are no contributions from the annulus with insertions on the same boundary or from the Möbius strip since such contact terms are absent by tadpole cancellation. By open-closed string duality, the $U(1)$ mass-terms are also given by some appropriate infrared (IR) closed string channel tadpoles.

(2) The mass-terms of $U(1)$ gauge bosons obtain volume independent corrections from $N = 1$ supersymmetric sectors, while $N = 2$ sectors give contributions dependent on the moduli of the corresponding fixed torus. Moreover, they are BPS saturated (given by the supertrace of the square of the four-dimensional helicity). Thus, mass-terms of

$N = 1$ sectors are localized in all six internal dimensions, while those of $N = 2$ sectors are six-dimensional, localized in four internal dimensions.

(3) $U(1)$'s that are free of four-dimensional anomalies can still be massive, if upon decompactification they suffer from six-dimensional anomalies.[2] This is expected since Kaluza–Klein (KK) states can contribute to (higher-dimensional) anomalies once there is a corner of moduli space where they become massless (decompactification limit). Uncancelled anomalies in six dimensions depend both on the localization of gauge fields (D-branes) and the localization of axions (coming from the bulk). Potentially anomalous sectors involve a six-dimensional coupling of a gauge field to an axion that extend in the same six dimensions.

As we already mentioned, the masses of $U(1)$'s arise through Green–Schwarz couplings involving RR axions. Moreover, at the orientifold point, the associated global symmetries remain unbroken to all orders in perturbation theory. Using the localization properties we described above, one can provide explicit realizations of all possible arrangements for the Abelian gauge bosons (A) and their corresponding axions (a):

$$(A, a) = (\text{brane, brane}), (\text{bulk, brane}), (\text{brane, bulk}), (\text{bulk, bulk}). \tag{1.1}$$

$N = 1$ sectors realize the first two possibilities, while $N = 2$ sectors realize the last two. Note that the axions can propagate at most in two internal dimensions, while $U(1)$ gauge bosons may propagate everywhere. It follows that the $U(1)$ mass M_A in these four cases is proportional to:

$$M_A \sim \mathcal{O}(1), \quad 1/\sqrt{V_A}, \quad 1/\sqrt{V_a}, \quad \sqrt{V_a/V_A}, \quad 1/\sqrt{V_a V_A} \tag{1.2}$$

in string units, where V_A and V_a stand for the internal volumes corresponding to the propagation of the $U(1)$ and the axion fields, respectively (V_a is two-dimensional). The last two possibilities correspond both to the (bulk, bulk) case and depend on whether V_a is part of V_A or orthogonal to it.

As a result, M_A can vary from the string scale M_s, up to much lower values that can attain M_s^2/M_{Planck}, in the two middle cases of (1.1), if the respective volume in Eq. (1.2) coincides with the total volume of the bulk. The gauge field exchange can then induce new (repulsive) forces at sub-millimeter distances (of the order of a few microns for M_s a few TeV). The third case, where the gauge field lives on the brane, is however experimentally excluded, since the corresponding gauge coupling g_A is of order unity. In the second case, the gauge field lives in the bulk and the four-dimensional $U(1)$ gauge coupling is infinitesimally small, $g_A \sim M_s/M_{\text{Planck}} \simeq 10^{-16}$. However, this value is still bigger that the gravitational coupling $\sim E/M_P$ for typical energies E of the order of the proton mass, and the strength of the new force would be 10^6–10^8 stronger than gravity. This an interesting region which will be soon explored in micro-gravity experiments [12]. Notice that the supernova constraints can exclude only the case where there are less than four large extra dimensions in the bulk, felt by the gauge field [9]. Finally, in the (bulk, bulk) case when V_a is part of V_A, the masses of all KK modes are shifted by a large amount according to

[2] Similar observations were made independently in [11].

Eq. (1.2) and the resulting force becomes effective at much smaller distances. On the other hand, when V_a is orthogonal to V_A the mass of the $U(1)_A$ is tiny $\mathcal{O}(M_s^2/M_{\text{Planck}})$ while its coupling is suppressed only by the volume of the bulk of A, $g_A \sim 1/\sqrt{V_A}$.

Of course, in all cases of (1.1), the $U(1)$ gauge bosons can be produced in particle accelerators at high energies leading to interesting signatures. Note that their masses are always lower than the string scale because of the (string) one-loop factor suppression. Moreover, their effective coupling is of order unity, if one takes into account the number of KK excitations that are produced at high energies.

The paper is organized as follows. In Section 2, we describe the effective action involving the anomalous $U(1)$ symmetries. In Section 3, we present the one-loop string computation and we give the general results for the contributions of $N = 1$ and $N = 2$ supersymmetric orbifold sectors. In Section 4, we study specific orientifold examples based on Z_3, Z_7, $Z_2 \times Z_3 = Z_6'$, Z_6 and $Z_3 \times Z_6$ orbifolds. Finally, Section 5 contains concluding remarks and a discussion of non-supersymmetric models.

2. The effective action

In four dimensions, there are two on-shell equivalent (dual) ways of describing the fields responsible for cancelling anomalies: as pseudoscalars or as two-index antisymmetric tensors. However, off-shell, the two descriptions are a priori different at the one-loop level.

Let us first consider the case of a pseudoscalar axion. The relevant part of the four-dimensional effective action (in the string frame) is:

$$S = \int d^4x \left[-\frac{1}{4g_A^2} F_A^2 - \frac{1}{2}(da + MA)^2 + \frac{a}{M} \sum_I k_I F_I \wedge F_I \right], \tag{2.1}$$

where F_A is the field strength of the anomalous $U(1)_A$, g_A is the corresponding gauge coupling, and k_I are the various mixed anomalies. Anomaly cancellation implies that a is shifted under $U(1)_A$ gauge transformation: $\delta A = d\Lambda$, $\delta a = -M\Lambda$, so that the action (2.1) changes by exactly the amount necessary to cancel the phase of the chiral fermion determinant. It follows that in the unitary gauge a vanishes and one is left over with a massive $U(1)_A$ with mass $M_A = g_A M$. Note that in the case where the gauge symmetry is not anomalous in four dimensions but the gauge field becomes massive due to a six-dimensional anomaly, all k_I vanish but still $M \neq 0$ and a transforms under gauge transformation.

In the type I string context, where the axion a comes from the RR closed string sector [4], the first and third terms of the above effective action appear at the level of the disk, while the second term is expanded into contributions corresponding to different orders of string perturbation theory; $(da)^2$ is a tree-level (sphere) term, the cross-product Ada appears at the disk level, while the mass-term A^2 is a one-loop contribution. Indeed, for this counting, the gauge kinetic terms have a dilaton factor $e^{-\phi}$ since g_A^2 is proportional to the string coupling $g_s \equiv e^{\phi}$, while a one-loop term is dilaton independent. On the other hand, every power of the RR field a absorbs a dilaton factor $e^{-\phi}$ which makes both the last two terms in (2.1) dilaton independent.

In a supersymmetric theory (at least), the above effective action is accompanied by a D-term potential

$$V = \int d^4x \, \frac{1}{g_A^2} \mathrm{D}^2, \qquad \mathrm{D} = Mm + \sum_i q_i |\Phi_i|^2, \tag{2.2}$$

where m is the twisted NS–NS blowing-up modulus that belongs in the same chiral multiplet with the RR axion a, while Φ_i denote the various open string charged scalars with $U(1)_A$ charges q_i. At the orientifold point, m vanishes and the global $U(1)_A$ symmetry remains unbroken despite the fact that the gauge field A becomes massive for $M \neq 0$ [5]. However, going away from the orientifold point when $m \neq 0$, vanishing of the D-term implies that some charged scalars should acquire non-zero VEVs, breaking the global $U(1)_A$ symmetry.

The above phenomenon can also be described in terms of an antisymmetric tensor $B_{\mu\nu}$. The corresponding effective action can be easily obtained from (2.1) by performing a standard Poincaré duality which exchanges equations of motion with Bianchi identities [13]:

$$S = \int d^4x \left[-\frac{1}{4g_A^2} F_A^2 - \frac{1}{12} \left(dB + \frac{k_I}{M} \Omega_I \right)^2 + (M\, dB + k_I \Omega_I) \wedge A \right], \tag{2.3}$$

where Ω_I are the various gauge Chern–Simons terms. Anomaly cancellation implies that $B_{\mu\nu}$ is shifted under gauge transformations: $\delta B = \sum_I \Lambda_I k_I F_I / M$, while the variation of the last term under $U(1)_A$ gauge transformation, cancels the anomalous contribution of the chiral fermion determinant.

The counting of the order of appearance of the various terms in type I string perturbation theory is similar as in the dual action (2.1). However, notice that in this representation there is no explicit mass-term for the $U(1)_A$ gauge field in the effective action. The reason is that the mass is now generated by a reducible diagram at the one-loop level. Inspection of (2.3) shows that a mixing between $B_{\mu\nu}$ and A_ρ arises at the level of the disk, corresponding to the vertex $\frac{M}{2} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu} \epsilon_\rho p_\sigma$, where p is the external momentum and ϵ's denote the polarization tensors. This generates a reducible contribution to the two-point function of the gauge field, given by the square of this vertex times the $B_{\mu\nu}$ propagator:

$$\frac{M^2}{2} \left[p^2 \epsilon^2 - (p \cdot \epsilon)^2 \right] / p^2 = \frac{M^2}{2} \epsilon^2, \tag{2.4}$$

where we have used the on-shell gauge-invariance condition $p \cdot \epsilon = 0$. Thus, a mass M for the anomalous gauge boson is generated by such a reducible one-loop diagram and no explicit mass-term is present in the effective Lagrangian.

Note that in the axion representation (2.1), the axion-gauge boson mixing at the disk-level generates a vertex proportional to $p \cdot \epsilon$ which vanishes on-shell and does not generate a reducible contribution for the gauge boson mass. Thus, an explicit one-loop mass-term must be introduced in the effective Lagrangian, consistently with the expression (2.1).

3. The calculation of the mass in orientifold models

The two possible diagrams that can contribute to terms quadratic in the gauge boson at the one-loop level are the annulus and the Möbius strip. Of those, only the annulus with the gauge field vertex operators inserted at the two opposite ends has the appropriate structure to contribute to the mass-term. Indeed, vertex operators inserted at the same boundary will be proportional to $\mathrm{Tr}[\gamma_k\lambda^a\lambda^b]$, where γ_k is the representation of the orientifold group element in the kth orbifold sector acting on the Chan–Paton (CP) matrices λ^a. On the other hand, for gauge fields inserted on opposite boundaries, the amplitude will be proportional to $\mathrm{Tr}[\gamma_k\lambda^a]\,\mathrm{Tr}[\gamma_k\lambda^b]$ and it is this form of traces that determines the anomalous $U(1)$'s [4]. The potential ultraviolet (UV) divergences that come from vertex operators inserted on the same boundary (both in cylinder and Möbius strip) cancel by tadpole cancellation [14].

Obviously, we must concentrate on the CP-even part of the amplitude which receives contributions only from even spin structures. This implies that we need the gauge boson vertex operators in the zero-ghost picture

$$V^a = \lambda^a\epsilon_\mu\big(\partial X^\mu + i(p\cdot\psi)\psi^\mu\big)e^{ip\cdot X}, \tag{3.1}$$

where λ is the Chan–Paton matrix and ϵ^μ is the polarization vector.

The world-sheet annulus is parameterized by $\tau = it/2$, where $\tau = \tau_1 + i\tau_2$ is the usual complex modular parameter of the torus, and corresponds to the rectangle $[0, t/2]\otimes[0, 1/2]$. The 2-point amplitude is then given by [15]

$$\mathcal{A} = -\frac{1}{4|G|}\int[d\tau][dz]\int\frac{d^4p}{(2\pi)^4}\sum_k\big\langle V(\epsilon_1, p_1, z)V(\epsilon_2, p_2, z_0)\big\rangle_k, \tag{3.2}$$

where the sum is over orientifold sectors, $|G|$ is the order of the orientifold group, and we fixed one of the positions of the vertex operators at $z_0 = 1/2$ using the translational symmetry of the annulus. The other vertex operator is located on the opposite boundary: $z = i\nu$ with $\nu\in[0, t/2]$. For notational simplicity, we set the Regge slope $\alpha' = 1/2$ so that the Virasoro Hamiltonian operator $L_0 = (p^2 + M^2)/2$.

Performing the contractions, we obtain

$$\begin{aligned}\mathcal{A} = {}&-\frac{1}{2|G|}\int[d\tau][dz]\int\frac{d^4p}{(2\pi)^4}\sum_k\big[(\epsilon_1\cdot\epsilon_2)(p_1\cdot p_2) - (\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)\big]\\ &\times\big(\mathrm{Tr}[\gamma_k\lambda]\big)^2 e^{-p_1\cdot p_2\langle X(z)X(z_0)\rangle}\big[\big\langle\psi(z)\psi(z_0)\big\rangle^2 - \big\langle X(z)\partial X(z_0)\big\rangle^2\big].\end{aligned} \tag{3.3}$$

It appears that the amplitude is $\mathcal{O}(p^2)$ and thus provides a correction only to the anomalous gauge boson coupling. We will see however, that after integration over the position z and the annulus modulus t, a term proportional to $1/p_1\cdot p_2$ appears from the ultraviolet (UV) region (as a result of the quadratic UV divergence in the presence of anomalous $U(1)$'s) that will provide the mass-term.

Strictly speaking, the amplitude above is zero on-shell if we enforce the physical state conditions $\epsilon\cdot p = p^2 = 0$ and momentum conservation $p_1 + p_2 = 0$. There is however a consistent off-shell extension, without imposing momentum conservation, that has given consistent results in other cases (see [16] for a discussion) and we adopt it here. We will thus impose momentum conservation only at the end of the calculation. We now define for convenience the reduced amplitude $\mathcal{A}_k$ by amputating the kinematical factors[3]

$$\mathcal{A}^{ab} = \int \frac{d^4 p}{(2\pi)^4} \big[(\epsilon_1 \cdot \epsilon_2)(p_1 \cdot p_2) - (\epsilon_1 \cdot p_2)(\epsilon_2 \cdot p_1)\big] \sum_k \mathrm{Tr}\big[\gamma_k \lambda^a\big] \mathrm{Tr}\big[\gamma_k \lambda^b\big] \mathcal{A}_k^{ab} \tag{3.4}$$

with

$$\mathcal{A}_k^{ab} = -\frac{1}{2|G|} \int [d\tau][dz] e^{-\delta \langle X(z) X(z_0)\rangle} \big[\langle \psi(z)\psi(z_0)\rangle^2 - \langle X(z)\partial X(z_0)\rangle^2\big] Z_k^{ab}, \tag{3.5}$$

where Z_k^{ab} is the annulus partition function in the kth orbifold sector, and we have set $\delta \equiv p_1 \cdot p_2$. The dependence of $\mathcal{A}_k^{ab}$ on the two gauge indices a, b is mild. It depends only on the type of brane the gauge fields come from. For instance, in standard supersymmetric Z_N orbifolds, there are three different cases corresponding to 99, 55 and 95 D-brane combinations.

We will need here the bosonic and fermionic propagators on the annulus. They can be obtained from those of the torus:

$$\langle X(e^{2\pi i \nu_1}) X(e^{2\pi i \nu_2})\rangle = -\frac{1}{4} \log \left| \frac{\vartheta_1(\nu_1 - \nu_2|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + \frac{\pi\, \mathrm{Im}^2(\nu_1 - \nu_2)}{2\tau_2}, \tag{3.6}$$

$$\langle \psi(e^{2\pi i \nu_1}) \psi(e^{2\pi i \nu_2})\rangle \binom{\alpha}{\beta} = \frac{i}{2} \frac{\vartheta\binom{\alpha}{\beta}(\nu_1 - \nu_2|\tau)\vartheta_1'(0|\tau)}{\vartheta\binom{\alpha}{\beta}(0|\tau)\vartheta_1(\nu_1 - \nu_2|\tau)} \tag{3.7}$$

by applying the world-sheet involution $z \to 1 - \bar{z}$ (see, for instance, Appendix of [17]). Thus, for example,

$$\begin{aligned} \langle X(z_1) X(z_2)\rangle\big|_{\text{annulus}} = \frac{1}{2}\Big(&\langle X(z_1)X(z_2)\rangle + \langle X(z_1)X(1-\bar{z}_2)\rangle + \langle X(1-\bar{z}_1)X(z_2)\rangle \\ &+ \langle X(1-\bar{z}_1)X(1-\bar{z}_2)\rangle\Big). \end{aligned} \tag{3.8}$$

In Eq. (3.7), α, β denote the fermionic spin structures and ϑ the Jacobi theta-functions. Setting $z_1 = e^{-2\pi\nu}$ and $z_2 = e^{i\pi}$ we obtain[4]

$$\begin{aligned} \langle X(z_1) X(z_2)\rangle\big|_{\text{annulus}} &= -\frac{1}{2} \log \left| \frac{\vartheta_1(i\nu - 1/2|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + \frac{\pi \nu^2}{\tau_2} \\ &= -\frac{1}{2} \log \left| \frac{\vartheta_2(i\nu|\tau)}{\vartheta_1'(0|\tau)} \right|^2 + \frac{\pi \nu^2}{\tau_2}. \end{aligned} \tag{3.9}$$

The fermionic propagator on the torus satisfies the identity [16]

$$\langle \psi(z_1)\psi(z_2)\rangle^2 \binom{\alpha}{\beta} = -\frac{1}{4}\mathcal{P}(z_1 - z_2) - \pi i \partial_\tau \log \frac{\vartheta\binom{\alpha}{\beta}(0|\tau)}{\eta(\tau)}, \tag{3.10}$$

[3] We consider in general insertions of different gauge fields on the different boundaries. The gauge fields can belong to different types of branes.

[4] Fixing the second position at a different point does not affect the result. This can be checked explicitly by shifting the integration measure.

where $\mathcal{P}(z_1 - z_2)$ is the Weierstrass function and η the Dedekind eta-function. The $\mathcal{P}$ term as well as the scalar correlator term in Eq. (3.5) are spin-structure independent and their contribution vanishes upon spin structure summation, because of space–time supersymmetry. The rest is position independent. Dropping the $\mathcal{P}$-piece, we effectively have

$$\left\langle \psi(z_1)\psi(z_2)\right\rangle^2 \binom{\alpha}{\beta}\bigg|_{\text{annulus}} = -2\pi i\,\partial_\tau \log \frac{\vartheta\binom{\alpha}{\beta}(0|\tau)}{\eta(\tau)}. \tag{3.11}$$

We should mention however that we expect this result to remain valid beyond supersymmetric vacua. In fact, in closed string threshold calculations for gauge couplings, there is a similar expression (see, for example, [16]) and the integral over the extra bosonic term $\langle X\partial X\rangle^2$ cancels against the integral over the Weierstrass function. We expect that a similar cancellation happens also here. Extra support for this conjecture is the structure of the open string partition function in the presence of magnetic fields, which is also used for the calculation of threshold corrections [14].

At this point we must be more explicit about the moduli integration measure. This is given by

$$\int [d\tau][dz] = \int_0^{i\infty} d\tau \int_0^{\tau} d(i\nu) = -\int_0^{\infty} \frac{dt}{2} \int_0^{t/2} d\nu. \tag{3.12}$$

The only dependence on ν comes from the bosonic propagator

$$e^{-\delta\langle X(z)X(z_0)\rangle} = \frac{(2\pi\eta^3(\tau))^\delta}{\vartheta_2(i\nu|\tau)^\delta} e^{\pi\nu^2\delta/\tau_2} = \tau_2^{\delta/2} \frac{(2\pi\eta^3(\tau))^\delta}{\vartheta_4(i\nu/\tau|-1/\tau)^\delta}, \tag{3.13}$$

where in the last step we performed a modular transformation on the ϑ-function. We are now in position to evaluate the ν integral. Since eventually we will set $\delta = 0$ we are interested in the leading term. We obtain

$$\int_0^{\tau_2} d\nu\, \tau_2^{\delta/2} \frac{(2\pi\eta^3(\tau))^\delta}{\vartheta_4(i\nu/\tau|-1/\tau)^\delta} = \tau_2^{1+\delta/2}\left[2\pi\eta^3(\tau)\right]^\delta + \mathcal{O}(\delta). \tag{3.14}$$

We can now proceed to parameterize the annulus contribution to the orientifold partition function as

$$Z_k^{ab} = \frac{1}{4\pi^4\tau_2^2} \sum_{\alpha,\beta=0,1} \frac{1}{2}(-1)^{\alpha+\beta+\alpha\beta} \frac{\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0|\tau)}{\eta^3(\tau)} Z_{\text{int},k}^{ab}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \tag{3.15}$$

where ab labels the type of branes at the two endpoints of the annulus, and $Z_{\text{int},k}^{ab}$ is the internal part of the annulus partition function, containing the contribution of the six compact (super)coordinates. For the ϑ-functions we use the notation and conventions of Appendix A in [16]. In particular there are some sign changes from the conventions of [18].

Putting everything together and assuming a supersymmetric ground state, we obtain

$$\mathcal{A}_k^{ab} = \frac{(2\pi)^\delta}{4\pi^4|G|}\int_0^\infty d\tau_2\,\tau_2^{-1+\delta/2}\eta^{3\delta}(\tau) \times \sum_{\alpha,\beta=0,1,\text{even}} \frac{1}{2}(-1)^{\alpha+\beta+\alpha\beta}\frac{i\pi\,\partial_\tau\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0|\tau)}{\eta^3(\tau)} Z_{\text{int},k}^{ab}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{(\sqrt{2}\pi)^\delta}{|G|}\int_0^\infty dt\,t^{-1+\delta/2}\eta^{3\delta}(it/2)F_k^{ab}(t), \tag{3.16}$$

where we have defined

$$F_k^{ab} = \tau_2^2 Z_k^{ab} = \frac{1}{4\pi^4}\sum_{\alpha,\beta=0,1,\text{even}}\frac{1}{2}(-1)^{\alpha+\beta+\alpha\beta}\frac{i\pi\,\partial_\tau\vartheta\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right](0|\tau)}{\eta^3(\tau)} Z_{\text{int},k}^{ab}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{3.17}$$

Note the similarity of this expression with the one appearing in the expression of the one-loop correction to gauge couplings (see, for instance, [16] and references therein). It follows that F_k^{ab} can be formally written as a supertrace over states from the open ab k-orbifold sector

$$F_k^{ab} = \frac{|G|}{(2\pi)^2}\,\text{Str}_{k,\text{open}}^{ab}\left[\frac{1}{12}-s^2\right]e^{-tM^2/2}, \tag{3.18}$$

where s is the four-dimensional helicity. As we mentioned above, we expect that this expression holds in the non-supersymmetric case, as well.

3.1. $N=1$ sectors

In this case there is no radius dependence of the integrand. The behavior of F_k^{ab} for large t is

$$\lim_{t\to\infty} F_k^{ab}(t) = C_k^{ab,IR} + \mathcal{O}\big(e^{-\pi t}\big) \tag{3.19}$$

with

$$C_k^{ab,\text{IR}} = \frac{|G|}{(2\pi)^2}\,\text{Str}_k\left[\frac{1}{12}-s^2\right]_{\text{open}}, \tag{3.20}$$

where the supertrace is restricted over massless states in the open channel k-sector of the orbifold. This expression is essentially the same with the one that appears in the evaluation of the one-loop beta-function and can be expressed in terms of the massless content of the kth sector as

$$\text{Str}_k\left[\frac{1}{12}-s^2\right] = -\frac{3}{2}N_V + \frac{1}{2}N_C, \tag{3.21}$$

where $N_{V,C}$ is the number of vector, respectively, chiral multiplets appearing in the kth sector.

For small t we have instead

$$\lim_{t\to 0} F_k^{ab}(t) = \frac{1}{t}\big[C_k^{ab,\mathrm{UV}} + \mathcal{O}\big(e^{-\pi/t}\big)\big], \tag{3.22}$$

where

$$C_k^{ab,\mathrm{UV}} = \frac{|G|}{(2\pi)^2}\,\mathrm{Str}_k\left[\frac{1}{12} - s^2\right]_{\mathrm{closed}}. \tag{3.23}$$

The relevant helicity supertrace is now in the transverse closed k-sector mapped from the open k-sector by a modular transformation. Here also, this can be written as $-\frac{3}{2}N_V + \frac{1}{2}N_C$ where now the states are from the closed kth string sector (transverse channel). Note that both in the direct and transverse channel all states contribute. We should stress that this result is valid for $N=1$ sectors only. Moreover, the trivial sector $k=0$, does not contribute to the supertrace due to its enhanced $N=4$ supersymmetry ($N_V = 3N_C$).

As shown explicitly in Appendix A, the t-integral has a logarithmic divergence in δ in the IR and a pole in the UV (reflecting the UV tadpole of the anomalous $U(1)$):

$$\mathcal{A}_k^{ab} = \frac{2C_k^{ab,\mathrm{UV}}}{\pi\delta|G|} + \mathcal{O}(\log\delta). \tag{3.24}$$

The on-shell limit can be obtained by setting $\epsilon_1 = \epsilon_2$, so that:

$$\big[(\epsilon_1\cdot\epsilon_2)(p_1\cdot p_2) - (\epsilon_1\cdot p_2)(\epsilon_2\cdot p_1)\big]\big/(p_1\cdot p_2) \to \epsilon\cdot\epsilon. \tag{3.25}$$

It follows that the contribution to the (unormalized) mass matrix from $N=1$ sectors reads:

$$\begin{aligned}\frac{1}{2}M_{ab}^2\bigg|_{N=1} &= \frac{2}{\pi|G|}\sum_{N=1\ \mathrm{sectors}} \mathrm{Tr}\big[\gamma_k\lambda^a\big]\,\mathrm{Tr}\big[\gamma_k\lambda^b\big]C_k^{ab,\mathrm{UV}}\\ &= \frac{1}{2\pi^3}\sum_{N=1\ \mathrm{sectors}} \mathrm{Tr}\big[\gamma_k\lambda^a\big]\,\mathrm{Tr}\big[\gamma_k\lambda^b\big]\,\mathrm{Str}_k\left[\frac{1}{12} - s^2\right]_{\mathrm{closed\ channel}}.\end{aligned} \tag{3.26}$$

We should remind to the reader that the multiplicities in the open channel of the annulus partition function have a direct particle interpretation (the projections happen at the CP factors of the boundaries). Such an interpretation does not seem possible in the closed string channel. Thus, our result does not seem expressible in terms of field theory data.

We now describe the explicit form of this contribution for Z_N orientifolds. The internal partition function of the kth sector is [18]

$$Z_{\mathrm{int},k}^{99} = Z_{\mathrm{int},k}^{55} = \prod_{j=1}^{3}\frac{(2\sin[\pi k v_j])\,\vartheta\big[{\alpha \atop \beta+2kv_j}\big]}{\vartheta\big[{1 \atop 1-2kv_j}\big]}, \tag{3.27}$$

$$Z_{\mathrm{int},k}^{95} = -2\big(2\sin[\pi k v_1]\big)\frac{\vartheta\big[{\alpha \atop \beta+2kv_1}\big]}{\vartheta\big[{1 \atop 1-2kv_1}\big]}\prod_{j=2}^{3}\frac{\vartheta\big[{\alpha+1 \atop \beta+2kv_j}\big]}{\vartheta\big[{0 \atop 1-2kv_j}\big]}, \tag{3.28}$$

where k runs over the orientifold $N=1$ sectors, (v_1, v_2, v_3) is the generating rotation vector of the orbifold satisfying $v_1+v_2+v_3=0$ in order to preserve at least $N=1$ supersymmetry and the 5-branes are stretched along the first torus by convention. To compare with other works, one should use the identities:

$$\vartheta\begin{bmatrix} 1 \\ 1+2kv_j \end{bmatrix} = -\vartheta\begin{bmatrix} 1 \\ 1-2kv_j \end{bmatrix}, \qquad \vartheta\begin{bmatrix} 0 \\ 1+2kv_j \end{bmatrix} = \vartheta\begin{bmatrix} 0 \\ 1-2kv_j \end{bmatrix}. \tag{3.29}$$

As shown in Appendix B, we can directly compute

$$C_k^{99,\mathrm{UV}} = C_k^{55,\mathrm{UV}} = -\frac{1}{2\pi^2}\prod_{i=1}^{3}\bigl|\sin[\pi k v_j]\bigr|, \tag{3.30}$$

$$C_k^{95,\mathrm{UV}} = \frac{\sin(\pi k v_1)}{2\pi^2}\eta_k, \tag{3.31}$$

where

$$\eta_k \equiv \sum_{i=1}^{3}\left[\{kv_i\} - \frac{1}{2}\right] = \frac{1}{2}\prod_{i=1}^{3}\frac{\sin[\pi k v_j]}{|\sin[\pi k v_j]|}. \tag{3.32}$$

Thus, the contribution to the mass from $N=1$ sectors of Z_N orbifolds is

$$\begin{aligned} \frac{1}{2}M^2_{99,ab}\bigg|_{N=1} &= \frac{1}{2}M^2_{55,ab}\bigg|_{N=1} \\ &= \sum_{\substack{k \\ N=1 \text{ sectors}}} -\frac{1}{\pi^3|G|}\prod_{i=1}^{3}\bigl|\sin[\pi k v_j]\bigr| \operatorname{Tr}\bigl[\gamma_k\lambda^a\bigr]\operatorname{Tr}\bigl[\gamma_k\lambda^b\bigr], \end{aligned} \tag{3.33}$$

$$\frac{1}{2}M^2_{95,ab}\bigg|_{N=1} = \sum_{\substack{k \\ N=1 \text{ sectors}}} \frac{\sin(\pi k v_1)}{2\pi^3|G|}\eta_k \operatorname{Tr}\bigl[\gamma_k\lambda^a\bigr]\operatorname{Tr}\bigl[\gamma_k\lambda^b\bigr], \tag{3.34}$$

where we have divided the 59 contribution by two, to avoid overcounting.

3.2. $N=2$ sectors

$N=2$ sectors are present when a two-torus remains invariant under the action of the appropriate orientifold element. Only massless states and their KK descendants survive the helicity supertrace (3.18). In this case, the function $F_k^{ab}(t)$ is given by

$$F_k^{ab}(t) = C_k^{ab,\mathrm{IR}}\varGamma_2(t), \tag{3.35}$$

where $C_k^{ab,\mathrm{IR}}$ is still given by (3.20). $\varGamma_2(t)$ is either the appropriate momentum lattice when these directions are NN (Neumann boundary conditions), or the winding lattice when these directions are DD (Dirichlet boundary conditions) [18]. No lattice sum can appear along ND directions.

For normalization purposes, the general closed string lattice sum containing both windings and momenta can be written as

$$Z_2 = \sum_{m_i,n_i\in Z} e^{-\frac{\pi\tau_2\alpha'}{V_2U_2}|m_1+Um_2+T(n_1+Un_2)/\alpha'|^2-2\pi\tau_1(m_1n_1+m_2n_2)}, \tag{3.36}$$

where $T = B + iV_2$ and $U = (G_{12} + iV_2)/G_{11}$ are, respectively, the Kähler and complex structure moduli of the torus, expressed in terms of the two-index antisymmetric tensor $B_{IJ} = B\epsilon_{IJ}$ and the 2×2 metric G_{IJ} ($V_2 = \sqrt{G}$). Setting the windings to zero, and $\alpha' = 1/2$, we obtain the open string momentum sum relevant in the NN case

$$\Gamma_2(t) = \sum_{m,n \in Z} e^{-\pi t \frac{|m+nU|^2}{2U_2 V_2}} = \frac{2V_2}{t} \sum_{m,n \in Z} e^{-\frac{2\pi V_2}{t} \frac{|m+nU|^2}{U_2}}, \tag{3.37}$$

while the open string (DD) winding sum is

$$\widetilde{\Gamma}_2(t) = \sum_{m,n \in Z} e^{-2\pi t V_2 \frac{|m+nU|^2}{U_2}} = \frac{1}{2V_2 t} \sum_{m,n \in Z} e^{-\frac{\pi}{2tV_2} \frac{|m+nU|^2}{U_2}}. \tag{3.38}$$

The normalizations above are in agreement with [19,20]. Note that the open and closed channel supertraces (3.18) are now the same, since massive string oscillator contributions cancel and one is left over with the lattice sum (BPS states).

Using the results of Appendix A, we obtain the pole contribution

$$I_k^{\mathrm{UV}} = \frac{4V_2 C_k^{ab,\mathrm{IR}}}{\pi \delta} + \mathcal{O}(\log \delta). \tag{3.39}$$

Consequently, the contribution to the mass is

$$\begin{aligned} \frac{1}{2} M_{ab}^2 \Big|_{N=2} &= \frac{4V_2}{\pi |G|} \sum_{N=2 \text{ sectors}} \mathrm{Tr}\big[\gamma_k \lambda^a\big] \mathrm{Tr}\big[\gamma_k \lambda^b\big] C_k^{ab,\mathrm{IR}} \\ &= -\frac{V_2}{\pi^3} \sum_{N=2 \text{ sectors}} \mathrm{Tr}\big[\gamma_k \lambda^a\big] \mathrm{Tr}\big[\gamma_k \lambda^b\big] \mathrm{Str}_{\tilde{k}} \left[\frac{1}{12} - s^2\right]_{\text{open channel}}. \end{aligned} \tag{3.40}$$

In the DD case, relevant for mass matrix elements coming from $D_{p<9}$ branes, the mass is similar as above with $\frac{V_2}{\alpha'} \to \frac{\alpha'}{V_2}$ ($V_2 \to 1/(4V_2)$ for $\alpha' = 1/2$).

We now proceed to evaluate the contributions to the mass coming from $N = 2$ sectors of Abelian orientifolds. For such sectors, one of the kv_i is integer. We will choose without loss of generality $kv_1 =$ integer. The internal partition function is then

$$Z_{\mathrm{int},k}^{99} = \Gamma_2 \frac{\vartheta\big[{\alpha \atop \beta + 2kv_1}\big]}{\eta^3} \prod_{j=2}^{3} \frac{(2\sin[\pi k v_j])\vartheta\big[{\alpha \atop \beta + 2kv_j}\big]}{\vartheta\big[{1 \atop 1-2kv_j}\big]} \tag{3.41}$$

and we can straightforwardly compute

$$C_k^{ab,\mathrm{IR}} = C_k^{ab,\mathrm{UV}} = \frac{(-1)^{kv_1}}{2\pi^2} \prod_{j=2}^{3} \sin[\pi k v_j] = -\frac{1}{2\pi^2} \prod_{j=2}^{3} \big|\sin[\pi k v_j]\big| \tag{3.42}$$

and

$$\frac{1}{2} M_{ab,\mathrm{NN}}^2 \Big|_{N=2} = \sum_{\substack{k \\ N=2 \text{ sectors}}} -\frac{2V_2}{\pi^3 |G|} \prod_{j=2}^{3} \big|\sin[\pi k v_j]\big| \mathrm{Tr}\big[\gamma_k \lambda^a\big] \mathrm{Tr}\big[\gamma_k \lambda^b\big], \tag{3.43}$$

$$\frac{1}{2}M^2_{ab,\mathrm{DD}}\Big|_{N=2} = \sum_{\substack{k\\ N=2\text{ sectors}}} -\frac{1}{2V_2\pi^3|G|}\prod_{j=2}^{3}\big|\sin[\pi k v_j]\big|\,\mathrm{Tr}\big[\gamma_k\lambda^a\big]\,\mathrm{Tr}\big[\gamma_k\lambda^b\big]. \quad (3.44)$$

Finally, for the 59 case, the relevant $N = 2$ sector is when the longitudinal torus is untwisted. In this case, the internal partition function is given by

$$Z^{95}_{\mathrm{int},k} = 2\Gamma_2 \frac{\vartheta\big[{\alpha \atop \beta+2kv_1}\big]}{\eta^3}\prod_{j=2}^{3}\frac{\vartheta\big[{\alpha+1 \atop \beta+2kv_j}\big]}{\vartheta\big[{0 \atop 1-2kv_j}\big]} \quad (3.45)$$

and we obtain

$$C_k^{95,ab,\mathrm{IR}} = (-1)^{kv_1}\frac{1}{4\pi^2} \quad (3.46)$$

and

$$\frac{1}{2}M^2_{ab,\mathrm{DN}}\Big|_{N=2} = \sum_{\substack{k\\ N=2\text{ sectors}}} (-1)^{kv_1}\frac{V_2}{2\pi^3|G|}\,\mathrm{Tr}\big[\gamma_k\lambda^a\big]\,\mathrm{Tr}\big[\gamma_k\lambda^b\big]. \quad (3.47)$$

As earlier, we have divided the 59 contribution by an additional factor of two. In the case where the two-torus corresponds to DD boundary conditions (in a D7–D3 configuration for instance), one should replace $V_2 \to 1/4V_2$.

4. Explicit orientifold examples

$N = 1$, Z_N orientifolds are generated by a rotation that acts as

$$gX^i = e^{2\pi i v_i}X^i, \qquad g\overline{X}^i = e^{-2\pi i v_i}\overline{X}^i, \quad (4.1)$$

where $X^i, \overline{X}^i$ are the complex coordinates of the three two-tori. The parameters v_i determining the fundamental Z_N rotation satisfy $Nv_i \in Z$ and $v_1 + v_2 + v_3 = 0$ in order to preserve space–time supersymmetry.

The action on the Chan–Paton indices is determined by the matrices $\gamma_k = (\gamma_1)^k$ representing the action of the orbifold element g^k, as

$$\gamma_k = e^{-2\pi i \hat{v}\cdot H}, \quad (4.2)$$

where H_I, $I = 1, \ldots, 16$, are the Cartan generators of $SO(32)$ and $\hat{v}^I$ is a rational vector specific to any given orbifold. A basis for the Cartan generators is given by diagonal matrices having the σ^3 Pauli matrix somewhere in the diagonal and zero everywhere else (so that $\mathrm{Tr}[H_I^2] = 2$). There is a vector $\hat{v}_9$ for D9-branes and different vectors ($\hat{v}_5$) for every potential set of D5-branes.

4.1. *The Z_3 orientifold*

Here there are no D5-branes [22]. The orbifold rotation vector is $(v_1, v_2, v_3) = (1, 1, -2)/3$ and the Chan–Paton projection vector is

$$\hat{v}_9 = \frac{1}{3}(1,1,1,1,1,1,1,1,1,1,1,1,0,0,0,0) \quad (4.3)$$

with $\mathrm{Tr}[\gamma_1] = \mathrm{Tr}[\gamma_2] = -4$. This breaks $SO(32)$ to $U(12) \times SO(8)$. The $U(1)$ factor of $U(12)$ is anomalous. The normalized generator of the anomalous $U(1)$ is

$$\lambda = \frac{1}{4\sqrt{3}} \sum_{i=1}^{12} H_I, \qquad \mathrm{tr}\big[\lambda^2\big] = \frac{1}{2}. \tag{4.4}$$

Thus, we can compute

$$\begin{aligned} \mathrm{Tr}[\gamma_1\lambda] &= -2\sqrt{3}\, i \sin(2\pi/3) = -3i, \\ \mathrm{Tr}[\gamma_2\lambda] &= -2\sqrt{3}\, i \sin(4\pi/3) = 3i, \end{aligned} \tag{4.5}$$

$$\mathrm{Tr}\big[\gamma_1\lambda^2\big] = \frac{1}{2}\cos(2\pi/3) = -\frac{1}{2}, \qquad \mathrm{Tr}\big[\gamma_2\lambda^2\big] = \frac{1}{2}\cos(4\pi/3) = -\frac{1}{2}. \tag{4.6}$$

Using (3.33), we can now evaluate the anomalous gauge boson mass:

$$\frac{1}{2}M^2 = -\frac{1}{3\pi^3}\big[\sin^3(\pi/3)\,\mathrm{Tr}[\gamma_1\lambda]^2 + \sin^3(2\pi/3)\,\mathrm{Tr}[\gamma_2\lambda]^2\big] = \frac{9\sqrt{3}}{4\pi^3}. \tag{4.7}$$

Putting back $M_s^2 = 1/\alpha'$ from the $2\alpha' = 1$ convention and taking into account the normalization of the F^2 kinetic terms $2\,\mathrm{Tr}[\lambda^2]/4g_A^2$, we obtain for the normalized gauge boson mass

$$M_{\mathrm{phys}}^2 = \frac{9\sqrt{3}}{4\pi^3} g_A^2 M_s^2. \tag{4.8}$$

Note that this example can be used to realize two out of the four possible configurations for the Abelian gauge bosons and their corresponding axions, displayed in Eq. (1.1), namely the cases (brane, brane) and (bulk, brane). Indeed, the RR axions from the twisted closed string sector are localized in all six internal dimensions, while the anomalous $U(1)$ can be either in the bulk (on the D9-branes), or on the brane with respect to directions that are T-dualized, so that one has Dp-branes with $p < 9$. Moreover, the $U(1)$ gauge coupling in Eq. (4.8) is given in general by $g_A^2 = g_s V_\parallel$, with $V_\parallel$ the internal volume (in string units) of the $p - 3$ compactified directions along the Dp-brane.

4.2. The Z_7 orientifold

The orbifold rotation vector is $(v_1, v_2, v_3) = (1, 2, -3)/7$. Tadpole cancellation implies the existence of 32 D9-branes. The Chan–Paton vector is

$$\hat{v}_9 = \frac{1}{7}(1, 1, 1, 1, 2, 2, 2, 2, -3, -3, -3, -3, 0, 0, 0, 0) \tag{4.9}$$

which implies

$$\mathrm{Tr}[\gamma_k] = 4, \quad k = 1, 2, 3, 4, 5, 6. \tag{4.10}$$

The gauge group is $U(4)^3 \times SO(8)$ and there are only $N = 1$ sectors.

The potentially anomalous $U(1)$'s are the Abelian factors of the gauge group and the relevant CP matrices are

$$\lambda_1 = \frac{1}{4}\sum_{I=1}^{4} H_I, \qquad \lambda_2 = \frac{1}{4}\sum_{I=5}^{8} H_I, \qquad \lambda_3 = \frac{1}{4}\sum_{I=9}^{12} H_I, \tag{4.11}$$

which satisfy $\mathrm{tr}[\lambda_i\lambda_j] = \frac{1}{2}\delta_{ij}$. The four-dimensional mixed non-Abelian anomalies of these $U(1)$'s are proportional to the matrix

$$\begin{pmatrix} 2 & 0 & -4 \\ -4 & 2 & 0 \\ 0 & -4 & 2 \\ 4 & 4 & 4 \end{pmatrix}, \tag{4.12}$$

where the columns label the $U(1)$'s while the rows label the non-Abelian factors $SU(4)^3 \times SO(8)$. It follows that all three $U(1)$'s are anomalous. We also have $\eta_1 = \eta_2 = -\eta_3 = \eta_4 = -\eta_5 = -\eta_6 = -1/2$ (see Eq. (3.32)). The contributions to the mass matrix are:

$$\frac{1}{2}M_{ij}^2 = -\frac{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}{7\pi^3}\sum_{k=1}^{6}\mathrm{Tr}[\gamma_k\lambda_i]\,\mathrm{Tr}[\gamma_k\lambda_j] = 2\frac{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}{7\pi^3}\delta_{ij} \tag{4.13}$$

and there is no mixing in this case.

4.3. The Z_6' orientifold

The orbifold rotation vector is $(v_1, v_2, v_3) = (1, -3, 2)/6$. There is an order two twist ($k = 3$) and we must have one set of D5-branes. Tadpole cancellation then implies the existence of 32 D9-branes and 32 D5-branes that we put together at one of the fixed points of the Z_2 action (say the origin). The Chan–Paton vectors are

$$\hat{v}_9 = \hat{v}_5 = \frac{1}{12}(1,1,1,1,5,5,5,5,3,3,3,3,3,3,3,3) \tag{4.14}$$

which imply

$$\mathrm{Tr}[\gamma_k] = 0, \quad k = 1,3,5, \qquad \mathrm{Tr}[\gamma_2] = -8, \qquad \mathrm{Tr}[\gamma_4] = 8. \tag{4.15}$$

The gauge group has a factor of $U(4) \times U(4) \times U(8)$ coming from the D9-branes and an isomorphic factor coming from the D5-branes. The $N = 1$ sectors correspond to $k = 1, 5$, while for $k = 2, 3, 4$ we have $N = 2$ sectors.

The potentially anomalous $U(1)$'s are the Abelian factors of the gauge group and the relevant CP matrices for the D9-branes are

$$\lambda_1 = \frac{1}{4}\sum_{I=1}^{4} H_I, \qquad \lambda_2 = \frac{1}{4}\sum_{I=5}^{8} H_I, \qquad \lambda_3 = \frac{1}{4\sqrt{2}}\sum_{I=9}^{16} H_I, \tag{4.16}$$

which satisfy $\mathrm{tr}[\lambda_i\lambda_j] = \frac{1}{2}\delta_{ij}$. Similar formulae apply to the other three $U(1)$ matrices $\tilde{\lambda}_i$ coming from the D5-sector. The four-dimensional anomalies of these $U(1)$'s (and their cancellation mechanism) were computed in [4]. The mixed anomalies with the six non-Abelian groups are given by the matrix[5]

$$\begin{pmatrix} 2 & 2 & 4\sqrt{2} & -2 & 0 & -2\sqrt{2} \\ -2 & -2 & -4\sqrt{2} & 0 & 2 & 2\sqrt{2} \\ 0 & 0 & 0 & 2 & -2 & 0 \\ -2 & 0 & -2\sqrt{2} & 2 & 2 & 4\sqrt{2} \\ 0 & 2 & 2\sqrt{2} & -2 & -2 & -4\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.17}$$

where the columns label the $U(1)$'s while the rows label the non-Abelian factors $SU(4)_9^2 \times SU(8)_9 \times SU(4)_5^2 \times SU(8)_5$. The upper 3×3 part corresponds to the 99 sector and the lower one to the 55 sector. As can be seen by this matrix, the two linear combinations $\sqrt{2}\,(A_1 + A_2) - A_3$ and $\sqrt{2}\,(\tilde{A}_1 + \tilde{A}_2) - \tilde{A}_3$ are free of mixed non-Abelian anomalies. It can also be shown that they are also free of mixed $U(1)$ anomalies. We can now compute:

$$\mathrm{Tr}[\gamma_k \lambda_1] = -2i \sin\left(\frac{\pi k}{6}\right), \qquad \mathrm{Tr}[\gamma_k \lambda_2] = -2i \sin\left(\frac{5\pi k}{6}\right),$$
$$\mathrm{Tr}[\gamma_k \lambda_3] = -2i\sqrt{2}\, \sin\left(\frac{\pi k}{2}\right), \tag{4.18}$$

$$\mathrm{Tr}[\gamma_k \tilde{\lambda}_1] = -2i \sin\left(\frac{\pi k}{6}\right), \qquad \mathrm{Tr}[\gamma_k \tilde{\lambda}_2] = -2i \sin\left(\frac{5\pi k}{6}\right),$$
$$\mathrm{Tr}[\gamma_k \tilde{\lambda}_3] = -2i\sqrt{2}\, \sin\left(\frac{\pi k}{2}\right) \tag{4.19}$$

$$\mathrm{Tr}\big[\gamma_k \lambda_1^2\big] = \frac{1}{2} \cos\left(\frac{\pi k}{6}\right), \qquad \mathrm{Tr}\big[\gamma_k \lambda_2^2\big] = \frac{1}{2} \cos\left(\frac{5\pi k}{6}\right),$$
$$\mathrm{Tr}[\gamma_k \lambda_3^2] = \frac{1}{2} \cos\left(\frac{\pi k}{2}\right) \tag{4.20}$$

while $\mathrm{Tr}[\gamma_k \lambda_i \lambda_j] = 0$ for $i \neq j$. We also have $\eta_1 = \eta_2 = \eta_4 = -\eta_5 = -1/2$.

The contribution to the mass matrix from $N = 1$ sectors is

$$\frac{1}{2} M_{99,ij}^2 = -\frac{\sqrt{3}}{24\pi^3} \big(\mathrm{Tr}[\gamma_1 \lambda_i]\, \mathrm{Tr}[\gamma_1 \lambda_j] + \mathrm{Tr}[\gamma_5 \lambda_i]\, \mathrm{Tr}[\gamma_5 \lambda_j]\big) \tag{4.21}$$

and similarly for $M_{55,ij}$, while

$$\frac{1}{2} M_{95,ij}^2 = -\frac{\sqrt{3}}{48\pi^3} \big(\mathrm{Tr}[\gamma_1 \lambda_i]\, \mathrm{Tr}[\gamma_1 \tilde{\lambda}_j] + \mathrm{Tr}[\gamma_5 \lambda_i]\, \mathrm{Tr}[\gamma_5 \tilde{\lambda}_j]$$
$$+ \mathrm{Tr}[\gamma_2 \lambda_i]\, \mathrm{Tr}[\gamma_2 \tilde{\lambda}_j] - \mathrm{Tr}[\gamma_4 \lambda_i]\, \mathrm{Tr}[\gamma_4 \tilde{\lambda}_j]\big). \tag{4.22}$$

[5] Note that here we use a different normalization for the $U(1)$ generators than in [4].

On the other hand, the contributions from $N=2$ sectors read

$$\frac{1}{2M^2_{99,ij}} = -\frac{V_2}{4\pi^3}\big(\mathrm{Tr}[\gamma_2\lambda_i]\,\mathrm{Tr}[\gamma_2\lambda_j] + \mathrm{Tr}[\gamma_4\lambda_i]\,\mathrm{Tr}[\gamma_4\lambda_j]\big) - \frac{V_3}{3\pi^3}\,\mathrm{Tr}[\gamma_3\lambda_i]\,\mathrm{Tr}[\gamma_3\lambda_j], \tag{4.23}$$

$$\frac{1}{2}M^2_{55,ij} = -\frac{1}{16V_2\pi^3}\big(\mathrm{Tr}[\gamma_2\tilde\lambda_i]\,\mathrm{Tr}[\gamma_2\tilde\lambda_j] + \mathrm{Tr}[\gamma_4\tilde\lambda_i]\,\mathrm{Tr}[\gamma_4\tilde\lambda_j]\big) - \frac{V_3}{3\pi^3}\,\mathrm{Tr}[\gamma_3\lambda_i]\,\mathrm{Tr}[\gamma_3\lambda_j] \tag{4.24}$$

and

$$\frac{1}{2}M^2_{95,ij} = -\frac{V_3}{12\pi^3}\,\mathrm{Tr}[\gamma_3\lambda_i]\,\mathrm{Tr}[\gamma_3\tilde\lambda_j]. \tag{4.25}$$

Thus, the unormalized mass matrix has eigenvalues and eigenvectors

$$m_1^2 = 6V_2, \quad -A_1 + A_2; \tag{4.26}$$

$$m_2^2 = \frac{3}{2V_2}, \quad -\tilde A_1 + \tilde A_2; \tag{4.27}$$

$$m^2_{3,4} = \frac{5\sqrt{3} + 48V_3 \pm \sqrt{3(25 - 128\sqrt{3}\,V_3 + 768V_3^2)}}{12}, \tag{4.28}$$

with respective eigenvectors

$$\pm a_\pm(A_1 + A_2 - \tilde A_1 - \tilde A_2) - A_3 + \tilde A_3, \tag{4.29}$$

where

$$a_\pm = \frac{\mp 3 + \sqrt{25 - 128\sqrt{3}\,V_3 + 768V_3^2}}{4\sqrt{2}\,(4\sqrt{3}\,V_3 - 1)}, \tag{4.30}$$

$$m^2_{5,6} = \frac{15\sqrt{3} + 80V_3 \pm \sqrt{5(135 - 384\sqrt{3}\,V_3 + 1280V_3^2)}}{12}, \tag{4.31}$$

with respective eigenvectors

$$\pm b_\pm(A_1 + A_2 + \tilde A_1 + \tilde A_2) + A_3 + \tilde A_3, \tag{4.32}$$

where

$$b_\pm = \frac{\pm 9\sqrt{3} - \sqrt{5(135 - 384\sqrt{3}\,V_3 + 1280V_3^2)}}{4\sqrt{2}\,(20V_3 - 3\sqrt{3}\,)}. \tag{4.33}$$

Note that the eigenvalues are always positive. They are also invariant under the T-duality symmetry of the theory $V_2 \to 1/4V_2$. Thus, all $U(1)$'s become massive, including the two anomaly free combinations. The reason is that these combinations are anomalous in six dimensions. Observe however that in the limit $V_3 \to 0$, the two linear combinations that are free of four-dimensional anomalies become massless. This is consistent with the fact that the six-dimensional anomalies responsible for their mass cancel locally in this limit.

To obtain the normalized mass matrix, we must also take into account the kinetic terms of the $U(1)$ gauge bosons which are

$$S_{\text{kinetic}} = -\frac{1}{4g_s}\left[V_1 V_2 V_3\left(F_1^2 + F_2^2 + F_3^2\right) + V_3\left(\widetilde{F}_1^2 + \widetilde{F}_2^2 + \widetilde{F}_3^2\right)\right]. \tag{4.34}$$

This implies $M_{99}^2 \to M_{99}^2/(V_1 V_2 V_3)$, $M_{55}^2 \to M_{55}^2/V_3$ and $M_{95}^2 \to M_{95}^2/(\sqrt{V_1 V_2}\, V_3)$. The resulting eigenvalues are too complicated and not illuminating to produce here.

Strictly speaking the formulae we presented should be used for $V_3 \geqslant 1$. When $V_3 < 1$ we can T-dualize and rewrite the theory in terms of D3–D7 branes. Then the unormalized mass remains as above with $V_3 \to 1/4V_3$ but the kinetic terms of the gauge bosons are no longer multiplied by V_3.

Using the Z_6' orientifold, one can realize the remaining two possible configurations for the anomalous $U(1)$ gauge fields and their corresponding axions, namely the (bulk, bulk) and (brane, bulk) cases of Eq. (1.1); the other two were realized for instance in the context of Z_3 orientifold, as we described before. In fact, identifying the second torus with the bulk, the two configurations correspond to the cases (4.26) and (4.27), respectively, that receive contributions from the corresponding $N = 2$ sector only. Furthermore, in the (bulk, bulk) case there are two possibilities as spelled out in the introduction: Eq. (4.26) as it stands realizes a normalized mass $\sim \sqrt{V_a/V_A} = 1/\sqrt{V_1 V_3}$ In this case $V_a = V_2$ is a subspace of $V_A = V_1 V_2 V_3$. Upon a T-duality in V_2 it realizes the other possibility, namely a normalized mass $\sim 1/\sqrt{V_a V_A} = 1/\sqrt{V_1 V_2 V_3}$. Here $V_a = V_2$ and $V_A = V_1 V_3$.

4.4. The Z_6 orientifold

The orbifold rotation vector is $(v_1, v_2, v_3) = (1, 1, -2)/6$. There is an order two twist ($k = 3$) and we must have one set of D5-branes. Tadpole cancellation then implies the existence of 32 D9-branes and 32 D5-branes, as in the previous example, that we put together at the origin of the internal space. The Chan–Paton vectors are

$$\hat{v}_9 = \hat{v}_5 = \frac{1}{12}(1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 3, 3, 3, 3) \tag{4.35}$$

implying

$$\mathrm{Tr}[\gamma_k] = 0 \quad \text{for} \quad k = 1, 3, 5, \qquad \mathrm{Tr}[\gamma_2] = 4, \qquad \mathrm{Tr}[\gamma_4] = -4. \tag{4.36}$$

The gauge group has a factor of $U(6) \times U(6) \times U(4)$ coming from the D9-branes and an isomorphic factor coming from the D5-branes. The $N = 1$ sectors correspond to $k = 1, 2, 4, 5$, while $k = 3$ is an $N = 2$ sector.

The potentially anomalous $U(1)$'s are the Abelian factors of the gauge group and the relevant CP matrices for the D9-branes are

$$\lambda_1 = \frac{1}{2\sqrt{6}} \sum_{I=1}^{6} H_I, \qquad \lambda_2 = \frac{1}{2\sqrt{6}} \sum_{I=7}^{12} H_I, \qquad \lambda_3 = \frac{1}{4} \sum_{I=13}^{16} H_I, \tag{4.37}$$

that satisfy $\mathrm{tr}[\lambda_i\lambda_j] = \frac{1}{2}\delta_{ij}$. Similar formulae apply to the other three $U(1)$ matrices $\tilde{\lambda}_i$ coming from the D5-sector. The four-dimensional mixed non-Abelian anomalies of these $U(1)$'s are proportional to the matrix

$$\begin{pmatrix} 6 & -3 & \sqrt{6} & 3 & 0 & \sqrt{6} \\ 3 & -6 & -\sqrt{6} & 0 & -3 & -\sqrt{6} \\ -9 & 9 & 0 & -3 & 3 & 0 \\ 3 & 0 & \sqrt{6} & 6 & -3 & \sqrt{6} \\ 0 & -3 & -\sqrt{6} & 3 & -6 & -\sqrt{6} \\ -3 & 23 & 0 & -9 & 9 & 0 \end{pmatrix}. \tag{4.38}$$

The columns label the $U(1)$'s, while the rows label the non-Abelian factors $SU(6)_9^2 \times SU(4)_9 \times SU(6)_5^2 \times SU(4)_5$. The upper 3×3 part corresponds to the 99 sector and the lower one to the 55 sector. As can be seen by this matrix, there are three linear combinations $A_1 + A_2 - \sqrt{\frac{3}{2}}\,A_3$, $\tilde{A}_1 + \tilde{A}_2 - \sqrt{\frac{3}{2}}\,\tilde{A}_3$ and $A_3 - \tilde{A}_3$ that are free of mixed non-Abelian anomalies. It can be shown that they are also free of mixed $U(1)$ anomalies.

We can now compute

$$\begin{aligned} &\mathrm{Tr}[\gamma_k\lambda_1] = -i\sqrt{6}\,\sin\frac{\pi k}{6}, \qquad \mathrm{Tr}[\gamma_k\lambda_2] = (-1)^k i\sqrt{6}\,\sin\frac{\pi k}{6}, \\ &\mathrm{Tr}[\gamma_k\lambda_3] = -2i\,\sin\frac{\pi k}{2} \end{aligned} \tag{4.39}$$

and similarly for $\tilde{\lambda}i$. Also

$$\mathrm{Tr}\big[\gamma_k\lambda_1^2\big] = \frac{1}{4}\cos\frac{\pi k}{6}, \qquad \mathrm{Tr}\big[\gamma_k\lambda_2^2\big] = \frac{(-1)^k}{4}\cos\frac{\pi k}{6}, \qquad \mathrm{Tr}\big[\gamma_k\lambda_3^2\big] = \frac{1}{4}\cos\frac{\pi k}{2}, \tag{4.40}$$

while $\mathrm{Tr}[\gamma_k\lambda_i\lambda_j] = 0$ for $i \neq j$. Finally $\eta_1 = \eta_2 = \eta_3 = -\eta_4 = -\eta_5 = -1/2$.

The various contributions to the mass matrix are

$$\begin{aligned} \frac{1}{2}M^2_{99,ij} = &-\frac{\sqrt{3}}{48\pi^3}\big[\mathrm{Tr}[\gamma_1\lambda_i]\,\mathrm{Tr}[\gamma_1\lambda_j] + \mathrm{Tr}[\gamma_5\lambda_i]\,\mathrm{Tr}[\gamma_5\lambda_j] \\ &+ 3\big(\mathrm{Tr}[\gamma_2\lambda_i]\,\mathrm{Tr}[\gamma_2\lambda_j] + \mathrm{Tr}[\gamma_4\lambda_i]\,\mathrm{Tr}[\gamma_4\lambda_j]\big)\big] - \frac{V_3}{3\pi^3}\,\mathrm{Tr}[\gamma_3\lambda_i]\,\mathrm{Tr}[\gamma_3\lambda_j] \end{aligned} \tag{4.41}$$

and similarly for $M_{55,ij}$, while

$$\begin{aligned} \frac{1}{2M^2_{95,ij}} = &-\frac{\sqrt{3}}{48\pi^3}\big(\mathrm{Tr}[\gamma_1\lambda_i]\,\mathrm{Tr}[\gamma_1\lambda_j] + \mathrm{Tr}[\gamma_5\lambda_i]\,\mathrm{Tr}[\gamma_5\lambda_j] \\ &+ \mathrm{Tr}[\gamma_2\lambda_i]\,\mathrm{Tr}[\gamma_2\lambda_j] + \mathrm{Tr}[\gamma_4\lambda_i]\,\mathrm{Tr}[\gamma_4\lambda_j]\big) - \frac{V_3}{12\pi^3}\,\mathrm{Tr}[\gamma_3\lambda_i]\,\mathrm{Tr}[\gamma_3\lambda_j]. \end{aligned} \tag{4.42}$$

This mass matrix has the following eigenvalues and eigenvectors:

$$m_1^2 = 0, \qquad A_1 + A_2 - \tilde{A}_1 - \tilde{A}_2 + \sqrt{6}\,(A_3 - \tilde{A}_3), \tag{4.43}$$

$$m_2^2 = \frac{3\sqrt{3}}{2}, \qquad A_1 - A_2 - \tilde{A}_1 + \tilde{A}_2, \tag{4.44}$$

$$m_3^2 = 3\sqrt{3}, \qquad A_1 - A_2 + \tilde{A}_1 - \tilde{A}_2, \tag{4.45}$$

$$m_4^2 = \frac{40}{3} V_3, \qquad -\sqrt{\frac{3}{2}}(A_1 + A_2 - \tilde{A}_1 - \tilde{A}_2) - A_3 + \tilde{A}_3, \tag{4.46}$$

$$m_\pm^2 = \frac{7\sqrt{3} + 80 V_3 \pm \sqrt{147 - 1040\sqrt{3}\, V_3 + 6400 V_3^2}}{12},$$
$$a_\pm (A_1 + A_2 + \tilde{A}_1 + \tilde{A}_2) + A_3 + \tilde{A}_3 \tag{4.47}$$

with

$$a_\pm = \frac{40 V_3 - \sqrt{3} \pm \sqrt{147 - 1040\sqrt{3}\, V_3 + 6400 V_3^2}}{12\sqrt{2} - 40\sqrt{6}\, V_3}. \tag{4.48}$$

In the limit $V_3 \to 0$ two more masses become zero (m_4 and m_-). It is straightforward to check that the appropriate linear combinations of $U(1)$'s are anomaly-free in four dimensions.

4.5. *The $Z_3 \times Z_6$ orientifold*

The orbifold rotation vectors are $v_\theta = (1, 0, -1)/3$ and $v_h = (1, -1, 0)/6$. There is an order two twist h^3. Tadpole cancellation implies the existence of 32 D9-branes and 32 D5-branes that we put together at the origin of the internal space. The Chan–Paton vectors are

$$\hat{v}_9^\theta = \hat{v}_5^\theta = \frac{1}{3}(2, 2, 0, 0, 1, 1, 0, 0, 1, 1, 2, 2, 0, 0, 0, 0) \tag{4.49}$$

and

$$\hat{v}_9^h = \hat{v}_5^h = \frac{1}{12}(1, 1, 1, 1, 5, 5, 5, 5, 3, 3, 3, 3, 3, 3, 3, 3). \tag{4.50}$$

The gauge group has a factor of $U(2)^6 \times U(4)$ coming from the D9-branes and an isomorphic factor coming from the D5-branes. Sectors are labelled by the group elements $\theta^k h^l$. The $N = 2$ sectors in the 99 and 55 configurations are $(k, l) \in \{(1, 0), (2, 0), (0, 1), (0, 2), (2, 2), (0, 3), (0, 4), (1, 4), (0, 5)\}$. In the 95 configuration we have fewer $N = 2$ sectors, namely $(k, l) \in \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5)\}$.

The potentially anomalous $U(1)$'s are the fourteen Abelian factors of the gauge group and the relevant CP matrices for the D9-branes are

$$\lambda_1 = \frac{1}{2}\sum_{I=1}^{2} H_I, \qquad \lambda_2 = \frac{1}{2}\sum_{I=3}^{4} H_I, \qquad \lambda_3 = \frac{1}{2}\sum_{I=5}^{6} H_I, \qquad \lambda_4 = \frac{1}{2}\sum_{I=7}^{8} H_I, \tag{4.51}$$

$$\lambda_5 = \frac{1}{2}\sum_{I=9}^{10} H_I, \qquad \lambda_6 = \frac{1}{2}\sum_{I=11}^{12} H_I, \qquad \lambda_7 = \frac{1}{2}\sum_{I=13}^{16} H_I. \tag{4.52}$$

Similar formulae apply to the other seven $U(1)$ matrices $\tilde{\lambda}_i$ coming from the D5-sector. The four-dimensional mixed non-Abelian anomalies of these $U(1)$'s are proportional to the matrix

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0
\end{pmatrix}. \tag{4.53}$$

The columns label the $U(1)$'s while the rows label the non-Abelian factors $SU(2)_9^6 \times SU(4)_9 \times SU(2)_5^6 \times SU(4)_5$. The upper 7×7 part corresponds to the 99 sector and the lower one to the 55 sector. As can be seen by this matrix, there are six linear combinations

$$A_1 - A_3 - A_5 + A_6, \qquad A_2 - A_4 + A_5 - A_6, \qquad 2(A_5 + A_6) + A_7, \tag{4.54}$$

$$\tilde{A}_1 - \tilde{A}_3 + \tilde{A}_5 - \tilde{A}_6, \qquad \tilde{A}_2 - \tilde{A}_4 + \tilde{A}_5 - \tilde{A}_6, \qquad 2(\tilde{A}_5 + \tilde{A}_6) + \tilde{A}_7 \tag{4.55}$$

that are free of mixed non-Abelian anomalies. Mixed $U(1)$ anomalies also cancel. We can also compute:

$$\begin{aligned}
&\eta_{(1,1)} = \eta_{(2,1)} = \eta_{(1,2)} = \eta_{(1,3)} = -\eta_{(2,3)} = -\eta_{(2,4)} = -\eta_{(1,5)} = -\eta_{(2,5)} = \frac{1}{2}, \\
&\eta_{(2,2)} = \eta_{(1,4)} = \eta_{(1,0)} = \eta_{(2,0)} = 0.
\end{aligned} \tag{4.56}$$

The mass matrix is given by

$$\begin{aligned}
\frac{1}{2} M^2_{99,ij} = &- \sum_{\substack{k,l \\ N=1 \text{ sectors}}} \frac{s[k,l]}{18\pi^3} \operatorname{Tr}[\gamma_{k,l}\lambda_i] \operatorname{Tr}[\gamma_{k,l}\lambda_j] \\
&- \frac{V_3}{9} \sum_{l=1}^{5} \sin^2\left[\frac{\pi l}{6}\right] \operatorname{Tr}[\gamma_{0,l}\lambda_i] \operatorname{Tr}[\gamma_{0,l}\lambda_j] \\
&- \frac{V_1}{9} \sin\left[\frac{\pi}{3}\right] \sin\left[\frac{2\pi}{3}\right] \left(\operatorname{Tr}[\gamma_{2,2}\lambda_i] \operatorname{Tr}[\gamma_{2,2}\lambda_j] + \operatorname{Tr}[\gamma_{1,4}\lambda_i] \operatorname{Tr}[\gamma_{1,4}\lambda_j]\right) \\
&- \frac{V_2}{9} \sum_{k=1}^{2} \sin^2\left[\frac{\pi k}{3}\right] \operatorname{Tr}[\gamma_{k,0}\lambda_i] \operatorname{Tr}[\gamma_{k,0}\lambda_j],
\end{aligned} \tag{4.57}$$

where

$$s[k,l] \equiv \left|\sin\left[\pi\left(\frac{2k+l}{6}\right)\right]\sin\left[\pi\frac{l}{6}\right]\sin\left[\pi\frac{k}{3}\right]\right|, \tag{4.58}$$

and similarly for $M_{55,ij}$ with $V_1 \to 1/4V_1$, $V_2 \to 1/4V_2$, while

$$\frac{1}{2}M^2_{95,ij} = \sum_{\substack{k,l \\ N=1 \text{ sectors}}} \frac{\eta_{k,l}}{36\pi^3}\sin\left[\frac{\pi k}{3}\right]\mathrm{Tr}[\gamma_{k,l}\lambda_i]\,\mathrm{Tr}[\gamma_{k,l}\lambda_j] + \frac{V_3}{36}\sum_{l=1}^{5}\sin^2\left[\frac{\pi l}{6}\right]\mathrm{Tr}[\gamma_{0,l}\lambda_i]\,\mathrm{Tr}[\gamma_{0,l}\lambda_j]. \tag{4.59}$$

It follows that there are no massless gauge bosons. The mass-squared matrix has a double eigenvalue $4\sqrt{3}$ and a double eigenvalue $6\sqrt{3}$. It has six eigenvalues that depend on V_3 and the rest depend on all three internal volumes. At $V_3 = 0$ there are two zero eigenvalues corresponding to the last linear combinations in (4.54), (4.55), a double eigenvalue $4\sqrt{3}$ and a double eigenvalue $6\sqrt{3}$, double eigenvalues $(49\sqrt{3} \pm \sqrt{5259})/18$ and the rest are

$$4\left(V_1 + V_2 \pm \sqrt{V_1^2 + V_2^2 - V_1V_2}\right) \tag{4.60}$$

with eigenvectors purely on the D9-branes and their duals with eigenvectors only on the D5-branes.

5. Conclusions

In this work we did an explicit one-loop string computation of the $U(1)$ masses in four-dimensional orientifolds and studied their localization properties in the internal compactified space. We have shown that non-vanishing mass-terms appear for all $U(1)$'s that are anomalous in four dimensions, but also for apparent anomaly free combinations if they acquire anomalies in a six-dimensional decompactification limit. In both cases, the global $U(1)$ symmetry remains unbroken at the orientifold point, to all orders in perturbation theory.

For supersymmetric compactifications, we found that $N = 1$ sectors lead to contributions to $U(1)$ masses that are localized in all six internal dimensions, while those of $N = 2$ sectors are localized only in four internal dimensions. All these mass terms are described as Green–Schwarz couplings involving axions coming from the RR closed string sector, that transform under the corresponding $U(1)$ gauge transformations. One can thus provide explicit realizations in brane world models of all possible configurations (1.1) for the gauge field and the axion, propagating in the bulk of large extra dimensions, or being localized on a brane. $N = 1$ sectors describe axions localized on a 3-brane, while $N = 2$ sectors describe axions propagating in two extra dimensions.

Our results can in principle easily be generalized to non-supersymmetric orientifolds. A particularly interesting class of non-supersymmetric constructions is given in the context of "brane supersymmetry breaking", where supersymmetry is broken only in the open string sector while it remains exact (to lowest order) in the closed string bulk [21]. In the simplest case, the breaking of supersymmetry arises only from combinations of D-branes with (anti)orientifold planes which affect only the Möbius amplitude and thus do not change the expression for the mass. Indeed, the latter appears as a contact term of the annulus that remains supersymmetric. On the other hand, in the case where the supersymmetry breaking arises also from configurations of branes with antibranes, there is an additional contribution to the mass that can be easily computed following our general method.

Our analysis has direct implications for model building [23]. In particular, special care is needed to guarantee that the $U(1)$ hypercharge remains massless despite the fact that it is anomaly free. An additional condition should be satisfied, namely that it remains anomaly free in any six-dimensional decompactification limit. On the other hand, anomalous $U(1)$'s could be used to reduce the rank of the low-energy gauge group and guarantee the conservation of global symmetries, such as the baryon and lepton number. Finally, the associated $U(1)$ gauge bosons could be produced in particle accelerators with new interesting experimental signals. Their masses are always lighter than the string scale, varying from a loop factor to a much bigger suppression by the volume of the bulk, giving rise to possible new (repulsive) forces at sub-millimeter distances, much stronger than gravity.

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Appendix A. Ultraviolet poles and infrared logarithms

In this appendix we calculate the UV tadpole (pole in δ). To this end, we split the integral of Eq. (3.16) into UV and IR parts:

$$\mathcal{A}_k^{ab} = I_k^{ab,\mathrm{IR}} + I_k^{ab,\mathrm{UV}}. \tag{A.1}$$

We will first consider $N = 1$ sectors, where no lattice sum appears in the internal partition function. The behavior in the IR is

$$\begin{aligned} I_k^{ab,\mathrm{IR}} &= \frac{(\sqrt{2}\pi)^\delta}{|G|} \int_1^\infty dt\, t^{-1+\delta/2} \eta^{3\delta}(it/2) F_k^{ab}(t) \\ &= \frac{(\sqrt{2}\pi)^\delta C_k^{ab,\mathrm{IR}}}{|G|} \int_1^\infty dt\, t^{-1+\delta/2} e^{-\frac{\pi t \delta}{8}} + \text{finite}. \end{aligned} \tag{A.2}$$

Changing variables, we obtain

$$I_k^{ab,\mathrm{IR}} = \left(\frac{16\pi}{\delta}\right)^{\delta/2} \frac{C_k^{ab,\mathrm{IR}}}{|G|} \int\limits_{\pi\delta/8}^{\infty} du\, u^{-1+\delta/2} e^{-u} + \text{finite}$$

$$= \left(\frac{16\pi}{\delta}\right)^{\delta/2} \frac{C_k^{ab,\mathrm{IR}}}{|G|} \Gamma(\delta/2, \pi\delta/8), \tag{A.3}$$

where $\Gamma(a,x)$ is the incomplete Γ-function with asymptotic expansion for small argument x:

$$\Gamma(a,x) = \Gamma(a) - \frac{x^a}{a} - \sum_{n=1}^{\infty} \frac{(-1)^n x^{a+n}}{n!(n+a)}. \tag{A.4}$$

We thus obtain

$$I_k^{ab,\mathrm{IR}} = -\frac{C_k^{ab,\mathrm{IR}}}{|G|} \log \frac{\pi\delta}{8} + \text{finite}. \tag{A.5}$$

To study the UV behavior, we use $\eta(it/2) = (t/2)^{-1/2}\eta(2/t)$ and consider

$$I_k^{ab,\mathrm{UV}} = \frac{(\sqrt{2}\pi)^\delta}{|G|} \int\limits_0^1 dt\, t^{-1+\delta/2} \eta^{3\delta}(it/2) F_k^{ab}(t)$$

$$= \frac{(4\pi)^\delta C_k^{ab,\mathrm{UV}}}{|G|} \int\limits_0^1 dt\, t^{-2-\delta} e^{-\pi\delta/2t} + \text{finite}$$

$$= \frac{C_k^{ab,\mathrm{UV}}}{|G|} \left(\frac{8}{\delta}\right)^\delta \frac{2}{\pi\delta} \Gamma(\delta+1, \pi\delta/2) + \text{finite} = \frac{2C_k^{ab,\mathrm{UV}}}{\pi\delta|G|} + \text{finite}, \tag{A.6}$$

leading to the pole, as advertised.

We will now focus on the $N=2$ sectors. Here

$$F_k^{ab}(t) = C_k^{ab,\mathrm{IR}} \Gamma_2(t), \tag{A.7}$$

where $C_k^{ab,\mathrm{IR}}$ is given by (3.20). The lattice sum is given by (3.37) in the NN case, and by (3.38) in the DD case. To obtain the UV contribution, we have to use the second form of the lattice sums in (3.37) and (3.38). We then find

$$I_k^{\mathrm{UV}} = (4\pi)^\delta C_k^{ab,\mathrm{IR}} \int\limits_0^1 dt\, t^{-2-\delta} e^{-\pi\delta/2t} \Gamma_2(t) + \text{finite}$$

$$= 2V_2 C_k^{ab,\mathrm{IR}} (4\pi)^\delta \left(\frac{2}{\pi\delta}\right)^{\delta+1} \Gamma(\delta+1, \pi\delta/2)$$

$$+ C_k^{ab,\mathrm{IR}} \sum_{(m,n)\neq(0,0)} \frac{2U_2}{\pi|m+nU|^2} \Gamma\left(1, \frac{\pi V_2|m+nU|^2}{2U_2}\right) + \cdots. \tag{A.8}$$

We have set $\delta = 0$ to all terms with non-zero momentum. This is justified because we will show that apart from the first term, the rest of the sum (in the second term) is finite. Indeed, the sum over non-zero momenta is finite because it is cutoff by the incomplete Γ-function. In fact, for large values of x

$$\Gamma(1,x) = e^{-x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right] \tag{A.9}$$

and the momentum sum is bounded by

$$\sum_{(m,n)\neq(0,0)} \frac{2U_2}{\pi|m+nU|^2} e^{-\frac{\pi V_2|m+nU|^2}{2U_2}} \tag{A.10}$$

which is convergent for $V_2 > 1$. It has a logarithmic divergence $\sim \log V_2$ when $V_2 \to 0$ but we always keep $V_2 \geqslant 1$ in our conventions. Thus, the pole is given by the first term only

$$I_k^{\mathrm{UV}} = \frac{4V_2 C_k^{ab,\mathrm{IR}}}{\pi\delta} + \mathcal{O}(\log\delta). \tag{A.11}$$

Appendix B. Calculation of the UV tadpoles for standard orientifolds

In this appendix we compute the asymptotic values C_k^{UV} and C_k^{IR} of Z_N orientifolds. The relevant $N = 1$ sector partition functions are

$$Z_{\mathrm{int},k}^{99} = Z_{\mathrm{int},k}^{55} = \prod_{j=1}^{3} \frac{(2\sin[\pi k v_j])\vartheta\left[{\alpha \atop \beta+2kv_j}\right]}{\vartheta\left[{1 \atop 1-2kv_j}\right]}, \tag{B.1}$$

$$Z_{\mathrm{int},k}^{95} = -2\big(2\sin[\pi k v_1]\big)\frac{\vartheta\left[{\alpha \atop \beta+2kv_1}\right]}{\vartheta\left[{1 \atop 1-2kv_1}\right]} \prod_{j=2}^{3} \frac{\vartheta\left[{\alpha+1 \atop \beta+2kv_j}\right]}{\vartheta\left[{0 \atop 1-2kv_j}\right]}, \tag{B.2}$$

where k runs over $N = 1$ sectors, (v_1, v_2, v_3) is the generating rotation vector of the orbifold satisfying $v_1 + v_2 + v_3 = 0$ in order to preserve $N = 1$ supersymmetry and the 5-branes are stretching along the first torus.

Using the property that on ϑ-functions $i\pi\partial_\tau = \frac{1}{4}\partial_v^2$, and the Riemann identity

$$\begin{aligned}
&\frac{1}{2}\sum_{\alpha,\beta=0,1} (-1)^{\alpha+\beta+\alpha\beta}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(v)\prod_{i=1}^{3}\vartheta\begin{bmatrix}\alpha+h_i\\ \beta+g_i\end{bmatrix}(0)\\
&\quad = \vartheta\begin{bmatrix}1\\ 1\end{bmatrix}(v/2)\prod_{i=1}^{3}\vartheta\begin{bmatrix}1-h_i\\ 1-g_i\end{bmatrix}(v/2)
\end{aligned} \tag{B.3}$$

in (3.16), we obtain

$$F_k^{99} = F_k^{55} = \frac{1}{16\pi^3}\prod_{i=1}^{3}\big(2\sin[\pi k v_j]\big)\sum_{i=1}^{3}\frac{\vartheta'\left[{1 \atop 1-2kv_i}\right](0)}{\vartheta\left[{1 \atop 1-2kv_i}\right](0)}, \tag{B.4}$$

$$F_k^{59} = -\frac{\sin(\pi k v_1)}{4\pi^3}\left[\frac{\vartheta'\big[{1 \atop 1-2kv_1}\big](0)}{\vartheta\big[{1 \atop 1-2kv_1}\big](0)} + \frac{\vartheta'\big[{0 \atop 1-2kv_2}\big](0)}{\vartheta\big[{0 \atop 1-2kv_2}\big](0)} + \frac{\vartheta'\big[{0 \atop 1-2kv_3}\big](0)}{\vartheta\big[{0 \atop 1-2kv_3}\big](0)}\right]. \tag{B.5}$$

Using now

$$\frac{\vartheta'\big[{1 \atop 1-2kv_i}\big](0)}{\vartheta\big[{1 \atop 1-2kv_i}\big](0)} = 2\pi \cot(\pi k v_i) + \mathcal{O}\big(e^{-\pi t}\big), \qquad \frac{\vartheta'\big[{0 \atop 1-2kv_i}\big](0)}{\vartheta\big[{0 \atop 1-2kv_i}\big](0)} = \mathcal{O}\big(e^{-2\pi t}\big), \tag{B.6}$$

we obtain

$$C_k^{99,\mathrm{IR}} = C_k^{55,\mathrm{IR}} = \frac{1}{\pi^2}\prod_{i=1}^{3}\big(\sin[\pi k v_j]\big)\sum_{i=1}^{3}\cot(\pi k v_i), \qquad C^{95,\mathrm{IR}} = -\frac{\cos(\pi k v_1)}{2\pi^2}. \tag{B.7}$$

For the mass computation, we are interested in the modular transform of F_k. Using

$$\vartheta'\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0,\tau) = -\frac{1}{\tau\sqrt{-i\tau}}e^{i\pi\frac{ab}{2}}\vartheta'\begin{bmatrix}\beta\\ -\alpha\end{bmatrix}\left(0,-\frac{1}{\tau}\right), \tag{B.8}$$

we can rewrite (B.4) and (B.5) as

$$F_k^{99} = F_k^{55} = -\frac{1}{2\pi^3\tau}\prod_{i=1}^{3}\big(\sin[\pi k v_j]\big)\sum_{i=1}^{3}\frac{\vartheta'\big[{1-2kv_i \atop -1}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_i \atop -1}\big]\big(0,-\frac{1}{\tau}\big)}, \tag{B.9}$$

$$\begin{aligned} F_k^{95} = \frac{\sin(\pi k v_1)}{4\pi^3\tau}\Bigg[&\frac{\vartheta'\big[{1-2kv_1 \atop -1}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_1 \atop -1}\big]\big(0,-\frac{1}{\tau}\big)} + \frac{\vartheta'\big[{1-2kv_1 \atop 0}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_1 \atop 0}\big]\big(0,-\frac{1}{\tau}\big)} \\ &+ \frac{\vartheta'\big[{1-2kv_3 \atop 0}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_3 \atop 0}\big]\big(0,-\frac{1}{\tau}\big)}\Bigg]. \end{aligned} \tag{B.10}$$

Defining by $\{kv_i\}$ to be the (positive) fractional part of kv_i, then

$$\frac{\vartheta'\big[{1-2kv_i \atop -1}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_i \atop -1}\big]\big(0,-\frac{1}{\tau}\big)} = 2\pi i\left[\{kv_i\} - \frac{1}{2}\right] + \mathcal{O}\big(e^{-\pi/t}\big) \tag{B.11}$$

and

$$\frac{\vartheta'\big[{1-2kv_i \atop 0}\big]\big(0,-\frac{1}{\tau}\big)}{\vartheta\big[{1-2kv_i \atop 0}\big]\big(0,-\frac{1}{\tau}\big)} = 2\pi i\left[\{kv_i\} - \frac{1}{2}\right] + \mathcal{O}\big(e^{-\pi/t}\big). \tag{B.12}$$

In the second case, when $\{kv_i\} \in Z$ the limit gives zero. We must have $\left|\{kv_i\} - \frac{1}{2}\right| < \frac{1}{2}$. Using now

$$\eta_k \equiv \sum_{i=1}^{3}\left[\{kv_i\} - \frac{1}{2}\right] = \frac{1}{2}\prod_{i=1}^{3}\frac{\sin[\pi k v_j]}{|\sin[\pi k v_j]|}, \tag{B.13}$$

we can directly compute (replacing $\tau = it/2$)

$$C_k^{99,\mathrm{UV}} = C_k^{55,\mathrm{UV}} = -\frac{1}{\pi^2}\prod_{i=1}^{3}\left|\sin[\pi k v_j]\right|, \tag{B.14}$$

$$C_k^{95,\mathrm{UV}} = \frac{\sin(\pi k v_1)}{\pi^2}\eta_k. \tag{B.15}$$

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