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Classification of heterotic Pati–Salam models

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Abstract

We extend the classification of free fermionic heterotic-string models to vacua in which the $SO(10)$ GUT symmetry is broken at the string level to the Pati–Salam subgroup. Using our classification method we recently presented the first example of a quasi-realistic heterotic-string vacuum that is free of massless exotic states. Within this method we are able to derive algebraic expressions for the Generalised GSO (GGSO) projections for all sectors that appear in the models. This facilitates the programming of the entire spectrum analysis in a computer code. The total number of vacua in the class of models that we classify is $2^{51} \sim 10^{15}$. We perform a statistical sampling in this space of models and extract 10^{11} GGSO configurations with Pati–Salam gauge group. Our results demonstrate that one in every 10^6 vacua correspond to a three generation exophobic model with the required Higgs states, needed to induce spontaneous breaking to the Standard Model.

Keywords: Heterotic-string; Pati–Salam; Fractionally charged states

1. Introduction

The heterotic-string models constructed in the free fermionic formulation [1] are among the most realistic string models constructed to date [2–7]. These models correspond to $Z_2 \times Z_2$

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(asymmetric)-orbifold compactifications, based on $\mathcal{N} = (2,0)$ super-conformal symmetry on the world-sheet. The fermionic construction is set at a special extended symmetry point in the moduli space, and where all compact dimensions are represented in terms of two dimensional free fermions propagating on the string world-sheet [8,9]. Marginal deformations from the free fermionic point can then be explored by incorporating Thirring interactions among the world-sheet fermions [10]. The free fermionic construction provides a set of rules that enables straightforward extraction of the massless states and interactions, and is therefore particularly suited to explore the phenomenological properties of string vacua. The quasi-realistic free fermionic $Z_2 \times Z_2$ orbifolds preserve the $SO(10)$ GUT embedding of the Standard Model spectrum. The matter states arise from spinorial 16 representations, and the Higgs states arise from the vectorial 10 representation. It should be noted that in these models the $SO(10)$ symmetry is broken directly at the string level, rather than in the effective low energy quantum field theory. The manifest symmetry in the effective low energy field theory is therefore a subgroup of $SO(10)$.

Early examples of quasi-realistic free fermionic constructions were obtained in the late eighties [2–5]. Over the past few years tools for the systematic classification of free fermionic $Z_2 \times Z_2$ orbifolds were developed. In the orbifold language [11] the free fermionic construction corresponds to symmetric, asymmetric or freely acting orbifolds [8,9,13,17]. A subclass of them corresponds to symmetric $Z_2 \times Z_2$ orbifold compactifications at enhanced symmetry points in the toroidal moduli space [8,9]. The chiral matter spectrum arises from twisted sectors and thus does not depend on the moduli. This facilitates the complete classification of the topological sectors of the $Z_2 \times Z_2$ symmetric orbifolds. For type II string $N = 2$ supersymmetric vacua the general free fermionic classification techniques were developed in Ref. [12]. The method was extended in Refs. [13–17] for the classification of heterotic $Z_2 \times Z_2$ free fermionic orbifolds, with unbroken $SO(10)$ and E_6 GUT symmetries. The classification of heterotic $N = 1$ (and $N = 2$) vacua revealed a symmetry in the distribution of $Z_2 \times Z_2$ (and Z_2) string vacua under exchange of vectorial, and spinorial plus anti-spinorial, representations of $SO(10)$ [14–18], akin to mirror symmetry [19].

Our classification methodology entails the expression of the Generalised GSO (GGSO) projections in terms of generic algebraic equations for the states that arise in the twisted sectors. The equations are incorporated in a computer code that allows scanning a large number of models. In Ref. [13] models with $N = 1$ space–time supersymmetry that produce spinorial states from all three distinct twisted sectors of the $Z_2 \times Z_2$ orbifold, were classified with respect to the number of chiral 16 representations. Such models were dubbed S^3 models. This was extended in Ref. [14] to models that may produce twisted vectorial 10 representations. Such models were dubbed S^2V, SV^2 and V^3 models, corresponding to vacua in which two, one and none, of the twisted sectors produce spinorial representations. The novelty of Ref. [14] was that a single basis is used to generate the different classes of models, which substantially simplifies the classification. All the different classes of models are generated by choices of the GGSO projection coefficients. This can be compared with the method of Ref. [20] that uses different basis sets to generate the S^2V, SV^2 and V^3 type of models. In Ref. [15] the classification was extended to include vectorial 10 representations in the data output. This enabled the observation of the spinor–vector duality over the entire space of $N = 1$ models. Ref. [16] demonstrated the existence of spinor–vector duality in $N = 2$ models. Ref. [17] elaborated further on the spinor–vector duality, in particular in terms of the operational interpretation of the GGSO free phases, and the breaking of the $N = 2$ right-moving world-sheet supersymmetry.

Absence of adjoint Higgs representations in heterotic-string models with unbroken $SO(10)$ GUT symmetries realised as level one Kac–Moody algebras implies that the models classified in

[13,14] cannot be spontaneously broken to the Standard Model in the effective field theory level. Thus, the $SO(10)$ GUT gauge symmetry must be broken directly at the string level. In the free fermionic models the GUT gauge symmetry generated by untwisted vector bosons is $SO(10)$, and can be enhanced to a larger gauge group by gauge bosons arising from other sectors. Phenomenologically the most appealing case is that of $SO(10)$ by itself, and therefore it is reasonable to demand that gauge bosons which enhance the $SO(10)$ symmetry be projected out by the Generalised GSO (GGSO) projections. The $SO(10)$ symmetry must therefore be broken to one of its subgroups. The cases with $SU(5) \times U(1)$ (flipped $SU(5)$) [2], $SO(6) \times SO(4)$ (Pati–Salam) [4], $SU(3) \times SU(2) \times U(1)^2$ (standard-like) [3,5] and $SU(3) \times SU(2)^2 \times U(1)$ (left–right symmetric) [7] were shown to produce quasi-realistic examples.

The Pati–Salam models obtained via the free fermionic construction of the heterotic-string utilise only periodic and anti-periodic boundary conditions, whereas all the other cases necessarily use fractional boundary conditions as well. The Pati–Salam case [21] therefore represents the simplest extension of the classification program of [13–17] to quasi-realistic models. The Pati–Salam string models contain sectors that preserve the underlying $SO(10)$ symmetry, as well as sectors that break that symmetry to the Pati–Salam subgroup. In general, the $SO(10)$ breaking sectors may contain massless exotic states that carry fractional electric charge [22,23]. The existence of such states is severely constrained by observations [24].

In Ref. [25] our classification method was used to demonstrate the existence of quasi-realistic string models that do not contain massless exotic states, which carry fractional electric charge. In this paper we extend the classification to Pati–Salam heterotic string models. The primary benefit of our method is in the representation of the GGSO projections in algebraic form for all the twisted sectors that a priori produce massless states. We can readily extract the full massless spectrum of these models. The algebraic formulas are incorporated in a computer code which enables us to scan a large space of models.

2. Pati–Salam heterotic-string models

The free fermionic formulation of the four dimensional heterotic string in the light-cone gauge is described by 20 left moving and 44 right moving real fermions. A large number of models can be constructed by choosing different phases picked up by fermions ($f_A, A = 1, \ldots, 44$) when transported along the torus non-contractible loops. Each model corresponds to a particular choice of fermion phases consistent with modular invariance that can be generated by a set of basis vectors $v_i, i = 1, \ldots, N$

$$v_i = \{\alpha_i(f_1), \alpha_i(f_2), \alpha_i(f_3), \ldots\}$$

describing the transformation properties of each fermion

$$f_A \to -e^{i\pi\alpha_i(f_A)}\, f_A, \quad A = 1, \ldots, 44 \tag{2.1}$$

The basis vectors span a space Ξ which consists of 2^N sectors that give rise to the string spectrum. Each sector is given by

$$\xi = \sum N_i v_i, \quad N_i = 0, 1 \tag{2.2}$$

The spectrum is truncated by a generalised GSO projection whose action on a string state $|S\rangle$ is

$$e^{i\pi v_i \cdot F_S}|S\rangle = \delta_S C \begin{bmatrix} S \\ v_i \end{bmatrix} |S\rangle \tag{2.3}$$

where F_S is the fermion number operator and $\delta_S = \pm 1$ is the space–time spin statistics index. Different sets of projection coefficients $c\left[\begin{smallmatrix} S \\ v_i \end{smallmatrix}\right] = \pm 1$ consistent with modular invariance give rise to different models. Summarising: a model can be defined uniquely by a set of basis vectors v_i, $i = 1, \ldots, N$ and a set of $2^{N(N-1)/2}$ independent projections coefficients $c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right]$, $i > j$.

The free fermions in the light-cone gauge in the usual notation are: $\psi^\mu, \chi^i, y^i, \omega^i, i = 1, \ldots, 6$ (left-movers) and $\bar{y}^i, \bar{\omega}^i, i = 1, \ldots, 6, \psi^A, A = 1, \ldots, 5, \bar{\eta}^B, B = 1, 2, 3, \bar{\phi}^\alpha, \alpha = 1, \ldots, 8$ (right-movers). The class of models we investigate, is generated by a set of thirteen basis vectors

$$B = \{v_1, v_2, \ldots, v_{13}\},$$

where

$$\begin{aligned}
v_1 = 1 &= \{\psi^\mu, \chi^{1,\ldots,6}, y^{1,\ldots,6}, \omega^{1,\ldots,6} | \bar{y}^{1,\ldots,6}, \bar{\omega}^{1,\ldots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\ldots,5}, \bar{\phi}^{1,\ldots,8}\} \\
v_2 = S &= \{\psi^\mu, \chi^{1,\ldots,6}\} \\
v_{2+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\}, \quad i = 1, \ldots, 6 \\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\ldots,5}\} \\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\ldots,5}\} \\
v_{11} = z_1 &= \{\bar{\phi}^{1,\ldots,4}\} \\
v_{12} = z_2 &= \{\bar{\phi}^{5,\ldots,8}\} \\
v_{13} = \alpha &= \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}
\end{aligned} \tag{2.4}$$

The first twelve vectors in this set are identical to those used in [13,14]. The vectors 1, S generate an $N = 4$ supersymmetric model, with $SO(44)$ gauge symmetry. The vectors e_i, $i = 1, \ldots, 6$ give rise to all possible symmetric shifts of the six internal fermionized coordinates ($\partial X^i = y^i \omega^i$, $\bar{\partial} X^i = \bar{y}^i \bar{\omega}^i$). Their addition breaks the $SO(44)$ gauge group, but preserves $N = 4$ supersymmetry. The vectors b_1 and b_2 define the $SO(10)$ gauge symmetry and the $Z_2 \times Z_2$ orbifold twists, which break $N = 4$ to $N = 1$ supersymmetry. The z_1 and z_2 basis vectors reduce the untwisted gauge group generators from $SO(16)$ to $SO(8)_1 \times SO(8)_2$. Finally v_{13} is the additional new vector that breaks the $SO(10)$ GUT symmetry to $SO(6) \times SO(4)$, and the $SO(8)_1$ hidden symmetry to $SO(4)_1 \times SO(4)_2$.

The second ingredient that is needed to define the string vacuum are the GGSO projection coefficients that appear in the one-loop partition function, $c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right]$, spanning a 13×13 matrix. Only the elements with $i > j$ are independent while the others are fixed by modular invariance. A priori there are therefore 78 independent coefficients corresponding to 2^{78} string vacua. Eleven coefficients are fixed by requiring that the models possess $N = 1$ supersymmetry. Without loss of generality we impose the associated GGSO projection coefficients

$$\begin{gathered}
c\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c\begin{bmatrix} S \\ 1 \end{bmatrix} = c\begin{bmatrix} S \\ e_i \end{bmatrix} = c\begin{bmatrix} S \\ b_m \end{bmatrix} = c\begin{bmatrix} S \\ z_n \end{bmatrix} = c\begin{bmatrix} S \\ \alpha \end{bmatrix} = -1 \\
i = 1, \ldots, 6, \ m = 1, 2, \ n = 1, 2
\end{gathered} \tag{2.5}$$

leaving 66 independent coefficients,

$$
\begin{array}{l}
c\begin{bmatrix} e_i \\ e_j \end{bmatrix}, \quad i \geqslant j, \quad c\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad c\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad c\begin{bmatrix} 1 \\ b_A \end{bmatrix}, \quad c\begin{bmatrix} 1 \\ z_A \end{bmatrix} \\
c\begin{bmatrix} e_i \\ z_n \end{bmatrix}, \quad c\begin{bmatrix} e_i \\ b_A \end{bmatrix}, \quad c\begin{bmatrix} b_A \\ z_n \end{bmatrix}, \quad c\begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \quad c\begin{bmatrix} e_i \\ \alpha \end{bmatrix}, \quad c\begin{bmatrix} b_A \\ \alpha \end{bmatrix}, \quad c\begin{bmatrix} z_A \\ \alpha \end{bmatrix} \\
\quad i, j = 1, \ldots, 6, A, B, m, n = 1, 2
\end{array}
$$

since all of the remaining projection coefficients are determined by modular invariance [1]. Each of the 66 independent coefficients can take two discrete values ± 1 and thus a simple counting gives 2^{66} (that is approximately $10^{19.9}$) models in the class of superstring vacua under consideration. We remark here that there may exist some degeneracies in this space of physical vacua with respect to the properties of the effective low energy field theory, i.e. in particular with respect to the massless spectra. For example, there exists a cyclic permutation symmetry among the three twisted sectors of the $Z_2 \times Z_2$ orbifold. However, many of the vacua that may seem equivalent from the point of view of the effective field theory limit of the observable massless spectra, may differ by other properties, like, for example: hidden sector matter states; the massive spectrum; superpotential couplings; and are therefore distinct. The important question that we address by a statistical analysis in this paper is the frequency by which exophobic vacua occur in the total space of configurations.

The vector bosons from the untwisted sector generate an

$$SO(6) \times SO(4) \times U(1)^3 \times SO(4)^2 \times SO(8)$$

gauge symmetry. Depending on the choices of the projection coefficients, extra gauge bosons may arise from the following ten sectors:

$$\mathbf{G} = \left\{ \begin{array}{llllll} z_1, & z_2, & \alpha, & \alpha + z_1, & & \\ x, & z_1 + z_2, & \alpha + z_2, & \alpha + z_1 + z_2, & \alpha + x, & \alpha + x + z_1 \end{array} \right\} \tag{2.6}$$

where

$$x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2 = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\} \tag{2.7}$$

Vector bosons that arise from these sectors enhance the untwisted gauge symmetry. We impose the condition that the only space–time vector bosons that remain in the spectrum are those that arise from the untwisted sector. This restricts further the number of phases, leaving a total of 51 independent GGSO phases. The gauge group in these models is therefore:

$$
\begin{array}{ll}
\text{observable:} & SO(6) \times SO(4) \times U(1)^3 \\
\text{hidden:} & SO(4)^2 \times SO(8)
\end{array}
$$

where the hidden $SO(4)^2 \sim SO(4)_1 \times SO(4)_2 \sim SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4$.

The untwisted matter is common in these models and is composed of three pairs of vectorial representations of the observable $SO(6)$ symmetry, and 12 states that are singlets under the non-Abelian gauge groups. The chiral matter spectrum arises from the twisted sectors. The chiral spinorial representations of the observable $SO(6) \times SO(4)$ arise from the sectors:

$$
\begin{aligned}
B^{(1)}_{pqrs} &= S + b_1 + p e_3 + q e_4 + r e_5 + s e_6 \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\
&\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^1, \bar{\psi}^{1..5}\}
\end{aligned}
$$

$$B^{(2)}_{pqrs} = S + b_2 + pe_1 + qe_2 + re_5 + se_6$$
$$B^{(3)}_{pqrs} = S + b_3 + pe_1 + qe_2 + re_3 + se_4 \tag{2.8}$$

where $p, q, r, s = 0, 1$; $b_3 = b_1 + b_2 + x = 1 + S + b_1 + b_2 + \sum_{i=1}^{6} e_i + \sum_{n=1}^{2} z_n$ and x is given in Eq. (2.7). These sectors give rise to **16** and $\overline{\mathbf{16}}$ representations of $SO(10)$ decomposed under $SO(6) \times SO(4) \equiv SU(4) \times SU(2)_L \times SU(2)_R$

$$\mathbf{16} = (\mathbf{4}, \mathbf{2}, \mathbf{1}) + (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$$
$$\overline{\mathbf{16}} = (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}) + (\mathbf{4}, \mathbf{1}, \mathbf{2})$$

The following sectors give rise to states that transform as representations of the hidden gauge group, and are singlets under the observable $SO(10)$ GUT symmetry. These states are therefore hidden matter states that arise in the string model, but are not exotic with respect to electric charge. The following 48 sectors produce the representations $((\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{1}))$ of $SU(2)^4 = SO(4)_1 \times SO(4)_2$:

$$\begin{aligned}
B^{(4)}_{pqrs} &= B^{(1)}_{pqrs} + x + z_1 = S + b_1 + pe_3 + qe_4 + re_5 + se_6 + x + z_1 \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\
&\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{1..4}\} \\
B^{(5)}_{pqrs} &= B^{(2)}_{pqrs} + x + z_1 = S + b_2 + pe_1 + qe_2 + re_5 + se_6 + x + z_1 \\
B^{(6)}_{pqrs} &= B^{(3)}_{pqrs} + x + z_1 = S + b_3 + pe_1 + qe_2 + re_3 + se_4 + x + z_1
\end{aligned} \tag{2.9}$$

There are 48 sectors producing spinorial **8** and anti-spinorial $\bar{\mathbf{8}}$ representations of the hidden $SO(8)$ gauge group:

$$\begin{aligned}
B^{(7)}_{pqrs} &= B^{(1)}_{pqrs} + x + z_2 = S + b_1 + pe_3 + qe_4 + re_5 + se_6 + x + z_2 \\
&= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\
&\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{5..8}\} \\
B^{(8)}_{pqrs} &= B^{(2)}_{pqrs} + x + z_2 = S + b_2 + pe_1 + qe_2 + re_5 + se_6 + x + z_2 \\
B^{(9)}_{pqrs} &= B^{(3)}_{pqrs} + x + z_2 = S + b_3 + pe_1 + qe_2 + re_3 + se_4 + x + z_2
\end{aligned} \tag{2.10}$$

We note that in these models there are three $SO(4)$ group factors, related with a cyclic symmetry. We could have therefore defined one of the other two $SO(4)$ group as the observable one, and the other two as the hidden ones. We follow here the convention that keeps the group generated by the world-sheet fermions $\bar{\psi}^{4,5}$ as the observable $SO(4)$ and the ones generated by $\bar{\phi}^{1,2}$ and $\bar{\phi}^{3,4}$ as hidden. The models then give rise to a multitude of sectors that produce exotic states with fractional electric charge, given by:

$$Q_{em} = \frac{1}{\sqrt{6}} T_{15} + \frac{1}{2} I_{3_L} + \frac{1}{2} I_{3_R} \tag{2.11}$$

where T_{15} is the diagonal generator of $SU(4)/SU(3)$ and I_{3_L}, I_{3_R} are the diagonal generators of $SU(2)_L$, $SU(2)_R$, respectively. The models then contain the exotic states in the representations:

$(\mathbf{4},\mathbf{1},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{1})$: $\pm\frac{1}{6}$ exotic coloured particles and singlets

$(\mathbf{1},\mathbf{2},\mathbf{1})$: $\pm\frac{1}{2}$ leptons

$(\mathbf{1},\mathbf{1},\mathbf{2})$: $\pm\frac{1}{2}$ singlets

We now enumerate the sectors that give rise to exotic states. The states corresponding to the representations $(\mathbf{4},\mathbf{2},\mathbf{1})$, $(\mathbf{4},\mathbf{1},\mathbf{2})$, $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$, $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ where $\mathbf{4}$ and $\bar{\mathbf{4}}$ are spinorial (anti-spinorial) representations of the observable $SO(6)$, and the $\mathbf{2}$ are doublet representations of the hidden $SU(2)\times SU(2)=SO(4)_1$, arise from the following sectors:

$$\begin{aligned}
B^{(10)}_{pqrs} &= B^{(1)}_{pqrs}+\alpha = S+b_1+pe_3+qe_4+re_5+se_6+\alpha\\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4,\\
&\quad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^1,\bar{\psi}^{1..3},\bar{\phi}^{1..2}\}\\
B^{(11)}_{pqrs} &= B^{(2)}_{pqrs}+\alpha = S+b_2+pe_1+qe_2+re_5+se_6+\alpha\\
B^{(12)}_{pqrs} &= B^{(3)}_{pqrs}+\alpha = S+b_3+pe_1+qe_2+re_3+se_4+\alpha
\end{aligned}\tag{2.12}$$

Similar states $B^{(13,14,15)}_{pqrs}$ arise from the sectors $B^{(10,11,12)}_{pqrs}+z_1$ and they correspond to the representations $(\mathbf{4},\mathbf{2},\mathbf{1})$, $(\mathbf{4},\mathbf{1},\mathbf{2})$, $(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$, $(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$ of $SO(6)_{obs}\times SO(4)_2$.

The states corresponding to the representations $((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$, $((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))$, $((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$ and $((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))$ transforming under $SU(2)_L\times SU(2)_R\times SO(4)_1$ arise from the sectors:

$$\begin{aligned}
B^{(16)}_{pqrs} &= B^{(1)}_{pqrs}+\alpha+x = S+b_1+pe_3+qe_4+re_5+se_6+\alpha+x\\
&= \{\psi^\mu,x^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4,\\
&\quad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^2,\bar{\eta}^3,\bar{\psi}^{4..5},\bar{\phi}^{1..2}\}\\
B^{(17)}_{pqrs} &= B^{(2)}_{pqrs}+\alpha+x = S+b_2+pe_1+qe_2+re_5+se_6+\alpha+x\\
B^{(18)}_{pqrs} &= B^{(3)}_{pqrs}+\alpha+x = S+b_3+pe_1+qe_2+re_3+se_4+\alpha+x
\end{aligned}\tag{2.13}$$

Similar states $B^{(19,20,21)}_{pqrs}$ arise from the sectors $B^{(16,17,18)}_{pqrs}+z_1$ and they produce analogous representations under $SU(2)_L\times SU(2)_R\times SO(4)_2$.

Finally states that transform in vectorial representations are obtained from sectors that contain four periodic world-sheet right-moving complex fermions. Massless states are obtained in such sectors by acting on the vacuum with a Neveu–Schwarz right-moving fermionic oscillator. Vectorial representations arise from the sectors:

$$\begin{aligned}
B^{(1)}_{pqrs}+x &= S+b_1+pe_3+qe_4+re_5+se_6+x\\
&= \{\psi^\mu,\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4,\\
&\quad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^2,\bar{\eta}^3\}\\
B^{(2)}_{pqrs}+x &= S+b_2+pe_1+qe_2+re_5+se_6+x\\
B^{(3)}_{pqrs}+x &= S+b_3+pe_1+qe_2+re_3+se_4+x
\end{aligned}\tag{2.14}$$

and produce the following representations:

- $\{\bar{\psi}^{123}\}|R\rangle^{(i)}_{pqrs}$, $i = 1, 2, 3$, where $|R\rangle^{(i)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(i)}_{pqrs}$ sector. These states transform as a vectorial representation of $SO(6)$.
- $\{\bar{\psi}^{45}\}|R\rangle^{(i)}_{pqrs}$, $i = 1, 2, 3$, where $|R\rangle^{(i)}_{pqrs}$ is the degenerated Ramond vacuum of the $B^{(i)}_{pqrs}$ sector. These states transform as a vectorial representation of $SO(4)$.
- $\{\bar{\phi}^{12}\}|R\rangle^{(i)}_{pqrs}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SO(4)$.
- $\{\bar{\phi}^{34}\}|R\rangle^{(i)}_{pqrs}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SO(4)$.
- $\{\bar{\phi}^{5..8}\}|R\rangle^{(i)}_{pqrs}$, $i = 1, 2, 3$. These states transform as a vectorial representation of $SO(8)$.
- The remaining states in those sectors transform as singlets of the non-Abelian group factors.

It is important to note that the states arising from the sectors in Eq. (2.14) are standard states from the point of view of the Standard Model charge assignments and grand unification embeddings. The term "exotic states" applies only to states that arise due to the "Wilson line" breaking of the non-Abelian GUT symmetries in string theory. In the Pati–Salam models these are the states that arise from the sectors that contain the basis vector α, which breaks the $SO(10)$ GUT symmetry to the Pati–Salam subgroup. States which arise from sectors that do not contain the basis vector α are standard from the point of view of the Standard Model charge assignments and grand unification representations. Thus, for example, the colour triplets appearing in Eq. (2.14) arise from the vectorial 10 representation of the underlying $SO(10)$ GUT symmetry. They are usually termed leptoquarks in the literature, and are counted as n_6 in our analysis. The experimental constraints on these "standard" states are not severe and contemporary experiments are actively seeking their discovery. The experimental constraints on the "exotic" fractionally charged states are far more restrictive. The lightest fractionally charged state is necessarily stable and will be overproduced in a thermal evolution of the early universe. Due to its charge it continues to scatter and cannot decouple from the evolving plasma. Consequently, fractionally charged states must be sufficiently massive and diluted to avoid constraints from contemporary searches and early universe dynamics. It is expected that all non-chiral states receive mass terms along flat directions at the high scale, or when the flat directions are lifted by the SUSY breaking mechanism.

3. The twisted matter spectrum

The counting of spinorials and vectorials is realised by utilising the so-called projectors. Each sector B^i_{pqrs}, corresponds to a projector $P^i_{pqrs} = 0, 1$ which is an entity expressed in terms of GGSO coefficients and determines the survival or not of a sector. The computational analysis and manipulation of the projectors becomes more feasible when rewritten in an analytic form.

3.1. Observable spinorial states and projectors

In order to get the particle content for the representations for the sectors of (2.8) we utilised the following normalisations for the hypercharge and the electromagnetic charge:

$$Y = \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5) \tag{3.1}$$

$$Q_{em} = Y + \frac{1}{2}(Q_4 - Q_5) \tag{3.2}$$

where the Q_i charges of a state, arise due to ψ^i for $i = 1, \ldots, 5$.

The following table summarises the eigenvalues of the electroweak $SU(2) \times U(1)$ Cartan generators, in respect to states which fall into the chiral observable Pati–Salam representations:

representation	$\bar{\psi}^{1,2,3}$	$\bar{\psi}^{4,5}$	Y	Q_{em}
	$(+,+,+)$	$(+,+)$	1	1
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+,+,+)$	$(-,-)$	0	0
	$(+,-,-)$	$(+,+)$	$1/3$	$1/3$
	$(+,-,-)$	$(-,-)$	$-2/3$	$-2/3$
	$(-,-,-)$	$(-,-)$	-1	-1
$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(-,-,-)$	$(+,+)$	0	0
	$(+,+,-)$	$(-,-)$	$-1/3$	$-1/3$
	$(+,+,-)$	$(+,+)$	$2/3$	$2/3$
$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(+,+,+)$	$(+,-)$	$1/2$	$1, 0$
	$(+,-,-)$	$(+,-)$	$-1/6$	$1/3, -2/3$
$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(-,-,-)$	$(+,-)$	$-1/2$	$-1, 0$
	$(+,+,-)$	$(+,-)$	$1/6$	$-1/3, 2/3$

In the previous table, "+" and "−" label the contribution of an oscillator with fermion number $F = 0$ or $F = -1$ to the degenerate vacuum. The case of $(+,-,-)$ under $\bar{\psi}^{1,2,3}$ for example, corresponds to a part of the Ramond vacuum formed by one oscillator with fermion number $F = 0$ and two oscillators with fermion numbers $F = -1$. Families and anti-families in the context of these models, can be formed only if we combine the surviving states of two different sectors:

$$\begin{aligned} \mathbf{16} &= (\mathbf{4},\mathbf{2},\mathbf{1}) + (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) = N_{4L} + N_{\bar{4}R} \\ \overline{\mathbf{16}} &= (\mathbf{4},\mathbf{1},\mathbf{2}) + (\bar{\mathbf{4}},\mathbf{2},\mathbf{1}) = N_{4R} + N_{\bar{4}L} \end{aligned} \tag{3.3}$$

A phenomenologically viable model, must of course consist of only 3 families:

$$N_{4L} - N_{\bar{4}L} = N_{\bar{4}R} - N_{4R} = 3 \tag{3.4}$$

In order to be able to distinguish between N_{4L}, $N_{\bar{4}L}$, $N_{\bar{4}R}$ and N_{4R}, one has to define Representation Operators that will determine the representations in which the states of each observable sector, will fall into. The operators $X_{pqrs}^{i_{SU(4)}} = \pm 1$ that define the $SU(4)$ chirality ($\mathbf{4}$ or $\bar{\mathbf{4}}$) for B^1_{pqrs}, B^2_{pqrs} and B^3_{pqrs} respectively are:

$$\begin{aligned} X_{pqrs}^{(1)_{SU(4)}} &= C\begin{pmatrix} B_{pqrs}^{(1)} \\ S + b_2 + \alpha + (1-r)e_5 + (1-s)e_6 \end{pmatrix} \\ X_{pqrs}^{(2)_{SU(4)}} &= C\begin{pmatrix} B_{pqrs}^{(2)} \\ S + b_1 + \alpha + (1-r)e_5 + (1-s)e_6 \end{pmatrix} \\ X_{pqrs}^{(3)_{SU(4)}} &= C\begin{pmatrix} B_{pqrs}^{(3)} \\ S + b_2 + \alpha + (1-p)e_1 + (1-q)e_2 \end{pmatrix} \end{aligned} \tag{3.5}$$

The representation operators $X_{pqrs}^{(i)_{SU(2)_{L/R}}} = \pm 1$ determine the $SU(2)_{L/R}$ representations $((\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1}))$ for $B^{(1)}_{pqrs}$, $B^{(2)}_{pqrs}$ and $B^{(3)}_{pqrs}$ respectively. In the following expressions $V_i = S + b_i + \alpha + x$.

$$
\begin{aligned}
X_{pqrs}^{(1)SU(2)_{L/R}} &= C\begin{pmatrix} B_{pqrs}^{(1)} \\ V_1+(1-p)e_3+(1-q)e_4+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X_{pqrs}^{(2)SU(2)_{L/R}} &= C\begin{pmatrix} B_{pqrs}^{(2)} \\ V_2+(1-p)e_1+(1-q)e_2+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X_{pqrs}^{(3)SU(2)_{L/R}} &= C\begin{pmatrix} B_{pqrs}^{(3)} \\ V_3+(1-p)e_1+(1-q)e_2+(1-r)e_3+(1-s)e_4 \end{pmatrix}
\end{aligned} \tag{3.6}
$$

The explicit expressions for the 48 projectors related to the observable chiral matter are:

$$
\begin{aligned}
P_{pqrs}^{(1)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_1 \\ B_{pqrs}^{(1)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B_{pqrs}^{(1)} \end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B_{pqrs}^{(1)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(1)} \end{pmatrix}\right) \\
P_{pqrs}^{(2)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_3 \\ B_{pqrs}^{(2)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B_{pqrs}^{(2)} \end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B_{pqrs}^{(2)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(2)} \end{pmatrix}\right) \\
P_{pqrs}^{(3)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_5 \\ B_{pqrs}^{(3)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B_{pqrs}^{(3)} \end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B_{pqrs}^{(3)} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(3)} \end{pmatrix}\right)
\end{aligned} \tag{3.7}
$$

Using the appropriate formalism these projectors can be expressed as a system of linear equations with p, q, r and s as unknowns. The solutions of a specific system of equations, yield the different combinations of p, q, r, s for which sectors survive the GSO projections. This formalism is more suitable and much more flexible for a computer-oriented analysis. In order to achieve the transition to this formalism, the following notation is introduced

$$
c\begin{bmatrix} a_i \\ a_j \end{bmatrix} = e^{i\pi(a_i|a_j)}, \qquad (a_i|a_j)=0,1 \tag{3.8}
$$

where a_i and a_j refer to the basis vectors, and the GGSO projection coefficients are defined in Eq. (2.3). The new expression implies properties which can be easily derived after performing standard algebraic methods involving the GGSO coefficients

$$
(a_i|a_j+a_k) = (a_i|a_j)+(a_i|a_k), \quad \forall a_i\colon \{\psi^\mu\}\cap a_i = \emptyset \tag{3.9}
$$

$$
(a_i|a_j) = (a_j|a_i.), \quad \forall a_i, a_j\colon a_i\cdot a_j = 0 \bmod 4 \tag{3.10}
$$

where $\#(a_i\cdot a_j)\equiv\#[a_i\cup a_j - a_i\cap a_j]$.

The analytic expressions for each different projector $P_{pqrs}^{1,2,3}$ respectively, are given in a matrix form $\Delta^i W^i = Y^i$.

$$
\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{pmatrix}
$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{pmatrix} \tag{3.11}$$

The corresponding algebraic expressions for the states from the remaining sectors above are given in Appendix A. We note that although the hidden sector states can play a crucial phenomenological role, like for example in SUSY breaking, their classification is not done in the analysis here, which focuses exclusively on states that are charged under the Standard Model group factors. Our aim in the present paper in particular is the classification in respect to the fractionally charged states. Experimental observations demand that the low energy exotic states should be truncated from the spectrum or accommodate heavy mass. The projectors shown in Appendix A are crucial in this regard since their values determines the number of surviving exotic representations in each model.

4. The four dimensional gauge group

The untwisted spectrum is common in all the Pati–Salam vacua that we classify. The models differ by the states that arise from the sectors in Eq. (2.6). In our classification method the GGSO projections are encoded in algebraic equations that depend on the GGSO projection coefficients, and are applied to all the sectors listed in Section 2.

If the gauge bosons of a sector transform under a subgroup of the Neveu–Schwarz gauge group, the NS gauge group is enhanced. We restrict the class of vacua to the cases without enhancement. We therefore find the conditions under which the gauge bosons of a specific sector survive. Below we present the type of enhancements that can occur from different sectors, assuming that only one set of conditions is satisfied in each distinct case.

4.1. Enhancements of the observable gauge group

- $x = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\}$ is the only sector which can enlarge the observable gauge group. Enhancement takes place when the following conditions are satisfied The pre-stated conditions

Enhancement conditions	Resulting enhancement
$(x\|e_i) = (x\|z_n) = 0$	$SU(4)_{obs} \times SU(2)_{L/R} \times U(1)' \to SU(6)$

  hold for all $i = 1, \ldots, 6$, $n = 1, 2$, and $U(1)'$ is a linear combination of the $U(1)_i$ where $i = 1, 2, 3$. In the case that any of the previous conditions is not satisfied, the enlargement of the gauge group is not possible.

4.2. Enhancements of the hidden gauge group

- $z_1 + z_2 = \{\bar{\phi}^{12345678}\}$ is the only sector that enlarges only the hidden gauge group when all of the following conditions are met:

Enhancement conditions	Resulting enhancement
$(e_i\|z_1+z_2)=(b_k\|z_1+z_2)=0 \quad \forall i=1,\dots,6,\ k=1,2$	$SU(2)_{1/2}\times SU(2)_{3/4}\times SO(8)_{hid}\to SO(12)$

4.3. Mixed gauge group enhancements

Parts of the observable and hidden gauge group can be enhanced simultaneously in the following cases.

- $\alpha+z_1+z_2=\{\bar\psi^{45},\bar\phi^{34},\bar\phi^{5678}\}$

Enhancement conditions	Resulting enhancement
$(e_i\|\alpha+z_1+z_2)=0$ $(b_1\|\alpha+z_1+z_2)=(b_2\|\alpha+z_1+z_2)=(\alpha\|\alpha+z_1+z_2)$ $(1\|\alpha+z_1+z_2)=1+(b_k\|\alpha+z_1+z_2)$	$SU(2)_{L/R}\times SU(2)_{3/4}\times SO(8)_{hid}\to SO(12)$

The conditions of the previous table hold for all $i=1,\dots,6$.

- $\alpha+x+z_1=\{\bar\eta^{123},\bar\psi^{123},\bar\phi^{34}\}$

Enhancement conditions	Resulting enhancement
$(e_i\|\alpha+x+z_1)=(z_2\|\alpha+x+z_1)=0$	$SU(4)_{obs}\times SU(2)_{1/2}\times U(1)'\to SU(6)$

The conditions above hold for all $i=1,\dots,6$.

- $\alpha+x=\{\bar\eta^{123},\bar\psi^{123},\bar\phi^{12}\}$

Enhancement conditions	Resulting enhancement
$(e_i\|\alpha+x)=(z_2\|\alpha+x)=0,\quad \forall i=1,\dots,6$ $(z_1\|\alpha+x)=(\alpha\|\alpha+x)$	$SU(4)_{obs}\times SU(2)_{1/2}\times U(1)'\to SU(6)$

- $\alpha+z_2=\{\bar\psi^{45},\bar\phi^{12},\bar\phi^{5678}\}$

Enhancement conditions	Resulting enhancement
$(e_i\|\alpha+z_2)=0,\quad \forall i=1,\dots,6$ $(b_1\|\alpha+z_2)=(b_2\|\alpha+z_2)$ $(b_k\|\alpha+z_2)+(z_1\|\alpha+z_2)=(\alpha\|\alpha+z_2)$	$SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)_{hid}\to SO(12)$

- $z_1=\{\bar\phi^{1234}\}$ produces the following enhancements:

Survival conditions	Resulting enhancement
$(e_i\|z_1)=(z_2\|z_1)=0$ $(b_k\|z_1)=1$	$SU(4)_{obs}\times SU(2)_{1/2}\times SU(2)_{3/4}\to SO(10)$
$(e_i\|z_1)=(z_2\|z_1)=0$ $(b_k\|z_1)=1$	$SU(2)_L\times SU(2)_R\times SU(2)_{2/1}\times SU(2)_{4/3}\to SO(8)$
$(e_i\|z_1)=(z_2\|z_1)=(b_2\|z_1)=0$ $(b_1\|z_1)=1$	$SU(2)_{1/2}\times SU(2)_{3/4}\times U(1)\to SO(6)$
$(e_i\|z_1)=(z_2\|z_1)=(b_1\|z_1)=0$ $(b_2\|z_1)=1$	$SU(2)_{1/2}\times SU(2)_{3/4}\times U(1)\to SO(6)$
$(e_i\|z_1)=(z_2\|z_1)=(b_k\|z_1)=0$	$SU(2)_{1/2}\times SU(2)_{3/4}\times U(1)\to SO(6)$
$(e_j\|z_1)=(z_2\|z_1)=0$ $(e_i\|z_1)=1$ AND	$SU(2)_{1/2}\times SU(2)_{3/4}\to SO(5)$

Survival conditions	Resulting enhancement
$(b_1\|z_1) = 0,\ (b_2\|z_1) = 1, \quad i = 1,2$ or $(b_1\|z_1) = 1,\ (b_2\|z_1) = 0, \quad i = 3,4$ or $(b_1\|z_1) = 1,\ (b_2\|z_1) = 1, \quad i = 5,6$	
$(e_j\|z_1) = (z_2\|z_1) = 0$ $(e_i\|z_1) = 1$ $(b_k\|z_1) = 0$	$SU(2)_{1/2} \times SU(2)_{3/4} \to SO(5)$
$(e_i\|z_1) = (b_k\|z_1) = 0$ $(z_2\|z_1) = 1$	$SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)_{hid} \to SO(12)$

The relations above, hold for all $i, j = 1, \dots, 6$ where $i \neq j$ and $k = 1, 2$. We note that while z_2 produces two cases in which only the hidden $SO(8)$ gauge group is enhanced to $SO(9)$, in other cases it leads to enhancements that mix the hidden and observable gauge groups.

- $z_2 = \{\bar{\phi}^{5678}\}$ can generate enhancements in the following cases:

Survival conditions	Resulting enhancement
$(e_i\|z_2) = (z_1\|z_2) = (\alpha\|z_2) = 0$ $(b_k\|z_2) = 1$	$SU(4)_{obs} \times SO(8)_{hid} \to SO(14)$
$(e_i\|z_2) = (z_1\|z_2) = 0$ $(b_k\|z_2) = (\alpha\|z_2) = 1$	$SU(2)_L \times SU(2)_R \times SO(8)_{hid} \to SO(12)$
$(e_i\|z_2) = (z_1\|z_2) = (b_2\|z_2) = (\alpha\|z_2) = 0$ $(b_1\|z_2) = 1$	$U(1) \times SO(8)_{hid} \to SO(10)$
$(e_i\|z_2) = (z_1\|z_2) = (b_1\|z_2) = (\alpha\|z_2) = 0$ $(b_2\|z_2) = 1$	$U(1) \times SO(8)_{hid} \to SO(10)$
$(e_i\|z_2) = (z_1\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$	$U(1) \times SO(8)_{hid} \to SO(10)$
$(e_j\|z_2) = (z_1\|z_2) = (\alpha\|z_2) = 0$ $(e_i\|z_2) = 1$ AND $(b_1\|z_2) = 0,\ (b_2\|z_2) = 1, \quad i = 1,2$ or $(b_1\|z_2) = 1,\ (b_2\|z_2) = 0, \quad i = 3,4$ or $(b_1\|z_2) = 1,\ (b_2\|z_2) = 1, \quad i = 5,6$	$SO(8)_{hid} \to SO(9)$
$(e_j\|z_2) = (z_1\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$ $(e_i\|z_2) = 1$	$SO(8)_{hid} \to SO(9)$
$(e_i\|z_2) = (b_k\|z_2) = 0$ $(\alpha\|z_2) = (z_1\|z_2) = 1$	$SO(4)_1 \times SO(8)_{hid} \to SO(12)$
$(e_i\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$ $(z_1\|z_2) = 1$	$SO(4)_2 \times SO(8)_{hid} \to SO(12)$

The relations above, hold for all $i, j = 1, \dots, 6$ where $i \neq j$ and $k = 1, 2$.

- $\alpha = \{\bar{\psi}^{45}\bar{\phi}^{12}\}$ can also present numerous potential enhancements.

Survival conditions	Resulting enhancement
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_1\|\alpha) = (b_2\|\alpha)$ $(1\|\alpha) = 1 + (b_k\|\alpha) + (z_1\|\alpha)$	$SU(4)_{obs} \times SU(2)_{L/R} \times SU(2)_{1/2} \to SO(10)$ AND $SU(2)_{L/R} \times SU(2)_{1/2} \times SU(2)_3 \times SU(2)_4 \to SO(8)$
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_1\|\alpha) = 1 + (b_2\|\alpha)$ $(1\|\alpha) = (b_1\|\alpha) + (z_1\|\alpha)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{1/2} \to SO(6)$
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_2\|\alpha) = 1 + (b_1\|\alpha)$ $(1\|\alpha) = (b_2\|\alpha) + (z_1\|\alpha)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{1/2} \to SO(6)$

Survival conditions	Resulting enhancement
$(e_i\|\alpha)=(z_2\|\alpha)=0$ $(b_1\|\alpha)=(b_2\|\alpha)$ $(1\|\alpha)=(b_2\|\alpha)+(z_1\|\alpha)$	$U(1)\times SU(2)_{L/R}\times SU(2)_{1/2}\to SO(6)$
$(e_j\|\alpha)=(z_2\|\alpha)=0$ $(e_i\|\alpha)=1$ AND $(b_1\|\alpha)=1+(b_2\|\alpha)$ and $(1\|\alpha)=(b_1\|\alpha)+(z_1\|\alpha), \quad i=1,2$ or $(b_1\|\alpha)=1+(b_2\|\alpha)$ and $(1\|\alpha)=(b_2\|\alpha)+(z_1\|\alpha), \quad i=3,4$ or $(b_1\|\alpha)=(b_2\|\alpha)$ and $(1\|\alpha)=1+(b_k\|\alpha)+(z_1\|\alpha), \quad i=5,6$	$SU(2)_{L/R}\times SU(2)_{1/2}\to SO(5)$
$(e_j\|\alpha)=(z_2\|\alpha)=0$ $(e_i\|\alpha)=1$ $(b_1\|\alpha)=(b_2\|\alpha)$ $(1\|\alpha)=(b_k\|\alpha)+(z_1\|\alpha)$	$SU(2)_{L/R}\times SU(2)_{1/2}\to SO(5)$
$(e_i\|\alpha)=0$ $(z_2\|\alpha)=1$ $(b_1\|\alpha)=(b_2\|\alpha)$ $(1\|\alpha)=(b_k\|\alpha)+(z_1\|\alpha)$	$SU(2)_{L/R}\times SU(2)_{1/2}\times SO(8)_{hid}\to SO(12)$

The relations above, hold for all $i, j=1,\dots,6$ where $i\neq j$ and $k=1,2$.

- $\alpha+z_1=\{\bar\psi^{45}\bar\phi^{34}\}$ gives rise to enhancements in the following occasions:

Survival conditions	Resulting enhancement
$(e_i\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $(b_1\|\alpha+z_1)=(b_2\|\alpha+z_1)$ $(\alpha\|\alpha+z_1)=1+(b_k\|\alpha+z_1)$	$SU(4)_{obs}\times SU(2)_{L/R}\times SU(2)_{3/4}\to SO(10)$ AND $SU(2)_{L/R}\times SO(4)_1\times SU(2)_{3/4}\to SO(8)$
$(e_i\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $1+(b_1\|\alpha+z_1)=(b_2\|\alpha+z_1)=(\alpha\|\alpha+z_1)$	$U(1)\times SU(2)_{L/R}\times SU(2)_{3/4}\to SO(6)$
$(e_i\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $1+(b_2\|\alpha+z_1)=(b_1\|\alpha+z_1)=(\alpha\|\alpha+z_1)$	$U(1)\times SU(2)_{L/R}\times SU(2)_{3/4}\to SO(6)$
$(e_i\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $(b_1\|\alpha+z_1)=(b_2\|\alpha)=(\alpha\|\alpha+z_1)$	$U(1)\times SU(2)_{L/R}\times SU(2)_{3/4}\to SO(6)$
$(e_j\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $(e_i\|\alpha+z_1)=1$ AND $(b_1\|\alpha+z_1)=1+(b_2\|\alpha+z_1)=(\alpha\|\alpha+z_1), \quad i=1,2$ or $(b_1\|\alpha+z_1)=1+(b_2\|\alpha+z_1)=1+(\alpha\|\alpha+z_1), \quad i=3,4$ or $(b_1\|\alpha+z_1)=(b_2\|\alpha+z_1)=1+(\alpha\|\alpha+z_1), \quad i=5,6$	$SU(2)_{L/R}\times SU(2)_{3/4}\to SO(5)$
$(e_j\|\alpha+z_1)=(z_2\|\alpha+z_1)=0$ $(e_i\|\alpha+z_1)=1$ $(b_1\|\alpha+z_1)=(b_2\|\alpha+z_1)$ $(1\|\alpha+z_1)=(b_k\|\alpha+z_1)+(z_1\|\alpha+z_1)$	$SU(2)_{L/R}\times SU(2)_{1/2}\to SO(5)$
$(e_i\|\alpha+z_1)=0$ $(z_2\|\alpha+z_1)=1$ $(b_1\|\alpha+z_1)=(b_2\|\alpha+z_1)=(\alpha\|\alpha+z_1)$	$SU(2)_{L/R}\times SU(2)_{3/4}\times SO(8)\to SO(12)$

5. Results

Using the algebraic expressions presented in the previous sections we can analyse the entire massless spectrum for a given choice of GGSO projection coefficients that completely specify a specific string model. These formulas are inputted into a computer program which is used to scan the space of string vacua produced by random generation of the one-loop GGSO projection coefficients. The number of possible configurations is $2^{51} \sim 10^{15}$, which is too large for a complete classification. For this reason a random generation algorithm is utilised,[2] and the characteristics of the model for each set of random GGSO projection coefficients are extracted. In this manner a model with some desired phenomenological criteria can be fished from the sample generated. In Ref. [25] this procedure was followed and produced a three generation Pati–Salam string model that does not contain any exotic massless states with fractional electric charge. In this paper we use this methodology to classify the Pati–Salam free fermionic string models with respect to some phenomenological criteria. The observable sector of a heterotic-string Pati–Salam model is characterized by 9 integers $(n_g, k_L, k_R, n_6, n_h, n_4, n_{\bar{4}}, n_{2L}, n_{2R})$, where

$$
\begin{aligned}
& n_{4L} - n_{\bar{4}L} = n_{\bar{4}R} - n_{4R} = n_g = \#\text{ of generations} \\
& n_{\bar{4}L} = k_L = \#\text{ of non-chiral left pairs} \\
& n_{4R} = k_R = \#\text{ of non-chiral right pairs} \\
& n_6 = \#\text{ of } (\mathbf{6}, \mathbf{1}, \mathbf{1}) \\
& n_h = \#\text{ of } (\mathbf{1}, \mathbf{2}, \mathbf{2}) \\
& n_4 = \#\text{ of } (\mathbf{4}, \mathbf{1}, \mathbf{1}) \quad \text{(exotic)} \\
& n_{\bar{4}} = \#\text{ of } (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{1}) \quad \text{(exotic)} \\
& n_{2L} = \#\text{ of } (\mathbf{1}, \mathbf{2}, \mathbf{1}) \quad \text{(exotic)} \\
& n_{2R} = \#\text{ of } (\mathbf{1}, \mathbf{1}, \mathbf{2}) \quad \text{(exotic)}
\end{aligned}
$$

Using the methodology outlined in Section 3 we obtain analytic formulas for all these quantities. The spectrum of a viable Pati–Salam heterotic string model should have $n_g = 3$,

$$
\begin{aligned}
& n_g = 3 \quad \text{three light chiral of generations} \\
& k_L \geqslant 0 \quad \text{heavy mass can be generated for non-chiral pairs} \\
& k_R \geqslant 1 \quad \text{at least one Higgs pair to break the PS symmetry} \\
& n_6 \geqslant 1 \quad \text{at least one required for missing partner mechanism} \\
& n_h \geqslant 1 \quad \text{at least one light Higgs bi-doublet} \\
& n_4 = n_{\bar{4}} \geqslant 0 \quad \text{heavy mass can be generated for vector-like exotics} \\
& n_{2L} = 0 \bmod 2 \quad \text{heavy mass can be generated for vector-like exotics} \\
& n_{2R} = 0 \bmod 2 \quad \text{heavy mass can be generated for vector-like exotics}
\end{aligned}
$$

A minimal model which is free of exotics has $k_L = 0$, $k_R = 1$, $n_6 = 1$, $n_h = 3$, $n_4 = n_{\bar{4}} = 0$, $n_{2L} = 0$ and $n_{2R} = 0$. The model given by the following GGSO coefficients matrix:

[2] We note that analysis of large sets of string vacua has also been performed by other groups [26].

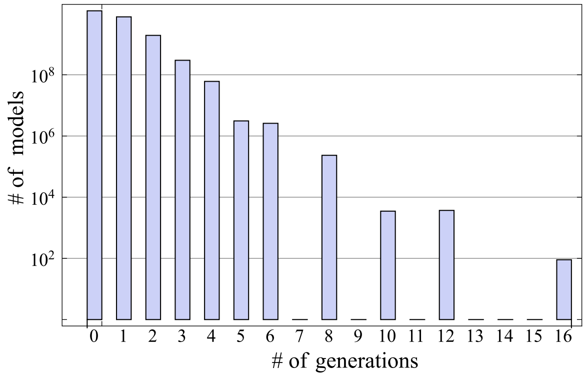

Fig. 1. Number of models versus number of generations (n_g) in a random sample of 10^{11} GGSO configurations.

$$[v_i|v_j] = e^{i\pi(v_i|v_j)} \tag{5.1}$$

$$(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array} \begin{array}{c} \begin{array}{ccccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\ \left(\begin{array}{ccccccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{array}\right) \end{array}. \tag{5.2}$$

was presented in Ref. [25] and produces the desired spectrum. The twisted sectors in this model produce three chiral generations; one pair of heavy Higgs states; one light Higgs bi-doublet; one vector sextet of $SO(6)$; and is completely free of massless exotic fractionally charged states. Additionally the model contains three pairs of untwisted $SO(6)$ sextets, which can obtain string scale mass along flat directions. The full massless spectrum of this model was presented in Ref. [25].

We next explore the space of Pati–Salam free fermionic heterotic string vacua. We perform a statistical sampling in a space of 10^{11} models out of the total of 2^{51}. Using a computer FORTRAN95 program running on a single node of the Theoretical Physics Division of University of Ioannina, HPC cluster, we were able to obtain the relative data within a period of one week. This corresponds to examining approximately 1:20 000 models in this class. Increasing the sample by one order of magnitude is within the cluster capabilities, however, as already checked by using a 10^9 and a 10^{10} random sample, the results obtain are similar to the ones presented below. Some of the results are presented in Figs. 1–6 and Table 1.

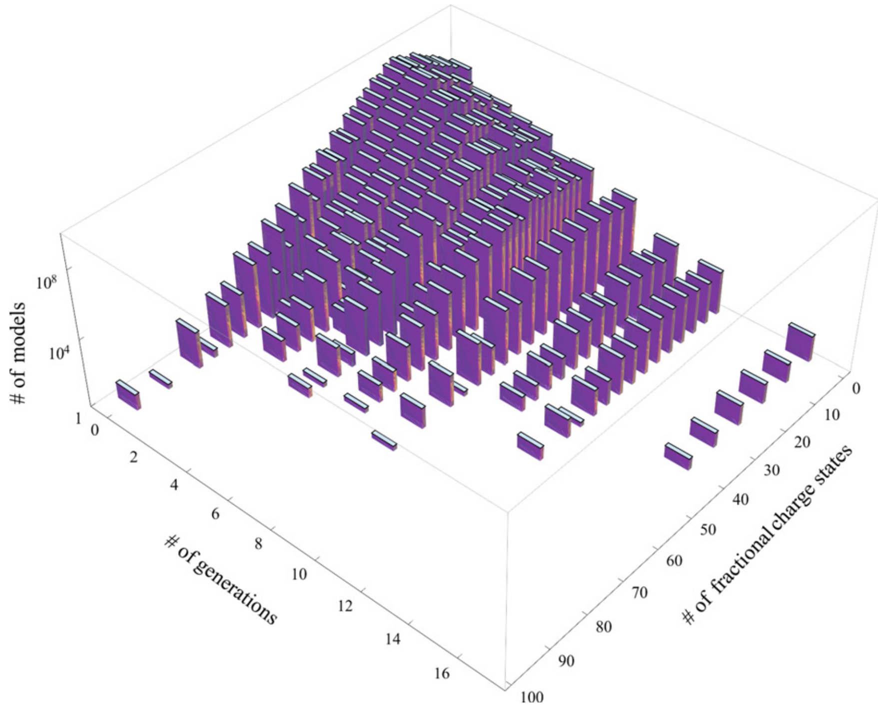

Fig. 2. Number of models versus number of generations (n_g) and total number of exotic multiplets in a random sample of 10^{11} GGSO configurations.

In Fig. 1 the number of models versus the number of generations is displayed. In agreement with the results of Refs. [14,13] the number of models has a peak for models with vanishing number of generations, and decreases with increasing number of generations. Of note in Fig. 1 is the absence of any models with 7, 9, 11, 13, 14 and 15 generations. This may indicate that this cases are completely forbidden or are extremely unlikely cases in the space of all possibilities.

In Fig. 2 we display a three dimensional plot of the number of models versus the number of generations and the total number of exotic fractionally charged states. As seen from the figure the distribution exhibits a peak for models with zero chiral generations and a nonvanishing number of exotic multiplets, and decreases with increasing and decreasing number of exotics. Moreover, we find no correlation between the absence of fractionally charge exotic states and the number of generations. We can have exophobic models for all values of n_g.

However, in the case of models without any exotic multiplets we observe the following relation between the number of chiral generations (n_g), the number of Higgs bi-doublets (n_h) and sextets (n_6)

$$n_g \mod 2 = n_h \mod 2 = n_6 \mod 2 \tag{5.3}$$

This empirical observation is in accord with the data of the exophobic model presented in Ref. [25], and is corroborated by the data of Table 1 where we display the multiplicities of models with respect to n_g, n_h and n_6. As noted from the table the number of Higgs bi-doublets and sextets is indeed odd or even depending on the number of generations. Another important phenomenological point to note from Table 1 is the existence of exophobic models with a varying

Table 1
Multiplicities of massless fractional charge free models with respect to: the number of generations n_g, the number of Higgs bi-doublets n_h, and the number of colour sextets n_6, in a random sample of 10^{11} PS models.

n_g	n_h	n_6	# of models
0	0	0	7 389 484
0	0	2	1 645 466
0	0	4	1 000 290
0	0	6	7964
0	0	8	35 156
0	0	12	125
0	0	16	48
0	2	0	1 772 537
0	2	2	3 370 245
0	2	4	282 693
0	2	6	101 806
0	2	8	240
0	2	10	1425
0	4	0	1 281 766
0	4	2	314 402
0	4	4	1 272 994
0	4	6	41 240
0	4	8	26 600
0	4	12	695
0	4	16	3
0	6	0	32 801
0	6	2	162 980
0	6	4	42 929
0	6	6	197 305
0	6	10	1077
0	8	0	83 905
0	8	2	891
0	8	4	44 391
0	8	8	53 896
0	8	10	667
0	8	12	198
0	8	16	38
0	10	0	948
0	10	2	3951
0	10	6	1650
0	10	8	716
0	10	10	2681
0	10	14	7
0	12	0	1657
0	12	4	2207
0	12	8	322
0	12	12	2458
0	14	2	14
0	14	10	4
0	16	0	336
0	16	4	37
0	16	8	98
0	16	16	121
0	18	2	3
0	20	0	2
0	20	4	1
0	20	12	2
0	24	0	2
0	24	8	1
0	24	24	1
1	1	1	690 074
1	1	3	50 495
1	3	1	54 719
1	3	3	701 850
1	3	5	47 239
1	5	3	51 664
1	5	5	91 419
1	5	7	2408
1	7	5	2636
1	7	7	2283
2	0	0	159 209
2	0	4	2935
2	2	2	1 060 873
2	2	6	15 898
2	2	10	243
2	4	0	4435
2	4	4	220 673
2	4	8	1180
2	6	2	25 966
2	6	6	53 586
2	6	10	52
2	8	0	526
2	8	4	1631
2	8	8	5419
2	10	2	824
2	10	6	61
2	10	10	629
3	1	1	240 224
3	1	3	19 086
3	3	1	20 709
3	3	3	238 714
3	3	5	14 007
3	5	3	14 932
3	5	5	56 886
3	5	7	539
3	7	5	591
3	7	7	3135
4	0	0	105 365
4	0	4	3234
4	0	8	114
4	0	12	3
4	2	2	145 699
4	2	6	2159

Table 1 (*continued*)

n_g	n_h	n_6	# of models
4	2	10	14
4	4	0	4757
4	4	4	118 796
4	4	8	1546
4	4	12	42
4	6	2	2660
4	6	6	27 834
4	6	10	84
4	8	0	556
4	8	4	2484
4	8	8	7942
4	10	2	24
4	10	6	81
4	10	10	22
4	12	0	37
4	12	4	124
4	12	12	234
4	16	0	1
5	1	1	5743
5	3	3	24 930
5	5	5	16 949
5	7	7	656
6	0	0	9339
6	0	4	162
6	2	2	34 884
6	2	6	55
6	4	0	184
6	4	4	10 612
6	4	8	26
6	6	2	62
6	6	6	7539
6	6	10	10
6	8	4	34
6	8	8	781
6	10	6	20
6	10	10	187
8	0	0	2543
8	0	8	35
8	2	2	2529
8	4	4	7055
8	4	12	3
8	6	6	1742
8	8	0	19
8	8	8	3328
8	8	16	1
8	10	10	134
8	12	4	4
8	12	12	100
8	16	8	3
8	16	16	4
10	0	0	124
10	2	2	219
10	4	4	112
10	6	6	187
10	8	8	23
12	0	0	47
12	2	2	22
12	4	4	122
12	8	8	145
12	10	10	3
12	12	12	43
16	0	0	7
16	4	4	17
16	8	8	7
16	12	12	4

number of Higgs bi-doublets representations. The Pati–Salam models face the potential problem of doublet–doublet splitting due the coupling of the Higgs bi-doublet to both the up and down quarks, and resulting flavor changing neutral currents transitions. One way to alleviate the problem is by having several Higgs bi-doublets representations, where one gives mass to up-type quarks and another generate masses to the down-type quarks.

In Fig. 3 we display the multiplicities of models versus the number of generations in the case of exotic free models. As seen from the figure the number of models decreases with increasing number of generations. The same exclusion of models with some number of generations noted in Fig. 1 is also seen in Fig. 2 for the same cases.

Fig. 4 displays the total number of three generation models versus the number of exotic fractionally charged states in a given three generation model. As seen from the figure the total number of exophobic three generation models is slightly less than 10^6, which is roughly $1/10^5$ from the entire sample. Hence we can surmise that exophobia is a common feature in the sampled space of string vacua. Having established a quasi-realistic spectrum the next stage is to analyse the

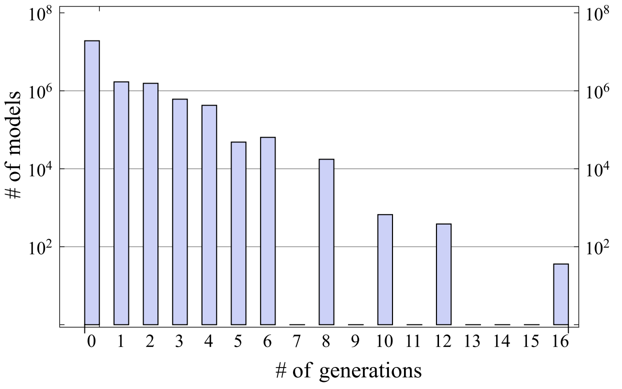

Fig. 3. Number of exotic free models versus number of generations (n_g) in a random sample of 10^{11} GGSO configurations.

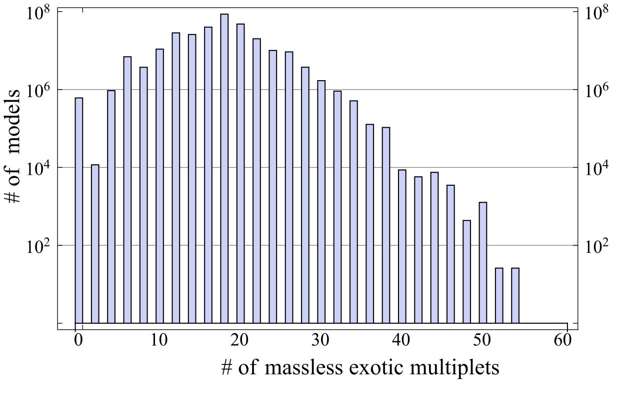

Fig. 4. Number of 3-generation models versus total number of exotic multiplets in a random sample of 10^{11} GGSO configurations.

Yukawa couplings in the models. The abundance of exotic free three generation models suggests that models with viable Yukawa and fermion mass spectrum do exist in this space of string vacua.

In Fig. 5 we display in a three dimensional plot the total number of three generation models versus the number of exotic $SU(4)$ **4**-plets and number of exotic $SO(4)$ $\mathbf{2_L}$ and $\mathbf{2_R}$ doublets. In Fig. 6 we display in a three dimensional plot the number of three generation models versus the number of additional non-chiral representations in the $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2_R}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2_R})$ and $(\mathbf{4}, \mathbf{2_L}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{2_L}, \mathbf{1})$ and additional $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ multiplets of $SU(4) \times SU(2)_L \times SU(2)_R$. Finally in Table 2 we tabulate the number of models with sequential imposition of phenomenological constraints. The total number of models in the sample is 10^{11}. We first impose that there is no enhancement of the four dimensional gauge symmetry. Roughly 80% percent of the models satisfy this criteria. Next we impose that the generations form complete families. That is there is no chiral representation of the Pati–Salam gauge group that is not accompanied by either the representation that completes it to a representation of $SO(10)$ or renders it non-chiral. So the entire chiral spectrum is contained in complete representations of $SO(10)$ decomposed under the Pati–Salam subgroup. Roughly 1/5 of the previous set satisfy this criterion. The restriction to three chiral generations reduces further the number of models by two orders of magnitude.

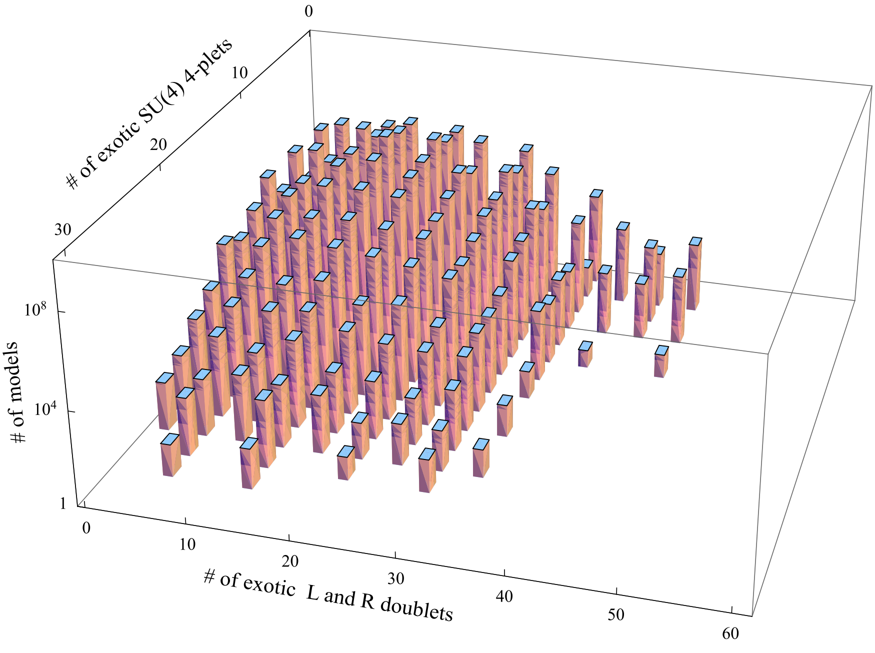

Fig. 5. Number of 3-generation models versus number of exotic $SU(4)$ multiplets and total number of L plus R exotic $SU(2)$ doublets in a random sample of 10^{11} GGSO configurations. We note that the exophobic cases correspond to the upper left column.

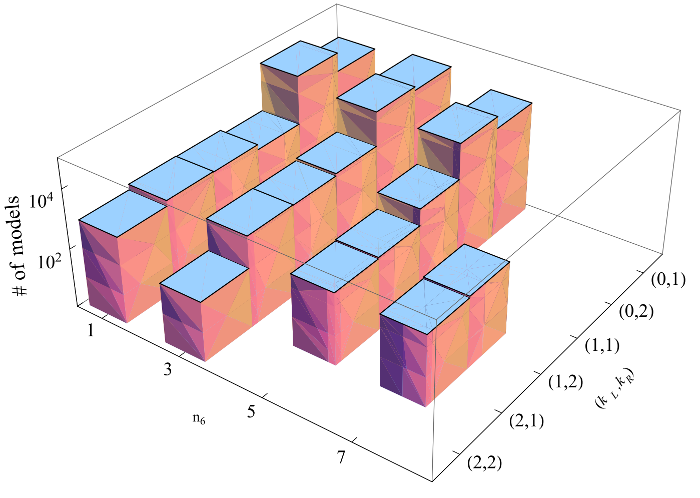

Fig. 6. Number of 3-generation models versus number of additional non-chiral left and right pairs (k_L, k_R) and additional $(6, 1, 1)$ $SU(4)$ representations (n_6) in a random sample of 10^{11} GGSO configurations. We note that accommodating the heavy Higgs states necessitates $k_R = 1$. By Eq. (5.3) the minimal case in realistic models also requires $n_6 = 1$.

Imposing the existence of heavy string states to break the Pati–Salam gauge symmetry to the Standard Model gauge group leads to a reduction by another order of magnitude. The requirement of Standard Model Higgs doublets does not lead to a further reduction because as noted

Table 2
Pati–Salam models statistics with respect to phenomenological constraints imposed on massless spectrum. Constraints in second column act additionally. Omitting constraint (e) does not change the results of (f), (g) since all massless exotic free models have an odd number of pairs of SM Higgs doublets.

	Constraint	# of models in sample	Probability	Estimated # of models in class
	None	100 000 000 000	1	2.25×10^{15}
(a)	+ No gauge group enhancements.	78 977 078 333	7.90×10^{-1}	1.78×10^{15}
(b)	+ Complete families	22 497 003 372	2.25×10^{-1}	5.07×10^{14}
(c)	+ 3 generations	298 140 621	2.98×10^{-3}	6.71×10^{12}
(d)	+ PS breaking Higgs	23 694 017	2.37×10^{-4}	5.34×10^{11}
(e)	+ SM breaking Higgs	19 191 088	1.92×10^{-4}	4.32×10^{11}
(f)	+ No massless exotics	121 669	1.22×10^{-6}	2.74×10^{9}
(g)	+ Minimal PS Higgs	31 804	3.18×10^{-7}	7.16×10^{8}

above in Eq. (5.3) the total number of Higgs bi-doublets is equal to the number of chiral generations modulo 2. Therefore, existence of three chiral generations necessarily implies a non-zero number of Higgs bi-doublets to be in the spectrum. Finally, imposing the absence of massless exotics reduces the number of models by further two orders of magnitude. Therefore, the reduction from the initial sample is by roughly six order of magnitude, i.e. one in every 10^6 models satisfy all of these constraints. Given that the total number of vacua in the space of models scanned is of the order of 10^{15}, we expect that 10^9 of the models satisfy these criteria, which leaves a substantial number to accommodate further phenomenological constraints. For example, requiring minimal number of PS breaking Higgs ($k_L = 0$, $k_R = 1$) truncates further by 4 the number of models as seen in line (g). Furthermore, approximately $1/4$ of these models have also minimal Standard Model Higgs sector with ($n_h = 1$).

6. Conclusions

The Standard Model data supports the embedding of its matter spectrum into spinorial 16 representations of $SO(10)$. Indeed, the augmentation of the Standard Model by the right-handed neutrinos, proposed originally by Pati and Salam [21], was corroborated by terrestrial and astrophysical neutrino experiments. String theory enables the construction of phenomenological models that provide the arena to explore the synthesis of gravity and the gauge interaction within a self-consistent framework. It is desirable that such phenomenological string models preserve the $SO(10)$ embedding of the Standard Model matter states, while its Higgs representations are obtained from the vectorial 10 representation.

Absence of adjoint Higgs representations in models with level one Kac–Moody algebras necessitates that the $SO(10)$ symmetry is broken directly at the string level. Heterotic string models in the free fermionic formulation produce such three generation models that preserve the $SO(10)$ embedding of the Standard Model spectrum. Early constructions of such models, constructed in the late eighties, consisted of isolated examples. During the last few years systematic methods to classify large classes of symmetric free fermionic models were developed. Initially these methods were applied to the classification of models with unbroken $SO(10)$ GUT symmetry, with respect to the number of generations, i.e. of the difference between spinorial and anti-spinorial representations, and subsequently also with respect to vectorial representations. The classification revealed a new duality symmetry in the space of vacua under exchange of spinor and vector representations.

In this paper we extended the classification to models in which the $SO(10)$ GUT symmetry is broken to the Pati–Salam subgroup. A generic feature of such string models in which the $SO(10)$ symmetry is broken and that preserve the canonical GUT embedding of the weak hypercharge, is the appearance of exotic fractionally charged states in the string spectrum. Such states are severely constrained by experimental observations. The reason being that the lightest of these states is stable due to electric charge conservation, and must be sufficiently massive and diluted in a viable model. One possibility is that the harmful states only exist in the massive string spectrum. In Ref. [25] we presented an explicit example of such an exophobic quasi-realistic Pati–Salam heterotic-string model. It is of interest to study whether such exophobic string models are also obtained in other classes of orbifold models [27]. We also note that, provided that they satisfy all the observational constraints, the exotic states may produce stable string relics [28] that are of further interest.

Furthermore, we elaborated on the classification method that enabled the discovery of the exophobic model in [25]. The key to obtaining this result is the extension of the algebraic expressions derived in Ref. [14] for spinorial and vectorial $SO(10)$ representations to all the sectors in the string models. This enables the derivation of algebraic formulas for the entire spectrum that arises in the string models. These formulas are used in a computer code, and enables us to scan a space of 2^{51} models. This number of vacua is too large for a complete classification and we performed a statistical analysis that samples 10^{11} models in this class of vacua. Imposing various phenomenological criteria we find that roughly one in 10^6 of the models pass similar phenomenological impositions as the exophobic model of Ref. [25]. This suggests that sufficient freedom remains in the space of vacua to satisfy the additional constraints required by the Standard Model data.

Having at our disposal a plethora of semi-realistic $N = 1$ string vacua with the full massless and massive spectrum give us the possibility to study not only their phenomenological properties, but also their cosmological implications, once supersymmetry breaking is incorporated. Following the lines of Ref. [29] the cosmological evolution of all these models can be studied since the exact one-loop free energy and effective potential can be calculated at the string level, at least for models in which supersymmetry breaking is achieved via geometrical fluxes [29]. This will lead to a cosmological evolutionary behavior at least for temperature below the Hagedron era and before the electroweak phase transition, thanks to the attractor mechanism valid in this intermediate cosmological regime [29].

Another direction along these lines is to check the possible deformations induced by the moduli participating in the supersymmetry breaking [30], and to select the low energy vacua, which lead to Hagedron and initial singularity free models at early cosmological times [30].

Finally, after the electroweak phase transitions, one can derive in full generality the soft supersymmetry breaking terms [31] in the low effective field theory, once the SUSY breaking fluxes (geometrical or not) are suitably fixed.

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Appendix A. Hidden matter states, representations and projectors

The expressions for the projectors corresponding to $B^{(4,5,6)}_{pqrs}$ from (2.9) are given below

$$\begin{aligned}
P^{(4)}_{pqrs} &= \frac{1}{8}\left(1 - c\begin{pmatrix} e_1 \\ B^{(4)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} e_2 \\ B^{(4)}_{pqrs} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - c\begin{pmatrix} z_2 \\ B^{(4)}_{pqrs} \end{pmatrix}\right) \\
P^{(5)}_{pqrs} &= \frac{1}{8}\left(1 - c\begin{pmatrix} e_3 \\ B^{(5)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} e_4 \\ B^{(5)}_{pqrs} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - c\begin{pmatrix} z_2 \\ B^{(5)}_{pqrs} \end{pmatrix}\right) \\
P^{(6)}_{pqrs} &= \frac{1}{8}\left(1 - c\begin{pmatrix} e_5 \\ B^{(6)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} e_6 \\ B^{(6)}_{pqrs} \end{pmatrix}\right) \\
&\quad \cdot \left(1 - c\begin{pmatrix} z_2 \\ B^{(6)}_{pqrs} \end{pmatrix}\right)
\end{aligned} \tag{A.1}$$

Their corresponding analytic expressions are

$$\begin{aligned}
&\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + x + z_1) \\ (e_2|b_1 + x + z_1) \\ (z_2|b_1 + x + z_1) \end{pmatrix} \\
&\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2 + x + z_1) \\ (e_2|b_2 + x + z_1) \\ (z_2|b_2 + x + z_1) \end{pmatrix} \\
&\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_3 + x + z_1) \\ (e_2|b_3 + x + z_1) \\ (z_2|b_3 + x + z_1) \end{pmatrix}
\end{aligned} \tag{A.2}$$

The remaining 48 projectors corresponding to hidden sectors given in (2.10) are given by

$$\begin{aligned}
P^{(7)}_{pqrs} &= \frac{1}{4}\left(1 - c\begin{pmatrix} e_1 \\ B^{(7)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} e_2 \\ B^{(7)}_{pqrs} \end{pmatrix}\right) \\
&\quad \cdot \frac{1}{4}\left(1 - c\begin{pmatrix} z_1 \\ B^{(7)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} \alpha \\ B^{(7)}_{pqrs} \end{pmatrix}\right) \\
P^{(8)}_{pqrs} &= \frac{1}{4}\left(1 - c\begin{pmatrix} e_3 \\ B^{(8)}_{pqrs} \end{pmatrix}\right) \cdot \left(1 - c\begin{pmatrix} e_4 \\ B^{(8)}_{pqrs} \end{pmatrix}\right)
\end{aligned}$$

$$\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(8)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha \\ B^{(8)}_{pqrs}\end{pmatrix}\right)$$

$$P^{(9)}_{pqrs}=\frac{1}{4}\left(1-c\begin{pmatrix} e_5 \\ B^{(9)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B^{(9)}_{pqrs}\end{pmatrix}\right)$$

$$\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(9)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha \\ B^{(9)}_{pqrs}\end{pmatrix}\right) \tag{A.3}$$

The analytic expressions for $P^{7,8,9}_{p,q,r,s}$ are given below:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}=\begin{pmatrix} (e_1|b_1+x+z_2) \\ (e_2|b_1+x+z_2) \\ (z_1|b_1+x+z_2) \\ (\alpha|b_1+x+z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}=\begin{pmatrix} (e_1|b_2+x+z_2) \\ (e_2|b_2+x+z_2) \\ (z_1|b_2+x+z_2) \\ (\alpha|b_2+x+z_2) \end{pmatrix}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (\alpha|e_1) & (\alpha|e_2) & (\alpha|e_3) & (\alpha|e_4) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}=\begin{pmatrix} (e_1|b_3+x+z_2) \\ (e_2|b_3+x+z_2) \\ (z_1|b_3+x+z_2) \\ (\alpha|b_3+x+z_2) \end{pmatrix} \tag{A.4}$$

A.1. Exotic states, representations and projectors

The representations and observable charges of $B^{10,11,12}_{p,q,r,s}$ in (2.12) and $B^{13,14,15}_{p,q,r,s}$ are given below:

representation	$\bar\psi^{1,2,3}$	$\bar\phi^{1,2}$ or $\bar\phi^{3,4}$	Y	Q_{em}
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+,+,+)$	$(+,+)$	$1/2$	$1/2$
	$(+,+,+)$	$(-,-)$	$1/2$	$1/2$
	$(+,-,-)$	$(+,+)$	$-1/6$	$-1/6$
	$(+,-,-)$	$(-,-)$	$-1/6$	$-1/6$
$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(-,-,-)$	$(-,-)$	$-1/2$	$-1/2$
	$(-,-,-)$	$(+,+)$	$-1/2$	$-1/2$
	$(+,+,-)$	$(-,-)$	$1/6$	$1/6$
	$(+,+,-)$	$(+,+)$	$1/6$	$1/6$
$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(+,+,+)$	$(+,-)$	$1/2$	$1/2$
	$(+,-,-)$	$(+,-)$	$-1/6$	$-1/6$
$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(-,-,-)$	$(+,-)$	$-1/2$	$-1/2$
	$(+,+,-)$	$(+,-)$	$1/6$	$1/6$

We can therefore summarise all the previous results by saying that sectors coming from $B^{10,11,12,13,14,15}_{p,q,r,s}$, give rise to $(\mathbf{4},1,1)$ and $(\bar{\mathbf{4}},1,1)$ representations under the SM gauge group, with fractional electric charges: $\pm\frac{1}{2}$ and $\pm\frac{1}{6}$.

The projectors corresponding to $B^{10,11,12}_{p,q,r,s}$ are:

$$
\begin{aligned}
P_{pqrs}^{(10)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_1 \\ B_{pqrs}^{(10)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B_{pqrs}^{(10)}\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(10)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B_{pqrs}^{(10)}\end{pmatrix}\right) \\
P_{pqrs}^{(11)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_3 \\ B_{pqrs}^{(11)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B_{pqrs}^{(11)}\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(11)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B_{pqrs}^{(11)}\end{pmatrix}\right) \\
P_{pqrs}^{(12)} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_5 \\ B_{pqrs}^{(12)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B_{pqrs}^{(12)}\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(12)}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B_{pqrs}^{(12)}\end{pmatrix}\right)
\end{aligned} \tag{A.5}
$$

We can get the expressions for $P^{13,14,15}$ if we substitute $B^{10,11,12} \to B^{13,14,15}$ and $\alpha + z_1 \to \alpha$.

The matrix formalism for the previous expressions is:

$$
\begin{aligned}
&\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \\ (\alpha+z_1|e_3) & (\alpha+z_1|e_4) & (\alpha+z_1|e_5) & (\alpha+z_1|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha) \\ (z_2|b_1+\alpha) \end{pmatrix} \\
&\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \\ (\alpha+z_1|e_1) & (\alpha+z_1|e_2) & (\alpha+z_1|e_5) & (\alpha+z_1|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2+\alpha) \\ (e_2|b_2+\alpha) \\ (z_1|b_2+\alpha) \\ (z_2|b_2+\alpha) \end{pmatrix} \\
&\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \\ (\alpha+z_1|e_1) & (\alpha+z_1|e_2) & (\alpha+z_1|e_3) & (\alpha+z_1|e_4) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_3+\alpha) \\ (e_2|b_3+\alpha) \\ (z_1|b_3+\alpha) \\ (z_2|b_3+\alpha) \end{pmatrix}
\end{aligned} \tag{A.6}
$$

We can get the analytical expressions for $P^{13,14,15}$ if we substitute $\alpha + z_1 \to \alpha$.

The representations and observable charges of $B^{16,17,18}_{p,q,r,s} + z_1$ in (2.13) and $B^{19,20,21}_{p,q,r,s}$ are given below:

representation	$\bar{\psi}^{4,5}$	$\bar{\phi}^{1,2}$ or $\bar{\phi}^{3,4}$	Y	Q_{em}
$((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$	$(+,+)$	$(+,+)$	$1/2$	$1/2$
	$(+,+)$	$(-,-)$	$1/2$	$1/2$
	$(-,-)$	$(+,+)$	$-1/2$	$-1/2$
	$(-,-)$	$(-,-)$	$-1/2$	$-1/2$
$((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))$	$(+,+)$	$(+,-)$	$1/2$	$1/2$
	$(-,-)$	$(+,-)$	$-1/2$	$-1/2$
$((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))$	$(+,-)$	$(+,+)$	0	$-1/2, 1/2$
	$(+,-)$	$(-,-)$	0	$-1/2, 1/2$
$((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$	$(+,-)$	$(+,-)$	0	$-1/2, 1/2$

The mixed states from $B^{16,17,18,19,20,21}_{p,q,r,s}$ give rise to $(\mathbf{1},\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{2})$ representations under the Standard Model gauge group with fractional electric charges: $\pm\frac{1}{2}$. The projectors corresponding to $B^{16,17,18}_{p,q,r,s}$ are:

$$
\begin{aligned}
P^{(16)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_1 \\ B^{(16)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B^{(16)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(16)}_{pqrs}\end{pmatrix}\right) \\
P^{(17)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_3 \\ B^{(17)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B^{(17)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(17)}_{pqrs}\end{pmatrix}\right) \\
P^{(18)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_5 \\ B^{(18)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B^{(18)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(18)}_{pqrs}\end{pmatrix}\right)
\end{aligned}
\tag{A.7}
$$

In order to get the expressions for $P^{19,20,21}_{p,q,r,s}$ we have to substitute $B^{16,17,18}_{p,q,r,s} \to B^{19,20,21}_{p,q,r,s}$.

$$
\begin{aligned}
&\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha+x) \\ (e_2|b_1+\alpha+x) \\ (z_2|b_1+\alpha+x) \end{pmatrix} \\
&\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2+\alpha+x) \\ (e_2|b_2+\alpha+x) \\ (z_2|b_2+\alpha+x) \end{pmatrix} \\
&\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_3+\alpha+x) \\ (e_2|b_3+\alpha+x) \\ (z_2|b_3+\alpha+x) \end{pmatrix}
\end{aligned}
\tag{A.8}
$$

We can get the analytical expressions for $P^{19,20,21}$ if we substitute $\alpha + x \to \alpha + x + z_1$.

A.2. Vectorial states, representations and projectors

The corresponding projectors to the vectorial representations of (2.14) are:

$$
\begin{aligned}
P^{(i)(\bar\psi_{123})}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_{2i-1} \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_{2i} \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{2}\left(1-c\begin{pmatrix} \alpha \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right) \\
P^{(i)(\bar\psi_{45})}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_{2i-1} \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_{2i} \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right) \\
&\quad\cdot\frac{1}{2}\left(1+c\begin{pmatrix} \alpha \\ B^{(i)}_{pqrs}+x\end{pmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
P_{pqrs}^{(i)(\bar{\Phi}_{12})} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_{2i} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1+c\begin{pmatrix} z_1 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{2}\left(1+c\begin{pmatrix} \alpha \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
P_{pqrs}^{(i)(\bar{\Phi}_{34})} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_{2i} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1+c\begin{pmatrix} z_1 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{2}\left(1-c\begin{pmatrix} \alpha \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
P_{pqrs}^{(i)(\bar{\Phi}_{5678})} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_{2i-1} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_{2i} \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)\cdot\left(1+c\begin{pmatrix} z_2 \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right) \\
&\quad \cdot\frac{1}{2}\left(1-c\begin{pmatrix} \alpha \\ B_{pqrs}^{(i)}+x \end{pmatrix}\right)
\end{aligned}
\tag{A.9}
$$

The explicit expressions for the 1st plane are the following:

$$
\Delta_v^{(1)} = \begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \\
(\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6)
\end{pmatrix}
$$

$$
Y_{\bar{\psi}^{123}}^{(1)} = \begin{pmatrix}
(e_1|b_1+x) \\ (e_2|b_1+x) \\ (z_1|b_1+x) \\ (z_2|b_1+x) \\ (\alpha|b_1+x)
\end{pmatrix}
$$

$$
Y_{\bar{\psi}^{45}}^{(1)} = \begin{pmatrix}
(e_1|b_1+x) \\ (e_2|b_1+x) \\ (z_1|b_1+x) \\ (z_2|b_1+x) \\ 1+(\alpha|b_1+x)
\end{pmatrix}
$$

$$
Y_{\bar{\phi}^{12}}^{(1)} = \begin{pmatrix}
(e_1|b_1+x) \\ (e_2|b_1+x) \\ 1+(z_1|b_1+x) \\ (z_2|b_1+x) \\ 1+(\alpha|b_1+x)
\end{pmatrix}
$$

$$Y^{(1)}_{\bar{\phi}^{34}} = \begin{pmatrix} (e_1|b_1+x) \\ (e_2|b_1+x) \\ 1+(z_1|b_1+x) \\ (z_2|b_1+x) \\ (\alpha|b_1+x) \end{pmatrix}$$

$$Y^{(1)}_{\bar{\phi}^{5..8}} = \begin{pmatrix} (e_1|b_1+x) \\ (e_2|b_1+x) \\ (z_1|b_1+x) \\ 1+(z_2|b_1+x) \\ (\alpha|b_1+x) \end{pmatrix} \tag{A.10}$$

The explicit expressions for the 2nd plane are the following:

$$\Delta^{(2)}_{v} = \begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \\ (\alpha|e_1) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\psi}^{123}} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ (z_1|b_2+x) \\ (z_2|b_2+x) \\ (\alpha|b_2+x) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\psi}^{45}} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ (z_1|b_2+x) \\ (z_2|b_2+x) \\ 1+(\alpha|b_2+x) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\phi}^{12}} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ 1+(z_1|b_2+x) \\ (z_2|b_2+x) \\ 1+(\alpha|b_2+x) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\phi}^{34}} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ 1+(z_1|b_2+x) \\ (z_2|b_2+x) \\ (\alpha|b_2+x) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\phi}^{5..8}} = \begin{pmatrix} (e_3|b_2+x) \\ (e_4|b_2+x) \\ (z_1|b_2+x) \\ 1+(z_2|b_2+x) \\ (\alpha|b_2 1+x) \end{pmatrix} \tag{A.11}$$

The explicit expressions for the 3rd plane are the following:

$$\Delta^{(2)}_v = \begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \\ (\alpha|e_1) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix}$$

$$Y^{(3)}_{\bar{\psi}^{123}} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ (z_1|b_3+x) \\ (z_2|b_3+x) \\ (\alpha|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar{\psi}^{45}} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ (z_1|b_3+x) \\ (z_2|b_3+x) \\ 1+(\alpha|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar{\phi}^{12}} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ 1+(z_1|b_3+x) \\ (z_2|b_3+x) \\ 1+(\alpha|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar{\phi}^{34}} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ 1+(z_1|b_3+x) \\ (z_2|b_3+x) \\ (\alpha|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar{\phi}^{5..8}} = \begin{pmatrix} (e_5|b_3+x) \\ (e_6|b_3+x) \\ (z_1|b_3+x) \\ 1+(z_2|b_3+x) \\ (\alpha|b_3+x) \end{pmatrix} \tag{A.12}$$

References

[1] I. Antoniadis, C. Bachas, C. Kounnas, Nucl. Phys. B 289 (1987) 87;
H. Kawai, D.C. Lewellen, S.H.-H. Tye, Nucl. Phys. B 288 (1987) 1;
I. Antoniadis, C. Bachas, Nucl. Phys. B 298 (1988) 586.
[2] I. Antoniadis, J. Ellis, J. Hagelin, D.V. Nanopoulos, Phys. Lett. B 231 (1989) 65.
[3] A.E. Faraggi, D.V. Nanopoulos, K. Yuan, Nucl. Phys. B 335 (1990) 347.
[4] I. Antoniadis, G.K. Leontaris, J. Rizos, Phys. Lett. B 245 (1990) 161;
G.K. Leontaris, J. Rizos, Nucl. Phys. B 554 (1999) 3.
[5] A.E. Faraggi, Phys. Lett. B 278 (1992) 131;
A.E. Faraggi, Nucl. Phys. B 387 (1992) 239.
[6] G.B. Cleaver, A.E. Faraggi, D.V. Nanopoulos, Phys. Lett. B 455 (1999) 135.
[7] G.B. Cleaver, A.E. Faraggi, C. Savage, Phys. Rev. D 63 (2001) 066001.
[8] A.E. Faraggi, Nucl. Phys. B 407 (1993) 57;
A.E. Faraggi, Phys. Lett. B 326 (1994) 62;
A.E. Faraggi, Phys. Lett. B 544 (2002) 207;
P. Berglund, et al., Phys. Lett. B 433 (1998) 269;
P. Berglund, et al., Int. J. Mod. Phys. A 15 (2000) 1345;
R. Donagi, A.E. Faraggi, Nucl. Phys. B 694 (2004) 187;
R. Donagi, K. Wendland, J. Geom. Phys. 59 (2009) 942.
[9] E. Kiritsis, C. Kounnas, P.M. Petropoulos, J. Rizos, hep-th/9605011;
E. Kiritsis, C. Kounnas, P.M. Petropoulos, J. Rizos, Nucl. Phys. B 483 (1997) 141;
E. Kiritsis, C. Kounnas, Nucl. Phys. B 503 (1997) 117;
A. Gregori, C. Kounnas, Nucl. Phys. B 560 (1999) 135.
[10] J. Bagger, D. Nemeschansky, N. Seiberg, S. Yankielowicz, Nucl. Phys. B 289 (1987) 53;
D. Chang, A. Kumar, Phys. Rev. D 38 (1988) 1893;
D. Chang, A. Kumar, Phys. Rev. D 38 (1988) 3734.
[11] L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Nucl. Phys. B 261 (1985) 678;
L.J. Dixon, J.A. Harvey, C. Vafa, E. Witten, Nucl. Phys. B 274 (1986) 285.
[12] A. Gregori, C. Kounnas, J. Rizos, Nucl. Phys. B 549 (1999) 16.
[13] A.E. Faraggi, C. Kounnas, S.E.M. Nooij, J. Rizos, hep-th/0311058;
A.E. Faraggi, C. Kounnas, S.E.M. Nooij, J. Rizos, Nucl. Phys. B 695 (2004) 41.
[14] A.E. Faraggi, C. Kounnas, J. Rizos, Phys. Lett. B 648 (2007) 84.
[15] A.E. Faraggi, C. Kounnas, J. Rizos, Nucl. Phys. B 774 (2007) 208.
[16] A.E. Faraggi, C. Kounnas, J. Rizos, Nucl. Phys. B 779 (2008) 19.
[17] T. Catelin-Julien, A.E. Faraggi, C. Kounnas, J. Rizos, Nucl. Phys. B 812 (2009) 103.
[18] C. Angelantonj, A.E. Faraggi, M. Tsulaia, arXiv:1003.5801.
[19] B.R. Greene, M.R. Plesser, Nucl. Phys. B 338 (1990) 15.
[20] S.E.M. Nooij, Oxford University DPhil thesis, hep-th/0603035.
[21] J.C. Pati, A. Salam, Phys. Rev. D 10 (1974) 275;
I. Antoniadis, G.K. Leontaris, Phys. Lett. B 216 (1989) 333.
[22] X.G. Wen, E. Witten, Nucl. Phys. B 261 (1985) 651;
G. Athanasiu, J. Atick, M. Dine, W. Fischler, Phys. Lett. B 214 (1988) 55.
[23] A.N. Schellekens, Phys. Lett. B 237 (1990) 363.
[24] See e.g.: V. Halyo, et al., Phys. Rev. Lett. 84 (2000) 2576.
[25] B. Assel, et al., Phys. Lett. B 683 (2010) 306.
[26] See e.g.: D. Senechal, Phys. Rev. D 39 (1989) 3717;
K.R. Dienes, Phys. Rev. Lett. 65 (1990) 1979;
K.R. Dienes, Phys. Rev. D 73 (2006) 106010;
M.R. Douglas, JHEP 0305 (2003) 046;
R. Blumenhagen, et al., Nucl. Phys. B 713 (2005) 83;
F. Denef, M.R. Douglas, JHEP 0405 (2004) 072;
T.P.T. Dijkstra, L. Huiszoon, A.N. Schellekens, Nucl. Phys. B 710 (2005) 3;
F. Gmeiner, R. Blumenhagen, G. Honecker, D. Lust, T. Weigand, JHEP 0601 (2006) 004;
B.S. Acharya, F. Denef, R. Valadro, JHEP 0506 (2005) 056;
P. Anastasopoulos, T.P.T. Dijkstra, E. Kiritsis, A.N. Schellekens, Nucl. Phys. B 759 (2006) 83;

M.R. Douglas, W. Taylor, JHEP 0701 (2007) 031;
K.R. Dienes, M. Lennek, D. Senechal, V. Wasnik, Phys. Rev. D 75 (2007) 126005;
O. Lebedev, et al., Phys. Lett. B 645 (2007) 88;
E. Kiritsis, M. Lennek, A.N. Schellekens, JHEP 0902 (2009) 030, arXiv:0909.0271;
Y.H. He, S.J. Lee, A. Lukas, JHEP 1005 (2010) 071;
B. Gato-Rivera, A.N. Schellekens, arXiv:1003.6075 [hep-th].
[27] T. Kobayashi, S. Raby, R.J. Zhang, Nucl. Phys. B 704 (2005) 3;
O. Lebedev, et al., Phys. Rev. D 77 (2008) 046013.
[28] A.E. Faraggi, Phys. Rev. D 46 (1992) 3204;
S. Chang, C. Coriano, A.E. Faraggi, Nucl. Phys. B 477 (1996) 65;
C. Coriano, A.E. Faraggi, M. Plümacher, Nucl. Phys. B 614 (2001) 233.
[29] T. Catelin-Jullien, C. Kounnas, H. Partouche, N. Toumbas, Nucl. Phys. B 820 (2009) 290;
F. Bourliot, C. Kounnas, H. Partouche, Nucl. Phys. B 816 (2009) 227;
F. Bourliot, J. Estes, C. Kounnas, H. Partouche, Nucl. Phys. B 830 (2010) 330;
F. Bourliot, J. Estes, C. Kounnas, H. Partouche, arXiv:0910.2814 [hep-th];
J. Estes, C. Kounnas, H. Partouche, arXiv:1003.0471 [hep-th].
[30] C. Kounnas, Fortschr. Phys. 56 (2008) 1143;
I. Florakis, C. Kounnas, Nucl. Phys. B 820 (2009) 237;
I. Florakis, C. Kounnas, N. Toumbas, Nucl. Phys. B 834 (2010) 273.
[31] S. Ferrara, C. Kounnas, F. Zwirner, Nucl. Phys. B 429 (1994) 589.