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Classification of the chiral $Z_2 \times Z_2$ fermionic models in the heterotic superstring

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Abstract

The first particle physics observable whose origin may be sought in string theory is the triple replication of the matter generations. The class of $Z_2 \times Z_2$ orbifolds of six-dimensional compactified tori, that have been most widely studied in the free fermionic formulation, correlate the family triplication with the existence of three twisted sectors in this class. In this work we seek an improved understanding of the geometrical origin of the three generation free fermionic models. Using fermionic and orbifold techniques we classify the $Z_2 \times Z_2$ orbifold with symmetric shifts on six-dimensional compactified internal manifolds. We show that perturbative three generation models are not obtained in the case of $Z_2 \times Z_2$ orbifolds with symmetric shifts on complex tori, and that the perturbative three generation models in this class necessarily employ an asymmetric shift. We present a class of three generation models in which the $SO(10)$ gauge symmetry cannot be broken perturbatively, while preserving the Standard Model matter content. We discuss the potential implications of the asymmetric shift for strong–weak coupling duality and moduli stabilization. We show that the freedom in the modular invariant phases in the $N = 1$ vacua that control the chiral content, can be interpreted as vacuum expectation values of background fields of the underlying $N = 4$ theory, whose dynamical components are projected out by the Z_2-fermionic projections. In this class of vacua the chiral content of the models is determined by the underlying $N = 4$ mother theory.

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1. Introduction

String theory is in a precarious state of affairs. On the one hand the theory shows great promise in its ability to provide a consistent framework for perturbative quantum gravity, while at the same time giving rise to the gauge and matter structures that are observed experimentally. However, the existence of a multitude of possible string vacua has led some authors to lose all hope and to advocate resorting to anthropic principles as the possible resolution for the contrived set of parameters that seem to govern our universe [1].

Our point of view is different. Ultimately the search for the principles that underly string theory and the vacuum selection will entail the conceptual resolution of the quantum gravity synthesis, and the fundamental understanding of quantum mechanics with its probabilistic interpretation when applied to the space–time arena.

A more pragmatic view of string theory suggests that the basic properties of the low energy data, as well as the basic properties of string theories should be utilized in trying to isolate vacua, or classes of vacua, that look most promising. From the low energy data point of view we may hypothesize that the viable string theory vacua should accommodate two pivotal ingredients: the existence of *three generations* and their *embedding in* an underlying $SO(10)$, or E_6 grand unified group structure. From the string theory point of view the basic properties that may serve as guides are the various T- and S-duality symmetries. In this respect it is also plausible that the self-dual points under these dualities may play a role in the vacuum selection principle.

A given set of string vacua that exhibits compelling properties must then be investigated in depth. In the least these can be viewed as case examples providing the concrete laboratories to study how the properties of the observed data may arise from quantum gravity, and to develop the tools to relate between the theory and experiment. However, there also exist the possibility that certain case examples capture some properties of the true string vacuum that may eventually prove relevant for the understanding of the low energy data. In any case, it is clear that different approaches must be pursued for better understanding of string theory and its possible connection with experimental data.

The first sector among the low energy experimental observables whose origin we may seek in a theory of quantum gravity is the flavor sector. In the context of the quantum field theories underlying the Standard Model of particle physics, this group of parameters does not arise from any physical principle, like the gauge principle. It is then encouraging that already from the early days of superstring phenomenology, it was observed that the flavor replication is related a topological property of the string compactifications, namely the Euler characteristic [2]. However, this observation does not yet explain the existence of three generations. The first particle physics observable whose origin we may seek to relate to string theory is therefore the replication of the three matter generations.

Among the most advanced string models to date are the three generation heterotic string models [3–8], constructed in the free-fermion formulation [9]. These models have been the subject of detailed studies, showing that they can, at least in principle, account for desirable physical features including the observed fermion mass spectrum, the longevity of the proton, small neutrino masses, the consistency of gauge-coupling unification with the experimental data from LEP and elsewhere, and the universality of the soft supersymmetry-breaking parameters [10]. An important property of the fermionic construction is the standard $SO(10)$ embedding of the Standard Model spectrum, which ensures natural consistency with the experimental values for $\alpha_s(M_Z)$ and $\sin^2\theta_W(M_Z)$. Furthermore, this class of models yielded the only known string model that reproduces in the low energy effective field theory solely the spectrum of the minimal supersymmetric standard model [11].

A vital property of the realistic fermionic models is their underlying $Z_2 \times Z_2$ orbifold structure. Many of the encouraging phenomenological characteristics of these models are rooted in this structure. In particular, the emergence of the three chiral generations in a large class of fermionic constructions is correlated with the existence of three twisted sectors in the $Z_2 \times Z_2$ orbifold of the six-dimensional internal manifold. Each twisted sector produces exactly one of the light chiral generations and there is no additional chiral matter. Thus, the fermionic construction offers a plausible and compelling explanation to the existence of three generations in nature.

To see more precisely the orbifold correspondence of the fermionic construction, we recall that the free-fermion models are generated by a set of basis vectors which define the transformation properties of the world-sheet fermions as they are transported around non-contractible loops of the string world sheet. A set of realistic fermionic models contains a subset of boundary conditions, the so-called extended NAHE-set, which can be seen to correspond to $Z_2 \times Z_2$ orbifold compactification with the standard embedding of the gauge connection [12]. The fermionic model constructed just with the basis vectors of the extended NAHE-set gives rise to 24 generations from the twisted sectors, as well as three additional generation/antigeneration pairs from the untwisted sector. At the $N = 4$ level the fermionic point in the moduli space corresponds to an $SO(12)$ enhancement of the internal lattice. The induced $Z_2 \times Z_2$ action gives rise to a model with $(h_{11}, h_{21}) = (27, 3)$, matching the data of the free-fermion model. We note that the data of this model differs from the $Z_2 \times Z_2$ orbifold at a generic point in the moduli space, which has $(h_{11}, h_{21}) = (51, 3)$. Alternatively, we can start with the $Z_2 \times Z_2$ orbifold at a generic point and produce the one at the free fermionic point by adding a freely acting shift on the internal lattice [13,14].

The above remarks make apparent the need to understand better the general structure of the realistic free fermionic models, and, in particular, the geometrical structure that underlies the three generation models.

In the framework of the fermionic construction the three generations are obtained by adding three, or four, additional boundary condition basis vectors beyond the minimal NAHE-set. The basis vectors reduce the number of generations to three generations, one from each of the twisted sectors of the $Z_2 \times Z_2$ orbifold.

In this paper we observe in some of the concrete quasi-realistic three generation models [6] that the action of two of the additional boundary condition basis vectors correspond to symmetric shifts on the internal coordinates, whereas the third corresponds to a fully asymmetric shift. We then proceed to classify all possible $Z_2 \times Z_2$ orbifolds with symmetric shifts, and demonstrate that three generations cannot be obtained solely with symmetric shifts on complex tori. This is one of the main results of the analysis and it reveals, at least in the context of the three generation models, that the geometrical structures that underly these models may not be simple Calabi–Yau manifolds, but it corresponds to geometries that are yet to be defined. This observation may eventually prove important for the issue of moduli stabilization.

Additionally, we will demonstrate the existence of three generation models with a perturbatively unbroken $SO(10)/E_6$ gauge group, in which the internal manifold is reduced to a product of six circles. This again demonstrates the possibility that the geometries relating to the viable vacua may not correspond to the complex geometries that have been more prevalent in the literature. Some of phenomenological difficulties that have been associated with symmetric compactifications, like supersymmetry breaking and moduli stabilization, may therefore be cured in the viable geometries. This class of models, while not viable with respect to perturbative phenomenology, produces one generation from a single fixed point in each twisted sector. Hence, realizing the $Z_2 \times Z_2$ geometric picture of the three chiral generations. Our classification demonstrates additionally that in a large class of $N = 1$ models the freedom in the phases appearing in the $N = 1$ partition function can be understood as the vacuum expectation value (VEV) of background fields of the $N = 4$ underlying theory, whose dynamical components are projected out by the extra $Z_2 \times Z_2$ projections. Thus, the information on the chiral content of the $N = 1$ models is already contained at the $N = 4$ level. Examples of this phenomenon are already noted in the case of the $Z_2 \times Z_2$ orbifold on $SO(12)$ versus $SO(4)^3$ lattices, as discussed above.

Our paper is organized as follows: in Section 2 we discuss the general structure of the models based on the fermionic construction. In a concrete model we show that the additional boundary vectors beyond the NAHE-set can be regarded as two symmetric shifts plus one fully asymmetric shift. The main aim of this section is to establish the connection of the analysis to follow with the phenomenological three generation models.

In Section 3 we present the setup of our analysis. We present the most general free fermionic model describing the heterotic string on a $Z_2 \times Z_2$ orbifold. In Section 4 we present our method to classify all possible symmetric shifts and proceed to perform the complete classification for gauge groups that descend from the $N = 4$ mother theory. We find that down to six generations the perturbative models can be described in terms of symmetric shifts and hence possess a geometrical interpretation in terms of $Z_2 \times Z_2$ symmetric orbifolds. However, the three generation perturbative models are not admitted in this classification and entail an additional shift which is necessarily asymmetric between the left and the right-movers. We demonstrate the existence of a class of three twisted generation models in which the GUT symmetry group cannot be broken perturbatively, while preserving complete twisted matter multiplets. Additionally, in this class of models the six-dimensional internal lattice is reduced to a product of six circles. Hence, one of the main conclusions of the analysis is that in the framework of $Z_2 \times Z_2$ orbifolds, three generations models are not obtained solely with symmetric shifts on complex tori, and suggests that the geometrical objects underlying the quasi-realistic free fermionic models are more esoteric than ordinary $Z_2 \times Z_2$ Calabi–Yau manifolds. In Section 5 we present our results and Section 6 concludes our paper.

2. General structure of realistic free fermionic models

In this section we recapitulate the main structure of the realistic free fermionic models. The notation and details of the construction of these models are given elsewhere [3–6,11, 15,16]. In the free fermionic formulation of the heterotic string in four dimensions all the world-sheet degrees of freedom required to cancel the conformal anomaly are represented in terms of free world-sheet fermions [9]. In the light-cone gauge the world-sheet field content consists of two transverse left- and right-moving space–time coordinate bosons, $X^{\mu}_{1,2}$ and $\bar{X}^{\mu}_{1,2}$, and their left-moving fermionic superpartners $\psi^{\mu}_{1,2}$, and additional 62 purely internal Majorana–Weyl fermions, of which 18 are left-moving, and 44 are right-moving. In the supersymmetric sector the world-sheet supersymmetry is realized non-linearly and the world-sheet supercurrent [17] is given by

$$T_F = \psi^{\mu}\partial X_{\mu} + i\chi^I y^I \omega^I \quad (I = 1, \ldots, 6). \tag{2.1}$$

The $\{\chi^I, y^I, \omega^I\}$ $(I = 1, \ldots, 6)$ are 18 real free fermions transforming as the adjoint representation of $SU(2)^6$. Under parallel transport around a non-contractible loop on the toroidal world-sheet the fermionic fields pick up a phase, $f \to -e^{i\pi\alpha(f)} f$, $\alpha(f) \in (-1, +1]$. Each set of specified phases for all world-sheet fermions, around all the non-contractible loops is called the spin structure of the model. Such spin structures are usually given in the form of 64-dimensional boundary condition vectors, with each element of the vector specifying the phase of the corresponding world-sheet fermion. The basis vectors are constrained by string consistency requirements and completely determine the vacuum structure of the model. The physical spectrum is obtained by applying the generalized GSO projections [9].

The boundary condition basis defining a typical realistic free fermionic heterotic string model is constructed in two stages. The first stage consists of the NAHE set, which is a set of five boundary condition basis vectors, $\{1, S, b_1, b_2, b_3\}$ [15,18]. The gauge group induced by the NAHE set is $SO(10) \times SO(6)^3 \times E_8$ with $N = 1$ supersymmetry. The space–time vector bosons that generate the gauge group arise from the Neveu–Schwarz sector and from the sector $\xi_2 \equiv 1 + b_1 + b_2 + b_3$. The Neveu–Schwarz sector produces the generators of $SO(10) \times SO(6)^3 \times SO(16)$. The ξ_2-sector produces the spinorial 128 of $SO(16)$ and completes the hidden gauge group to E_8. The NAHE set divides the internal world-sheet fermions in the following way: $\bar{\phi}^{1,\ldots,8}$ generate the hidden E_8 gauge group, $\bar{\psi}^{1,\ldots,5}$ generate the $SO(10)$ gauge group, and $\{\bar{y}^{3,\ldots,6}, \bar{\eta}^1\}$, $\{\bar{y}^1, \bar{y}^2, \bar{\omega}^5, \bar{\omega}^6, \bar{\eta}^2\}$, $\{\bar{\omega}^{1,\ldots,4}, \bar{\eta}^3\}$ generate the three horizontal $SO(6)$ symmetries. The left-moving $\{y, \omega\}$ states are divided into $\{y^{3,\ldots,6}\}$, $\{y^1, y^2, \omega^5, \omega^6\}$, $\{\omega^{1,\ldots,4}\}$ and χ^{12}, χ^{34}, χ^{56} generate the left-moving $N = 2$ world-sheet supersymmetry. At the level of the NAHE set the sectors b_1, b_2 and b_3 produce 48 multiplets, 16 from each, in the 16 representation of $SO(10)$. The states from the sectors b_j are singlets of the hidden E_8 gauge group and transform under the horizontal $SO(6)_j$ $(j = 1, 2, 3)$ symmetries. This structure is common to all known realistic free fermionic models.

The second stage of the construction consists of adding to the NAHE set three (or four) additional basis vectors. These additional vectors reduce the number of generations to three, one from each of the sectors b_1, b_2 and b_3, and simultaneously break the four-dimensional gauge group. The assignment of boundary conditions to $\{\bar{\psi}^{1,\ldots,5}\}$ breaks

$SO(10)$ to one of its subgroups $SU(5) \times U(1)$ [3], $SO(6) \times SO(4)$ [5], $SU(3) \times SU(2) \times U(1)^2$ [4,6,11], $SU(3) \times SU(2)^2 \times U(1)$ [16] or $SU(4) \times SU(2) \times U(1)$ [20]. Similarly, the hidden E_8 symmetry is broken to one of its subgroups, and the flavor $SO(6)^3$ symmetries are broken to $U(1)^n$, with $3 \leqslant n \leqslant 9$. For details and phenomenological studies of these three generation string models we refer interested readers to the original literature and review articles [10].

The correspondence of the free fermionic models with the orbifold construction is illustrated by extending the NAHE set, $\{1, S, b_1, b_2, b_3\}$, by at least one additional boundary condition basis vector [12]

$$\xi_1 = (0, \ldots, 0| \underbrace{1, \ldots, 1}_{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}}, 0, \ldots, 0). \tag{2.2}$$

With a suitable choice of the GSO projection coefficients the model possesses an $SO(4)^3 \times E_6 \times U(1)^2 \times E_8$ gauge group and $N = 1$ space–time supersymmetry. The matter fields include 24 generations in the 27 representation of E_6, eight from each of the sectors $b_1 \oplus b_1 + \xi_1$, $b_2 \oplus b_2 + \xi_1$ and $b_3 \oplus b_3 + \xi_1$. Three additional 27 and $\overline{27}$ pairs are obtained from the Neveu–Schwarz $\oplus \xi_1$ sector.

To construct the model in the orbifold formulation one starts with the compactification on a torus with nontrivial background fields [19]. The subset of basis vectors

$$\{1, S, \xi_1, \xi_2\} \tag{2.3}$$

generates a toroidally-compactified model with $N = 4$ space–time supersymmetry and $SO(12) \times E_8 \times E_8$ gauge group. The same model is obtained in the geometric (bosonic) language by tuning the background fields to the values corresponding to the $SO(12)$ lattice. The metric of the six-dimensional compactified manifold is then the Cartan matrix of $SO(12)$, while the antisymmetric tensor is given by

$$B_{ij} = \begin{cases} G_{ij}, & i > j, \\ 0, & i = j, \\ -G_{ij}, & i < j. \end{cases} \tag{2.4}$$

When all the radii of the six-dimensional compactified manifold are fixed at $R_I = \sqrt{2}$, it is seen that the left- and right-moving momenta $P^I_{R,L} = [m_i - \frac{1}{2}(B_{ij} \pm G_{ij})n_j]e_i^{I*}$ reproduce the massless root vectors in the lattice of $SO(12)$. Here $e^i = \{e_i^I\}$ are six linearly-independent vielbeins normalized so that $(e_i)^2 = 2$. The e_i^{I*} are dual to the e_i, with $e_i^* \cdot e_j = \delta_{ij}$.

Adding the two basis vectors b_1 and b_2 to the set (2.3) corresponds to the $Z_2 \times Z_2$ orbifold model with standard embedding. Starting from the $N = 4$ model with $SO(12) \times E_8 \times E_8$ symmetry [19], and applying the $Z_2 \times Z_2$ twist on the internal coordinates, reproduces the spectrum of the free-fermion model with the six-dimensional basis set $\{1, S, \xi_1, \xi_2, b_1, b_2\}$. The Euler characteristic of this model is 48 with $h_{11} = 27$ and $h_{21} = 3$.

It is noted that the effect of the additional basis vector ξ_1 of Eq. (2.2), is to separate the gauge degrees of freedom, spanned by the world-sheet fermions $\{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^1, \bar{\eta}^2, \bar{\eta}^3, \bar{\phi}^{1,\ldots,8}\}$, from the internal compactified degrees of freedom $\{y, \omega|\bar{y}, \bar{\omega}\}^{1,\ldots,6}$. In the realistic free fermionic models this is achieved by the vector 2γ [12], with

$$2\gamma = (0, \ldots, 0| \underbrace{1, \ldots, 1}_{\bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3} \bar{\phi}^{1,\ldots,4}}, 0, \ldots, 0), \tag{2.5}$$

which breaks the $E_8 \times E_8$ symmetry to $SO(16) \times SO(16)$. The $Z_2 \times Z_2$ twist induced by b_1 and b_2 breaks the gauge symmetry to $SO(4)^3 \times SO(10) \times U(1)^3 \times SO(16)$. The orbifold still yields a model with 24 generations, eight from each twisted sector, but now the generations are in the chiral 16 representation of $SO(10)$, rather than in the 27 of E_6. The same model can be realized with the set $\{1, S, \xi_1, \xi_2, b_1, b_2\}$, by projecting out the $16 \oplus \overline{16}$ from the ξ_1-sector taking

$$c\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \to -c\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{2.6}$$

This choice also projects out the massless vector bosons in the 128 of $SO(16)$ in the hidden-sector E_8 gauge group, thereby breaking the $E_6 \times E_8$ symmetry to $SO(10) \times U(1) \times SO(16)$. We can define two $N = 4$ models generated by the set (2.3), Z_+ and Z_-, depending on the sign in Eq. (2.6). The first, say Z_+, produces the $E_8 \times E_8$ model, whereas the second, say Z_-, produces the $SO(16) \times SO(16)$ model. However, the $Z_2 \times Z_2$ twist acts identically in the two models, and their physical characteristics differ only due to the discrete torsion Eq. (2.6).

This analysis confirms that the $Z_2 \times Z_2$ orbifold on the $SO(12)$ lattice is at the core of the realistic free fermionic models. To illustrate how the chiral generations are generated in the free fermionic models we consider the E_6 model which is produced by the extended NAHE-set $\{1, S, \xi_1, \xi_2, b_1, b_2\}$.

The chirality of the states from a twisted sector b_j is determined by the free phase $c\left[\begin{smallmatrix} b_j \\ b_i \end{smallmatrix}\right]$. Since we have a freedom in the choice of the sign of this free phase, we can get from the sector (b_i) either the 27 or the $\overline{27}$. Which of those we obtain in the physical spectrum depends on the sign of the free phase. The free phases $c\left[\begin{smallmatrix} b_j \\ b_i \end{smallmatrix}\right]$ also fix the total number of chiral generations. Since there are two b_i projections for each sector b_j, $i \neq j$ we can use one projections to project out the states with one chirality and the other projection to project out the states with the other chirality. Thus, the total number of generations with this set of basis vectors is given by

$$8\left(\frac{c\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} b_1 \\ b_3 \end{smallmatrix}\right]}{2}\right) + 8\left(\frac{c\left[\begin{smallmatrix} b_2 \\ b_1 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} b_2 \\ b_3 \end{smallmatrix}\right]}{2}\right) + 8\left(\frac{c\left[\begin{smallmatrix} b_3 \\ b_1 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} b_3 \\ b_1 \end{smallmatrix}\right]}{2}\right).$$

Since the modular invariance rules fix $c\left[\begin{smallmatrix} b_j \\ b_i \end{smallmatrix}\right] = c\left[\begin{smallmatrix} b_i \\ b_j \end{smallmatrix}\right]$ we get that the total number of generations is either 24 or 8. Thus, to reduce the number of generation further it is necessary to introduce additional basis vectors.

To illustrate the reduction to three generations in the realistic free fermionic models we consider the model in Table 1.

Here the vector ξ_1 (2.2) is replaced by the vector 2γ (2.5). At the level of the NAHE set we have 48 generations. One half of the generations is projected by the vector 2γ. Each of the three vectors in Table 1 acts non-trivially on the degenerate vacuum of the sectors b_1,

Table 1

	ψ^μ	χ^{12}	χ^{34}	χ^{56}	$\bar{\psi}^{1,\dots,5}$					$\bar{\eta}^1$	$\bar{\eta}^2$	$\bar{\eta}^3$	$\bar{\phi}^{1,\dots,8}$							
α	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0
β	0	0	0	0	1	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0
γ	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

	y^3y^6	$y^4\bar{y}^4$	$y^5\bar{y}^5$	$\bar{y}^3\bar{y}^6$	$y^1\omega^5$	$y^2\bar{y}^2$	$\omega^6\bar{\omega}^6$	$\bar{y}^1\bar{\omega}^5$	$\omega^2\omega^4$	$\omega^1\bar{\omega}^1$	$\omega^3\bar{\omega}^3$	$\bar{\omega}^2\bar{\omega}^4$
α	1	0	0	0	0	0	1	1	0	0	1	1
β	0	0	1	1	1	0	0	0	0	1	0	1
γ	0	1	0	1	0	1	0	1	1	0	0	0

Table 2

	y^3y^6	$y^4\bar{y}^4$	$y^5\bar{y}^5$	$\bar{y}^3\bar{y}^6$	$y^1\omega^5$	$y^2\bar{y}^2$	$\omega^6\bar{\omega}^6$	$\bar{y}^1\bar{\omega}^5$	$\omega^2\omega^4$	$\omega^1\bar{\omega}^1$	$\omega^3\bar{\omega}^3$	$\bar{\omega}^2\bar{\omega}^4$
$\alpha+\beta$	1	0	1	1	1	0	1	1	0	1	1	0
$\beta+\gamma$	0	1	1	0	1	1	0	1	1	1	0	1
$\alpha+\beta+\gamma$	1	1	1	0	1	1	1	0	1	1	1	0

b_2 and b_3 and reduces the number of generations in each step by a half. Thus, we obtain one generation from each sector b_1, b_2 and b_3.

The geometrical interpretation of the basis vectors beyond the NAHE set is facilitated by taking combinations of the basis vectors in Table 1, which entails choosing another set to generate the same vacuum. The combinations $\alpha+\beta$, $\alpha+\gamma$, $\alpha+\beta+\gamma$ produce the following boundary conditions under the set of internal real fermions.

It is noted that the two combinations $\alpha+\beta$ and $\beta+\gamma$ are fully symmetric between the left and right movers, whereas the third, $\alpha+\beta+\gamma$, is asymmetric. The action of the first two combinations on the compactified bosonic coordinates translates therefore to symmetric shifts. Thus, we see that reduction of the number of generations is obtained by further action of symmetric shifts.

Due to the presence of the third combination the situation, however, is more complicated. The third combination in Table 2 is asymmetric between the left and right movers and therefore does not have an obvious geometrical interpretation. Below we perform a complete classification of all the possible NAHE-based $Z_2 \times Z_2$ orbifold models with symmetric shifts on the complex tori, which reveals that three generations are not obtained in this manner. Three generations are obtained in the free fermionic models by the inclusion of the asymmetric shift in Table 2. This outcome has profound implications on the type of geometries that may be related to the realistic string vacua, as well as on the issue of moduli stabilization.

3. $N = 1$ heterotic orbifold constructions

In this section we revise the $Z_2 \times Z_2$ heterotic orbifold construction and relate this to the free fermionic construction. We isolate the individual conformal blocks that will facilitate the classification of the models and set up a procedure to analyse all possible

$N=1$ heterotic $Z_2 \times Z_2$ models. We start by describing the procedure to descend from $N=4$ to $N=1$ supersymmetric heterotic vacua.

3.1. The $N=4$ models

The partition function for any heterotic model via the fermionic construction is

$$Z=\frac{1}{\tau_2}\frac{1}{\eta^{12}\bar{\eta}^{24}}\sum_{a,b\in\Xi} c\begin{bmatrix} a\\ b\end{bmatrix}\frac{1}{2^M}\prod_{i=1}^{20}\theta\begin{bmatrix} a_i\\ b_i\end{bmatrix}^{\frac{1}{2}}\prod_{j=1}^{44}\bar{\theta}\begin{bmatrix} a_j\\ b_j\end{bmatrix}^{\frac{1}{2}}. \tag{3.1}$$

In the above equation M is the number of basis vectors and the parameters in the θ-functions represent the action of the vectors. In order to obtain a supersymmetric model we need at least two basis vectors $\{1, S\}$.

$$1=\{\psi^{1,2},\chi^{1,\dots,6},y^{1,\dots,6},\omega^{1,\dots,6}\bar{y}^{1,\dots,6},\bar{\omega}^{1,\dots,6},|\bar{\psi}^{1,\dots,6},\bar{\eta}^{1,2,3},\bar{\phi}^{1,\dots,8}\}, \tag{3.2}$$

$$S=\{\psi^{1,2},\chi^{1,\dots,6}\}. \tag{3.3}$$

The supersymmetric GSO projection is induced by the set S for any choice of the GSO coefficient

$$c\begin{bmatrix} S\\ 1\end{bmatrix}=\pm 1. \tag{3.4}$$

The corresponding partition function has a factorized left-moving contribution coming from the sector S,

$$Z_{1,S}=\frac{1}{\tau_2|\eta|^4}\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+\mu ab}\frac{\theta\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]^4}{\eta^4}\frac{\Gamma_{6,6+16}[SO(44)]}{\eta^6\bar{\eta}^{22}} \tag{3.5}$$

where

$$\Gamma_{6,6+16}[SO(44)]=\frac{1}{2}\sum_{c,d}\frac{\theta\left[\begin{smallmatrix} c\\ d\end{smallmatrix}\right]^6\bar{\theta}\left[\begin{smallmatrix} c\\ d\end{smallmatrix}\right]^{22}}{\eta^6\bar{\eta}^{22}}, \tag{3.6}$$

and

$$\mu=\frac{1}{2}\left(1-c\begin{bmatrix} S\\ 1\end{bmatrix}\right)$$

defines the chirality of $N=4$ supersymmetry. Therefore, the role of the boundary condition vector S is to factorize the left-moving contribution,

$$Z^L_{N=4}=\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+\mu ab}\theta\begin{bmatrix} a\\ b\end{bmatrix}(v)\theta\begin{bmatrix} a\\ b\end{bmatrix}^3(0)\sim v^4 \tag{3.7}$$

which is zero with the multiplicity of $N=4$ supersymmetry.

The above partition function gives rise to an $SO(44)$ right-moving gauge group and is the maximally symmetric point in the moduli space of the Narain $\Gamma_{6,6+16}$ lattice. The general $\Gamma_{6,6+16}$ lattice depends on 6×22 moduli, the metric G_{ij} and the antisymmetric tensor B_{ij} of the six-dimensional internal space, as well as the Wilson lines Y_i^I that appear in the 2d-world-sheet.

$$S = \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}g^{ab}G_{ij}\partial_a X^i\partial_b X^j + \frac{1}{4\pi}\int d^2\sigma\,\epsilon^{ab}B_{ij}\partial_a X^i\partial_b X^j + \frac{1}{4\pi}\int d^2\sigma\,\sqrt{g}\sum_I \psi^I\big[\bar\nabla + Y_i^I\bar\nabla X^i\big]\bar\psi^I. \tag{3.8}$$

Here i runs over the internal coordinates and I runs over the extra 16 right-moving degrees of freedom described by $\bar\psi^I$.

The compactified sector of the partition function is given by $\Gamma_{6,6+16}$

$$\Gamma_{6,6+16} = \frac{(\det G)^3}{\tau_2^3}\sum_{m,n}\exp\left\{-\pi\frac{T_{ij}}{\tau_2}\big[m^i+n^i\tau\big]\big[m^j+n^j\bar\tau\big]\right\} \times\frac{1}{2}\sum_{\gamma,\delta}\prod_{I=1}^{16}\exp\big[-i\pi n^i\big(m^j+n^j\bar\tau\big)Y_i^I Y_j^I\big] \times\bar\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\big(Y_i^I\big(m^i+n^i\bar\tau\big)|\tau\big), \tag{3.9}$$

where $T_{ij} = G_{ij} + B_{ij}$.

Eq. (3.9) is the winding mode representation of the partition function. Using a Poisson resummation we can put it in the momentum representation form:

$$\Gamma_{6,22} = \sum_{P,\bar P,Q}\exp\left\{\frac{i\pi\tau}{2}P_i G^{ij}P_j - \frac{i\pi\bar\tau}{2}\bar P_i G^{ij}\bar P_j - i\pi\bar\tau\hat Q^I\hat Q^I\right\}, \tag{3.10}$$

with

$$P_i = m_i + B_{ij}n^j + \frac{1}{2}Y_i^I Y_j^I n^j + Y_i^I Q^I + G_{ij}n^j, \tag{3.11}$$

$$\bar P_i = m_i + B_{ij}n^j + \frac{1}{2}Y_i^I Y_j^I n^j + Y_i^I Q^I - G_{ij}n^j, \tag{3.12}$$

$$\hat Q^I = Q^I + Y_i^I n^i. \tag{3.13}$$

The charge momenta Q^I are induced by the right-moving fermions $\bar\psi^I$ which appear explicitly in the θ-functions

$$\theta\begin{bmatrix}a^I\\ b^I\end{bmatrix} = \sum_{n\in Z} q^{\frac{(Q^I)^2}{2}} e^{2\pi i(v-\frac{b^I}{2})Q^I}, \tag{3.14}$$

where the charge momentum $Q^I = (n - \frac{a^I}{2})$.

For generic G_{ij}, B_{ij} and for vanishing values for Wilson lines, $Y_i^I = 0$ one obtains an $N = 4$ model with a gauge group $U(1)^6 \times SO(32)$. The $U(1)^6$ can be extended to $SO(12)$ by fixing the moduli of the internal manifold [12].

The $N = 4$ fermionic construction based on $\{1, S\}$ (3.5) has an extended gauge group, $SO(44)$. From the lattice construction point of view, an $N = 4$ model with a gauge group

$G \subset SO(44)$ can be generated by switching on Wilson lines and fine tune the moduli of the internal manifold. Moving from the $SO(44)$ to $U(1)^6 \times SO(32)$ heterotic point as well as to the $U(1)^6 \times E_8 \times E_8$ point can be realized continuously [23]. The partition function at the $U(1)^6 \times E_8 \times E_8$ point takes a simple factorized form

$$\begin{aligned}\Gamma_{6,6+16} = {} & \frac{(\det G)^3}{\tau_2^3} \sum_{m,n} \exp\left\{-\pi \frac{T_{ij}}{\tau_2}\left[m^i + n^i \tau\right]\left[m^j + n^j \bar{\tau}\right]\right\} \\ & \times \frac{1}{2} \sum_{\gamma,\delta} \bar{\theta} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}^8 \frac{1}{2} \sum_{\gamma,\delta} \bar{\theta} \begin{bmatrix} \gamma + h \\ \delta + g \end{bmatrix}^8 . \end{aligned} \tag{3.15}$$

3.2. *The $N = 1$ models*

To break the number of supersymmetries down from $N = 4$ to $N = 1$ in the fermionic formulation we need to introduce the vectors b_1 and b_2.

$$b_1 = \left\{\chi^{3,4}, \chi^{5,6}, y^{3,4}, y^{5,6} | \ldots\right\}, \tag{3.16}$$

$$b_2 = \left\{\chi^{1,2}, \chi^{5,6}, y^{1,2}, y^{5,6} | \ldots\right\}. \tag{3.17}$$

The b_1 twists the second and third complex planes $(3,4)$ and $(5,6)$ while b_2 twists the first and third $(1,2)$ and $(5,6)$ ones. Thus, b_1, b_2 separate the internal lattice into the three complex planes: $(1,2)$, $(3,4)$ and $(5,6)$.

The action of the b_i-twists fully determines the fermionic content for the left-moving sector. The dots $\ldots$ in b_1, b_2 stand for the n_1, n_2 right-moving fermions. To generate a modular invariant model we can distinguish four options. n_i are either 8, 16, 24 or 32 real right-moving fermions in the basis vector b_i.

Defining the basis vectors with 8 real right-moving fermions leads to massless states in the spectrum in vectorial representations of the gauge groups; 16 real right-moving fermions give rise to spinorial representations on each plane. Adding either 24 or 32 right-moving fermions would produce massive states in the spectrum. We therefore discard the last two options. We thus need to introduce 16 real fermions (8 complex) in the vectors b_1, b_2 for the existence of spinorial representations on the first and second plane.

A suitable choice is for instance,

$$b_1 = \left\{\chi^{3,4}, \chi^{5,6}, y^{3,4}, y^{5,6} | \bar{y}^{3,4}, \bar{y}^{5,6}, \bar{\eta}^1, \bar{\psi}^{1,\ldots,5}\right\}, \tag{3.18}$$

$$b_2 = \left\{\chi^{1,2}, \chi^{5,6}, y^{1,2}, y^{5,6} | \bar{y}^{1,2}, \bar{y}^{5,6}, \bar{\eta}^2, \bar{\psi}^{1,\ldots,5}\right\}. \tag{3.19}$$

We define the vectors $x = \{0, \ldots | \bar{\psi}^{1,\ldots,5}, \bar{\eta}^{1,2,3}\}$, and $\tilde{b}_{1,2} = b_{1,2} + x$. The $N = 1$ partition function based on $\{1, S, \tilde{b}_1, \tilde{b}_2\}$ takes the following form:

$$Z_{N=1} = \frac{1}{\tau_2|\eta|^4}\frac{1}{2}\sum_{\alpha,\beta} e^{i\pi(a+b+\mu ab)} \times \frac{1}{4}\sum_{h_1,h_2,g_1,g_2} \frac{\theta\left[{a \atop b}\right]}{\eta}\frac{\theta\left[{a+h_2 \atop b+g_2}\right]}{\eta}\frac{\theta\left[{a+h_1 \atop b+g_1}\right]}{\eta}\frac{\theta\left[{a-h_1-h_2 \atop b-g_1-g_2}\right]}{\eta} \times \frac{1}{2}\sum_{\gamma,\delta}\frac{\Gamma_{6,6}\left[{\gamma,h_1,h_2 \atop \delta,g_1,g_2}\right]}{\eta^6\bar{\eta}^6}\frac{Z_\eta\left[{\gamma,h_1,h_2 \atop \delta,g_1,g_2}\right]}{\bar{\eta}^3}\frac{Z_{26}\left[{\gamma \atop \delta}\right]}{\bar{\eta}}^{13} e^{i\pi\varphi_L}, \tag{3.20}$$

$$\Gamma_{6,6}\begin{bmatrix}\gamma,h_1,h_2\\ \delta,g_1,g_2\end{bmatrix} = \left|\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\theta\begin{bmatrix}\gamma+h_2\\ \delta+g_2\end{bmatrix}\right|^2\left|\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\theta\begin{bmatrix}\gamma+h_1\\ \delta+g_1\end{bmatrix}\right|^2 \times\left|\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\theta\begin{bmatrix}\gamma-h_1-h_2\\ \delta-g_1-g_2\end{bmatrix}\right|^2, \tag{3.21}$$

$$Z_\eta\begin{bmatrix}\gamma,h_1,h_2\\ \delta,g_1,g_2\end{bmatrix} = \bar{\theta}\begin{bmatrix}\gamma+h_2\\ \delta+g_2\end{bmatrix}\bar{\theta}\begin{bmatrix}\gamma+h_1\\ \delta+h_2\end{bmatrix}\bar{\theta}\begin{bmatrix}\gamma-h_1-h_2\\ \delta-g_1-g_2\end{bmatrix}, \tag{3.22}$$

$$Z_{26}\begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \bar{\theta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^{13}. \tag{3.23}$$

In Eq. (3.21) the internal manifold is twisted and thereby separated explicitly into three planes. The above model is the minimal $Z_2 \times Z_2$ with $N = 1$ supersymmetry and massless spinorial representations in the same $SO(10)$ group coming from the first and/or from the second plane. The number of families depends on the choice of the phase φ_L. The freedom of this phase arises from the different possible choices of the modular invariance coefficients $c\left[{v_i \atop v_j}\right]$. The maximal number of the families for this model is 32. Introducing internal shifts, associated to φ_L, can reduce this number as we will discuss below.

We could have chosen the boundary conditions for different right-moving fermions. This would lead to spinorial representations on each plane, but the group to which they would belong would differ in each plane. As we require spinors in the same group we have discarded this option. Choosing an overlap with more than 6 complex fermions in the right-moving sector between the vectors b_1 and b_2 leads to a $SO(14)$ gauge group, which does not have chiral fermions.

In order to have spinors in the spectrum on *all* three planes we need to separate at least an $SO(16)$ (or E_8) from the $\Gamma_{6,22}$ lattice. We therefore need to introduce the additional vector

$$z = \{\bar{\phi}^{1,\dots,8}\} \tag{3.24}$$

to the set. With this vector the partition function for the gauge sector (3.23) modifies to

$$Z_{26}\begin{bmatrix}\gamma,h_z\\ \delta,g_z\end{bmatrix} = \frac{1}{2}\sum_{h_z,g_z}\bar{\theta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^5\bar{\theta}\begin{bmatrix}\gamma+h_z\\ \delta+g_z\end{bmatrix}^8. \tag{3.25}$$

We can further separate out the internal $\Gamma_{6,6}$ lattice by introducing the additional vector,

$$e = \{y_{1,\dots,6}, \omega_{1,\dots,6}|\bar{y}_{1,\dots,6}, \bar{\omega}_{1,\dots,6}\}, \tag{3.26}$$

which modifies the $\Gamma_{6,6}$ in (3.21) by

$$\Gamma_{6,6}\begin{bmatrix}\gamma, h_1, h_2\\ \delta, g_1, g_2\end{bmatrix} = \frac{1}{2}\sum_{h_e,g_e}\left|\theta\begin{bmatrix}\gamma+h_e\\ \delta+g_e\end{bmatrix}\theta\begin{bmatrix}\gamma+h_e+h_2\\ \delta+g_e+g_2\end{bmatrix}\right|^2 \times\left|\theta\begin{bmatrix}\gamma+h_e\\ \delta+g_e\end{bmatrix}\theta\begin{bmatrix}\gamma+h_e+h_1\\ \delta+g_e+g_1\end{bmatrix}\right|^2 \times\left|\theta\begin{bmatrix}\gamma+h_e\\ \delta+g_e\end{bmatrix}\theta\begin{bmatrix}\gamma+h_e-h_1-h_2\\ \delta+g_e-g_1-g_2\end{bmatrix}\right|^2. \tag{3.27}$$

In the above $\{1, S, b_1, b_2, e\}$ construction the gauge group of the observable sector becomes either $SO(10)\times U(1)^3$ or $E_6\times U(1)^2$ and the hidden sector necessarily is $SO(16)$ or E_8 depending on the generalized GSO coefficients, (the choice of the phase φ_L), while the gauge group from the $\Gamma_{6,6}$ lattice becomes $G_L = SO(6)\times U(1)^3$.

So far the construction of the $N=1$ models is generic. The only requirement we are imposing is the presence of *spinors on all three planes*. We call this the S^3 subclass of models. In a general $N=1$ model the spinors could be replaced by vectorial representations of the observable gauge group. This replacement gives rise to three additional classes of models which we denote by S^2V, SV^2 and V^3. In this work we will focus on the S^3 class and we will deal with the other classes in a future work. The condition of spinorial representations arising from each one of the $Z_2\times Z_2$ orbifold planes together with the complete separation of the internal manifold is synonymous to having a well-defined hidden gauge group.

3.3. *The general S^3 $N=1$ model*

In the class of $Z_2\times Z_2$ orbifold models, the internal manifold is broken into three planes. The hidden gauge group is necessarily E_8 or $SO(16)$ broken to any subgroup by Wilson lines (at the $N=4$ level). In order to classify all possible S^3 models, it is necessary to consider all possible basis vectors consistent with modular invariance. Namely:

$$z_1 = \{\bar{\phi}^{1,\dots,4}\}, \tag{3.28}$$

$$z_2 = \{\bar{\phi}^{5,\dots,8}\}, \tag{3.29}$$

$$e_i = \{y_i, \omega_i \,|\, \bar{y}_i, \bar{\omega}_i\}, \quad i\in\{1,2,3,4,5,6\}. \tag{3.30}$$

The z_1, z_2 vectors allow for a breaking of hidden E_8 or $SO(16)$ to $SO(8)\times SO(8)$ depending on the modular coefficients. As we discuss below this splitting of the hidden gauge group has important consequences in the classification of the S^3 class of models by the number of generations. The introduction of e_i vectors is necessary in order to obtain all possible internal shifts which also induces all possible modification to the number of generations.

The general $N=1$, S^3 model based on $\{1, S, e_i, z_1, z_2, \tilde{b}_1, \tilde{b}_2\}$ is

$$
\begin{aligned}
Z_{N=1} = {} & \frac{1}{\tau_2|\eta|^4}\frac{1}{2}\sum_{\alpha,\beta} e^{i\pi(a+b+\mu ab)} \\
& \times \frac{1}{4}\sum_{h_1,h_2,g_1,g_2} \frac{\theta\left[{a \atop b}\right]}{\eta}\frac{\theta\left[{a+h_1 \atop b+g_1}\right]}{\eta}\frac{\theta\left[{a+h_2 \atop b+g_2}\right]}{\eta}\frac{\theta\left[{a-h_1-h_2 \atop b-g_1-g_2}\right]}{\eta} \\
& \times \frac{1}{2^6}\sum_{p_i,q_i} \frac{\Gamma_{2,2}\left[{h_1|p_1,p_2 \atop g_1|q_1,q_2}\right]}{\eta^2\bar{\eta}^2}\frac{\Gamma_{2,2}\left[{h_2|p_3,p_4 \atop g_2|q_3,q_4}\right]}{\eta^2\bar{\eta}^2}\frac{\Gamma_{2,2}\left[{-h_1-h_2|p_5,p_6 \atop -g_1-g_2|q_5,q_6}\right]}{\eta^2\bar{\eta}^2} \\
& \times \frac{1}{8}\sum_{\gamma,\gamma',\xi,\delta,\delta',\zeta} \frac{Z_\eta\left[{\gamma,h_1,h_2 \atop \delta,g_1,g_2}\right]}{\bar{\eta}^3}\frac{Z_{10}\left[{\gamma \atop \delta}\right]}{\bar{\eta}^5}\frac{Z_{16}\left[{\gamma',\xi \atop \delta',\zeta}\right]}{\bar{\eta}^8} e^{i\pi\varphi_L},
\end{aligned}
\tag{3.31}
$$

$$
Z_\eta\begin{bmatrix}\gamma, h_1, h_2 \\ \delta, g_1, g_2\end{bmatrix} = \bar{\theta}\begin{bmatrix}\gamma+h_2 \\ \delta+g_2\end{bmatrix}\bar{\theta}\begin{bmatrix}\gamma+h_1 \\ \delta+h_2\end{bmatrix}\bar{\theta}\begin{bmatrix}\gamma-h_1-h_2 \\ \delta-g_1-g_2\end{bmatrix},
\tag{3.32}
$$

$$
Z_{10}\begin{bmatrix}\gamma \\ \delta\end{bmatrix} = \bar{\theta}\begin{bmatrix}\gamma \\ \delta\end{bmatrix}^5,
\tag{3.33}
$$

$$
Z_{16}\begin{bmatrix}\gamma', \xi \\ \delta', \zeta\end{bmatrix} = \bar{\theta}\begin{bmatrix}\gamma' \\ \delta'\end{bmatrix}^4 \bar{\theta}\begin{bmatrix}\gamma'+\xi \\ \delta'+\zeta\end{bmatrix}^4.
\tag{3.34}
$$

The $\Gamma_{6,6}$ lattice of $N=4$ is twisted by h_i, g_i, thus in the $N=1$ case separated into three $(2,2)$ planes. The contribution of each of these planes in $N=1$ partition function is written in terms of twisted by h_i, g_i and shifted by p_i, q_i $\Gamma_{2,2}$ lattice. The expressions of those lattices at the self-dual point (fermionic construction point) is

$$
\begin{aligned}
& \Gamma_{2,2}\begin{bmatrix}h|p_i,p_j \\ g|q_i,q_j\end{bmatrix}\Bigg|_{f.p} \\
& \quad = \frac{1}{4}\sum_{a_i,b_i,a_j,b_j} e^{i\pi\phi_1+i\pi\phi_2}\left|\theta\begin{bmatrix}a_i \\ b_i\end{bmatrix}\theta\begin{bmatrix}a_i+h \\ b_i+g\end{bmatrix}\theta\begin{bmatrix}a_j \\ b_j\end{bmatrix}\theta\begin{bmatrix}a_j+h \\ b_j+g\end{bmatrix}\right|,
\end{aligned}
\tag{3.35}
$$

where the phases

$$
\phi_i = a_i q_i + b_i p_i + q_i p_i, \qquad \phi_j = a_j q_j + b_j p_j + q_j p_j
$$

define the two shifts of the $\Gamma_{2,2}$ lattice. At the generic point of the moduli space the shifted $\Gamma_{2,2}$ lattice depends on the moduli (T,U), keeping however identical modular transformation properties as those of the fermionic point.

For non-zero twist, $(h,g)\neq(0,0)$, $\Gamma_{2,2}$ is independent of the moduli T,U and thus it is identical to that of (3.35) constructed at the fermionic point [21,22]. Thus for non-zero twist, $(h,g)\neq(0,0)$,

$$
\Gamma_{2,2}\begin{bmatrix}h|p_i,p_j \\ g|q_i,q_j\end{bmatrix}_{(T,U)}\Bigg|_{(h,g)\neq(0,0)} = \Gamma_{2,2}\begin{bmatrix}h|p_i,p_j \\ g|q_i,q_j\end{bmatrix}\Bigg|_{f.p}.
\tag{3.36}
$$

For zero twist, $(h,g)=(0,0)$, the momentum and winding modes are moduli dependent and are shifted by q_i, q_j and p_i, p_j,

$$\begin{aligned} &\Gamma_{2,2}\begin{bmatrix} 0|p_i,p_j \\ 0|q_i,q_j \end{bmatrix}_{(T,U)} \\ &\quad = \sum_{\vec{m},\vec{n}\in Z} e^{i\pi\{m_1q_i+m_2q_j\}}\exp\left\{2\pi i\bar{\tau}\left[m_1\left(n_1+\frac{p_i}{2}\right)+m_2\left(n_2+\frac{p_j}{2}\right)\right]\right. \\ &\qquad \left. -\frac{\pi\tau_2}{T_2U_2}\left|m_1U-m_2+T\left(n_1+\frac{p_i}{2}\right)+TU\left(n_2+\frac{p_j}{2}\right)\right|^2\right\}. \end{aligned} \tag{3.37}$$

The phase φ_L is determined by the chirality of the supersymmetry as well as by the other modular coefficients

$$\varphi_L(a,b)=\frac{1}{2}\sum_{i,j}\left(1-c\begin{bmatrix} v_i \\ v_j \end{bmatrix}\right)\alpha_i\beta_j, \tag{3.38}$$

where α_i and β_j are the upper- and lower-arguments in θ-functions corresponding to the boundary conditions in the two directions of the world sheet torus and which are associated to the basis vectors v_i and v_j of the fermionic construction. The only freedom which remains in the general S^3 $N=1$ model is therefore the choice of the generalized GSO projection coefficients $c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right]=\pm 1$. The space of models is classified according to that choice which determines at the end the phase φ_L. We have in total 55 independent choices for $c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right]$ that can take the values ± 1. Thus, the total number of models in this restricted class of $N=1$ models is 2^{55}. Latter, we will classify all these models according to the values of the GSO coefficients.

The so-called NAHE models is a small subclass of the general S^3, $N=1$ deformed fermionic $N=1$ model. More precisely we can write the NAHE set basis vectors as a linear combination of basis vectors $\{1,S,e_i,z_1,z_2,b_1,b_2\}$ which define the general S^3 $N=1$ model:

$$b_1^{\text{NAHE}}=S+b_1, \tag{3.39}$$

$$b_2^{\text{NAHE}}=S+b_2+e_5+e_6, \tag{3.40}$$

$$b_3^{\text{NAHE}}=1+b_1+b_2+e_5+e_6+z_1+z_2. \tag{3.41}$$

We see that the NAHE set is included in these models as mentioned in Section 3.2.

3.4. *The $N=4$ gauge group*

We describe the gauge configuration of the models defined by the basis vectors $\{1,S,e_i,z_1,z_2,b_1,b_2\}$. For this purpose we start with a simplification and separate out the internal manifold using Eq. (3.26). As the twisting vectors b_1 and b_2 are used to break the $SO(16)\to SO(10)\times U(1)^3$ we will firstly describe the configuration without these vectors. The gauge group induced by the vectors $\{1,S,e,z_1,z_2\}$ without enhancements is

$$G=SO(16)\times SO(8)_1\times SO(8)_2\times SO(12), \tag{3.42}$$

Table 3
The configuration of the gauge group of the $N = 4$ theory. We have separated a priori the internal and the hidden and observable gauge group using the vectors e and z_i. Introducing the other vectors e_i and b_i only induce breaking of these groups

$c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} e \\ z_1 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} e \\ z_2 \end{smallmatrix}\right]$	Gauge group G
$+$	$+$	$+$	$E_8 \times SO(28)$
$+$	$-$	$+$	$SO(24) \times SO(20)$
$+$	$+$	$-$	$SO(24) \times SO(20)$
$+$	$-$	$-$	$SO(32) \times SO(12)$
$-$	$+$	$+$	$SO(16) \times SO(16) \times SO(12)$
$-$	$-$	$+$	$SO(16) \times SO(16) \times SO(12)$
$-$	$+$	$-$	$SO(16) \times SO(16) \times SO(12)$
$-$	$-$	$-$	$E_8 \times E_8 \times SO(12)$

where the internal manifold is described by $SO(12)$ and the hidden sector by $SO(8) \times SO(8)$ and the observable by $SO(16)$. By choosing the GSO coefficients the $SO(16)$ can enhance either to E_8 or mix with the other sectors producing either $SO(24)$ or $SO(32)$. Similarly the $SO(8) \times SO(8)$ can enhance either to $SO(16)$ or E_8 or mix with the observable or internal manifold gauge group. This leads to enhancements of the form $SO(20)$ or $SO(24)$. The exact form depends only on the three GSO coefficients $c\left[\begin{smallmatrix} e \\ z_1 \end{smallmatrix}\right]$, $c\left[\begin{smallmatrix} e \\ z_2 \end{smallmatrix}\right]$, $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$. We have shown the results in Table 3.

Proceeding to the complete model $\{1, S, e_i, z_1, z_2, b_1, b_2\}$ we break these gauge groups to their subgroups. Imposing the shifts e_i we can break the internal gauge group down to its Cartan generators by a suitable choice of the coefficients. By a suitable choice we can break $SO(20) \to SO(8) \times U(1)^6$.

When we also include the twists we break $SO(16) \to SO(10) \times U(1)^3$ and $E_8 \to E_6 \times U(1)^2$. Similarly we can break $SO(24) \to SO(10) \times U(1)^3 \times SO(8)$ and $SO(32) \to SO(10) \times U(1)^3 \times SO(8) \times SO(8)$. Enhancements can subsequently occur of the form $SO(8) \times U(1) \subset SO(32) \to SO(10)$ or $SO(8) \times SO(8) \times U(1) \subset SO(32) \to SO(18)$. We find possible enhancements of the form $SO(10) \times SO(8) \subset SO(32) \to SO(18)$.

In Table 3 we notice that the coefficient $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$ distinguishes between the $SO(32)$ models and the $E_8 \times E_8$ models. Since we require complete separation of the gauge group into a well-defined observable and hidden gauge group, we set the coefficient $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = -1$ in the classification.

4. Generic $Z_2 \times Z_2$ model in the free fermionic formulation

4.1. General formalism

In the free fermionic formulation of the heterotic superstring, a model is determined by a set of basis vectors, associated with the phases picked up by the fermions when parallelly transported along non-trivial loops and a set of coefficients associated with GSO projections. The free fermions in the light-cone gauge in the traditional notation are: $\psi^\mu, \chi^i, y^i, \omega^i$, $i = 1, \ldots, 6$ (left movers) and $\bar{y}^i, \bar{\omega}^i$, $i = 1, \ldots, 6$, ψ^A, $A = 1, \ldots, 5$, $\bar{\phi}^\alpha$, $\alpha = 1, 8$ (right movers). The class of models under consideration is generated by a set of

12 basis vectors

$$B = \{v_1, v_2, \dots, v_{12}\},$$

where

$$\begin{aligned}
v_1 &= 1 = \{\psi^\mu, \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\},\\
v_2 &= S = \{\psi^\mu, \chi^{1,\dots,6}\},\\
v_{2+i} &= e_i = \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\}, \quad i = 1, \dots, 6,\\
v_9 &= b_1 = \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\dots,5}\},\\
v_{10} &= b_2 = \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\dots,5}\},\\
v_{11} &= z_1 = \{\bar{\phi}^{1,\dots,4}\},\\
v_{12} &= z_2 = \{\bar{\phi}^{5,\dots,8}\}.
\end{aligned} \tag{4.1}$$

The vectors $1, S$ generate an $N = 4$ supersymmetric model. The vectors $e_i, i = 1, \dots, 6$ give rise to all possible symmetric shifts of internal fermions $(y^i, \omega^i, \bar{y}^i, \bar{\omega}^i)$ while b_1 and b_2 stand for the $Z_2 \times Z_2$ orbifold twists. The remaining fermions not affected by the action of the previous vectors are ϕ^i, $i = 1, \dots, 8$ which normally give rise to the hidden sector gauge group. The vectors z_1, z_2 divide these eight fermions in two sets of four which in the $Z_2 \times Z_2$ case is the maximum consistent partition function [9]. This is the most general basis, with symmetric shifts for the internal fermions, that is compatible with Kac–Moody level one $SO(10)$ embedding.

The associated projection coefficients are denoted by $c\left[{v_i \atop v_j}\right]$, $i, j = 1, \dots, 12$ and can take the values ± 1. They are related by modular invariance $c\left[{v_i \atop v_j}\right] = \exp\{i\frac{\pi}{2} v_i \cdot v_j\} c\left[{v_j \atop v_i}\right]$ and $c\left[{v_i \atop v_i}\right] = \exp\{i\frac{\pi}{4} v_i \cdot v_i\} c\left[{v_j \atop 1}\right]$ leaving 2^{66} independent coefficients. Out of them, the requirement of $N = 1$ supersymmetric spectrum fixes (up to a phase convection) all $c\left[{S \atop v_i}\right]$, $i = 1, \dots, 12$. Moreover, without loss of generality we can set $c\left[{1 \atop 1}\right] = -1$, and leave the rest 55 coefficients free. Therefore, a simple counting gives 2^{55} (that is approximately $10^{16.6}$) distinct models in the class under consideration. In the following we study this class of models by deriving analytic formulas for the gauge group and the spectrum and then using these formulas for the classification.

4.2. The gauge group

Gauge bosons arise from the following four sectors:

$$G = \{0, z_1, z_2, z_1 + z_2, x\},$$

where

$$x = 1 + S + \sum_{i=1}^{6} e_i + \sum_{k=1}^{2} z_k = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\}. \tag{4.2}$$

The 0 sector gauge bosons give rise to the gauge group

$$SO(10) \times U(1)^3 \times SO(8)^2.$$

Table 4
Typical enhanced gauge groups and associated projection coefficients for a generic model generated by the basis (4.1) (coefficients not included equal to $+1$ except those fixed by space–time supersymmetry and conventions)

$c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} b_1 \\ z_1 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} b_2 \\ z_1 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} b_1 \\ z_2 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} b_2 \\ z_2 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} e_1 \\ z_1 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} e_2 \\ z_2 \end{smallmatrix}\right]$	$c\left[\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right]$	Gauge group
$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$SO(10) \times SO(18) \times U(1)^2$
$+$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$SO(10) \times SO(9)^2 \times U(1)^3$
$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$SO(10)^2 \times SO(9) \times U(1)^2$
$+$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$SO(10)^3 \times U(1)$
$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$SO(26) \times U(1)^3$
$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$E_6 \times U(1)^2 \times E_8$
$-$	$-$	$+$	$-$	$+$	$+$	$+$	$+$	$E_6 \times U(1)^2 \times SO(16)$
$-$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$E_6 \times U(1)^2 \times SO(8) \times SO(8)$
$-$	$+$	$+$	$+$	$+$	$+$	$+$	$-$	$SO(10) \times U(1)^3 \times E_8$
$-$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$SO(10) \times U(1)^3 \times SO(16)$

The x gauge bosons when present lead to enhancements of the traditionally called observable sector (the sector that includes $SO(10)$) while the $z_1 + z_2$ sector can enhance the hidden sector ($SO(8)^2$). However, the z_1, z_2 sectors accept oscillators that can also give rise to mixed type gauge bosons and completely reorganize the gauge group. The appearance of mixed states is in general controlled by the phase $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right]$. The choice $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = +1$ allows for mixed gauge bosons and leads to the gauge groups presented in Table 4.

The choice $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = -1$ eliminates all mixed gauge bosons and there are a few possible enhancements: $SO(10) \times U(1) \to E_6$ and/or $SO(8)^2 \to \{SO(16), E_8\}$. The x sector gauge bosons survive only when

$$\sum_{j=1, i\neq j}^{6} (e_i|e_j) + \sum_{k=1}^{2} (e_i|z_k) = 0 \bmod 2, \quad i = 1, \ldots, 6, \tag{4.3}$$

$$\sum_{j=1}^{6} (e_j|z_k) = 0 \bmod 2, \quad k = 1, 2, \tag{4.4}$$

where we have introduced the notation

$$c\begin{bmatrix} v_i \\ v_j \end{bmatrix} = e^{i\pi(v_i|v_j)}, \quad (v_i|v_j) = 0, 1 \tag{4.5}$$

and one of the constraints in (4.3), (4.4) can be dropped because is linearly independent with the rest.

As far as the $SO(8)^2$ is concerned we have the following possibilities:

$$\text{(i)} \quad (e_i|z_1) = (b_a|z_1) = 0 \quad \forall i = 1, \ldots, 6, \ a = 1, 2, \tag{4.6}$$

$$\text{(ii)} \quad (e_i|z_2) = (b_a|z_2) = 0 \quad \forall i = 1, \ldots, 6, \ a = 1, 2, \tag{4.7}$$

$$\text{(iii)} \quad (e_i|z_1 + z_2) = (b_a|z_1 + z_2) = 0 \quad \forall i = 1, \ldots, 6, \ a = 1, 2. \tag{4.8}$$

Depending on which of the above equations are true the enhancement is

$$\text{both (i) and (ii)} \Longrightarrow E_8, \tag{4.9}$$

$$\text{one of (i) or (ii) or (iii)} \Longrightarrow SO(16), \tag{4.10}$$

$$\text{none of (i) or (ii) or (iii)} \Longrightarrow SO(8) \times SO(8). \tag{4.11}$$

In the sequel we will restrict to the case $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = -1$ as this is the more promising phenomenologically, we intent to examine $c\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] = +1$ in detail in a future publication.

4.3. Observable matter spectrum

The untwisted sector matter is common to all models and consists of six vectorials of $SO(10)$ and 12 non-Abelian gauge group singlets. In models where the gauge group enhances to E_6 extra matter comes from the x sector giving rise to six E_6 fundamental reps (**27**).

Chiral twisted matter arise from the following sectors

$$\begin{aligned}
B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 + (x), \\
B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6 + (x), \\
B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4 + (x),
\end{aligned} \tag{4.12}$$

where $b_3 = b_1 + b_2 + x$. These are 48 sectors (16 sectors per orbifold plane) and we choose to label them using the plane number i (upper index) and the integers $p_i, q_i, r_i, s_i = \{0, 1\}$ (lower index) corresponding to the coefficients of the appropriate shift vectors. Note that for a particular orbifold plane i only four shift vectors can be added to the twist vector b_i (the ones that have non empty intersection) the other two give rise to massive states. Each of the above sectors (4.12) can produce a single spinorial of $SO(10)$ (or fundamental of E_6 in the case of enhancement). Since the E_6 model spectrum is in one to one correspondence with the $SO(10)$ spectrum in the following we use the name spinorial meaning the **16** of $SO(10)$ and in the case of enhancement the **27** of E_6.

One of the advantages of our formulation is that it allows to extract generic formulas regarding the number and the chirality of each spinorial. This is important because it allow an algebraic treatment of the entire class of models without deriving each model explicitly. The number of surviving spinorials per sector (4.12) is given by

$$P^{(1)}_{pqrs} = \frac{1}{16} \prod_{i=1,2} \left(1 - c\begin{bmatrix} e_i \\ B^{(1)}_{pqrs} \end{bmatrix}\right) \prod_{m=1,2} \left(1 - c\begin{bmatrix} z_m \\ B^{(1)}_{pqrs} \end{bmatrix}\right), \tag{4.13}$$

$$P^{(2)}_{pqrs} = \frac{1}{16} \prod_{i=3,4} \left(1 - c\begin{bmatrix} e_i \\ B^{(2)}_{pqrs} \end{bmatrix}\right) \prod_{m=1,2} \left(1 - c\begin{bmatrix} z_m \\ B^{(2)}_{pqrs} \end{bmatrix}\right), \tag{4.14}$$

$$P^{(3)}_{pqrs} = \frac{1}{16} \prod_{i=5,6} \left(1 - c\begin{bmatrix} e_i \\ B^{(3)}_{pqrs} \end{bmatrix}\right) \prod_{m=1,2} \left(1 - c\begin{bmatrix} z_m \\ B^{(3)}_{pqrs} \end{bmatrix}\right), \tag{4.15}$$

where P^i_{pqrs} is a projector that takes values $\{0, 1\}$. The chirality of the surviving spinorials is given by

$$X^{(1)}_{pqrs} = c\begin{bmatrix} b_2 + (1-r)e_5 + (1-s)e_6 \\ B^{(1)}_{pqrs} \end{bmatrix}, \tag{4.16}$$

$$X^{(2)}_{pqrs} = c\begin{bmatrix} b_1+(1-r)e_5+(1-s)e_6 \\ B^{(2)}_{pqrs} \end{bmatrix}, \tag{4.17}$$

$$X^{(3)}_{pqrs} = c\begin{bmatrix} b_1+(1-r)e_3+(1-s)e_4 \\ B^{(3)}_{pqrs} \end{bmatrix}, \tag{4.18}$$

where $X^i_{pqrs} = +$ corresponds to a **16** of $SO(10)$ (or **27** in the case of E_6) and $X^i_{pqrs} = -$ corresponds to a $\overline{\mathbf{16}}$ (or $\overline{\mathbf{27}}$) and we have chosen the space–time chirality $C(\psi^\mu) = +1$. The net number of spinorials and thus the net number of families is given by

$$N_F = \sum_{i=1}^{3}\sum_{p,q,r,s=0}^{1} X^{(i)}_{pqrs} P^{(i)}_{pqrs}. \tag{4.19}$$

Similar formulas can be easily derived for the number of vectorials and the number of singlets and can be extended to the $U(1)$ charges but in this work we will restrict to the spinorial calculation.

Formulas (4.13)–(4.15) allow us to identify the mechanism of spinorial reduction, or in other words the fixed point reduction, in the fermionic language. For a particular sector $(B^{(i)}_{pqrs})$ of the orbifold plane i there exist two shift vectors (e_{2i-1}, e_{2i}) and the two zeta vectors (z_1, z_2) that have no common elements with $B^{(i)}_{pqrs}$. Setting the relative projection coefficients (4.15) to -1 each of the above four vectors acts as a projector that cuts the number of fixed points in the associated sector by a factor of two. Since four such projectors are available for each sector the number of fixed points can be reduced from 16 to 1 per plane.

The projector action (4.13)–(4.15) can be expanded and written in a simpler form

$$\Delta^{(i)} W^{(i)} = Y^{(i)}, \tag{4.20}$$

where

$$\Delta^{(1)} = \begin{bmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{bmatrix}, \qquad Y^{(1)} = \begin{bmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{bmatrix},$$

$$\Delta^{(2)} = \begin{bmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{bmatrix}, \qquad Y^{(2)} = \begin{bmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{bmatrix},$$

$$\Delta^{(3)} = \begin{bmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{bmatrix}, \qquad Y^{(3)} = \begin{bmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{bmatrix} \tag{4.21}$$

and

$$W^i = \begin{bmatrix} p_i \\ q_i \\ r_i \\ s_i \end{bmatrix}. \tag{4.22}$$

They form three systems of equations of the form $\Delta^i W^i = Y^i$ (one for each orbifolds plane). Each system contains 4 unknowns p_i, q_i, r_i, s_i which correspond to the labels of surviving spinorials in the plane i. We call the set of solutions of each system Ξ_i. The net number of families (4.19) can be written as

$$N_F = \sum_{i=1}^{3} \sum_{p,q,r,s \in \Xi_i} X^{(i)}_{pqrs}. \tag{4.23}$$

The chiralities (4.16)–(4.18) can be further expanded in the exponential form $X^{(i)}_{pqrs} = \exp(i\pi \chi^{(i)}_{pqrs})$

$$\begin{aligned}
\chi^{(1)}_{pqrs} = {} & 1 + (b_1|b_2) + (1-r)(e_5|b_1) + (1-s)(e_6|b_1) + p(e_3|b_2) + q(e_4|b_2) \\
& + r(e_5|b_2) + s(e_6|b_2) + p(1-r)(e_3|e_5) + p(1-s)(e_3|e_6) \\
& + q(1-r)(e_4|e_5) + q(1-s)(e_4|e_6) + (r+s)(e_5|e_6) \bmod 2,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\chi^{(2)}_{pqrs} = {} & 1 + (b_1|b_2) + (1-r)(e_5|b_2) + (1-s)(e_6|b_2) + p(e_1|b_1) + q(e_2|b_1) \\
& + r(e_5|b_1) + s(e_6|b_1) + p(1-r_2)(e_1|e_5) + q(1-r)(e_2|e_5) \\
& + p(1-s)(e_1|e_6) + q(1-s)(e_2|e_6) + (r+s)(e_5|e_6) \bmod 2,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\chi^{(3)}_{pqrs} = {} & 1 + (b_1|b_2) + (1-p)(e_1|b_1) + (1-q)(e_2|b_1) + (e_5 + e_6|b_1) \\
& + (1-r)(e_3|b_2) + (1-s)(e_4|b_2) + (1-r)(1-p)(e_3|e_1) \\
& + (1-r)(1-q)(e_3|e_2) + (1-r)(e_3|e_5) + (1-r)(e_3|e_6) \\
& + (1-s)(e_4|e_6) + (1-r)(e_3|z_1 + z_2) + (1-s)(e_4|z_1 + z_2) \\
& + (b_1|z_1 + z_2) \bmod 2.
\end{aligned} \tag{4.26}$$

We remark here that the projection coefficient $c\left[\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right]$ simply fixes the overall chirality and that our equations depend only on

$$\begin{aligned}
& (e_i|e_j), \qquad (e_i|b_A), \qquad (e_i|z_n), \qquad (z_n|b_A), \\
& \quad i = 1, \ldots, 6, \ A = 1, 2, \ n = 1, 2.
\end{aligned} \tag{4.27}$$

However, the following six parameters do not appear in the expressions $(e_1|e_2)$, $(e_3|e_4)$, $(e_3|b_1)$, $(e_4|b_1)$, $(e_1|b_2)$, $(e_2|b_2)$ and thus a generic model depends on 37 discrete parameters.

5. Results

5.1. Models

We apply here the formalism developed above in order to derive sample models in the free fermionic formulation.

5.1.1. The $Z_2 \times Z_2$ symmetric orbifold

The simplest example is the symmetric $Z_2 \times Z_2$ orbifold. Here we set all the free GSO phases (4.27) to zero. The full GSO phase matrix takes the form ($c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right] = \exp[i\pi(v_i|v_j)]$)

$$
(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}.
$$

With the above choice $\Delta^{(i)} = W^{(i)} = 0$ in Eq. (4.20). All projectors become inactive and thus the number of surviving twisted sector spinorials takes its maximum value which is 48 with all chiralities positive according to (4.24)–(4.26). Moreover three spinorials and three antispinorials arise from the untwisted sector. Following (4.3), (4.4) the gauge group enhances to $E_6 \times U(1)^2 \times E_8$ and the spinorials of $SO(10)$ combine with vectorials and singlets to produce $48 + 3 = 51$ families (**27**) and 3 antifamilies ($\overline{\mathbf{27}}$) of E_6.

5.1.2. A three generation E_6 model

We can obtain a three family E_6 model by choosing the following set of projection coefficients

$$
(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\
\left(\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}.
$$

The full gauge group is here $E_6 \times U(1)^2 \times SO(8)^2$. Three families (**27**), one from each plane, arise from the sectors $S + b_i + (x)$, $i = 1, 2, 3$. Another set of three families and three antifamilies arise from the untwisted sector. The hidden sector consists of nine 8-plets under each $SO(8)$. In addition there exist a number of non-Abelian gauge group singlets. The model could be phenomenologically acceptable provided one finds a way of breaking E_6. Since it is not possible to generate the E_6 adjoint (not in Kac–Moody level one), we need to realize the breaking by an additional Wilson-line like vector. However, a detailed investigation of acceptable basis vectors, shows that the E_6 breaking is accompanied by truncation of the fermion families. Thus this kind of perturbative E_6 breaking is not compatible with the presence of three generations. It would be interesting to utilize string dualities in order to study the non-perturbative aspects of such models.

5.1.3. A six generation E_6 model

Similarly a six family $E_6 \times U(1)^2 \times E_8$ model can be obtained using the following projection coefficients

$$
(v_i|v_j) = \begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}.
$$

In this model we have six families from the twisted sector, two from each plane together with three families and three antifamilies from the untwisted sector, accompanied by a number of singlets and 8-plets of both hidden $SO(8)$'s.

5.2. $N = 4$ liftable vacua

In the models considered above we have managed to separate the orbifold twist action (represented here by b_1, b_2) from the shifts (represented by e_i) and the Wilson lines (z_1, z_2). However, these actions are further correlated through the GSO projection coefficients $c\left[\begin{smallmatrix} v_i \\ v_j \end{smallmatrix}\right]$. Nevertheless, we remark that the twist action can be decoupled from the other two in the case

$$
c\begin{bmatrix} b_n \\ z_k \end{bmatrix} = c\begin{bmatrix} b_m \\ e_i \end{bmatrix} = +1, \quad i = 1, \ldots, 6,\ k = 1, 2,\ m, n = 1, 2, 3. \tag{5.1}
$$

The above relation defines a subclass of $N = 1$ four-dimensional vacua with interesting phenomenological properties and includes three generation models. Due to the decoupling of the orbifold twist action these vacua are direct descendants of $N = 4$ vacua so we will refer to these models as $N = 4$ liftable models. In this subclass of models some important phenomenological properties of the vacuum, as the number of generations, are predetermined at the $N = 4$ level as it is related to the $(e_i|e_j)$ and $(z_i|e_j)$ phases. The orbifold action reduces the supersymmetries and the gauge group and makes chirality apparent, however the number of generations is selected by the $N = 4$ vacuum structure. At the $N = 1$ level this is understood as follows: the $Z_2 \times Z_2$ orbifold has 48 fix points. Switching on some of the above phases correspond to a free action that removes some of the fixed points and thus reduces the number of spinorials. Moreover, in this case, the chirality of the surviving spinorials is again related as seen by (4.24)–(4.26) to the $(e_i|e_j)$ and $(e_i|z_k)$ coefficients, which are all fixed at the $N = 4$ level. The observable gauge group of liftable models is always E_6 and this can be easily seen by applying (5.1) to (4.3), (4.4).

Typical examples of such vacua are the three and six generation $E_6 \times U(1)^2 \times SO(8)^2$ models presented in Section 5.1. A careful counting, taking into account some symmetries among the coefficients, shows that this class of models consists of 2^{20} models, or 2^{21} if we include $(b_1|b_2)$. These vacua are interesting because they can admit a geometrical interpretation.

From the orbifold description we learn that all breakings of the hidden and observable gauge group are induced using Wilson lines. From the 4D point of view the internal gauge group is broken in a similar fashion using Wilson lines. The twisted planes in Eq. (3.36) describe the removal of the free moduli using twists. When a group is broken using Wilson lines the field corresponding to this Wilson line obtains a non-zero VEV. The fixing of the moduli using twists can be interpreted as the removal of the quantum fluctuations of the fields identified with the Wilson lines. These Wilson lines become discrete Wilson lines and the VEV becomes a fixed value.

5.3. *Classification*

As we discussed above, the free GSO phases of the $N = 1$ partition function control the number of chiral generations in a given model. In Section 3 we have given analytic formulas that enable the calculation of the number of generations for any given set of phases. To gain an insight into the structure of this class of vacua we can proceed with a computer evaluation of these formulas and thus classify the space of these vacua with respect to the number of generations. This also allows a detailed examination of the structure of these vacua and in particular how the generations are distributed among the three orbifold planes. The main obstacle to this approach is the huge number of vacua under consideration. As a first step in this direction we restrict ourselves to the class of liftable vacua that is physically appealing and contains representative models with the right number of generations. As stated above this class consists in principle of 2^{21} models and their complete classification takes a few minutes on a personal computer using an appropriate computer program. The program analyses all different options for the free GSO coefficients. The different configurations are then used to calculate the number of generations using formulae (4.13)–(4.19). For the analysis of the gauge group we use

Table 5
Inequivalent realistic liftable models with a $E_6 \times U(1)^2 \times SO(8) \times SO(8)$ gauge group. The chiral content of each model is listed per plane and numbered, '+' lists all the positive chiral states per plane while '−' lists all the negative states per plane. The total sum of all the planes is then listed and subsequently the net total number of chiral states. The list is ordered by the total net number of chiral states

No.	1		2		3		Total		Net
	+	−	+	−	+	−	+	−	
1	16	0	8	0	8	0	32	0	32
2	8	0	8	0	8	0	24	0	24
3	8	0	8	0	4	0	20	0	20
4	8	0	6	2	4	0	18	2	16
5	8	0	4	0	4	0	16	0	16
6	12	4	4	0	4	0	20	4	16
7	8	0	8	0	4	4	20	4	16
8	6	2	4	0	4	0	14	2	12
9	4	0	4	0	4	0	12	0	12
10	8	0	2	0	2	0	12	0	12
11	4	0	4	0	2	0	10	0	10
12	4	0	4	0	3	1	11	1	10
13	6	2	4	0	2	0	12	2	10
14	4	4	4	0	4	0	12	4	8
15	4	0	4	0	2	2	10	2	8
16	4	0	3	1	2	0	9	1	8
17	4	0	2	0	2	0	8	0	8
18	6	2	3	1	2	0	11	3	8
19	6	2	2	0	2	0	10	2	8
20	10	6	2	0	2	0	14	6	8
21	6	2	4	0	2	2	12	4	8
22	3	1	3	1	2	0	8	2	6
23	3	1	2	0	2	0	7	1	6
24	2	0	2	0	2	0	6	0	6
25	4	0	2	2	2	0	8	2	6
26	4	0	2	0	1	1	7	1	6
27	4	0	1	0	1	0	6	0	6
28	6	2	1	0	1	0	8	2	6
29	3	1	3	1	1	0	7	2	5
30	2	0	2	0	1	0	5	0	5
31	3	1	2	0	1	0	6	1	5
32	3	1	2	0	2	2	7	3	4
33	2	2	2	0	2	0	6	2	4
34	4	4	2	0	2	0	8	4	4
35	4	0	2	2	2	2	8	4	4
36	3	1	2	0	1	1	6	2	4
37	2	0	2	0	1	1	5	1	4
38	2	0	1	0	1	0	4	0	4
39	3	1	1	0	1	0	5	1	4
40	1	1	3	1	3	1	7	3	4
41	2	0	1	0	1	1	4	1	3
42	3	1	1	1	1	0	5	2	3
43	1	0	1	0	1	0	3	0	3
44	2	0	1	1	1	1	4	2	2
45	2	0	2	0	1	3	5	3	2
46	2	2	2	0	1	1	5	3	2
47	1	1	1	0	1	0	3	1	2
48	2	2	1	0	1	0	4	2	2
49	4	4	1	0	1	0	6	4	2
50	1	1	1	1	3	1	5	3	2
51	1	1	1	0	1	1	3	2	1
52	1	1	0	1	3	1	4	3	1
53	2	2	2	2	2	2	6	6	0
54	2	0	2	2	1	3	5	5	0
55	2	2	1	1	1	1	4	4	0
56	4	4	2	2	2	2	8	8	0
57	4	4	1	1	1	1	6	6	0
58	1	1	1	1	1	1	3	3	0
59	2	2	2	2	1	1	5	5	0
60	1	3	1	0	1	0	3	3	0
61	4	4	4	4	4	4	12	12	0

formulae (4.3)–(4.8). The results are presented in Tables 5–7. In these tables we list the number of generations coming from the twisted sectors. They are listed per plane. The number of positive chiral generations is separated from the number of negative chiral generations on each plane. The total number is then listed before listing the total net number of generations. As the sign of the chirality is determined by the coefficient $(b_1|b_2)$ (see (4.24)–(4.26)) we have included models that have a positive net number of generations. In order to maintain a complete separation of the hidden gauge group we have set $(z_1|z_2) = 1$. The tables are ordered by the total net number of chiral states.

We find that there are no liftable models with a $SO(10)$ observable gauge group, which is always extended to E_6, and the states from the vector x are not projected out. Since

Table 6
Inequivalent realistic liftable models with a $E_6 \times U(1)^2 \times SO(16)$ gauge group. The chiral content of each model is listed per plane and numbered, '+' lists all the positive chiral states per plane while '−' lists all the negative states per plane. The total sum of all the planes is then listed and subsequently the net total number of chiral states. The list is ordered by the total net number of chiral states

No.	1		2		3		Total		Net
	+	−	+	−	+	−	+	−	
1	16	0	8	0	8	0	32	0	32
2	8	0	8	0	8	0	24	0	24
3	8	0	6	2	4	0	18	2	16
4	8	0	4	0	4	0	16	0	16
5	12	4	4	0	4	0	20	4	16
6	8	0	8	0	4	4	20	4	16
7	6	2	4	0	4	0	14	2	12
8	4	0	4	0	4	0	12	0	12
9	4	4	4	0	4	0	12	4	8
10	4	0	4	0	2	2	10	2	8
11	4	0	3	1	2	0	9	1	8
12	4	0	2	0	2	0	8	0	8
13	6	2	3	1	2	0	11	3	8
14	6	2	2	0	2	0	10	2	8
15	10	6	2	0	2	0	14	6	8
16	6	2	4	0	2	2	12	4	8
17	3	1	3	1	2	0	8	2	6
18	3	1	2	0	2	0	7	1	6
19	2	0	2	0	2	0	6	0	6
20	3	1	2	0	2	2	7	3	4
21	2	2	2	0	2	0	6	2	4
22	4	4	2	0	2	0	8	4	4
23	4	0	2	2	2	2	8	4	4
24	3	1	2	0	1	1	6	2	4
25	2	0	2	0	1	1	5	1	4
26	1	1	3	1	3	1	7	3	4
27	2	0	1	1	1	1	4	2	2
28	1	1	1	1	3	1	5	3	2
29	2	2	2	2	2	2	6	6	0
30	2	0	2	2	1	3	5	5	0
31	2	2	1	1	1	1	4	4	0
32	4	4	2	2	2	2	8	8	0
33	4	4	1	1	1	1	6	6	0
34	1	1	1	1	1	1	3	3	0
35	4	4	4	4	4	4	12	12	0

the models admit a geometrical interpretation, it means that they must descend from the ten-dimensional $E_8 \times E_8$ heterotic-string on $Z_2 \times Z_2$ Calabi–Yau threefold.

In 3% of all the models the hidden gauge group is enhanced to $SO(8) \times SO(8) \to SO(16)$. We find that in total 1024 liftable models are enhanced to $SO(8) \times SO(8) \to E_8$. We find that the 24 generations NAHE model as explained in Section 2 is present in Table 5. The problem of a detailed investigation of the full class of vacua will be considered further in a future publication.

Table 7
Inequivalent realistic liftable models with a $E_6 \times U(1)^2 \times E_8$ gauge group. The chiral content of each model is listed per plane and numbered, '$+$' lists all the positive chiral states per plane while '$-$' lists all the negative states per plane. The total sum of all the planes is then listed and subsequently the net total number of chiral states. The list is ordered by the total net number of chiral states

No.	1		2		3		Total		Net
	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	
1	16	0	16	0	16	0	48	0	48
2	12	4	8	0	8	0	28	4	24
3	8	0	8	0	8	0	24	0	24
4	10	6	4	0	4	0	18	6	12
5	6	2	6	2	4	0	16	4	12
6	6	2	4	0	4	0	14	2	12
7	4	0	4	0	4	0	12	0	12
8	3	1	3	1	3	1	9	3	6
9	4	4	2	2	2	2	8	8	0
10	4	4	4	4	4	4	12	12	0
11	2	2	2	2	2	2	6	6	0

5.4. *General properties*

In Section 3 we discussed a direct translation between the bosonic formulation and the fermionic formulation of the heterotic string compactifications. $Z_2 \times Z_2$ orbifold compactifications are relevant for our class of models. These orbifolds contain three twisted sectors, or three twisted planes. A priori we may have the possibility that all three twisted planes produce spinorial $SO(10)$ representations. We refer to this subclass of models as S^3 models. The alternatives are models in which spinorial representations may be obtained from only two, one, or none, twisted planes, and the others produce vectorial representations. We refer to these cases as S^2V, SV^2 and V^3 models, respectively. The focus of the analysis in this paper is on the S^3 subclass of models, which also contains the NAHE-based three generation models. The S^3 subclass allows, depending on the one-loop GSO projection coefficients, the possibility of spinorials on each plane. In specific models in this subclass each Standard Model family is obtained from a distinct orbifold plane. Such models therefore produce three generation models and may be phenomenologically interesting. The only other phenomenologically viable option can come from the subclass S^2V models as this class of models may contain a model with for example 2 generations coming from the first plane and 1 generation coming form the second plane and none from the third. The SV^2 class of models cannot produce a physical model because it is not possible to reduce the number of families to 3 as they would have to be coming from one plane and 3 cannot be written as a power of 2. Similarly the V^3 subclass of models will not contain phenomenologically interesting models.

5.4.1. *3 generations realized only with twisted and shifted real manifolds*

Since the projectors are constructed using the complete separation of the internal manifold we see that three generation models are only possible when

$$\Gamma_{6,6} = \Gamma_{2,2}^{3} \to \Gamma_{1,1}^{6}. \tag{5.2}$$

These $\Gamma_{1,1}$ internal parts do not describe a complex manifold. They describe internal real circles. If we use solely complex manifolds, of the type $\Gamma_{6,6} = \Gamma_{2,2}^3$, and using only symmetric shifts, we find that there are no 3 generation models. We therefore conclude that the net number of generations can never be equal to three in the framework of $Z_2 \times Z_2$ Calabi–Yau compactification. This implies the necessity of non-zero torsion in CY $Z_2 \times Z_2$ compactifications in order to obtain semi-realistic three generation models.

In the realistic free fermionic models the reduction of the number of families together with the breaking of the observable $SO(10)$ is realized by isolating full multiplets at two fixed points on the internal manifold. In reducing the number of families down to one, different component of each family are attached to the two distinct fixed points. We remove one full multiplet and simultaneously break the $SO(10)$ symmetry. We therefore keep a full multiplet on each twisted plane. In the $SO(10)$ models described here a whole 16 or $\overline{16}$ of $SO(10)$ is attached to a fixed point. We are therefore not able to break the $SO(10)$, and simultaneously preserve the full Standard Model multiplets. For this reason we find that the observable $SO(10)$ cannot be broken perturbatively in this class of three generation models, and may only be broken non-perturbatively. It is therefore not possible to reduce both the number of families down to 3 and break the observable gauge group $SO(10)$ down to its subgroups perturbatively.

We conclude that there is a method to reduce the number of generations from 48 to 3. Since we need 4 projectors we need to separate the hidden gauge group using $SO(8)$ characters

$$\Gamma_{0,8} \to \Gamma_{0,4}\Gamma_{0,4} \tag{5.3}$$

and we need to break the internal complex manifold to an internal real manifold

$$\Gamma_{6,6} \to [\Gamma_{1,1}\Gamma_{1,1}]^3. \tag{5.4}$$

If we reduce the number of generations to 3 we cannot break the $SO(10)$ observable group to its subgroups, while maintaining a full multiplet. The $SO(10)$ observable gauge group cannot therefore be broken perturbatively. We can reduce the number of generations from 48 to 6 using 3 projectors. This entails that we can choose either to separate the hidden gauge group using $SO(16)$ characters, or to leave the internal manifold complex.

We argued above that we cannot break $SO(10)$ down to a subgroup perturbatively, while reducing the number of generations to 3. If we want to break the $SO(10)$ symmetry perturbatively, and keep a full $SO(10)$ multiplet from a given twisted plane, we can only reduce the number of generations to 6. This can be achieved if we define three different projectors like the ones defined in Eqs. (4.13)–(4.15). We are therefore left with two options.

- We can use $SO(16)$ characters for the separation of the hidden gauge group. We have then constructed only one projector which leaves us no other option than to break the complex structure using symmetric shifts

$$\Gamma_{6,6} \to \Gamma_{1,1}^6. \tag{5.5}$$

- We can use $SO(8)$ characters for the separation of the hidden gauge group. In doing so we have constructed two projectors. The third can be realized by the symmetric shifts that leave the complex structure of the internal manifold intact

$$\Gamma_{6,6} \to \Gamma_{2,2}^{3}. \tag{5.6}$$

6. Discussion and conclusions

String theory duly attracts wide interest. It provides a consistent approach to perturbative quantum gravity, while at the same time incorporating the gauge and matter structures that are relevant for particle physics phenomenology. However, the multitude of vacua that the theory admits and the lack of a dynamical principle to choose among them, hinders the prospects that the theory will yield unique experimental predictions. This has led some authors to advocate the anthropic principle as a possible resolution for understanding the contrived set of parameters that seem to govern our world.

The approach pursued in our work is different. In our view the understanding of the dynamical principles that underly quantum gravity and the vacuum selection must await the better conceptual understanding of the quantum gravity synthesis. It may well be that at the end of the day the probabilistic nature of quantum mechanics will emerge as a derived property rather than a fundamental property of quantum gravity. In this respect we should regard the string theories as merely providing a perturbative glimpse into the underlying properties of quantum gravity, and how it may relate to the gauge and matter observables. In this context we must utilize both the low energy data as well as the basic properties of string theory to isolate promising string vacua and develop the tools to discern between the experimental predictions of different classes. An example, is the $SO(10)$ embedding of the Standard Model spectrum, which is viable in the heterotic limit of M-theory, but not in its type I limit.

Given the Standard Model properties, we may hypothesize that the true string vacuum should accommodate two pivotal ingredients. One is the existence of three generations and the second is their embedding in an underlying $SO(10)$ or E_6 grand unified gauge group. In this context, the replication of the matter generations is the first particle physics observable whose origin may be sought in string theory. This follows from the fact that the flavor sector of the Standard Model does not arise from any physical principle, like the gauge principle, as well as from the fact that in certain classes of string compactifications the number of generations is related to a topological number of compact manifolds, the Euler characteristic.

A class of string compactifications that admit three generations, as well as their embedding in an underlying $SO(10)$ group structure are the NAHE-based free fermionic models. A subset of the boundary condition basis vectors that span these models can be seen to correspond to $Z_2 \times Z_2$ orbifold compactifications at special points in the moduli space. However, the geometrical understanding of the full three generation models is still lacking. The aim of the current work is therefore to advance the geometrical understanding of the NAHE-based free fermionic models. In this paper we showed that two of the boundary condition basis vectors beyond those that correspond to the $Z_2 \times Z_2$ orbifold correspond to symmetric shifts on the compact tori, whereas the third correspond to an asymmetric shift. We then proceeded to classify all possible symmetric shifts on complex tori and demonstrated that three generation models do not arise in this manner. Three generation models that realize the $Z_2 \times Z_2$ orbifold picture of the three chiral generations were found. In these cases the $SO(10)$ gauge group cannot be broken perturbatively, while preserving the full Standard Model matter content. Additionally in these cases the internal lattice is broken to $\Gamma^6_{1,1}$, i.e., to a product of six circles. In this class of models each of the chiral generations is attached to a single fixed point in each of the twisted orbifold planes. This should be contrasted with the case of the three generation free fermionic models in which the $SO(10)$ symmetry is broken perturbatively by Wilson lines. In those cases, each generation is obtained from a separate orbifold plane, but different components of each generation are attached to different fixed points of the corresponding twisted sector.

Additionally, we demonstrated in this paper that for a wide range of models, for which a geometrical origin is understood, there exist an interpretation of the phases that appear in the $N = 1$ partition function, in terms of vacuum expectation values of background fields of the $N = 4$ vacua. In these cases the dynamical components of the background fields are projected out, but their vacuum expectation value is retained and takes the form of the free GSO phases of the $N = 1$ partition function. These phases also control the chirality of the models. Thus, we have the situation in which the chirality of the models is already determined by the VEVs of the background fields of the $N = 4$ vacuum. In effect, the chiral content of the $N = 1$ vacua in these cases is determined by the Narain $N = 4$ lattice. An example of this phenomenon was already seen in the case of $Z_2 \times Z_2$ on $SO(12)$ lattice that yields 24 generations versus the $Z_2 \times Z_2$ orbifold on $SO(4)^3$ lattice that yields 48 generations. The interpretation of the chiral content of the $N = 1$ models in terms of the $N = 4$ vacua will be especially instrumental when seeking the strong coupling duals of the $N = 1$ models, due to the fact that the $N = 4$ duals can be obtained with relative ease. The understanding of the $N = 1$ duals will then entail the understanding of the corresponding $Z_2 \times Z_2$ operation on the dual side.

We discovered in this paper that the three generation free fermionic models necessarily employ an asymmetric shift on the internal compactified space. This observation has profound implications. In the first place, since the asymmetric shift can act only at enhanced symmetry points in the moduli space, it implies that some moduli are fixed and frozen. In fact in some cases it is seen that all the geometrical moduli are projected out. In those cases the geometrical moduli may be interchanged with twisted moduli which are much more difficult to identify, and hence their moduli spaces are more intricate. Additionally, the necessity of incorporating an asymmetric shift has important implications in the context of non-perturbative dualities. In the case of the duals of the heterotic models, a geometric moduli is interchanged with the dilaton. Hence, the fact that the geometric moduli are fixed around their self-dual value on the heterotic side implies that on the dual side the dilaton has to be fixed around its self-dual point. This is a fascinating possibility that we will return to in future work. However, we note that the low energy phenomenological data may point in the direction of esoteric compactifications that would have otherwise been overlooked. The results show that, in the framework of $Z_2 \times Z_2$ Calabi–Yau compactification, the net number of generation can never be equal to three. This implies the necessity of non-zero torsion in CY compactifications in order to obtain semi-realistic three generation models. Additionally, the necessity to incorporate an asymmetric shift in the reduction to three generations, has profound implications for the issues of moduli stabilization and vacuum selection. The reason being that it can only be implemented at enhanced symmetry points in the moduli space. In this context we envision that the self-dual point under T-duality plays a special role. In the context of non-perturbative dualities the dilaton and moduli are interchanged, with potentially important implications for the problem of dilaton stabilization. We will report on these aspects in future publications.

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