Dissipationless clustering of neutrinos in cosmic-string-induced wakes

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Velocity perturbations which form in the wake of long straight strings moving at a relativistic speed can generate significant large-scale structure in the Universe. With neutrinos as dark matter, a considerable fraction of all galaxies forms in large-scale sheets. Here we study the dissipationless clustering of neutrinos in cosmic-string-induced wakes using an analysis based on the Gilbert equation, i.e., taking into account the velocity dispersion of neutrinos. We compare the results with simple heuristic analyses.

I. INTRODUCTION

Recent observations indicate that there is much more structure on large scales (scales of $25h^{-1}$ Mpc and above) than was initially expected. Clusters are seen to congregate into superclusters;[1] filaments of the length larger than $50h^{-1}$ Mpc have been discovered;[2] pencil surveys have revealed voids of diameter $60h^{-1}$ Mpc;[3] and most recently a systematic redshift survey has produced evidence that a large fraction of galaxies is concentrated in sheets of length and width $50h^{-1}$ Mpc and thickness $5h^{-1}$ Mpc.[4]

These observations raise new questions for theoretical models of large-scale structure. Are there mechanisms which lead to sheetlike concentrations of galaxies, and if yes, is there a distinguished scale for these structures?

In this paper we shall show that in a model in which the energy-density perturbations are seeded by cosmic strings and in which the dark matter is hot (e.g., $100h^2$ eV neutrinos), there is an affirmative answer to both questions. This may be an important advantage of the cosmic-string model compared to other models such as the "canonical cold-dark-matter model" based on linear adiabatic density fluctuations with a flat Harrison-Zel'dovich spectrum and cold dark matter.

Let us briefly compare the model discussed in this paper, the cosmic-string model with hot dark matter, to other theories. Any cosmological model needs to specify the nature of the primordial energy-density perturbations and the characteristics of the dark matter. Nonbaryonic dark matter is either hot or cold depending on whether the thermal velocity of the dark particles at the time t_{eq} of equal matter and radiation is large or negligible. Two popular primordial energy-density perturbation spectra are linear adiabatic perturbations with a Harrison-Zel'dovich spectrum produced in an inflationary phase and nonadiabatic seed perturbations due to cosmic strings.

The current "standard model" of galaxy formation[5] is based on linear adiabatic perturbations and cold dark matter. It gives some encouraging agreement with observations on the scale of galaxies and clusters.[6] A model with adiabatic perturbations and hot dark matter, on the other hand, is very hard to reconcile with observation.[7] It predicts that galaxies should be younger than clusters, in conflict with the data. In addition, the predicted microwave-background anisotropies are barely compatible with present observational upper bounds. Hence it may be surprising that a model with hot dark matter and cosmic strings can be a viable model for galaxy formation.

Before discussing large-scale structure formation in the cosmic-string model with hot dark matter, we briefly recall why this model is viable whereas the hot-dark-matter model with purely adiabatic perturbations has severe problems.[8–10] The crucial issue is neutrino free streaming.[11] Since neutrinos have large thermal velocities, they cannot clump at early times on small scales. The comoving distance hot particles can move in one Hubble expansion time is

$$\lambda_J(t)=3v(t)tz(t)\ , \tag{1.1}$$

where $v(t)$ is the thermal velocity of the hot particles and $z(t)$ is the redshift. This distance is called the neutrino Jeans length. It attains a maximum at t_{eq} and decreases as $t^{-1/3}$ for $t>t_{eq}$. The maximal Jeans length is determined by v_{eq}, the mean velocity of the hot particles at t_{eq}. v_{eq} is determined by the neutrino mass which in turn is determined by demanding that neutrinos give the critical energy density for an $\Omega=1$ universe. The result is

$$v_{eq}=T_{\nu,eq}/m_\nu\simeq 0.05 \tag{1.2}$$

(where $T_{\nu,eq}$ is the neutrino temperature at t_{eq}), which leads to $\lambda_J(t_{eq})\simeq 6h_{50}^{-2}$ Mpc, where h_{50} is the Hubble constant in units of 50 km s^{-1} Mpc^{-1}.

In a model with hot dark matter and adiabatic density perturbations, all primordial perturbations on scales smaller than $\lambda_J(t_{eq})$ are erased. In particular, no perturbations on the scale of galaxies survive. Hence all galaxies must form by fragmentation of larger objects. However, this conflicts with observations which indicate that clusters are younger than galaxies. In addition, in order to form galaxies by a redshift 3, large-scale perturbations must go nonlinear early and must therefore have a large initial amplitude. This is only marginally consistent with the present upper bounds on microwave-background anisotropies.

With nonadiabatic seed perturbations, the above problems of the hot-dark-matter model disappear. The basic point is that the seeds survive neutrino free streaming.[8] Seeds with mean separation d start accreting matter as soon as $\lambda_J(t)$ drops below d. Hence, galaxies can form independent of clusters.

In Refs. 8–10, accretion of hot dark matter about cosmic-string loops was studied. It was found that spherical accretion leads to flat halo rotation curves. Also, assuming a one-loop–one-object correspondence, the resulting mass function of galaxies is $n(M) \sim M^{-1.5}$. These issues were studied in more detail numerically in Refs. 12 and 13. For some related work on neutrino clustering see also Ref. 14.

The scaling solution[15] describing the scale-invariant distribution of cosmic strings states that the network of cosmic strings consists of both loops and infinite strings. So far, most work on cosmic strings and hot dark matter has concentrated on the effects of loops. However, in Ref. 16 it was pointed out that wakes which form behind long, straight, rapidly moving strings can also give rise to structure. In Ref. 17 it was suggested that this mechanism may explain some of the recent observations of large-scale structure. In the context of cold dark matter, the formation of structure in cosmic-string-induced wakes was studied in detail in Ref. 18. This model was studied numerically in Ref. 19. It was discovered that accretion of matter onto wakes gives rise to planar density perturbations. The most numerous, most prominent, and most stable of these have a planar extent of about 40×40 Mpc^2. However, with cold dark matter only a small fraction of the total nonlinear mass in the Universe ends up in wakes; most clusters about small loops.

With hot dark matter, accretion onto small loops is suppressed by neutrino free streaming. Hence, it is possible that wakes are much more important. In order to address this issue it is necessary to study the accretion of neutrinos in the wakes induced by long moving strings. In this paper we study this problem and show that, provided the mass per unit length μ exceeds a limiting value

$$G\mu > 5 \times 10^{-7} , \tag{1.3}$$

the neutrino perturbations become nonlinear in the wakes. We show that in this model, a substantial fraction of all nonlinear mass ends up in wakes.

The outline of this paper is as follows. In Sec. II we explain the formation of wakes and present an analytical toy model[20] to study the induced growth of neutrino perturbations. The model is an adaptation of the Zel'dovich approximation to hot dark matter. In Sec. III we summarize an improved calculation which takes into account the fact that the initial perturbation is a velocity rather than a density perturbation and which includes the finite-velocity dispersion of the dark particles. We calculate the thickness of the wakes and the conditions on $G\mu$ which must be satisfied for any nonlinear structures to form. In Sec. IV we analyze the stability of wakes. Our conclusions are summarized in Sec. V.

Throughout the paper we use units in which $c = k_B = \hbar = 1$. G is Newton's constant, t_0 denotes the present time, t_{eq} is the time of equal matter and radiation, and $z(t)$ is the redshift at time t. h is the present Hubble expansion parameter in units of 100 $\mathrm{km\,s^{-1}\,Mpc^{-1}}$. We shall in general take $h = \frac{1}{2}$ and for convenience use $h_{50} = 2h$. $a(t)$ is the scale factor of the Universe, and for convenience we use $\Omega = 1$.

II. ACCRETION ONTO WAKES: THE ZEL'DOVICH APPROXIMATION

The network of cosmic strings at time t consists of a collection of infinite strings with mean curvature radius $c_1 t$ and mean separation $c_2 t$, where c_1 and c_2 are constants of the order 1 whose precise value must be determined in numerical simulations,[21–23] and of a distribution of loops with radii $R < t$. Here we shall focus on the effects of the infinite strings.

Since the infinite strings are approximately straight when viewed on a distance scale $\ll t$, there is no local gravitational force. However, globally, the hypersurface perpendicular to the tangent vector along the string is not a flat plane but a cone or, when unwrapped onto the plane, $\mathcal{R}^2 \setminus$(wedge). The angle of the wedge (the "missing angle") is $8\pi G\mu$ (Ref. 24) [Fig. 1(a)].

The long strings typically move at relativistic speeds and form velocity perturbations in their wake as indicated in Fig. 1(b). For $t \geq t_{\mathrm{eq}}$, the velocity perturbations will induce density perturbations. We present here an analytical toy model[20] to calculate the growth of these perturbations. The analysis is based on the Zel'dovich[25] approximation.

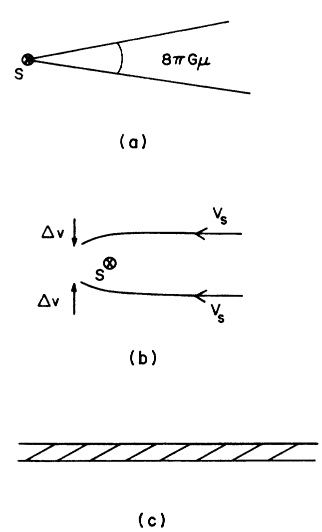

FIG. 1. (a) The deficit angle of a long straight cosmic string. (b) The formation of the velocity perturbations. (c) The induced density perturbations.

As indicated in Fig. 1(c), the velocity perturbations induce density perturbations in the wake of the moving strings. For $t > t_{eq}$, a wedge with opening angle $8\pi G\mu$ and twice the background energy density $\rho(t)$ forms. Our approximation will consist of treating the perturbation as a planar density perturbation with surface density $\sigma(t)$. For a wake formed at t_i from a string moving with velocity v_s, we have

$$\sigma(t)=4\pi G\mu t_i v_s \gamma_s \left[\frac{t}{t_i}\right]^{2/3}\rho(t) . \quad (2.1)$$

γ_s is the relativistic γ factor corresponding to v_s. ($4\pi G\mu t_i v_s \gamma_s$ is the mean thickness of the wedge.) It can be shown that this method gives the same contribution to the growing mode of density perturbations as could be obtained by setting $\sigma=0$ and using the initial displacement $\psi(t_s)=\delta v t_i z(t_i)=4\pi G\mu v_s \gamma_s t_i z(t_i)$.

The aim of the calculation is to determine if any mass goes nonlinear about the wakes (obviously a necessary condition for galaxies to form), and if the answer to this question is affirmative, to calculate the thickness of the resulting nonlinear sheets.

The calculation proceeds by considering the physical distance $h(q,t)$ of a dark-matter particle above the wake (Fig. 2). q is the initial comoving distance. Initially, $h(q,t)$ will be increasing as $a(t)$ as a consequence of the expansion of the Universe. Gradually, however, as a consequence of gravitational attraction, a comoving displacement $\psi(q,t)$ will develop:

$$h(q,t)=a(t)[q-\psi(q,t)] . \quad (2.2)$$

Eventually, the dark particle will "turn around," $\dot{h}(q,t)=0$, and start to collapse onto the wake. The thickness of the nonlinear sheet is determined by the maximal q for which "turn around" has occurred by the present time.

We analyze the evolution of $\psi(q,t)$ using the Zel'dovich approximation which means linearizing the gravitational perturbation equations in ψ and treating the source in the Newtonian limit:

$$\ddot{h}=-\nabla_h\Phi , \quad (2.3)$$

where the Newtonian potential Φ satisfies Poisson's equation

$$\nabla_h^2\Phi=4\pi G[\rho+\sigma\delta(h)] \quad (2.4)$$

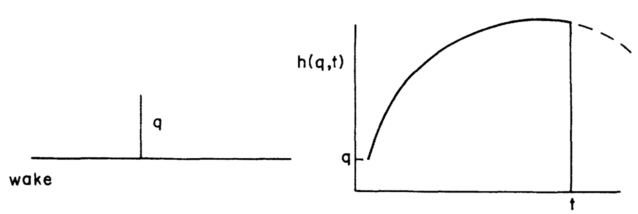

FIG. 2. The physical height of a dark particle above the wake.

(using coordinates with the center of the wake at $h=0$). The linearized equation for ψ becomes

$$\ddot{\psi}+2\frac{\dot{a}}{a}\dot{\psi}+3\frac{\ddot{a}}{a}\psi=4\pi G a^{-1}\sigma\theta(z) . \quad (2.5)$$

This approach works, strictly speaking, only for cold particles, since no thermal velocities are included. For cold particles, the initial conditions for (2.5) would be $\psi(t_i)=\dot{\psi}(t_i)=0$. We adapt the method to hot dark matter by a simple trick. On the average, the dark particles will only start to respond to the gravitational attraction caused by the wake once the neutrino Jeans length $\lambda_J(t)$ has fallen below q. We shall denote this time by $t_s(q)$. From (1.1) and introducing the variable λ_0,

$$\lambda_0=v_{eq}z_{eq}^{1/2}t_{eq} , \quad (2.6)$$

we obtain

$$z(t_s)=\left[\frac{q}{\lambda_0}\right]^2 . \quad (2.7)$$

There is a maximal q, $q_{max}(t_i)$, for which (2.7) applies: t_s cannot be earlier than the time t_i when the wake is formed. From (2.7),

$$q_{max}(t_i)=\lambda_0 z(t_i)^{1/2} \quad (2.8)$$

and $t_s(q)=t_i$ for $q>q_{max}$. Hence, for hot dark matter we shall use (2.5) with initial conditions

$$\psi(t_s(q))=\dot{\psi}(t_s(q))=0 . \quad (2.9)$$

Again the use of an explicit surface density can be avoided by using $\sigma=0$ and the nonvanishing initial displacement $\psi(t_s)=\delta v\, t_i z(t_i)(t_s/t_i)^{2/3}$. Equation (2.5) with initial conditions (2.9) can be solved by the Green's function method. We obtain

$$\psi(t)=\frac{18\pi G}{5}\sigma(t_s)\left[\frac{t_s}{t_0}\right]^{2/3}\left[\frac{t}{t_0}\right]^{2/3}t_0^2 \quad (2.10)$$

for $t \gg t_s$. From (2.2), (2.10), and (2.1) it follows that the comoving coordinate $q_{nl}(t)$ which is turning around at time t is given by

$$q_{nl}(t)=\psi_0\left[\frac{t}{t_s}\right]^{2/3} \quad (2.11)$$

with

$$\psi_0=\frac{24\pi G\mu}{5}v_s\gamma_s z(t_i)^{-1/2}t_0 . \quad (2.12)$$

For cold dark matter, $t_s(q)=t_i$, and hence we get the expected result that the thickness of the nonlinear sheet seeded by the wake grows monotonically as $a(t)$. For hot dark matter, (2.11) is an implicit equation for q_{nl} since for $q<q_{max}(t_i)$, t_s depends on q. From (2.7) and (2.11) we obtain

$$q(t)=\lambda_0\frac{\lambda_0}{\psi_0}z(t) . \quad (2.13)$$

This implies that with hot dark matter, the nonlinear structure does not grow monotonically. In fact, nothing goes nonlinear until the redshift

$$z_{\max}(t_i) \equiv z(q_{\max}(t_i)) , \tag{2.14}$$

where $z(q)$ is given by (2.13). At that time, the sheet with $q = q_{\max}(t_i)$ turns around. According to (2.13), sheets with smaller q would only turn around later. However, as soon as an outer sheet turns around, nonlinear and hydrodynamical effects will become important for smaller q's. We expect that at $z_{\max}(t_i)$ the entire layer $0 < q < q_{\max}(t_i)$ will become nonlinear. From then on, the region of nonlinearity will expand outward as described by (2.11) with $t_s = t_i > t_{eq}$. Note that for $t_i \geq t_{eq}$, $z_{\max}$ is independent of t_i. This means that the onset of nonlinearity is independent of t_i (as was already noted in Ref. 18). However, the thickness of the wakes decreases as t_i increases.

In Fig. 3 we sketch the thickness of the nonlinear sheet at a fixed time as a function of t_i. Velocity perturbations for $t < t_{eq}$ damp out due to the ambient pressure (in addition velocity perturbations in neutrinos dissipate by free streaming). Hence, long strings do not generate wakes for $t_i < t_{eq}$. Thus, the most prominent and most numerous nonlinearities were formed at t_{eq}. Their length is determined by the long string curvature radius

$$l \sim c_1 t_{eq} z_{eq} \sim c_1 40 h_{50}^{-2} \text{ Mpc} ; \tag{2.15}$$

their width is determined by the string velocity (which typically obeys $v_s \gamma_s \sim 1$)

$$w \sim c_1 v_s t_{eq} z_{eq} \sim c_1 40 h_{50}^{-2} \text{ Mpc} \tag{2.16}$$

(c_1 enters since the velocity of the long string is only coherent over the time scale $c_1 t_{eq}$) and their thickness is given by $q_{\max}$, for $G\mu$ equal to the critical value, determined by (2.19):

$$q_{\max} \sim v_{eq} t_{eq} z_{eq} \sim 2 h_{50}^{-2} \text{ Mpc} . \tag{2.17}$$

The typical separation of sheets is determined by c_2:

$$s \sim c_2 t_{eq} z_{eq} \sim c_2 40 h_{50}^{-2} \text{ Mpc} . \tag{2.18}$$

Note, in particular, that none of these dimensions depends on $G\mu$. However, the time when the nonlinear structures form does depend on $G\mu$. In order for any

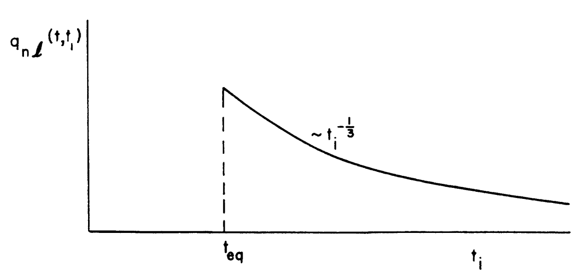

FIG. 3. The first scale that goes nonlinear at a fixed time t, as a function of the wake formation time t_i.

nonlinearities to form by redshift 1, $G\mu$ must exceed a certain bound determined by (2.14). We obtain

$$G\mu > \frac{5}{24\pi} v_{eq} (v_s \gamma_s)^{-1} z_{eq}^{-1} \sim 5 \times 10^{-7} (v_s \gamma_s)^{-1} h_{50}^{-2} . \tag{2.19}$$

This bound is consistent with the initial scenario of cluster formation with cosmic strings[26] based on the one-loop–one-cluster approximation. The compatibility of (2.19) with other constraints on the cosmic-string model will be discussed in Sec. V.

As mentioned before, the model presented in this section is incomplete in that it treats the seed perturbation as a density rather than as a velocity perturbation. It also neglects the free streaming of the particles which make up the initial wake, and neglects the velocity dispersion of the neutrinos which are accreting onto the wake. Nevertheless, most of the important physical concepts are contained in the model. In the following section we present an analysis which includes the three points mentioned above. We shall see that even quantitatively the toy model results are fairly close to the real ones.

III. AN IMPROVED ANALYSIS USING THE GILBERT EQUATION

A more exact calculation of accretion onto wakes is based on the Gilbert equation.[27,9,10] We will follow the derivation of Ref. 9. The starting point is the observation that since neutrinos interact very weakly, their phase-space density $f(\mathbf{r},\mathbf{p})$ is conserved ($\mathbf{r}$ and $\mathbf{p}$ are physical distance and momentum, respectively). Thus, $f(\mathbf{r},\mathbf{p})$ obeys the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \nabla f + \dot{\mathbf{p}} \cdot \nabla_p f = 0 . \tag{3.1}$$

In order to determine the density perturbation $\delta\rho(\mathbf{x})$ in the wake of a long straight-moving string, we solve the collisionless Boltzmann equation for the initial velocity perturbation induced by the string. The velocity perturbation corresponds to an initial phase-space density perturbation f_1. By integrating over momenta we obtain a differential equation for the energy-density perturbation $\delta\rho$. It proves useful to go to Fourier space where we eventually obtain a simple integral equation for $\delta\rho$, the Gilbert equation.

We now briefly review the derivation of the Gilbert equation and adapt its form to our context. First we note that $\dot{\mathbf{p}}$ is given by the Newtonian gravitational potential Φ,

$$\dot{\mathbf{p}} = -m \nabla \Phi , \tag{3.2}$$

where Φ obeys the Poisson equation

$$\nabla_r^2 \Phi = 4\pi G \rho . \tag{3.3}$$

The first step in the derivation is to rewrite the collisionless Boltzmann equation in comoving coordinates $\mathbf{x}$ and $\mathbf{q}$:

$$\mathbf{x} = a^{-1} \mathbf{r}, \quad \mathbf{q} = a^2 m \dot{\mathbf{x}} = a\mathbf{p} - \dot{a} m \mathbf{r} . \tag{3.4}$$

Since the equations of motion are unchanged when adding a time derivative to the Lagrangian L, we can work with

$$L' = L - \frac{d}{dt}\left(\tfrac{1}{2} m a \dot{a} \mathbf{x}^2\right) , \tag{3.5}$$

where

$$L = \tfrac{1}{2}\dot{\mathbf{r}}^2 - m\Phi(\mathbf{r},t) . \tag{3.6}$$

We can introduce an effective potential $\Phi_{\text{eff}}(\mathbf{x},t)$ by

$$L' = \tfrac{1}{2} m a^2 \dot{\mathbf{x}}^2 - m\Phi_{\text{eff}}(\mathbf{x}) . \tag{3.7}$$

From (3.5) and (3.7),

$$\Phi_{\text{eff}}(\mathbf{x}) = (\Phi + \tfrac{1}{2} a\ddot{a}\mathbf{x}^2) . \tag{3.8}$$

Hence, the collisionless Boltzmann equation in comoving coordinates is

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}}\frac{\partial f}{\partial \mathbf{x}} - m\frac{\partial \Phi_{\text{eff}}}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{q}} = 0 . \tag{3.9}$$

or, equivalently, in comoving time τ given by $d\tau = a^{-1}dt$,

$$a^{-1}\frac{\partial f}{\partial \tau} + \dot{\mathbf{x}}\frac{\partial f}{\partial \mathbf{x}} - m\frac{\partial \Phi}{\partial \mathbf{x}}\frac{\partial f}{\partial \mathbf{q}} - m a\ddot{a}\mathbf{x}\frac{\partial f}{\partial \mathbf{q}} = 0 . \tag{3.10}$$

We now consider small perturbations from a homogeneous configuration:

$$f(\mathbf{x},\mathbf{q}) = f_0(\mathbf{q}) + f_1(\mathbf{x},\mathbf{q}) \tag{3.11}$$

and

$$\Phi(\mathbf{x}) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x}) . \tag{3.12}$$

Using

$$\nabla_x \Phi_0 = \frac{4\pi}{3} G(\rho + 3p) a^2 \mathbf{x} , \tag{3.13}$$

it follows immediately that for $f_1 = \Phi_1 = 0$, (3.10) is satisfied. To first order, (3.10) becomes

$$a^{-1}\frac{\partial f_1}{\partial \tau} + a^{-2} m^{-1}\mathbf{q}\nabla_x f_1 - 4\pi G m a^2 \nabla_x^{-1}(\delta\rho + 3\delta p)\nabla_q f_0 = 0 . \tag{3.14}$$

Since for neutrinos in the matter-dominated era $\delta p \ll \delta\rho$, we shall drop δp in the following.

In Fourier space we obtain an ordinary first-order differential equation

$$a\frac{\partial \tilde{f}_1}{\partial \tau} + \frac{i\mathbf{k}\cdot\mathbf{q}}{m}\tilde{f}_1 = -4\pi G m a^4 \delta\tilde{\rho}\frac{i\mathbf{k}\cdot\nabla_q f_0}{k^2} . \tag{3.15}$$

It proves convenient to change the time variable to $\hat{\lambda}$ given by $a\,d\hat{\lambda} = d\tau$. It is also a useful trick to combine the first two terms as

$$\exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{m}\hat{\lambda}\right]\frac{d}{d\hat{\lambda}}\left[\exp\left[i\frac{\mathbf{k}\cdot\mathbf{q}}{m}\hat{\lambda}\right]\tilde{f}_1\right] . \tag{3.16}$$

We can then integrate (3.15) to obtain

$$\exp\left[i\frac{\mathbf{k}\cdot\mathbf{q}}{m}\hat{\lambda}\right]\tilde{f}_1(\hat{\lambda})\Big|_{\hat{\lambda}_i}^{\hat{\lambda}} = -4\pi G\frac{i}{k^2}\mathbf{k}\cdot\nabla_q f_0 m\int_{\hat{\lambda}_i}^{\hat{\lambda}} d\hat{\lambda}'\delta\tilde{\rho}a^4(\hat{\lambda}')\exp\left[i\frac{\mathbf{k}\cdot\mathbf{q}}{m}\hat{\lambda}'\right] . \tag{3.17}$$

When integrating over $\mathbf{q}$, (3.17) becomes an integral equation for $\delta\tilde{\rho}(\mathbf{x})$, the Gilbert equation.

The unperturbed neutrino phase-space density is

$$f_0(q) = (e^{q/q_*} + 1)^{-1} . \tag{3.18}$$

$q_* = T_\nu a$ is the characteristic value of the neutrino momentum. The comoving distance $\lambda(t)$ traveled by a neutrino in an unperturbed universe is proportional to $\hat{\lambda}$:

$$\lambda(t) = \frac{q_*}{m}\hat{\lambda}(t) . \tag{3.19}$$

In terms of λ,

$$\tilde{f}_1(\lambda) = \exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{q_*}(\lambda - \lambda_i)\right]\tilde{f}_1(\lambda_i) - 4\pi i G k^{-2}\left[\frac{m}{q_*}\right]^2\frac{\mathbf{k}\cdot\mathbf{q}}{q}\int_{\lambda_i}^{\lambda} d\lambda' a^4(\lambda')\delta\tilde{\rho}(\lambda') f_0' \exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{q_*}(\lambda - \lambda')\right] , \tag{3.20}$$

where f_0' is the derivative of f_0 with respect to its argument q/q_*.

The neutrino energy density perturbation in physical space is given by

$$\delta\rho(\mathbf{x},\lambda) = \frac{2m}{(2\pi a)^3}\int d^3q\, f_1(\mathbf{x},\mathbf{q},\lambda) . \tag{3.21}$$

The relative neutrino energy density perturbation $\delta_\nu(\mathbf{x},\lambda)$ is obtained by dividing by the neutrino background energy density:

$$\bar{\rho}_\nu = \frac{2m}{(2\pi a)^3}\int d^3q\, f_0(q) . \tag{3.22}$$

From (3.20) we obtain the integral equation for $\tilde{\delta}_\nu(\mathbf{k},\lambda)$:

$$\tilde{\delta}_\nu(\mathbf{k},\lambda) = \frac{\int d^3q \exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{q_*}(\lambda - \lambda_i)\right]\tilde{f}_1(\mathbf{k},\mathbf{q},\lambda_i)}{4\pi\int_0^\infty dq\, q^2 f_0(q)} - \frac{iG}{\pi^2}\frac{m}{k}\left[\frac{m}{q_*}\right]^2\int d\lambda' a(\lambda')\tilde{\delta}_\nu(\lambda') I(k(\lambda - \lambda')) \tag{3.23}$$

with

$$I(k(\lambda - \lambda')) = \int d^3q\frac{\mathbf{k}\cdot\mathbf{q}}{kq} f_0' \exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{q_*}(\lambda - \lambda')\right] . \tag{3.24}$$

Using an approximation which preserves the number of neutrinos,

$$f_0\left[\frac{q}{q_*}\right]\simeq\eta_3 e^{-q/q_*}\ , \tag{3.25}$$

with

$$\eta_3=\frac{1}{2}\int_0^\infty dx\, x^2(e^x+1)^{-1}\simeq 0.9\ , \tag{3.26}$$

we can evaluate I analytically (see Appendix) and obtain

$$I(\beta)=8\pi i\eta_3 q_*^3\beta\frac{1}{(1+\beta^2)^2}\equiv 8\pi i\eta_3 q_*^3 H(\beta)\ . \tag{3.27}$$

The approximation (3.25) overestimates the number of neutrinos with small momenta relative to the "hotter" ones. Consequently we slightly overestimate the number of neutrinos that fall into the wake. By evaluating (3.27) numerically for $\beta=1$ using the Fermi-Dirac distribution, we conclude that the error is smaller than 1.2 (for smaller β the error is smaller and for larger β the contribution is suppressed [see (3.33)]).

Equation (3.27) inserted into (3.23) gives the general form of the Gilbert equation. It differs from the equation derived in Ref. 9 in that there is an initial phase-space density perturbation $f_1(\mathbf{x},\mathbf{q},\lambda_i)$ but no external density perturbation. Thus (3.23) applies in the case of initial velocity perturbations.

Let us now consider the specific case of velocity perturbations due to string wakes. We will restrict ourselves to perturbations close enough to the wake so that curvature as well as acceleration or deceleration of the long strings are not important. In this case the string may be approximated as sweeping out a plane (which we take to be the x-y plane) as it moves with constant velocity v_s. It is sufficient to assume that the neutrinos remain stationary as the string passes by. In this approximation each neutrino is given a velocity impulse

$$\delta\mathbf{v}=-4\pi G\mu v_s\gamma_s\,\mathrm{sgn}(z)\hat{z} \tag{3.28}$$

toward the plane of the wake. $\gamma_s\equiv(1-v_s^2)^{-1/2}$ is the usual relativistic γ factor. Hence, a long string at time t_i induces a comoving momentum perturbation

$$\delta\mathbf{q}=a(t_i)m\,\delta\mathbf{v}\ . \tag{3.29}$$

Thus, the initial phase-space density perturbation $f_1(\mathbf{x},\mathbf{q},\lambda_i)$ is

$$f_1(\mathbf{x},\mathbf{q},\lambda_i)=\frac{1}{q_*}\frac{\mathbf{q}\cdot\delta\mathbf{q}}{q}f_0'\left[\frac{q}{q_*}\right] \tag{3.30}$$

or, Fourier transforming,

$$\tilde{f}_1(\mathbf{k},\mathbf{q},\lambda_i)=\frac{1}{q_*}f_0'\left[\frac{q}{q_*}\right](2\pi)^{1/2}\delta(k_x)\delta(k_y)\frac{q_z}{q}\left[2\pi\delta(k_z)-\frac{2i}{k_z}\right]\delta q\ , \tag{3.31}$$

where δq is the magnitude of $\delta\mathbf{q}$.

Since $I(0)=0$ we may ignore the first term in (3.31) and evaluate the q integral in (3.23) which we denote by $\tilde{J}(\mathbf{k},\lambda)$:

$$\tilde{J}(\mathbf{k},\lambda)=-\frac{1}{8\pi\eta_3 q_*^3}(2\pi)^{1/2}\delta(k_x)\delta(k_y)2i\frac{\delta q}{q_*}\frac{1}{k_z}I(k_z(\lambda-\lambda_i))\ . \tag{3.32}$$

Combining (3.23), (3.27), and (3.32) we finally obtain

$$\tilde{\delta}_\nu(\lambda)=2(2\pi)^{1/2}\frac{\delta q}{q_*}\delta(k_x)\delta(k_y)\frac{1}{k_z}H(k_z(\lambda-\lambda_i))+\frac{8}{\pi}Gm^3 q_*\eta_3\frac{1}{k}\int d\lambda' a(\lambda')\tilde{\delta}(\lambda')H(k(\lambda-\lambda'))\ . \tag{3.33}$$

Obviously, no modes with $k_x\neq 0$ or $k_y\neq 0$ get excited. Hence we define $\tilde{\delta}(k_z,\lambda)$ by

$$\tilde{\delta}_\nu(\mathbf{k},\lambda)=2\pi\delta(k_x)\delta(k_y)\tilde{\delta}(k_z,\lambda)\ . \tag{3.34}$$

In order to eliminate some of the constants from (3.33) it proves convenient to introduce as the time variable the rescaled conformal time ξ,

$$\xi=\frac{\tau}{4\tau_*}\quad\text{with } \tau_*=\left[\frac{8\pi G}{3}\rho_{\nu,\mathrm{eq}}\right]^{-1/2}, \tag{3.35}$$

and to normalize the scale factor at t_{eq}:

$$a(t_{\mathrm{eq}})=1\ . \tag{3.36}$$

$\lambda=(q_*/m)\hat{\lambda}$ can be expressed in terms of ξ, taking into account the smooth transition between the radiation-dominated era to the matter-dominated phase. From Ref. 9 we have

$$\lambda-\lambda'=\frac{q_*\tau_*}{m}F(\xi',\xi) \tag{3.37}$$

with

$$F(\xi',\xi)=\ln\left[1+\frac{1}{\xi'}\right]-\ln\left[1+\frac{1}{\xi}\right]\ . \tag{3.38}$$

Finally, combining (3.33), (3.34), and (3.37) we obtain

$$\tilde{\delta}(k_z,\xi)=\frac{2}{(2\pi)^{1/2}}\delta v\,\tau_* a(t_i)g(k_z,\xi_i,\xi)+6\int_{\xi_i}^{\xi}d\xi'\tilde{\delta}(k_z,\xi')g(k_z,\xi',\xi) \tag{3.39}$$

with

$$g(k_z,\xi',\xi)=\frac{F(\xi',\xi)}{\left[1+\left[k_z\frac{q_*\tau_*}{m}\right]^2F(\xi',\xi)^2\right]^2}\ . \tag{3.40}$$

Equations (3.39) and (3.40) will now be applied to study the growth of nonlinear structures seeded by wakes. From the denominator in (3.40) it follows that modes with

$$k_z \frac{q_* \tau_*}{m} \equiv k_z L \equiv \alpha(k_z) > 1 \qquad (3.41)$$

will initially be heavily damped. This is a consequence of neutrino free streaming. The comoving length scale L hence plays the role of the maximal neutrino Jeans length for our problem.

Equation (3.39) can be solved numerically. With $c_0 = 2(2\pi)^{-1/2}\delta v \tau_* a(t_i)$, and $\xi \gg 1$ the solution can be fit to

$$\tilde{\delta}(k_z,\xi) = c_0 \xi^2 \frac{A(\xi_i)}{B(\xi_i) + \alpha(k_z)^{n(\xi_i)}} . \qquad (3.42)$$

For $10^{-2} < \xi_i < 10$, $n(\xi_i) = 4$. Note that this power is different from the value $n=2$ obtained using a planar stationary density perturbation.[9] The reason for this difference is that any perturbation which builds up in the center of the wake is itself subject to free streaming. In Fig. 4 we plot the ξ_i dependence of the constants A and B.

We can now calculate the height h of the nonlinear sheet seeded by the wake. In linear perturbation theory h is determined by

$$\delta M(h)/M(h) = 1 , \qquad (3.43)$$

where $\delta M(h)$ is the mass perturbation per unit area a distance h from the center of the wake:

$$\begin{aligned} \frac{\delta M(h)}{M(h)}(\xi) &= \int_{-h}^{h} dh' \delta_\nu(h',\xi) \frac{1}{2h} = (2\pi)^{-1/2} \int_{-h}^{h} dh' \int_{-\infty}^{\infty} dk_z e^{ik_z h'} \tilde{\delta}(k_z,\xi) \frac{1}{2h} \\ &= (2\pi)^{-1/2} c_0 \xi^2 A(\xi_i) \int_{-h}^{h} dh' \int_{-\infty}^{\infty} dk_z e^{ik_z h'} \frac{1}{B(\xi_i) + \alpha(k_z)^{n(\xi_i)}} \frac{1}{2h} \\ &= (2\pi)^{-1/2} c_0 \xi^2 \frac{A(\xi_i)}{B(\xi_i)} I(h,\xi_i) \frac{1}{h} , \end{aligned} \qquad (3.44)$$

where

$$I(h,\xi_i) = \int_{-\infty}^{\infty} dk_z \frac{\sin k_z h}{k_z} \left[1 + \frac{\alpha(k_z)^{n(\xi_i)}}{B(\xi_i)} \right]^{-1} . \qquad (3.45)$$

The condition for anything to go nonlinear by the present time is

$$\lim_{h \to 0} \frac{\delta M}{M}(h, \xi^2 = \tfrac{1}{4} z_{eq}) = 1 \qquad (3.46)$$

for wakes formed at t_{eq}. With v_{eq} defined by

$$T_\nu(t_{eq}) = v_{eq} m \qquad (3.47)$$

(this gives $v_{eq} \simeq 0.05$) we obtain the condition

$$\begin{aligned} &\frac{\pi}{\sqrt{2}} v_s \gamma_s G\mu z_{eq} v_{eq}^{-1} \frac{A(\xi_i)}{B(\xi_i)^{1-1/n}} a(\xi_i) \\ &\qquad \equiv \frac{\pi}{\sqrt{2}} v_s \gamma_s G\mu z_{eq} v_{eq}^{-1} f(\xi_i) > 1 \end{aligned} \qquad (3.48)$$

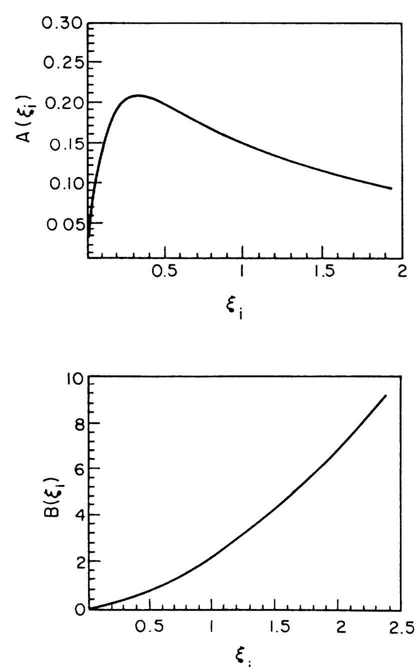

FIG. 4. The ξ_i dependence of the constants A and B.

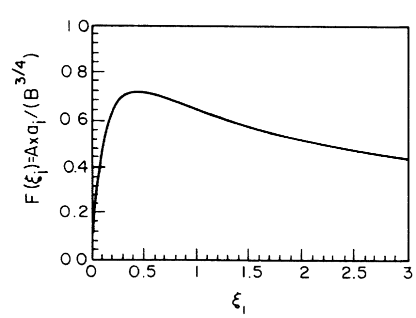

FIG. 5. The ξ_i dependence of the function $f(\xi_i) = A(\xi_i) a_i / B(\xi_i)^{3/4}$.

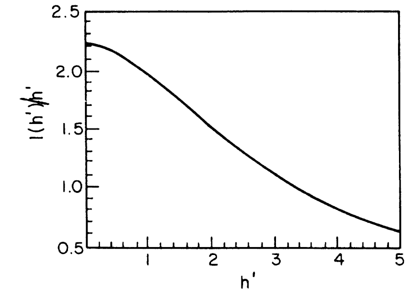

FIG. 6. The h dependence of the function $I(h,\xi_i)/h$. To avoid picking a specific ξ_i in this graph we have used a rescaled version of h: $h'=(h/L)B(\xi_i)$. This rescaling factors out the ξ_i dependence of $I(h,\xi_i)/h$ and allows this dependence to be included in $f(\xi_i)$.

or

$$G\mu > \frac{\sqrt{2}}{\pi}(v_s\gamma_s)^{-1}v_{\rm eq}z_{\rm eq}^{-1}f(\xi_i)^{-1}$$
$$\simeq 2(v_s\gamma_s)^{-1}h_{50}^{-2}f(\xi_i)^{-1}10^{-6} \ . \quad (3.49)$$

In Fig. 5 we plot $f(\xi_i)$ as a function of ξ_i. As predicted, $f(\xi_i)$ peaks for $\xi_i \simeq 0.25 \equiv \xi_{\rm eq}$ for $t_i \simeq t_{\rm eq}$.

From (3.49) we see that long strings seed nonlinear structures only if $G\mu$ is sufficiently large:

$$G\mu > 2\times 10^{-6}h_{50}^{-2} \ . \quad (3.50)$$

If this condition is satisfied, then nonlinearities form from wakes with $\xi_i \simeq \xi_{\rm eq}$. From (3.45) it follows that their thickness at the time the nonlinearity first appears rapidly grows to

$$h \sim L \sim 2h_{50}^{-2}\ \text{Mpc} \ . \quad (3.51)$$

To render this statement more precise we have evaluated (3.45) numerically. In Fig. 6 we plot $I(h)h^{-1}$. Using (3.44) we can obtain the time $t(h)$ when the nonlinear structure extends to height h. We can also combine Fig. 6 and (3.44) to obtain the thickness of a wake at a fixed time as a function of ξ_i (Fig. 7). This gives us the range of sizes of the sheets which we predict in the cosmic string model with hot dark matter. The bound (3.50) is stricter than the bound obtained by the modified Zel'dovich approximation by a factor of about 4. This may be the effect of the free streaming of the initial perturbation. For $h_{50}=1$ this lower bound is uncomfortably close to the upper bounds from the microwave background anisotropies[28] and gravitational radiation[29]

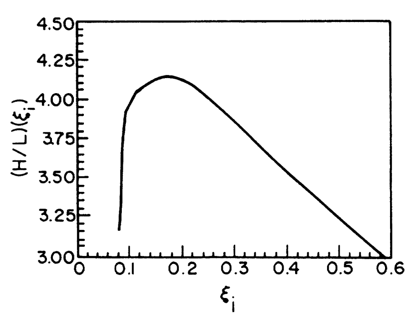

FIG. 7. The ξ_i dependence of the wake thickness for $G\mu=2(G\mu)_c$, normalized by the maximum free streaming length L.

IV. STABILITY OF WAKES

We have seen in the previous section that wakes produced at about $t_{\rm eq}$ are the thickest and also most numerous ones. Their planar dimension is about $40h_{50}^{-1}$ Mpc, the comoving horizon at $t_{\rm eq}$. Here we shall demonstrate that these wakes are stable towards disruption by loops.

To analyze the stability of wakes toward the disruptive effect of close loops we can compare the comoving displacement of a test particle an initial (comoving) distance q above the wake toward the wake (from Sec. III) and toward the loop (from Ref. 9). For q we choose the edge of the wake h determined by $\delta M/M=1$, and hence the displacement toward the wake is

$$\delta h = h = \lambda_J(t_{\rm eq}) = v_{\rm eq}t_0 z_{\rm eq}^{-1/2} \ . \quad (4.1)$$

The displacement toward a loop of radius R a distance $d(R)$ from the test particle δd, can also be determined in linear theory. The linear perturbation theory growth of the mass about a loop viewed from distance $d(R)$ is

$$\delta M = \beta\mu R z(t_R) \ , \quad (4.2)$$

where $z(t_R)$ is the redshift when $\lambda_J(t)=d$, when the loop is formed, or $z_{\rm eq}$, whichever is smaller. Then, δM is given by

$$\delta M = 4\pi\rho d(R)^2\delta d \ . \quad (4.3)$$

Hence,

$$\delta d = \tfrac{3}{2}\beta G\mu z(t_R)\frac{Rt_0^2}{d(R)^2} \ . \quad (4.4)$$

For $d(R)$ we use the mean separation of loops at radius R. The scaling density of loops is[15]

$$n(R,t_0) = \nu R^{-2}t_0^{-2} \quad (4.5)$$

for loops created after $t_{\rm eq}$. ν is a constant whose value is uncertain by at least a factor of 10:[21–23]

$$d(R) = (n(R,t_0)R)^{-1/3} = \nu^{-1/3}R^{1/3}t_0^{2/3} \ . \quad (4.6)$$

For loops formed after $t_{\rm eq}$, $z(t_R)=t_0^{2/3}R^{-2/3}$ and hence

$$\delta d = \tfrac{3}{2}\beta G\mu\nu^{+2/3}R^{-1/3}t_0^{4/3} \ . \quad (4.7)$$

For loops formed before $t_{\rm eq}$, δd is largest for the largest R, namely, $R \sim t_{\rm eq}$. For these

$$\frac{\delta d}{\delta h} \sim 10^{-1}\beta_{10}(G\mu)_6\nu_{-2}^{2/3}v_{{\rm eq}_{0.05}}^{-1}h_{50}^{2} \ , \quad (4.8)$$

where β_{10} is β in units of 10, $(G\mu)_6$ is $G\mu$ in units of 10^{-6},

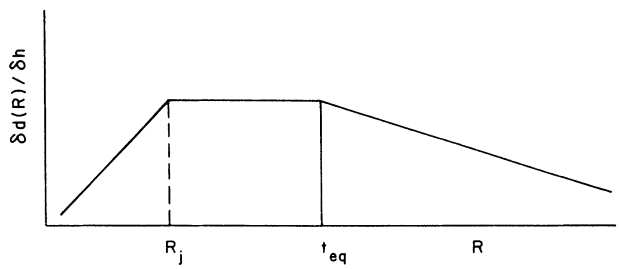

FIG. 8. The ratio of displacements toward the wake (δh) and toward the closest loop of radius $R(\delta d(R))$. The amplitude A is given in (4.8).

ν_{-2} is ν in units of 10^{-2}, and $v_{\mathrm{eq}_{0.05}}$ is v_{eq} in units of 0.05. We conclude that for the "standard" values of the string parameters, wakes are stable toward the disruptive effects of large loops.

For loops formed before t_{eq} (but while their mean separation at the time of formation is larger than the Jeans length at the t_{eq}), $z(t_R)=z_{\mathrm{eq}}$, $d(R)\sim R^{1/2}$, and hence $\delta d/\delta h$ is independent of R. Thus, small loops do not have any disruptive effects on wakes, either. In fact, for very small R [so small that $d(R)$ is smaller than λ_J at t_{eq}], $z(t_R)$ starts to decrease as R decreases: using $d(R)$ for loops created in the radiation-dominated era and the time dependence of $\lambda_J(t)$ we find

$$z(t_R)=z_{\mathrm{eq}}R\nu^{-2/3}t_{\mathrm{eq}}^{-1/3}t_0^{4/3}L^{-2}\sim R\ . \tag{4.9}$$

Thus, very small loops have a negligible effect on wake disruption. In Fig. 8 we plot $\delta d(R)/\delta h$ as function R.

An equivalent way to study disruption would be to compare the forces toward the wake and toward the loop at a distance h from the wake and $d(R)$ from the loop. The displacement toward the loop could also be calculated using the Zel'dovich approximation. We have done this calculation and the result is essentially identical.

V. DISCUSSION AND CONCLUSIONS

We have studied the accretion of hot dark matter in wakes triggered by the velocity perturbations due to long moving strings. We find that, provided $G\mu$ exceeds a lower bound which is about 10^{-6} [the exact formula is given in (3.49)], nonlinear sheet perturbations form which will correspond to regions of high galaxy number density. Long strings present at t_{eq} give rise to the thickest wakes. Strings at a much later or much earlier time do not have the strength to produce nonlinear structures. Hence, there is a distinguished scale for these sheets, namely, the correlation length of the string at t_{eq} which is roughly the comoving horizon at t_{eq}, $40h_{50}^{-1}$ Mpc.

Thus, we predict that (provided $G\mu$ exceeds the lower bound mentioned above) wakes will give rise to a network of overdense sheets of galaxies. The mean planar size and separation of the wakes can be determined by the results from the cosmic-string evolution simulations. These indicate that the scaling solution is characterized by a correlation length of about $\frac{1}{3}$ of the horizon.[22,23] It would be very interesting to compare the detailed statistics of sizes and shapes of the wakes with those of the observed large-scale structures in the Universe.

Our calculations are based on the Gilbert equation, an equation for the energy density of neutrinos which follows from the Boltzmann equation for collisionless particles. This approach correctly describes the initial perturbation in the wake of a long moving string as a velocity perturbation. It also takes into account the velocity dispersion of neutrinos. However, as we demonstrate in this paper, most of the results are described correctly using a naive model based on a modification of the Zel'dovich approximation.[20]

An important result of our analysis is that, provided $G\mu$ exceeds the lower bound mentioned above, the resulting wakes in the hot dark matter are very similar, both in planar extent and thickness, to those in the cold-dark-matter model. Since in the cold-dark-matter model wakes forming from strings at $t\gg t_{\mathrm{eq}}$ are unstable to disruption, even the statistics will be similar. However, the relative importance of loops and wakes is very different in the two models. If we assume a scaling solution for the distribution of loops, then we can compare the energy density ρ_w of wakes and ρ_v of matter accreted onto loops. We shall parametrize the scaling solution as

$$\rho_\infty(t)=c\mu t^{-2}\ , \tag{5.1}$$

where ρ_∞ is the energy density in long strings and c is a constant, and

$$n(R,t)=\nu R^{-5/2}t_{\mathrm{eq}}^{1/2}t^{-2} \tag{5.2}$$

for loops produced before z_{eq}. ν is given by the level c of the scaling solution and by the mean radius of the loop R_f at the time of its formation t_f. Assuming a delta function loop produced function with $R_f=\alpha t_f$ we get

$$\nu=\tfrac{1}{2}c\beta^{-1}\alpha^{1/2}\ . \tag{5.3}$$

The energy density ρ_w in wakes can be determined from the equations in Sec. II. For hot dark matter we obtain

$$\rho_w(t_0)=c\xi v_s\gamma_s\tfrac{8}{15}\mu z_{\mathrm{eq}}t_0^{-2}\ , \tag{5.4}$$

where the correlation length of an infinite string is $\xi(t)\equiv\xi t$. The energy density ρ_l has been determined in Ref. 20:

$$\rho_l(t_0)=\tfrac{16}{15}\nu\beta\mu v_{\mathrm{eq}}^{-3/2}z_{\mathrm{eq}}^{3/2}(\tfrac{9}{5}\beta G\mu)^{1/2}t_0^{-2}\ , \tag{5.5}$$

where βR is the mean length of a loop with radius R. Thus

$$\frac{\rho_w}{\rho_l}=\xi v_s\gamma_s\alpha^{-1/2}v_{\mathrm{eq}}^{3/2}z_{\mathrm{eq}}^{-1/2}(\tfrac{9}{5}\beta G\mu)^{-1/2}\ . \tag{5.6}$$

With $v_s\gamma_s=1$, $\alpha=10^{-2}\alpha_2$, $\beta=10$, and $v_{\mathrm{eq}}=0.05$ we obtain

$$\frac{\rho_w}{\rho_l}\simeq\frac{\xi}{5}\alpha_2^{-1/2}h_{50}^{-1}(G\mu)_6^{-1/2}\ . \tag{5.7}$$

For cold dark matter, ρ_w is similar to (5.4). ρ_l, however, is much larger,

$$\rho_l(t_0)=c\alpha^{1/2}(\gamma G\mu)^{-1/2}z_{\rm eq}\mu t_0^{-2} , \tag{5.8}$$

and hence

$$\frac{\rho_w}{\rho_l}=\tfrac{8}{15}\xi v_s\gamma_s\alpha^{-1/2}(\gamma G\mu)^{1/2}$$
$$\simeq \xi\alpha_2^{-1/2}(G\mu)_6^{1/2}\tfrac{1}{25} , \tag{5.9}$$

where $\gamma \simeq 50$ (Ref. 30).

We conclude that with hot dark matter, the relative contribution of the energy density in nonlinear matter from wakes is larger by a factor of about 10 than with cold dark matter. According to the first numerical simulations by Albrecht and Turok,[21] $c\sim 1$ and $\alpha\sim 1$. Therefore, with cold dark matter wakes would have been unimportant for structure formation compared to wakes.[18] Even with hot dark matter, only a small fraction of the mass would have been accreted onto wakes.

At the present time all cosmic-string simulations agree with $c \gg 1$ and $\alpha \ll 1$ (if there is a scaling solution at all). Hence, wakes are much more important. Evidence from the Bennett and Bouchet[22] and Allen and Shellard[23] simulations suggest that $\alpha < 10^{-2}$. Bennett and Bouchet[31] in fact find no evidence that α is above the gravitational cutoff limit $\gamma G\mu\sim 10^{-4}$.

To be specific we shall use $\alpha=10^{-2}$. In this case, with hot-dark-matter wakes are predicted to give about half the nonlinear mass, whereas with cold dark matter the contributions from loops dominates. For $\alpha \ll 10^{-2}$, loops are unimportant both with hot and cold dark matter.

For $c\sim 10$ (Refs. 22 and 23), a large fraction of the mass of the Universe accretes onto wakes. To get significant bias, i.e., Ω in wakes $\ll 1$, we need $G\mu < 10^{-6}$ or $\xi < 1$.

The cosmic-string model with hot dark matter might run into problems in explaining galaxy formation. If there is no loop scaling solution, then it might be difficult to form galaxies isolated from the large-scale structure, at least in the one-loop–one-object scenario. However, galaxies might form from concentrations of small loops in a way suggested in Ref. 32. We note that if all galaxies form by fragmentation of the wakes, we may recover the age problem of the standard hot-dark-matter model.

If there is a scaling solution of loops with $\alpha\sim 10^{-2}$, it might still be difficult to get sufficient power on galaxy scales due to loop motion[33] and rocket effect.[34] These issues deserve further study.

We have not incorporated baryons into our analysis. This will be crucial in order to understand the fragmentation of wakes. Dark baryon perturbations created in these wakes may allow the formation of dwarf galaxies with dark haloes having a high phase-space density (see, e.g., Ref. 35).

The main upshot of this analysis is that the cosmic-string model with hot dark matter produces a lot of structure on large scales and seems to fit large-scale structure data better than the standard cold-dark-matter model. However the model has these nice features only in a very narrow range of values of $G\mu$. If $G\mu \le 2\times 10^{-6}$, then no nonlinear structures form from wakes, but if $G\mu \ge 5\times 10^{-6}$, then observable step discontinuities in the microwave background are predicted,[28] and if $G\mu \ge 8\times 10^{-6}$, then cosmic strings would give observable timing residuals for the millisecond pulsar.[29]

APPENDIX: EVALUATION OF THE INTEGRAL $I(k(\lambda-\lambda'))$

Here we analytically evaluate the integral $I(k(\lambda-\lambda'))$ of (3.24):

$$I(k(\lambda-\lambda'))=\int d^3q \exp\left[-i\frac{\mathbf{k}\cdot\mathbf{q}}{q_*}(\lambda-\lambda')\right]\frac{\mathbf{k}\cdot\mathbf{q}}{k_q}f_0'(q) . \tag{A1}$$

Let θ be the angle spanned by the vectors $\mathbf{k}$ and $\mathbf{q}$. Then

$$I(k(\lambda-\lambda'))=2\pi\int_0^\infty dq\, q^2 f_0'(q)\int_{-1}^{+1} d\cos\theta \cos\theta \exp\left[-i\frac{kq}{q_*}(\lambda-\lambda')\cos\theta\right] . \tag{A2}$$

With $u\equiv\cos\theta$ and $w\equiv(kq/q_*)(\lambda-\lambda')$ this becomes

$$I(k(\lambda-\lambda'))=2\pi\left[\frac{q_*}{k(\lambda-\lambda')}\right]^3\int_0^\infty d\omega\,\omega^2 f_0'\left[\frac{q_*\omega}{k(\lambda-\lambda')}\right]\int_{-1}^{+1}du\, ue^{-i\omega u} . \tag{A3}$$

We can evaluate the u integral explicitly:

$$\int_{-1}^{+1}du\, ue^{-i\omega u}=-i\int_{-1}^{+1}du\, u\sin\omega u=2i\omega^{-2}(\omega\cos\omega-\sin\omega) . \tag{A4}$$

Thus

$$I(k(\lambda-\lambda'))=4\pi i\left[\frac{q_*}{k(\lambda-\lambda')}\right]^3\int_0^\infty d\omega(\omega\cos\omega-\sin\omega)f_0'\left[\frac{q_*\omega}{k(\lambda-\lambda')}\right] . \tag{A5}$$

Using the approximation (3.25) for $f_0(q)$ we obtain

$$I(k(\lambda-\lambda'))=-4\pi i\eta_3\left[\frac{q_*}{k(\lambda-\lambda')}\right]^3\int_0^\infty d\omega(\omega\cos\omega-\sin\omega)e^{-\omega/k(\lambda-\lambda')} . \tag{A6}$$

Let us denote the integrand by B. To evaluate B we reduce it to a combination of terms of the form

$$B_1=\int_0^{\infty}dx\, xe^{ax}=\frac{1}{a^2}\ , \tag{A7}$$

$$B_2=\int_0^{\infty}dx\, e^{ax}=\frac{1}{a}\ , \tag{A8}$$

where $\mathcal{R}(a)\leq 0$ was assumed. In our case

$$a_{\pm}=\frac{-1}{k(\lambda-\lambda')}\pm i\ . \tag{A9}$$

Thus

$$I(k(\lambda-\lambda'))=-4\pi i\eta_3\left[\frac{q_*}{k(\lambda-\lambda')}\right]^3\left[\frac{1}{2}\left[\frac{1}{a_+^2}+\frac{1}{a_-^2}\right]-\frac{1}{2i}\left[-\frac{1}{a_+}+\frac{1}{a_-}\right]\right]. \tag{A10}$$

Combining the coefficients involving $a_{\pm}$ we find

$$I(k(\lambda-\lambda'))=8\pi i q_*^3\eta_3 H(k(\lambda-\lambda')) \tag{A11}$$

with

$$H(x)=\frac{x}{(1+x^2)^2}\ . \tag{A12}$$

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