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Dynamical simulations of semilocal strings

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We explore the stability of semilocal string numerically. Our results confirm the conclusions of Hindmarsh and also show a range of asymptotic field configurations that will relax to the semilocal string configuration rather than to the trivial vacuum. We also study field configurations corresponding to two semilocal strings with different colours and find that the colour rapidly equilibrates and has little effect on the dynamics of the strings. This result and its limitations are discussed and implications for the evolution of the semilocal string network are drawn. In addition, we construct models that contain monopoles labelled by colour indices.

1. Introduction

In a recent paper [1], it was shown that the simultaneous presence of global and gauge symmetries in a field theory can result in the formation of a stable string-like defects even when the first homotopy group of the vacuum is trivial. This new variety of strings was called "semilocal". In the present paper we shall study the properties of semilocal strings in more detail and offer additional insight into the existence and formation of these defects.

Specifically, in ref. [1], the generalization of the abelian Higgs model to an $SU(2)_{\text{global}} \times U(1)_{\text{local}}$ symmetry was considered. The Higgs mechanism broke this symmetry down to $U(1)_{\text{global}}$. The vacuum manifold V, which is the coset space $SU(2)_{\text{global}} \times U(1)_{\text{local}}/U(1)_{\text{global}}$ is simply connected. Nevertheless, this model was shown to admit vortex solutions which were found to be stable [2] for $\beta \leqslant 1$ where β is the ratio of the square of the scalar to the vector mass. (For $\beta = 1$ there is a one-parameter family of configurations with the same energy and varying core radius).

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Semilocal strings have two properties that make them unusual and interesting:

(i) If V is the vacuum manifold then $\pi_1(V) = 0$ and hence, semilocal strings provide a counterexample to the usual criterion for the formation of string-like defects. In addition, since the strings are stable only for $\beta \leqslant 1$, it is clear that topology is not sufficient to determine the stability of the string solution (which exists for all values of β). Rather, the stability of the semilocal string is a dynamical question.

(ii) Each string is labelled by continuous "colour" parameters. While the colour of an isolated string can be freely chosen, the relative colour of two strings is fixed. Two strings of different colour may have a long-range interaction via the "colour" field.

An added motivation for the study of semilocal strings comes from the recent discovery [3] that the string solution exists even when the $SU(2)_{global}$ group is gauged. But, in this case, the semilocal model is precisely the bosonic sector of the Weinberg–Salam model! The stability of the strings in the Weinberg–Salam model is currently under investigation by two of us [4].

In this paper, after reviewing the semilocal string solution, we numerically study the formation of a single string for a range of the parameter β. Our approach is different from that of Hindmarsh [2]: rather than looking at the perturbation equations, we lay down a configuration of fields that is close to the semilocal string solution and simulate its evolution numerically to watch it evolve into or dissipate away from the semilocal string solution. Our results confirm the findings in ref. [2] that the string solutions are stable for $\beta < 1$ and are unstable for $\beta > 1$. An advantage of our method is that we can also study the dynamics of a decaying string solution or that of a string while it is forming. In fact, we have set up a class of initial conditions that are at different "distances" (in functional space) from the semilocal string configuration. This allows us to get an idea of the range of initial conditions that will lead to strings. This study is essential if we are interested in the cosmology of semilocal strings since the formation of a network of semilocal strings is likely to be much more complicated than the formation of ordinary U(1) strings [5].

In a cosmological setting, we expect some sort of network of semilocal strings to form at a suitable phase transition [6]; different strings in the network will, in general, have different colours. The question then is: what effect do colour gradients have on the evolution of the semilocal string network. We shall take a first step towards answering this question when we study the interaction of two strings with different colour. Our results are negative in the sense that the colour gradients smooth out very rapidly and have little effect on the dynamics of the strings themselves. However, we cannot conclude from this that the evolution of the semilocal string network will be similar to that of ordinary strings: the fact that semilocal strings can terminate is likely to be a crucial difference, and will have to be studied through three-dimensional simulations.

In ref. [1], it was shown under some assumptions that semilocal monopoles cannot exist. However, it is still of some interest to see whether we can construct monopoles that are "coloured" in the same way that semilocal strings are coloured. We give some examples of field-theoretic models containing coloured monopoles in sect. 4. Here we also note that if the colour interaction between monopoles is important for their dynamics then it could affect the annihilation rate of primordial monopoles.

2. Semilocal string review

Let us briefly review the Nielsen–Olesen [7] string solutions. They are cylindrically symmetric, topologically stable solutions of the abelian Higgs model. This model is described by the lagrangian density

$$L_{\mathrm{AH}} = \left|\left(\partial_\mu + \tfrac{1}{2}ieA_\mu\right)\phi\right|^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda\left(\phi^*\phi - \tfrac{1}{2}\eta^2\right)^2. \tag{2.1}$$

For static solutions where the string lies on the z-axis, the energy functional obtained from the lagrangian (2.1) is

$$E_{\mathrm{AH}} = \int \mathrm{d}^2x\left[\tfrac{1}{4}\left(\partial_i A_j - \partial_j A_i\right)^2 + \left|\left(\partial_j + \tfrac{1}{2}ieA_j\right)\phi\right|^2 + \lambda\left(\phi^*\phi - \tfrac{1}{2}\eta^2\right)^2\right]. \tag{2.2}$$

The Nielsen–Olesen solution that extremizes the energy functional (2.2) is of the form

$$\phi_{\mathrm{NO}} = f_{\mathrm{NO}}(r)\,\mathrm{e}^{im\theta}, \qquad A_{\mathrm{NO}} = -\frac{v(r)}{r}\hat{e}_\theta, \tag{2.3}$$

where $(r,\ \theta)$ are cylindrical coordinates in the xy-plane and m is the winding number of the string which we will henceforth take to be one. For $m=1$, the Nielsen–Olesen solution is a minimum of the energy for any value of λ. The functions $f(r)$ and $v(r)$, are not known in closed form and are solutions of the following differential equations:

$$f'' + \frac{f'}{r} - \left(1 - \tfrac{1}{2}ev\right)^2\frac{f}{r^2} - 2\lambda\left(f^2 - \tfrac{1}{2}\eta^2\right)f = 0, \tag{2.4}$$

$$v'' - \frac{v'}{r} + e\left(1 - \tfrac{1}{2}ev\right)f^2 = 0, \tag{2.5}$$

with boundary conditions

$$f(0) = 0 = v(0), \qquad f(\infty) = \frac{\eta}{\sqrt{2}}, \qquad v(\infty) = \frac{2}{e} \tag{2.6}$$

(primes denote differential with respect to r). These solutions have been the focus of extensive investigation in the literature. Some of the relevant papers are given in ref. [8].

Following Bogomol'nyi [9], we rescale fields and coordinates as follows:

$$\bar{x}^{\mu} = \frac{e\eta}{2\sqrt{2}} x^{\mu}, \qquad \bar{\phi} = \frac{\sqrt{2}}{\eta}\phi, \qquad \bar{A}_{\mu} = \frac{\sqrt{2}}{\eta} A_{\mu}. \tag{2.7}$$

With this rescaling the number of parameters in the model is reduced to one:

$$\beta \equiv \frac{m_{\mathrm{s}}^2}{m_{\mathrm{v}}^2} = \frac{8\lambda}{e^2}, \tag{2.8}$$

where the mass of the scalar field is given by $m_{\mathrm{s}}^2 = 2\lambda\eta^2$ and the mass of the vector field by $m_{\mathrm{v}}^2 = e^2\eta^2/4$.

The semilocal string [1] may be viewed as a generalization of the Nielsen–Olesen string. The semilocal model, in terms of the rescaled, dimensionless variables and coordinates, is given by the lagrangian

$$L_{\mathrm{sl}} = \left(\frac{e\eta^2}{4}\right)^2 \left[\left|(\partial_{\mu} + iA_{\mu})\mathbb{1}\boldsymbol{\Phi}\right|^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\beta(\boldsymbol{\Phi}^{\dagger}\boldsymbol{\Phi} - 1)^2\right], \tag{2.11}$$

where we have dropped the bars over the coordinates and variables for notational simplicity. Notice that $\boldsymbol{\Phi}$ is now a *complex doublet* of fields and $\mathbb{1}$ is the 2×2 unit matrix. The symmetry breaking described by the lagrangian (2.11) is $\mathrm{SU}(2)_{\mathrm{gl}} \times \mathrm{U}(1)_{\mathrm{l}} \to \mathrm{U}(1)_{\mathrm{gl}}$ and the vacuum manifold is $\mathrm{SU}(2)_{\mathrm{gl}} \times \mathrm{U}(1)_{\mathrm{l}}/\mathrm{U}(1)_{\mathrm{gl}} = \mathrm{S}^3$ with a distinguished gauged S^1 embedded in it. Since $\pi_1(\mathrm{S}^3) = 1$ there are no topological string solutions. However, it may be seen that there are still stable vortex solutions for semi-topological reasons. It we only look at the gauged part of the symmetry, the breaking is identical to that of the abelian Higgs model and this tells us that we should have local strings.

The important effect of the global symmetry is to eliminate the topological reason for the existence of the string. The stability of the string now depends on the dynamics: if the Nielsen–Olesen string is weakly stable, the global symmetry will be effective in destabilizing it. If, on the other hand, the Nielsen–Olesen string is strongly stable, the global symmetry will not be able to destabilize it and the semilocal model will continue to have stable string solutions in spite of the absence of (non-trivial) topology. Therefore the stability of the semilocal string solution depends on the single parameter β that enters into the model.

The energy functional for the semilocal model can be written down from the lagrangian. For static, translationally invariant configurations, the rescaled (dimensionless) energy per unit length is

$$E_{\mathrm{sl}} = \int \mathrm{d}^2x \left[\tfrac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + \left|(\partial_j + iA_j)\mathbb{1}\Phi\right|^2 + \tfrac{1}{2}\beta(\Phi^\dagger\Phi - 1)^2 \right], \quad (2.12)$$

The (unit winding) semilocal solution that minimizes E_{sl} is

$$\Phi_{\mathrm{sl}} = f_{\mathrm{NO}}\, \mathrm{e}^{i\theta}\Phi_0, \qquad \Phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A_{\mathrm{sl}} = A_{\mathrm{NO}}. \quad (2.13)$$

Note that the choice of the constant doublet Φ_0 is arbitrary since we can always (globally) rotate it to a different value by performing a global SU(2) rotation. By having chosen a certain Φ_0, the semilocal string solution is such that Φ_0 is constant in space. Hindmarsh [2] has shown that the semilocal string is stable to perturbations in the direction orthogonal to Φ_0 when $\beta < 1$.

The choice of Φ_0 is arbitrary for an isolated string and one value of Φ_0 can be rotated into another value without any cost in energy. However, if these are two or more strings with different values of Φ_0 then a global transformation cannot eliminate this difference. That is, the *relative* value of Φ_0 is significant while the *absolute* value is not. Hence, we must label each string by its value of Φ_0 or its "colour". Multi-vortex solutions corresponding to parallel semilocal strings with the same colour have been studied by Gibbons et al. [10].

Another property of semilocal strings is that they can end in a "cloud" of energy or what has also been called a global monopole [2]. This is because there is no topological index that forces the string to continue indefinitely or form a closed loop. However, the energy barrier that stabilizes the semilocal string also provides an energy barrier to ending the string [11].

The motion of a string end is also related to the notion of colour. It may happen that a string ends in a cloud of energy from which a string of a different colour might emerge. In other words, strings of different colour could have junctions at which the change in colour carries energy. It is still not clear to us if such colour junctions have any consequences for the dynamics of a network of semilocal strings.

3. Numerical simulations

In this section we use numerical simulations to

(1) Test the stability of semilocal strings as a function of the parameter β.

(2) Find the sector of S^3 such that initial conditions starting asymptotically in that sector relax to a semilocal string configuration rather than to the vacuum. We also find the dependence on β of that sector.

(3) Study the evolution of a pair of semilocal strings with nonzero relative $SU(2)_{\mathrm{global}}$ phase.

The simulations are performed by numerically solving the equations of motion for the fields $\boldsymbol{\Phi}$ and A_μ obtained by varying the lagrangian (2.11). In the temporal gauge $A_0 = 0$ these equations are of the form:

$$\ddot{\boldsymbol{\Phi}} - \nabla^2\boldsymbol{\Phi} + \beta\boldsymbol{\Phi}(\boldsymbol{\Phi}\boldsymbol{\Phi}^\dagger - 1) - 2iA\cdot\nabla\boldsymbol{\Phi} - i\boldsymbol{\Phi}\nabla A + A^2\boldsymbol{\Phi} = 0, \tag{3.1}$$

$$-\partial_0^2 A_j + \partial^i F_{ij} - 2\,\mathrm{Im}\left(\boldsymbol{\Phi}^\dagger\partial_j\boldsymbol{\Phi}\right) - 2A_j\boldsymbol{\Phi}^\dagger\boldsymbol{\Phi} = 0. \tag{3.2}$$

The condition $A_0 = 0$ implies the constraint

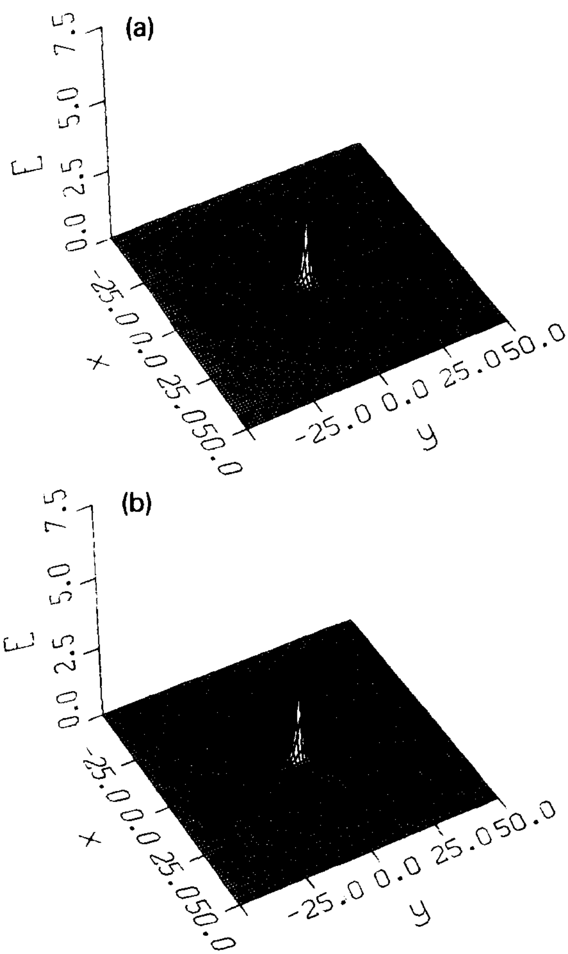

Fig. 1. The evolution of an isolated string $\Phi = (g(r), f(r)\,e^{i\theta})$, for $\beta = 0.9$. The energy density. (a) $T = 0$, (b) $T = 100$, (c) $T = 400$ and (d) $T = 500$.

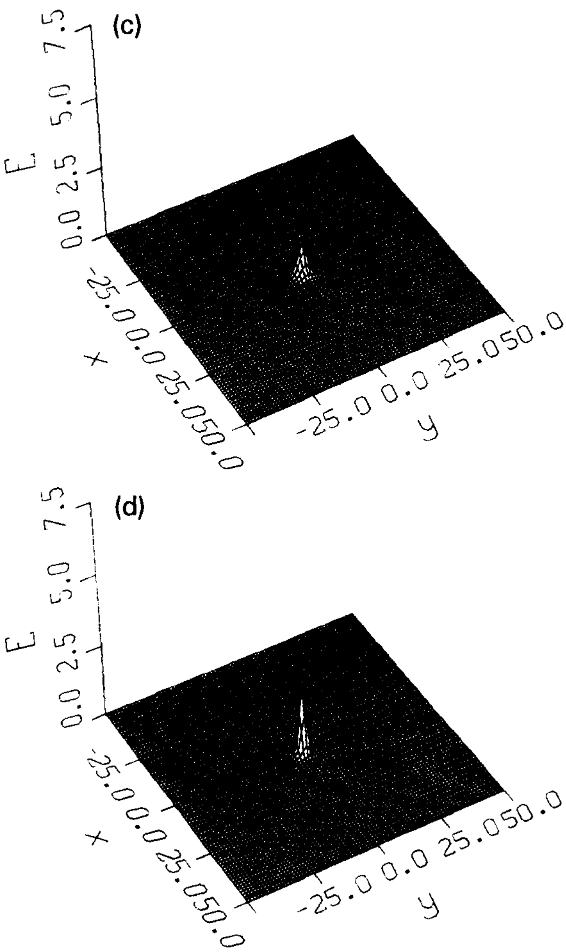

Fig. 1. Continued.

$$2\,\mathrm{Im}(\dot{\Phi}\Phi^{\dagger}) + \partial_i \dot{A}_i = 0. \tag{3.3}$$

The conservation of the constraint (3.3), in addition to energy conservation, was used as a test during the evolution. The algorithm used for the solution of eqs. (3.1), (3.2) was a second-order accurate staggered leapfrog scheme on an 80×80 lattice with spacing $\mathrm{d}x = 0.6$ and timestep $\mathrm{d}t = 0.2$. The ratio $\mathrm{d}t/\mathrm{d}x$ was slightly smaller than the Courant number $1/\sqrt{2}$ thus enhancing stability at the cost of introducing a small numerical viscosity which however had no significant effects for the evolution timescales considered. The boundary conditions we used were to set the field derivatives across the boundaries to zero, thus smoothing the field configurations there.

In order to test the stability of semilocal strings we perturb the semilocal string

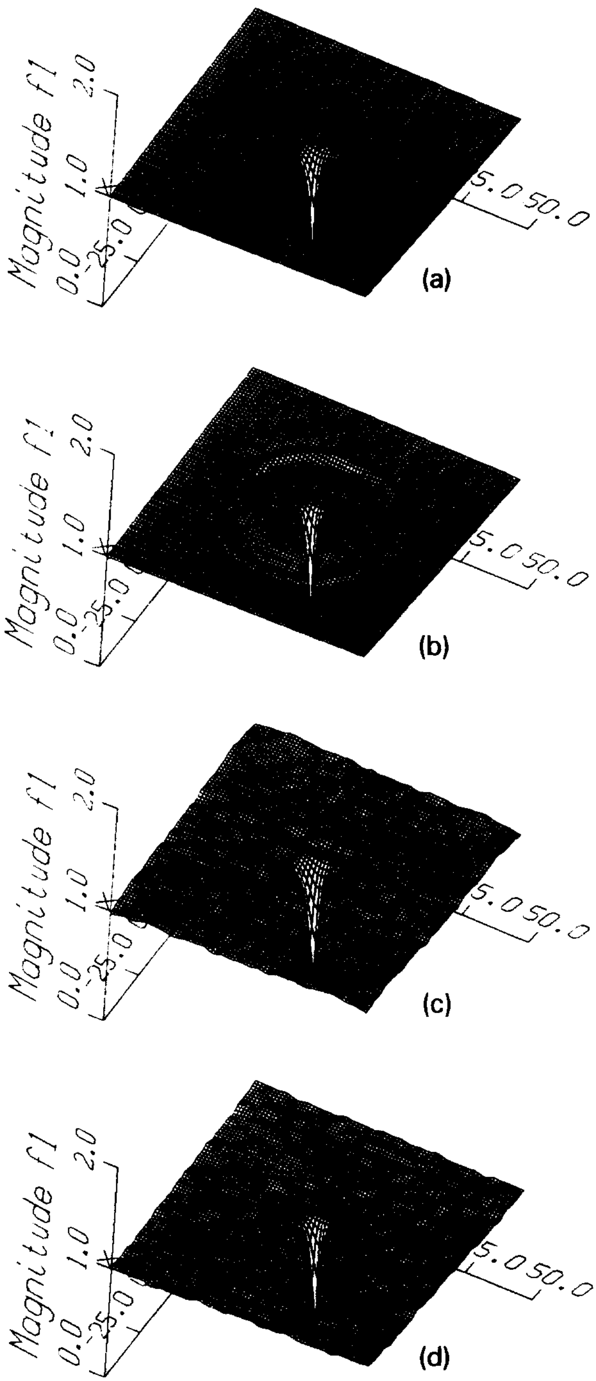

Fig. 2. The evolution of an isolated string $\Phi = (g(r), f(r)\,e^{i\theta})$, for $\beta = 0.9$. The function $f(r)$. (a) $T = 0$, (b) $T = 100$, (c) $T = 400$ and (d) $T = 500$.

solution and evolve the perturbed configuration for different values of the parameter β. The unperturbed semilocal string solution may be written as:

$$\Phi = \left(0, f_0(r)\ \mathrm{e}^{i\theta}\right), \qquad A_\mu = \frac{v_0(r)}{er}\delta_{\mu\theta}, \tag{3.5}$$

where $f_0(r)$ has the asymptotic behaviour:

$$f_0(r \ll 1) \sim r, \qquad v_0(r \ll 1) \sim r^2, \tag{3.6}$$

$$f_0(r \gg 1) \sim 1, \qquad v_0(r \gg 1) \sim 1. \tag{3.7}$$

The perturbed configuration whose evolution we considered here is:

$$\Phi = \left(g(r), f(r)\ \mathrm{e}^{i\theta}\right), \tag{3.8}$$

$$A_\mu = \frac{v(r)}{er}\delta_{\mu\theta} \tag{3.9}$$

with the initial ansatz [2] $f(r) = r/(1+r^2)^{1/2}$, $g(r) = 1/(1+r^2)^{1/2}$ and $v(r) = r^2/(1+r^2)$. We evolved the above configuration for several different values of the parameter β. A critical change in the evolution occured at $\beta = 1$. In fig. 1 we show the evolution of the energy density for $\beta = 0.9$. The energy maximum which is associated with the semilocal string persists during the whole evolution while the perturbation $g(r)$ rapidly decreases and oscillates around zero (fig. 3). The corresponding evolution of the lower component is shown in fig. 2. The fact that

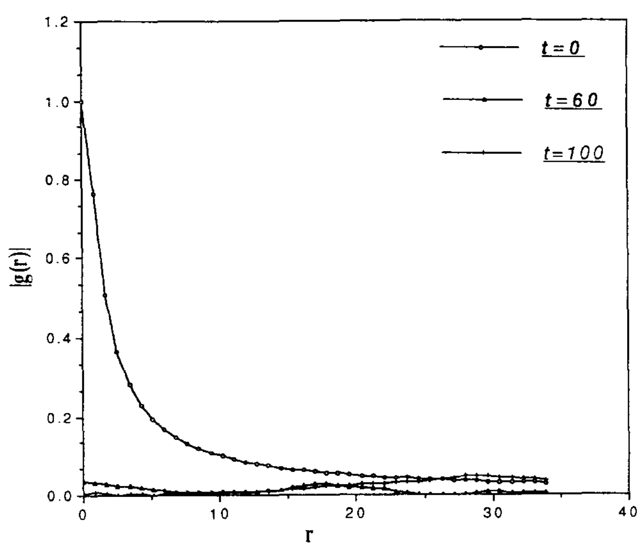

Fig. 3. The evolution of an isolated string $\Phi = (g(r), f(r)\,\mathrm{e}^{i\theta})$, for $\beta = 0.9$. The function $|g(r)|$ for $T = 0$, 60 and 100.

our initial conditions are not exact solutions of the equations of motion explains the observed oscillations of the maximum of the energy density associated with the string. Similar evolution was obtained for several values of β between 0 and 1.

Fig. 4 shows the evolution of the energy density for $\beta = 1.1$. In this case the semilocal string maximum rapidly dissipates while the perturbation $g(r)$ increases and tends to relax around the value 1 (fig. 6). The bottom component shows the increase in the string width (fig. 5). This type of evolution proceeds from smaller to larger r. The configuration eventually relaxes to the vacuum. Similar evolution was seen for several $\beta > 1$.

This behaviour implies stability (instability) of semilocal strings for $\beta < 1$ ($\beta > 1$) thus confirming the conclusion of ref. [2]. As mentioned in ref. [2] this result may be obtained alternatively by looking for negative eigenvalues of the fluctuation

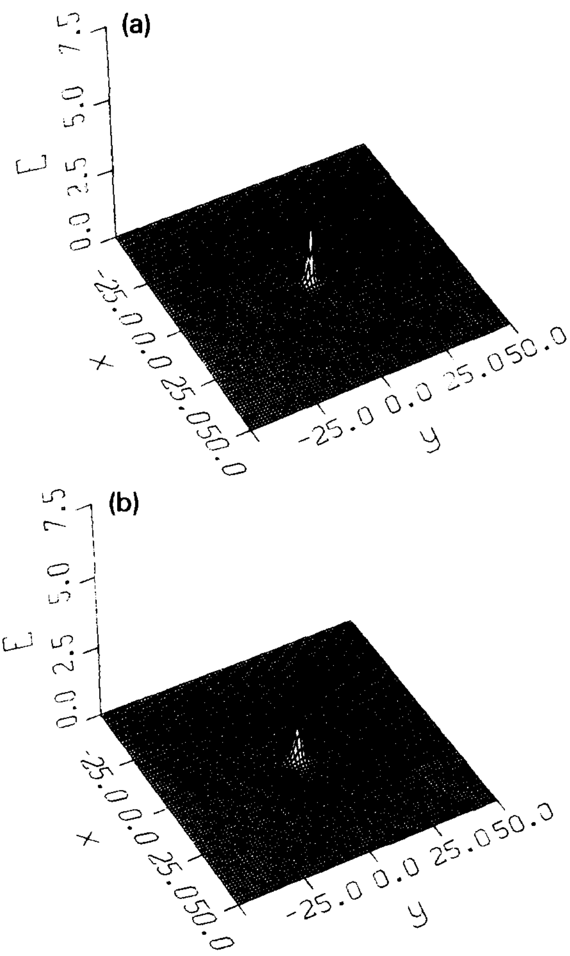

Fig. 4. The evolution of an isolated string $\Phi = (g(r), f(r)\,e^{i\theta})$, for $\beta = 1.1$. The energy density. (a) $T = 0$, (b) $T = 100$, (c) $T = 400$ and (d) $T = 500$.

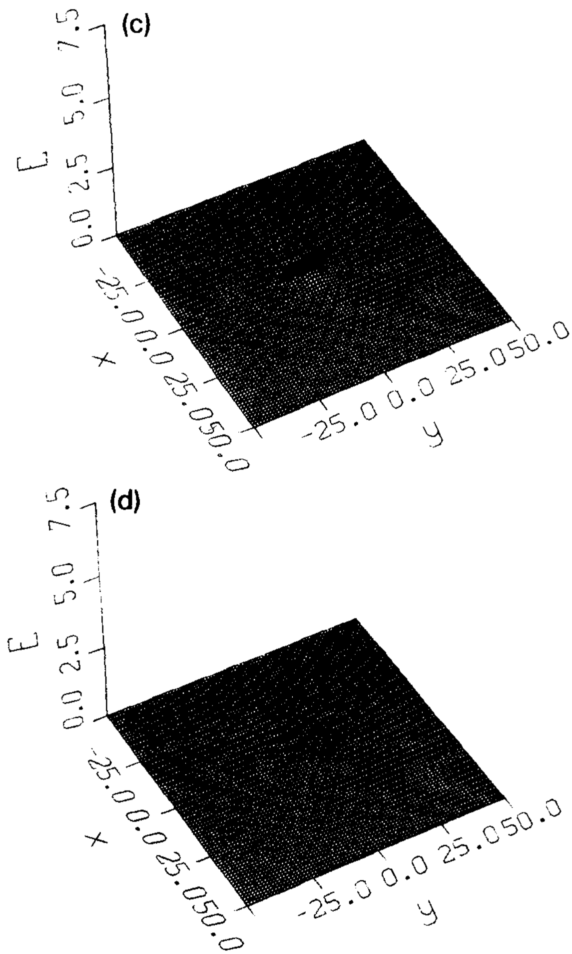

Fig. 4. Continued.

operator $(-D^2 + V(\Phi)'')|_{g=0}$. We have pursued this eigenvalue method in the more general case of electroweak strings. The results, when reduced to the special case of semilocal strings, are in good agreement with the ones presented here [4].

After having verified that there is a parameter region for which semilocal strings are stable we address the question of string formation. In particular we look for a sector of S^3 such that any initial condition starting asymptotically in that sector leads to semilocal string formation rather than relaxation to the vacuum. In order to address this issue we evolve the following configuration for several values of β:

$$\Phi = (\omega,\ f(r)\ e^{i\theta}), \tag{3.10}$$

$$A_\mu = \frac{v(r)}{er}\delta_{\mu\theta}, \tag{3.11}$$

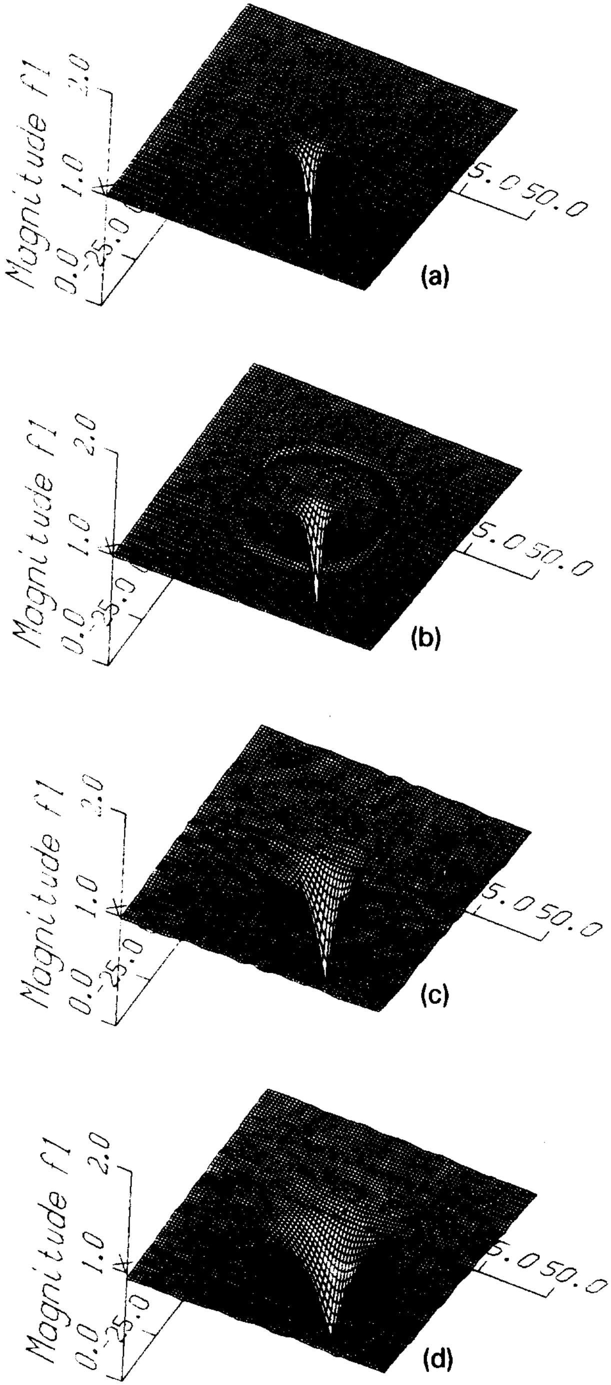

Fig. 5. The evolution of an isolated string $\Phi = (g(r), f(r)\,e^{i\theta})$, for $\beta = 1.1$. The function $f(r)$. (a) $T = 0$, (b) $T = 100$, (c) $T = 400$ and (d) $T = 500$.

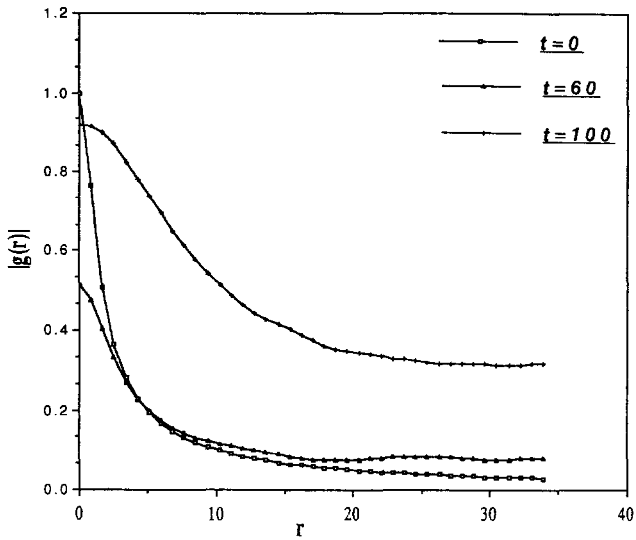

Fig. 6. The evolution of an isolated string $\Phi = (g(r), f(r)\,e^{i\theta})$, for $\beta = 1.1$. The function $|g(r)|$ for $T = 0$, 60 and 100.

with $f(r)$ and $v(r)$ same as above and ω a constant parametrizing the distance of the asymptotic value of the initial configuration from the circle of S^3 spanned by $U(1)_{\text{local}}$. Clearly, the configuration (3.10) has diverging energy and will relax to a

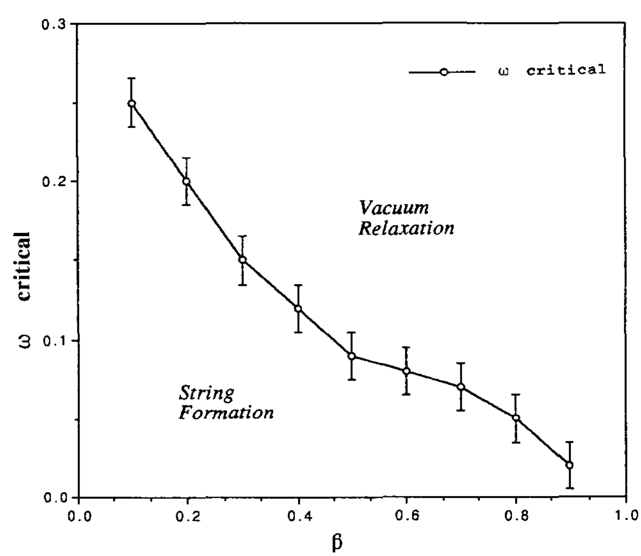

Fig. 7. The numerically obtained function ω_{crit}. Initial conditions with ω under the curve resulted in relaxation towards a semilocal string configuration.

finite energy which can be either the semilocal string $(0, f(r)\, e^{i\theta})$ or the vacuum $(1,0)$. As ω increases and goes to 1 it becomes easier for the configuration to relax to the vacuum for fixed β. On the other hand as β decreases, for fixed ω the semilocal string is favoured as its stability increases. This behaviour is indeed seen in our simulations. We fixed β and evolved the initial ansatz (3.10), (3.11) for several values of ω. For ω smaller than a critical value (which depends on β) the configuration relaxed to the semilocal string. For ω larger than the critical value, vacuum relaxation occurred. In fig. 7 we plot the critical value ω_{crit} as a function of β. For values of β within the error bars the fields did not relax within the evolution time. As anticipated, the sector of S^3 leading to semilocal string formation increases as β decreases. The numerically obtained function $\omega_{\text{crit}}(\beta)$ may now be used in a Monte-Carlo simulation to obtain the initial semilocal string network after a cosmological phase transition. This study will be presented in a separate publication.

The study of the interactions of semilocal strings is important in order to understand the evolution of a network of such strings. In ref. [1] it was argued that the evolution of semilocal strings could be different from that of gauged strings due to the long-range Goldstone boson field induced by the nontrivial relative orientation of the submanifolds spanned by $U(1)_{\text{local}}$ for two interacting semilocal strings. This conjecture relied on the assumption that the nontrivial relative orientation would persist during the evolution. In order to test this assumption, we introduce the "Hopf parametrization" [10]

$$\Phi = f\, e^{i\chi/2}\begin{pmatrix} \cos\alpha/2\, e^{i\gamma/2} \\ \sin\alpha/2\, e^{-i\gamma/2} \end{pmatrix} \tag{3.12}$$

and rewrite the action (2.11) as

$$S = \int\Big[\big|(\partial_\mu + iA_\mu)(f\, e^{i\chi/2})\big|^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\beta(f^2-1)^2 + \tfrac{1}{4}f^2(\partial_\mu\alpha)^2$$
$$+ \tfrac{1}{4}f^2(\partial_\mu\gamma)^2 + f^2\cos\alpha\big(\tfrac{1}{2}\partial_\mu\chi + A_\mu\big)(\partial^\mu\gamma)\Big]\, d^4x. \tag{3.13}$$

An estimate for the energy can be obtained by ignoring the phases γ and χ. Since $f=1$ outside strings, the energy per unit length between two static, parallel strings with colours α_1 and α_2 at a distance d is roughly

$$E = \int f^2(\nabla\alpha)^2\, d^2x \approx K(\alpha_2-\alpha_1)^2 \ln d/d_0, \tag{3.14}$$

where d_0 is the core width of the string, $d \gg d_0$ and K is some constant that can be calculated numerically. The (dimensionless) force between the strings is given by the slope

$$m_{d/d_0} = \frac{\delta E}{\delta(d/d_0)} = K(\Delta\alpha)^2 d_0/d, \tag{3.15}$$

with $\Delta\alpha = \alpha_2 - \alpha_1$, whereas the tendency to align the U(1) orbits of both strings in S^3 is measured by

$$m_{\Delta\alpha} = \frac{\delta E}{\delta(\Delta\alpha)} = 2K(\Delta\alpha) \ln d/d_0. \tag{3.16}$$

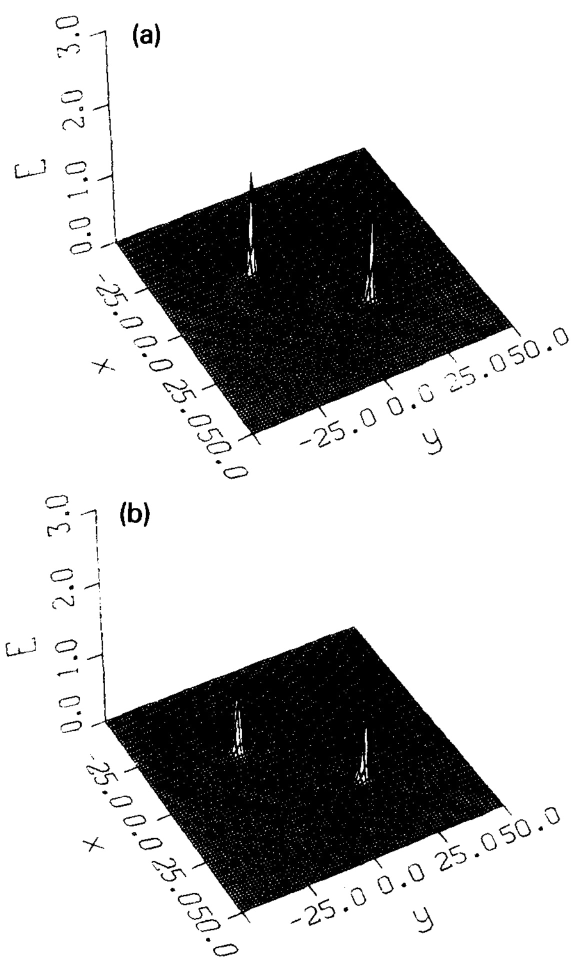

Fig. 8. The evolution of the energy density for the string pair given by (3.19). The string cores remain at their initial positions. (a) $T = 0$, (b) $T = 50$, (c) $T = 200$ and (d) $T = 250$.

Their ratio satisfies

$$\frac{m_{d/d_0}}{m_{\Delta\alpha}} = \frac{\Delta\alpha}{2(d/d_0)\ \ln(d/d_0)} \leqslant \frac{\pi}{2(d/d_0)\ \ln(d/d_0)} \tag{3.17}$$

and this suggests that, for $d \gg d_0$, colour alignment is a more efficient way to lower the energy. In general, the strings will radiate away their colour difference in the form of Goldstone bosons and there will be little or no interaction observed. Our simulations had a typical separation of $d/d_0 \approx 30$, which yields a ratio $m_{d/d_0}/m_{\Delta\alpha} \approx 0.02$, and confirm this result. (Note that a similar calculation for monopoles yields a slightly higher ratio $m_{d/d_0}/m_{\Delta\alpha} \sim d_0/d$; semilocal monopoles

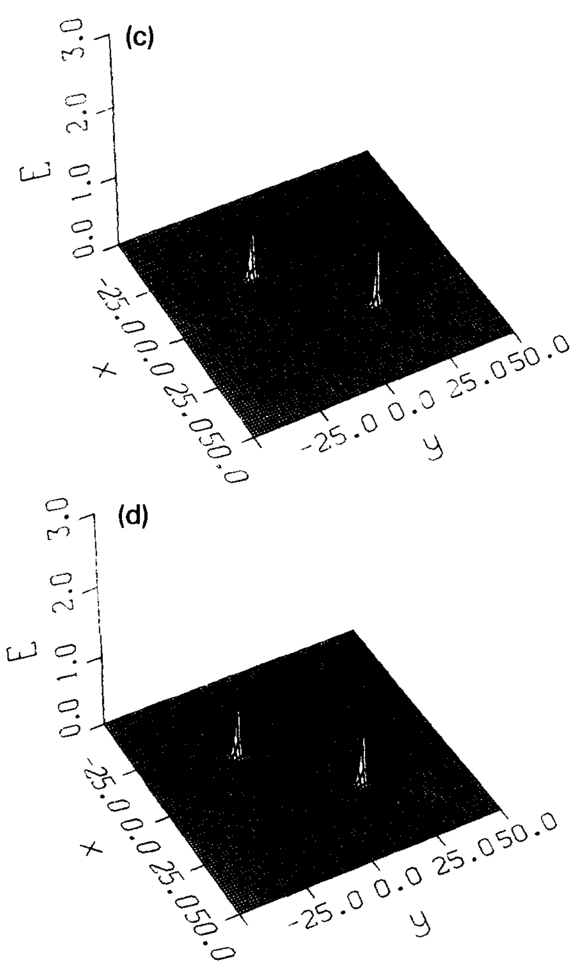

Fig. 8. Continued.

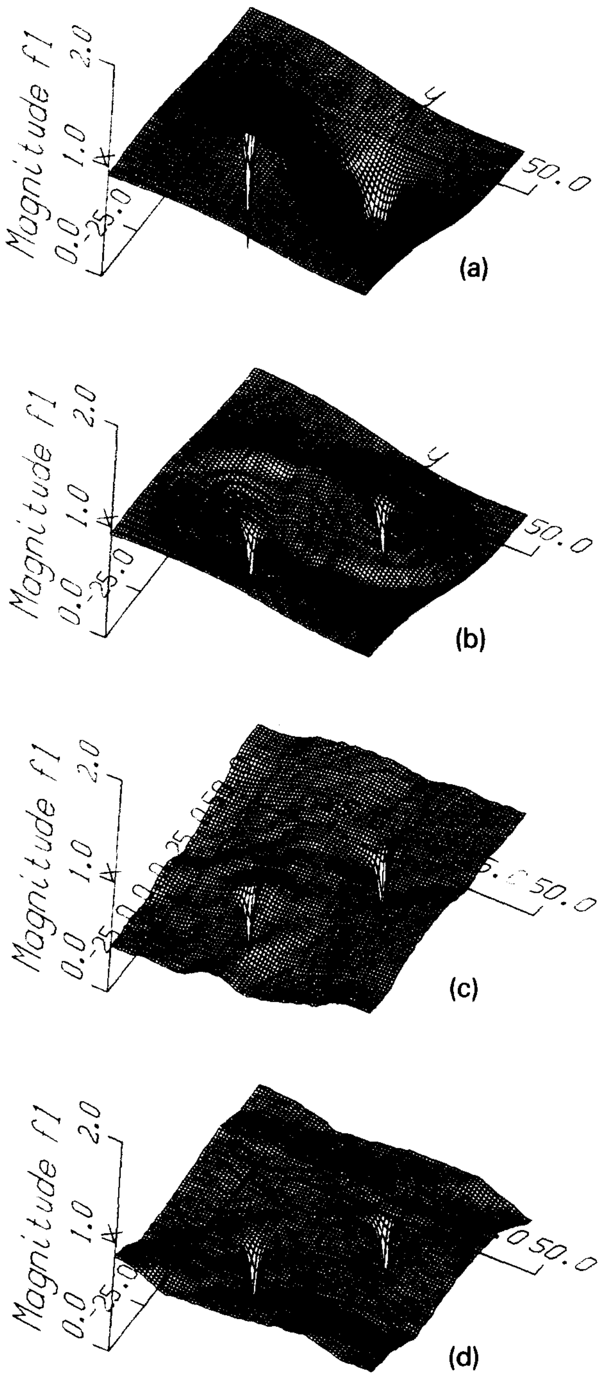

Fig. 9. The evolution of the upper component of the field Φ for the string pair (3.19). The relative SU(2) phase is minimized during the evolution. (a) $T = 0$, (b) $T = 50$, (c) $T = 200$ and (d) $T = 250$.

were shown not to exist within some fairly general assumptions in ref. [1] but we provide some alternative models with "coloured" monopoles in sect. (4).

We evolved several field configurations corresponding to pairs of semilocal strings. In order to minimize noise from Goldstone radiation emitted during the evolution we did not use an arbitrary initial ansatz, instead we first minimized numerically the energy functional subject to semilocal string pair boundary conditions and used the configuration minimizing the energy as initial condition in the numerical simulation of evolution. We used optimum interpolating initial conditions for the following types of string pairs:

$$\mathbf{\Phi} = \left(0,\, f(r)\, \mathrm{e}^{i\theta}\right) \leftrightarrow \mathbf{\Phi} = \left(f(r)\, \mathrm{e}^{i\theta},\, 0\right), \tag{3.18}$$

$$\mathbf{\Phi} = \left(0,\, f(r)\, \mathrm{e}^{i\theta}\right) \leftrightarrow \mathbf{\Phi} = \left(if(r)\, \mathrm{e}^{i\theta},\, 0\right), \tag{3.19}$$

$$\mathbf{\Phi} = \left(0,\, f(r)\, \mathrm{e}^{i\theta}\right) \leftrightarrow \mathbf{\Phi} = \left(0,\, if(r)\, \mathrm{e}^{i\theta}\right). \tag{3.20}$$

In all cases we observed the same pattern of behaviour; the position of the strings remained the same during the whole evolution while the fields tended to minimize the initial relative SU(2) phase. In fig. 8 we show the evolved energy density corresponding to the initial condition (3.19) while fig. 9 shows the corresponding evolution of the upper component of the field $\mathbf{\Phi}$ (the evolution of the lower component is similar, but with the initial minimum on the right). Clearly, the positions of the cores remained unchanged during the evolution while the field $\mathbf{\Phi}$ relaxed by equating the magnitudes of the upper and lower components of both strings thus minimizing the relative SU(2) orientation. This result implies that the interaction of semilocal strings is weak and it may be possible to use an effective action similar to the Nambu action to describe their evolution. This however, does not mean that semilocal strings evolve identically with gauged strings in three dimensions. For example semilocal strings can terminate while gauged strings cannot. Three dimensional simulations are clearly necessary to give a complete picture of the semilocal string evolution.

To summarize, in this section we have presented results of numerical simulations with semilocal strings. Our simulations show that semilocal strings are stable for $\beta < 1$ and that they can form with initial conditions of nonzero measure. They also show that semilocal string interactions are quite weak.

4. Generalizations

As was pointed out in ref. [1], the semilocal string can be generalized trivially by considering other global symmetry groups. Semilocal strings with symmetry group $\mathrm{SU(N)} \times \mathrm{U(1)}$ were discussed in [2], where it was shown that they correspond to

$\mathbb{C}\mathrm{P}^n$ instantons in two dimensions. The physical properties of these models are basically the same as for the SU(2) × U(1) string.

What about other semi-local defects? We define a semi-local defect as one for which

$$\pi_n(\mathrm{G}/\mathrm{H}) = 0 \quad \text{but} \quad \pi_n(\mathrm{G}_{\mathrm{local}}/\mathrm{H}_{\mathrm{local}}) \neq 0,$$

($n = 1$ for strings, $n = 2$ for monopoles and $n = 3$ for textures).

Under very general assumptions, it was shown in ref. [1] that semi-local monopoles cannot exist. However, if one relaxes the condition $\pi_n(\mathrm{V}) = 0$, it is possible to construct models with "colour"; we will call these defects "coloured", as opposed to semilocal, because their stability is really topological.

To illustrate the difference, consider a model where the Higgs field is the tensor product of the adjoint representation of SU(2) and the two-dimensional representation of U(1), that is,

$$\mathbf{\Phi} = \begin{pmatrix} \phi \\ \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_1, & \phi_2 \\ \psi_1, & \psi_2 \\ \chi_1, & \chi_2 \end{pmatrix}, \tag{4.1}$$

where the $\mathrm{SU}(2)_{\mathrm{global}}$ transformations are

$$\begin{pmatrix} \phi \\ \psi \\ \chi \end{pmatrix} \to \mathrm{e}^{i\alpha^a T^a} \begin{pmatrix} \phi \\ \psi \\ \chi \end{pmatrix}, \tag{4.2}$$

with

$$T^1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{4.3}$$

and the $\mathrm{U}(1)_{\mathrm{local}}$ transformations are given by

$$\begin{pmatrix} \phi_1, & \phi_2 \\ \psi_1, & \psi_2 \\ \chi_1, & \chi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha\ \phi_1 - \sin\alpha\ \phi_2, & \sin\alpha\ \phi_1 + \cos\alpha\ \phi_2 \\ \cos\alpha\ \psi_1 - \sin\alpha\ \psi_2, & \sin\alpha\ \psi_1 + \cos\alpha\ \psi_2 \\ \cos\alpha\ \chi_1 - \sin\alpha\ \chi_2, & \sin\alpha\ \chi_1 + \cos\alpha\ \chi_2 \end{pmatrix}. \tag{4.4}$$

The most general action compatible with these symmetries is

$$S = \int \mathrm{d}^4x \left[\tfrac{1}{2} \left| D_\mu \mathbf{\Phi} \right|^2 - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{4}\lambda \left(\mathbf{\Phi}^2 - \eta^2 \right)^2 - \gamma \left| \mathbf{\Phi}_1 \times \mathbf{\Phi}_2 \right|^2 \right] \tag{4.5}$$

with λ and γ positive to ensure that the energy is bounded from below. We are using here the obvious notation $\mathbf{\Phi}_1 = (\phi_1, \psi_1, \chi_1)$, and similarly for $\mathbf{\Phi}_2$. Note that $\mathbf{\Phi}_1 \times \mathbf{\Phi}_2$ is U(1)-invariant, and its modulus is also SU(2)-invariant.

The vacuum manifold is the set

$$\mathbf{\Phi}^2 - \eta^2 = \mathbf{\Phi}_1^2 + \mathbf{\Phi}_2^2 - \eta^2 = 0, \qquad \mathbf{\Phi}_1 \times \mathbf{\Phi}_2 = 0, \tag{4.6}$$

and it can be parametrized as

$$\mathbf{\Phi}_1 = \eta \cos\theta \, \boldsymbol{n}, \qquad \mathbf{\Phi}_2 = \eta \sin\theta \, \boldsymbol{n}, \tag{4.7}$$

where $\boldsymbol{n} = (n_1, n_2, n_3)$ lies on the two sphere $|\boldsymbol{n}| = 1$, provided we identify the points $(\theta, \boldsymbol{n})$ and $(\theta + \pi, -\boldsymbol{n})$. This is a non-trivial fibre bundle T with base space S^1 and fibre S^2 (or, alternatively, with base $\mathbb{R}P^2$ and fibre S^1). It can be shown that $\pi_1(T) = \mathbb{Z}$: the non-contractible loops are those starting at $(\theta, \boldsymbol{n})$ and finishing at $(\theta + \pi, -\boldsymbol{n})$ in the above parametrization.

Let us consider the symmetry breaking. When the Higgs field acquires a v.e.v., say,

$$\mathbf{\Phi} = \begin{pmatrix} 0, & 0 \\ 0, & 0 \\ 1, & 0 \end{pmatrix} \tag{4.8}$$

the stability group is given by

$$\begin{aligned} &\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1, \\ e^{i\pi}&\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{with} \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1, \end{aligned} \tag{4.9}$$

i.e. $H = U(1)_{\text{global}} \times \mathbb{Z}_2$. If we only consider the local symmetries, we have U(1) breaking down to $\mathbb{Z}_2$, and the homotopy group of the quotient is again $\mathbb{Z}$ (a circle with opposite points identified is still a circle) so strings will form.

Locally, the vacuum manifold looks like $S^2 \times S^1$ and therefore these strings have colour parameters living on S^2, but they are not semilocal because the vacuum is not simply connected. In particular, the strings will be stable even after gauging the global SU(2). We will call them "*coloured*" strings. (Note that this model also has global monopoles.)

In ref. [1] it was shown that semilocal monopoles do not exist within some very general assumptions. A semilocal monopole would occur in a theory where $\pi_2(G/H) = 0$ but $\pi_2(G_{\text{local}}/H_{\text{local}}) \neq 0$. Note that if one relaxes the condition $\pi_2(G/H) = 0$, one can easily construct coloured monopoles. For instance, to obtain a model with coloured monopoles, simply interchange the global and local symmetry group in the previous example. Then we have $SU(2)_{local} \times U(1)_{global} \to U(1)_{local}$, and $\pi_2(V) = \pi_2(F) = \mathbb{Z}$. These monopoles are topologically stable. One can also construct coloured textures in this way.

Finally, note that, as in the case of semilocal strings, there is an obvious generalization to other coloured models by considering different global groups.

5. Conclusions

We have started an investigation of the cosmological formation and subsequent evolution of string networks. In particular:

(1) we have shown that there is a non-zero measure set of asymptotic field configurations which will yield semilocal strings in a cosmological phase transition, and

(2) we have shown that the interactions between infinitely long, parallel, semilocal strings with different colours are essentially the same as for Nielsen–Olesen strings. The colour difference is radiated away in the form of Goldstone bosons.

The main difference between U(1) and semilocal strings is therefore the fact that the latter can terminate. The behavior of terminating strings is not well understood and should be clarified before network simulations can be attempted.

We have also confirmed by explicit numerical simulations Hindmarsh's analysis of the stability of semilocal cosmic strings: we find that strings are stable for $\beta < 1$ and unstable for $\beta > 1$.

Finally, we have given examples of topological defects with colour. These are not semilocal defects, because their stability is topological, but they have a colour degree of freedom similar to that of semilocal strings.

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