FERMION BOUND STATES IN LOCALIZED STATIC MAGNETIC FIELDS

E.G. FLORATOS
Physics Department, University of Crete, Herakleion, Greece

K. TAMVAKIS
Physics Deparment, University of Ioannina, Ioannina, Greece

and

G. TIKTOPOULOS
Physics Department, National Technical University, Athens, Greece

Received 6 September 1984

We investigate the class of static abelian magnetic fields which are localized in the sense $rA \to 0$ at infinity, A being the gauge potential. We find that those admitting fermion bound states form a wide subclass of "non-zero measure". We give a method to construct all such fields starting from appropriate conserved integrable fermionic currents $J(x)$. The motivation for this work is that fermions similarly bound in non-abelian gauge fields may induce chiral symmetry breakdown in QCD.

1. Introduction. A specific dynamical mechanism was recently proposed [1] for the formation of pseudoscalar quark–antiquark bound states in QCD as Goldstone bosons of spontaneously broken chiral symmetry. The idea is based on the observation that fermions might be bound in suitable localized gauge field configurations which need not be solutions of the classical field equations but may nevertheless be sufficiently numerous to saturate functional averages like $\langle \bar{q}q(x)\rangle$, $\langle \bar{q}q(x)\bar{q}q(y)\rangle$ etc. $q(x)$ being the quark field.

In ref. [2] it had already been shown that certain gauge field configurations $A_\mu(x)$ may be responsible for $\langle \bar{q}q\rangle \neq 0$. Specific examples of static fields A_μ were given in which the nonvanishing of the trace of the fermion propagator was related to the existence of fermion bound states (zero modes).

If the above suggestions are to have any relevance for chiral symmetry breakdown in QCD the class of gauge field configurations in which fermions can be "appropriately bound" must claim a substantial measure in function space. Obviously, one can not limit oneself to static fields; and for general time-dependent A_μ one cannot speak of bound states in the ordinary sense. Presumably, the required property then would be the existence of solutions of the Dirac equation which remain spatially normalizable at all times.

In this paper we investigate in some detail the case of localized static abelian magnetic fields. We find that indeed a large class of such fields admits fermion bound states.

2. Threshold bound states. We shall work in the temportal gauge: $A_0 = 0$ in which we shall assume $\partial A/\partial t = 0$ and $|x|A(x) \to 0$ as $|x| \to \infty$. This last "localization" condition is intended to ensure that the Dirac equation is approximated asymptotically by the free Dirac equation. We shall also assume that the cartesian components of A are smooth functions of the coordinates x_1, x_2, x_3 in the sense of having continuous partial derivatives of all required orders.

The Dirac equation for a fermion of mass m and unit charge

$$[\gamma_\mu(\partial/\partial x_\mu - iA_\mu) + m]\psi = 0 ,$$

for stationary solutions

0370-2693/84/$03.00 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

$$\psi(x) = \exp(-iEt)\begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix},$$

reads as

$$(m-E)u_1 + \boldsymbol{\sigma}\cdot \mathbf{D}_A u_2 = 0\,,$$
$$-\boldsymbol{\sigma}\cdot \mathbf{D}_A u_1 + (m+E)u_2 = 0\,, \tag{1}$$

where $\mathbf{D}_A \equiv -i\partial - \boldsymbol{A}$.

We are interested in bound states i.e. solutions with normalizable u_1 and u_2. From the second-order equation that both u_1 and u_2 satisfy,

$$[\boldsymbol{\sigma}\cdot \mathbf{D}_A]^2 u_j = (E^2 - m^2)u_j \quad (j=1,2)\,,$$

we see that, since $[\boldsymbol{\sigma}\cdot \mathbf{D}_A]^2 \to -\partial^2$ as $|\boldsymbol{x}| \to \infty$, there can only be a continuum of nonnormalizable solutions for $|E| > m$. Furthermore, since $[\boldsymbol{\sigma}\cdot \mathbf{D}_A]^2$ is a nonnegative operator there can be no bound states for $|E| < m$ either. Therefore bound states can exist only for the two "threshold" values of the energy $E = \pm m$.

For $E = m$ eq. (1) yields $\boldsymbol{\sigma}\cdot \mathbf{D}_A u_2 = 0$, $u_2 = (2m)^{-1} \times \boldsymbol{\sigma}\cdot \mathrm{D}_A u_1$. But then $[\boldsymbol{\sigma}\cdot \mathrm{D}_A]^2 u_1 = 0$ and $\|\boldsymbol{\sigma}\cdot \mathrm{D}_A u_1\|^2 = 0$. Thus

$$\boldsymbol{\sigma}\cdot \mathbf{D}_A u_1 = 0\,, \quad u_2 = 0\,.$$

Similarly, for $E = -m$ we obtain

$$\boldsymbol{\sigma}\cdot \mathbf{D}_A u_2 = 0\,, \quad u_1 = 0\,.$$

Thus the Dirac equation (1) has bound state solutions if and only if the two component equation

$$\boldsymbol{\sigma}\cdot (i\partial + \boldsymbol{A}(\boldsymbol{x}))\, u(\boldsymbol{x}) = 0\,, \tag{2}$$

has normalizable solutions. Our primary goal in this paper is to characterize the class of localized vector potentials $\boldsymbol{A}(\boldsymbol{x})$ (i.e. potentials satisfying $|\boldsymbol{x}|\boldsymbol{A} \to 0$ at infinity) for which eq. (2) admits normalizable solutions.

3. Solving for the potential $\boldsymbol{A}$. We begin by noting that eq. (2) implies that the fermionic current

$$\boldsymbol{J}(\boldsymbol{x}) = u^+(\boldsymbol{x})\,\sigma u(\boldsymbol{x})\,, \tag{3}$$

is locally conserved i.e. $\partial\cdot \boldsymbol{J} = 0$. It is not difficult to see that, provided $\partial\cdot \boldsymbol{J} = 0$ and $J = u^+u \neq 0$, one can solve eq. (2) with respect to $\boldsymbol{A}$:

$$\boldsymbol{A} = [u^+\partial u - (\partial u^+)u]/2iu^+u + \sigma x (u^+\sigma u)/2u^+u\,. \tag{4}$$

Furthermore, using eq. (3) u can be expressed in terms of the three component of J and the phase χ of one of the components of u, say the upper one:

$$u(\boldsymbol{x}) = \frac{1}{\sqrt{2}}\begin{pmatrix} (J+J_3)^{1/2} \\ (J-J_3)^{1/2}\exp[i\tan^{-1}(J_2/J_1)] \end{pmatrix} e^{i\chi}. \tag{5}$$

Substituting this into eq. (4) we obtain $\boldsymbol{A}$ in terms of $\boldsymbol{J}$ and χ:

$$\boldsymbol{A} = \tfrac{1}{2}(1 - J_3/J)\,\partial \tan^{-1}(J_2/J_1) + (1/2J)(\partial \times \boldsymbol{J}) + \partial\chi\,. \tag{6}$$

Since $u^+u = J$, the normalizability condition reads

$$\int d^3x\; J(\boldsymbol{x}) < \infty\,. \tag{7}$$

Thus one can characterize the class of gauge fields $\boldsymbol{A}$ for which eq. (2) admits normalizable solutions by means of formula (6) where $\boldsymbol{J}(\boldsymbol{x})$ may be any conserved integrable vector field which is smooth and vanishes nowhere (expect, of course, at infinity). An additional important restriction on $\boldsymbol{J}$ is that its asymptotic behavior at infinity must be of the form

$$\boldsymbol{J} \to f^+\boldsymbol{\sigma} f \quad (|\boldsymbol{x}| \to \infty)\,, \tag{8}$$

where f is a solution to the free equation $\boldsymbol{\sigma}\cdot\partial f = 0$.

4. Axially symmetric fields. How restrictive are the above conditions on $\boldsymbol{J}$? Are there any such currents at all? To explore this matter further we shall focus now on the class of axially symmetric fields i.e. we shall take $\boldsymbol{J}$ and χ to be symmetric under rotations about, say, the 3-axis.

The most general divergenceless, axially symmetric vector field $\boldsymbol{J}$ is of the form $[\rho = (x_1^2 + x_2^2)^{1/2}]$:

$$\boldsymbol{J} = (-x_2, x_1, 0)(1/\rho)J_\varphi(\rho, x_3) + (x_1, x_2, 0)(1/\rho)J_\rho(\rho, x_3) + (0,0,1)J_3(\rho, x_3)\,, \tag{9}$$

where

$$J_\rho = \rho^{-1}\,\partial\Psi(\rho,x_3)/\partial x_3\,, \quad J_3 = -\rho^{-1}\,\partial\Psi(\rho,x_3)/\partial\rho\,. \tag{10}$$

Note that smoothness for the cartesian components of $\boldsymbol{J}$ implies that at $\rho = 0$ we have

$$\Psi(\rho, x_3) \underset{\rho\to 0}{\propto} \rho^2 \quad \text{(or faster)}\,, \tag{11}$$

$$J_\varphi(\rho, x_3) \underset{\rho\to 0}{\propto} \rho \quad \text{(or faster)}\,. \tag{12}$$

Because of the axial symmetry the spinor function u can be taken to be an eigenfunction of the three-component of the (total) angular momentum with eigenvalue $M_j = M - \frac{1}{2}$ where M is an integer. Accordingly, the φ-dependence of the components of u is given by

$$u = \begin{pmatrix} \exp[\mathrm{i}(M-1)\varphi] & g_1(\rho, x_3) \\ \exp[\mathrm{i}M\varphi] & g_2(\rho, x_3) \end{pmatrix}. \tag{13}$$

Asymptotically, $u \to u_{\mathrm{As}}$ where u_{As} satisfies the free equation $\boldsymbol{\sigma}\cdot\partial u_{\mathrm{As}} = 0$. Since $(\boldsymbol{\sigma}\cdot\partial)^2 = -\partial^2$ the components of u_{As} are harmonic functions, so for $|\boldsymbol{x}| \to 0$ they must be of the form r^{-l-1} times a spherical harmonic of order l for some $l > 0$. Taking (13) into account we have, for $M \geqslant 1$,

$$u_{\mathrm{As}} = \text{const.} \times r^{-l-1} \times \begin{pmatrix} -(l-M+1)P_l^{M-1}(\cos\theta)\exp[\mathrm{i}(M-1)\varphi] \\ P_l^M(\cos\theta)\exp[\mathrm{i}M\varphi] \end{pmatrix}. \tag{14}$$

The functions P_l^M and P_l^{M-1} are the usual associated Legendre functions [3]. The coefficient $-(l - M + 1)$ is the right one so $\boldsymbol{\sigma}\cdot\partial u_{\mathrm{As}}$ vanishes. The restriction $M \geqslant 1$ is no loss of generality because of the conjugation symmetry: to each solution u with angular momentum M_j for the potential A corresponds the solution $\sigma_2 u^*$ for the potential $-A$ with angular momentum $-M_j$.

From eq. (14) we calculate the asymptotic form of the current $\boldsymbol{J}^{\mathrm{As}} = u_{\mathrm{As}}^+ \boldsymbol{\sigma} u_{\mathrm{As}}$:

$$J_\varphi^{\mathrm{As}} = 0,$$
$$J_\rho^{\mathrm{As}} = \lambda r^{-2l-2} 2V_1V_2,$$
$$J_3^{\mathrm{As}} = \lambda r^{-2l-2}(V_1^2 - V_2^2), \tag{15}$$

where λ is some positive constant and

$$V_1 = -(l-M+1)P_l^{M-1}(\cos\vartheta),$$
$$V_2 = P_l^M(\cos\vartheta), \tag{16}$$
$$l = 1, 2, 3, \dots .$$

From eqs. (15), (16) and (10) we also obtain the asymptotic form of Ψ

$$\Psi^{\mathrm{As}} \underset{|x|\mapsto\infty}{\sim} (\rho/2l)(\rho J_3^{\mathrm{As}} - x_3 J_\rho^{\mathrm{As}}). \tag{17}$$

Thus in order to generate an acceptable fermionic current J to use in formula (6) for A we choose:

(i) The pair of integers $l (> 0)$ and M which determines u_{As}, $\boldsymbol{J}^{\mathrm{As}}$ and ψ^{As}.

(ii) A function $\Psi(\rho, x_3)$ having the asymptotic form Ψ^{As} and such that $\rho^{-2}\Psi$ is smooth.

(iii) A function $J_\varphi(\rho, x_3)$ vanishing asymptotically faster than r^{-2l-2} and such that $\rho^{-1}J_\varphi$ is smooth.

These choices of Ψ and J_φ must be made so that $\boldsymbol{J}$ does not vanish anywhere (except at infinity).

5. An example. For $l = M = 1$ we have

$$u_{\mathrm{As}} \sim \begin{pmatrix} \cos\theta/r^2 \\ \mathrm{e}^{\mathrm{i}\varphi}\sin\vartheta/r^2 \end{pmatrix},$$

$$J_\rho^{\mathrm{As}} \sim 2\rho x_3/r^6, \quad J_3^{\mathrm{As}} \sim (x_3^2 - \rho^2)/r^6,$$

$$\Psi^{\mathrm{As}} \sim -\rho^2/2r^4 .$$

For Ψ we choose the simple form

$$\Psi = -\xi\rho^2/2(r^2 + a^2)^2,$$

which gives

$$J_\rho = \xi 2\rho x_3/(r^2 + a^2)^3,$$

$$J_3 = \xi(a^2 + x_3^2 - \rho^2)/(r^2 + a^2)^3,$$

where the parameter $\xi > 0$ will be determined to normalize u. Note that $J_\rho = J_3 = 0$ at exactly one point of the ρx_3 plane: $(\rho = a, x_3 = 0)$. To prevent $\boldsymbol{J}$ from vanishing on the circle $\rho = a, x_3 = 0$ we simply pick $J_\varphi(\rho, x_3)$ so it does not also vanish at $(\rho = a, x_3 = 0)$. A simple choice is

$$J_\varphi = \xi 2a\rho/(r^2 + a^2)^3,$$

which also happens to lead to a simple expression for J (normalized for $\xi = (a/\pi^2)$):

$$J = (a/\pi^2)(r^2 + a^2)^{-2} .$$

Substitution into formula (6) gives the gauge field

$$\begin{aligned} A = {} & (-x_2, x_1, 0)6a^2/(r^2 + a^2)^2 \\ & + (x_1, x_2, 0)6ax_3/(r^2 + a^2)^2 \\ & + (0, 0, 1)3a(a^2 + x_3^2 - \rho^2)/(r^2 + a^2)^2 \\ & + \partial\tan^{-1}(x_3/a) + \partial\chi . \end{aligned}$$

Finally, with the choice $\chi = -\tan^{-1}(x_3/a)$ we obtain from eq. (5) the wave function

$$u = \frac{\sqrt{a}}{\pi(r^2 + a^2)^{3/2}} \begin{pmatrix} x_3 - \mathrm{i}a \\ x_1 + \mathrm{i}x_2 \end{pmatrix}.$$

6. A lower bound on the magnetic flux. The radial part of $\boldsymbol{J}$ is a vector field defined on the ρ_3 halfplane:

$$\boldsymbol{J}_r \equiv (J_\rho, J_3) = \rho^{-1}(\partial\Psi/\partial x_3, -\partial\Psi/\partial\rho) , \tag{18}$$

$\boldsymbol{J}_r$ is orthogonal to the gradient of Ψ and the zeroes of $\boldsymbol{J}_r$ are the stationary or critical points of Ψ, where J_φ must be chosen not to vanish. We will assume that they are nondegenerate, namely, that the matrix of second derivatives of Ψ is nonsingular at its stationary points. (This is not too severe a restriction since the thus excluded fields are "of measure zero".) Nondegenerate critical points can be maxima, minima of saddle points. Moreover, if N_+, N_- and N_s is the total number of maxima, minima and saddle points in the ρx_3 halfplane, respectively, the integer $N_+ + N_- - N_s$ must equal the index [‡1] of the vector field $\partial\Psi$ (which is the same as that of $\boldsymbol{J}_r$ since $\boldsymbol{J}_r \perp \partial\Psi$) for a semicircle centered at the origin whose radius is large enough so that $\Psi \sim \Psi^{\mathrm{As}}$ on its circumference. By looking at the number and the relative distribution of the zeroes of P_l^{M-1} and P_l^M one finds that the index equals $l - M + 1$. Thus we have (for $M \geqslant 1$):

$$N_+ + N_- - N_s = l - M + 1 . \tag{19}$$

Actually, for $M \geqslant 1$, there is always at least one minimum i.e. $N_- \geqslant 1$ since Ψ is always negative for small θ or small $\pi - \theta$ in the asymptotic region ($|x_3| \gg \rho, r \to \infty$). This is because, as eq. (17) shows, for $|x_3| \gg \rho$, Ψ_{As} has the sign of $-x_3 J_\rho^{\mathrm{As}}$ i.e. the sign of $-\cos\theta P_l^{M-1} P_l^M$ which is positive for $\cos\theta$ near ± 1. Thus Ψ being negative in the neighbourhood of the three-axis and being zero on it must necessarily have a minimum in ρx_3 halfplane (similarly for $M \leqslant 0$ we have $N_+ \geqslant 1$).

Suppose now $(\rho_0, \bar{x})$ is a zero of $\boldsymbol{J}_r$. Then the azimuthal component of $\partial \times \boldsymbol{J}$ at any point on the circle C: $(\rho = \rho_0, x_3 = \bar{x})$ is given by

$$(\partial \times \boldsymbol{J})_\varphi|_{\mathrm{C}} = \partial J_\rho/\partial x_3 - \partial J_3/\partial\rho$$
$$= \rho_0^{-1}(\partial^2\Psi/\partial\rho^2 + \partial^2\Psi/\partial x_3^2)_{\rho_0,\bar{x}} .$$

Noting that on C we have

$$\boldsymbol{J} = (-x_2, x_1, 0)\rho^{-1} J_\varphi ,$$

we may express the magnetic flux Φ_{c} through C as a line integral on C in the positive φ direction of $\boldsymbol{A}$ given by formula (6).

$$\Phi_{\mathrm{c}} = \oint_{\mathrm{C}} \mathrm{d}\boldsymbol{l} \cdot \boldsymbol{A} = \tfrac{1}{2} \oint_{\mathrm{C}} \mathrm{d} \tan^{-1}(x_2/x_1) + (\pi\rho_0/|J_\varphi|)(\partial \times \boldsymbol{J})_\varphi$$
$$= \pi + (\pi/|J_\varphi|)(\partial^2\Psi/\partial\rho^2 + \partial^2\Psi/\partial x_3^2)_{\rho_0,\bar{x}} . \tag{20}$$

Thus if $(\rho_0, \bar{x})$ is a minimum of Ψ the second term above is positive and we have the bound

$$\Phi_{\mathrm{c}} > \pi . \tag{21}$$

Thus a necessary condition for a localized magnetic field to have a fermion bound state is that the magnetic flux through some circle centered on the three-axis and whose plane is perpendicular to the three-axis be greater than π. It would be interesting to explore whether this condition is also sufficient.

7. Conclusion. We have studied the class of admissible fermion currents $\boldsymbol{J}$ which inserted into formula (6) yield nonsingular localized static magnetic fields which admit fermionic (threshold) bound states. Although we have not obtained a direct description of this class of gauge fields, it is encouraging that at least it is not "of measure zero" within the class of static fields. The choise of the conserved current $\boldsymbol{J}$ involves essentially two arbitrary functions and so does $\boldsymbol{B} = \partial \times \boldsymbol{A}$. In this paper we have ocnsidered only axially symmetric fields but it seems that the above qualitative statement should be true quite generally. One might perhaps expect some generalization of the magnetic flux condition $\Phi > \pi$ of section 6 which represents a minimum of strength for the magnetic field configuration to be able to sustain a fermion bound state.

‡1 The index of a closed curve C with respect to a vector field $\mathfrak{v}$ is defined as the total angle by which $\mathfrak{v}$ rotates as one moves on C once around counterclockwise (see ref. [4]).

References

[1] G. Tiktopoulos, Nucl. Phys. B234 (1984) 228.
[2] E. Floratos and J. Stern, Phys. Lett. 119B (1982) 419.
[3] I. Bateman, Higher transcendental functions (Mc Graw-Hill, New York, 1957).
[4] V.I. Arnold, Ordinary differential equations (MIT Press, Cambridge, MA, 1973).