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Instanton solutions of the self-dual membrane in various dimensions

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Abstract

We present some methods of determining explicit solutions for self-dual supermembranes in $4 + 1$ and $8 + 1$ dimensions with spherical or toroidal topology. For configurations of axial symmetry, the continuous $SU(\infty)$ Toda equation turns out to play a central role, and a specific method of determining all the periodic solutions are suggested. A number of examples are studied in detail. © 1998 Elsevier Science B.V.

Nowadays, a revived interest in membrane theory [1] has been spurred by the fact that M-theory, which is considered as the leading candidate theory for explaining the non-perturbative net of string dualities, contains membranes and their dual five-branes in eleven dimensions [2]. The main activity in recent literature has been the classification of the BPS spectra of various string compactifications which presumably M-theory owes to organize in a compact and intuitive way. Among the BPS states, there is an important class made up of the Euclidean solitons (instantons). This sector plays a role in the understanding of the non-perturbative vacuum structure of string compactifications.

Some years ago, we introduced, at the level of the bosonic membranes, a specific self-duality, which in modern language is nothing but S-duality for Euclidean instantons [3]. The self-dual membranes solve $SU(N)$ Nahm's equations for a specific $N \rightarrow \infty$ limit where $SU(N)$ becomes the area-preserving diffeomorphism group on the surface of the membrane, a symmetry that exists in the light-cone quantization of the membranes.

Recently, extensions of the self-duality of membranes in 7, 8, 9 dimensions have been introduced [4,5]. In the present work, we develop new methods for solving the self-duality equations in three and seven dimensions [6]. In the case of toroidal compactifications, the role of string excitations of self-dual membranes becomes visible and we exhibit explicit examples where analytic solutions are found. In three, and also in seven dimensions, and for the

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case of cylindrical symmetry, the self-duality equations reduce to continuous Toda equations which have been studied in order to determine self-dual Euclidean solutions of Einstein equations [7]. In the present work, we provide a first-order non-linear system, the axially-symmetric three-dimensional self-duality equations, which at the same time provide a Lax pair of the axially symmetric Toda equation. Inverting this non-linear system, we find a completely integrable linear system, which we explicitly solve and, thus, we present a method to determine all the solutions of the axially symmetric Toda equations [8,9].

We start our analysis by reviewing the salient features of the theory. In Ref. [10] it was pointed out that in the large- N limit, $SU(N)$ YM theories have, at the classical level, a simple geometrical structure with the $SU(N)$ matrix potentials $A_\mu(X)$ replaced by c-number functions of two additional coordinates θ, ϕ of an internal sphere S^2 at every space time point, while the $SU(N)$ symmetry is replaced by the infinite-dimensional algebra of area-preserving diffeomorphisms of the sphere S^2 called $SDiff(S^2)$. The $SU(N)$ fields are Hermitian $N \times N$ matrices which in the large N limit are written in terms of the spherical harmonics on S^2 , while commutators are replaced by the Poisson brackets on S^2 :

$$[A_\mu, A_\nu] \rightarrow \{A_\mu, A_\nu\} = \frac{\partial A_\mu}{\partial \sigma_1} \frac{\partial A_\nu}{\partial \sigma_2} - \frac{\partial A_\nu}{\partial \sigma_1} \frac{\partial A_\mu}{\partial \sigma_2}. \quad (1)$$

In three dimensions the self-duality relation is defined by the equation ²

$$E_i = B_i, \quad (2)$$

where E_i and B_i are the electric and the magnetic $SU(\infty)$ colour fields. Since

$$E_i = \frac{\partial A_i}{\partial t}, \quad i = 1, 2, 3 \quad (3)$$

and

$$B_i = \frac{1}{2} \varepsilon_{ijk} \{A_j, A_k\}, \quad (4)$$

where ε_{ijk} is the antisymmetric tensor in three dimensions, one obtains the following equations

$$\dot{A}_i = \frac{1}{2} \varepsilon_{ijk} \{A_j, A_k\}, \quad i, j, k = 1, 2, 3. \quad (5)$$

These equations solve the Gauss constraints and the second-order Euclidean equations of motion for the bosonic part of the supermembrane (fermionic DOF set to zero) in the light-cone gauge [11]. In what follows we discuss methods of solution of the above three-dimensional system.

It has been suggested in Ref. [3] that one can use quaternions to transform the above equations into a matrix differential one. We define the matrix

$$A = A_i \sigma_i, \quad i = 1, 2, 3, \quad (6)$$

where σ_i are the standard Pauli matrices. The matrix function A satisfies the equation

$$\dot{A} = -\frac{i}{2} \{A, A\}. \quad (7)$$

In the case of the sphere, which has been analysed in Ref. [3], the Darboux coordinates are $\xi_1 = \cos \theta$, $\xi_2 = \phi$. The infinite-dimensional group $SDiff(S^2)$ has $SO(3)$ as the only finite-dimensional subgroup which is generated by the three functions $e_1 = \cos \phi \sin \theta$, $e_2 = \sin \phi \sin \theta$, $e_3 = \cos \theta$:

$$\{e_i, e_j\} = -\varepsilon_{ijk} e_k \quad (8)$$

Looking for factorized $SO(3)$ -symmetric solutions, we set $A = T_i(t) e_i$, which implies

$$\dot{T}_i = \frac{i}{4} \varepsilon_{ijk} [T_j, T_k], \quad (9)$$

that is, the Nahm equation for an $SU(2)$ monopole of magnetic charge $k = 2$ [12]. We thus obtain for each choice of solution of the Nahm equations for magnetic charge $k = 2$ (eight-dimensional moduli space) a solution of the self-duality equations.

This system of equations is known to be integrable, and particular solutions for specific boundary conditions at $t = 0$, $t = 2$ (simple poles with $SU(2)$ matrices as residues) can be expressed in terms of elliptic functions [13]. In Ref. [3], zero total angular momentum (axially symmetric) solutions of the system (5) have been explicitly determined in terms of the functions e_i and the solutions of the $SU(2)$ Toda equation.

² The anti-self dual case $E_i = -B_i$ can be treated similarly.

In the following we will show that the requirement of axial symmetry on the above system leads to a first-order system for two functions, which plays the role of the Lax pair for the continuous axially symmetric Toda equation. Indeed, the ansatz

$$A_1 = R(\sigma_1, t) \cos \sigma_2, \quad A_2 = R(\sigma_1, t) \sin \sigma_2, \\ A_3 = z(\sigma_1, t) \tag{10}$$

leads to the system

$$\dot{z} = RR' \tag{11}$$

$$\dot{R} = -Rz', \tag{12}$$

where the prime now is used to declare differentiation with respect to σ_1 (i.e. $\frac{\partial}{\partial \sigma_1}$)³. Combining Eqs. (11) and (12) we obtain the axially symmetric continuous Toda equation

$$\frac{d^2 \Psi}{dt^2} + \frac{d^2 e^\Psi}{d\sigma_1^2} = 0, \tag{13}$$

where $R^2 = e^\Psi$. Solutions of this equation have been discussed in the literature in connection with the self-dual 4d Einstein metrics with rotational and axial Killing vectors [7,8]. Here though, we note that σ_1 runs in a compact interval $(0, 2\pi)$ for torus and $(-1, 1)$ for the sphere.

At this point, we want to provide a specific example of a solution with separation of variables of the Toda equation, in the case of spherical topology ($\sigma_1 = \cos \theta$, $\sigma_2 = \phi$). Separation of variables $R(\sigma_1, t) = R_1(\sigma_1)R_2(t)$ corresponds to $\Psi(\sigma_1, t) = \Theta(\sigma_1) + T(t)$ which leads to

$$\frac{d^2 T}{dt^2} - ke^T = 0 \tag{14}$$

$$\frac{d^2 e^\Theta}{d\sigma_1^2} + k = 0, \tag{15}$$

Multiplying (14) by \dot{T} we obtain

$$\frac{dT}{dt} = \sqrt{2k} \left(e^T + \frac{v}{k} \right)^{1/2}, \tag{16}$$

where v is a new constant. Eq. (16) is easily solved, making use of the transformation $e^T = \frac{v}{k} \phi^{-2}$, the final result being

$$R(\theta, t) = \kappa \frac{\sin \theta}{\sinh[\kappa(t_0 - t)]} \tag{17}$$

$$z(\theta, t) = \kappa \coth[\kappa(t_0 - t)] \cos \theta, \tag{18}$$

where κ is a new constant. Interestingly, this solution coincides with that of [3] representing axisymmetric ellipsoids, which was derived from the $SU(2)$ Toda equation (with respect to the time t).

We now exhibit a variation of the method of Ref. [14] where by inversion of the non-linear system (12) we construct a linear one and we determine all solutions. Indeed, by going from the pair of variables (R, z) to (S, T) , which we take to be the inverse mapping $(\sigma_1, t) \rightarrow (R, z)$, we find

$$\frac{\partial S}{\partial u} - \frac{\partial T}{\partial v} = 0 \tag{19}$$

$$\frac{\partial S}{\partial v} + u \frac{\partial T}{\partial u} = 0, \tag{20}$$

where $u = R^2$ and $v = 2z$. This system is linear and we can easily separate the variables u and v , $S = S_1(u)S_2(v)$, $T = T_1(u)T_2(v)$. We introduce two constants of separation,

$$\frac{\partial S_1}{\partial u} = \lambda T_1, \quad -u \frac{\partial T_1}{\partial u} = \mu S_1 \\ \frac{\partial T_2}{\partial v} = \lambda S_2, \quad \frac{\partial S_2}{\partial v} = \mu T_2. \tag{21}$$

We see that S_2 and T_2 are trigonometric (hyperbolic) functions of v depending on whether the sign of the product $\lambda \cdot \mu$ is minus (plus). Also, from the first order equations of S_1 and T_1 , assuming analyticity around $u = 0$, we obtain unique solutions $T_1 \propto J_0(k_0 R)$ and $S_1 \propto RJ_1(k_0 R)$, and $k_0 = \sqrt{\lambda \cdot \mu}$. By appropriate linear combinations of the solutions of S_1, T_1 and S_2, T_2 , we can determine functions S and T which, by inversion, give functions R, z , periodic in σ_1 . As a demonstration, consider the solution

$$S = \iota A \cos(k_0 z) RJ_1(\iota k_0 R) + k_1 z \tag{22}$$

$$T = A \sin(k_0 z) J_0(\iota k_0 R) - k_1 \ln R, \tag{23}$$

where A, k_1, k_0 are real constants. If space is compactified in the z -direction with length L , and we want σ_1 to range from 0 to 2π , we choose $k_1 =$

³ We observe that, if we replace σ_2 by $n\sigma_2$, n integer, then this implies that $t \rightarrow nt$ in the original solution.

$2\pi/L$ and $k_0 = nk_1$ for some integer n . The above then represents a membrane with n branches extending to $R = \infty$, which, at some critical time, collides with itself and separates into a finite piece with toroidal topology, exhibiting n ripples within the period L , and n infinite pieces that fly away. We leave the question of explicit constructions for a future work. We should note, though, that the linearization method of Ref. [14] should be examined in more detail in order to construct other interesting examples.

We finally discuss in three dimensions two sorts of toroidal compactifications where by double compactification we derive string self-dual solutions. First, when the three dimensional space topology is $R^2 \times S^1$, we doubly compactify the membrane [1]. We choose as an example $A_3 = n\sigma_2$ and $A_{1,2} = A_{1,2}(\sigma_1, t)$. Then it is straightforward to see that $A_1 + iA_2$ must be an analytic function of $\sigma_1 - int$, where n is the winding number. These are world-sheet string instantons.

The second compactification is on the three-dimensional torus T^3 , where windings for various embeddings of toroidal membrane lead to string excitations with non-zero center-of-mass momentum. We discuss this case below, where more general seven-dimensional embeddings are studied.

We now extend our discussion in seven dimensions, where the fully antisymmetric symbol of three dimensions ε_{ijk} in Eqs. (5) is replaced by the corresponding octonionic structure constants Ψ_{ijk} [4,5]:

$$\dot{X}_i = \frac{1}{2} \Psi_{ijk} \{X_j, X_k\}, \quad (24)$$

where the indices run from 1 to 7 while Ψ_{ijk} is completely antisymmetric and has the value 1 for the following combinations of indices:

$$\Psi_{ijk} = \begin{pmatrix} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \end{pmatrix}. \quad (25)$$

The second-order Euclidean equations and the Gauss law results automatically by making use of the Ψ_{ijk} cyclic symmetry $\{\dot{X}_i, X_i\} = 0$. In Ref. [5], one class of three-dimensional solutions which are embedded in the seven-dimensional system was found according to the identifications

$$X_3 \rightarrow A_3, \quad X_{\pm} \rightarrow A_{\pm}/\sqrt{3}, \quad (26)$$

where the seven coordinates $X_i, (i = 1, 2, \dots, 7)$ are grouped in terms of the complex coordinates $X_{\pm} = X_1 \pm iX_2$, $Y_{\pm} = X_4 \pm iX_5$, and $Z_{\pm} = X_6 \pm iX_7$ and we have made the ansatz that $X_+ = Z_+ = iY_-$, while $A_{\pm,3}$ is the three-dimensional solution. The seven-dimensional solution is essentially the three-dimensional one rotated by an orthogonal transformation in 7-space. Therefore, any three-dimensional self-dual solution automatically generates a corresponding 7 dimensional one.

The generalization to the string-like solution of the self-duality Eq. (24) in 7 dimensions is straightforward. We assume the form

$$X_i(\sigma_{1,2}, t) = A_i \sigma_1 + B_i \sigma_2 + P_i t + f_i(\sigma_1, \sigma_2, t), \quad (27)$$

with $i = 1, \dots, 7$, and f being a periodic function of $\sigma_{1,2}$ and A, B integer vectors. Then we obtain

$$P_i = \Psi_{ijk} A_j B_k \quad (28)$$

$$\dot{f}_i = \Psi_{ijk} \left(A_j \frac{\partial}{\partial \sigma_2} - B_j \frac{\partial}{\partial \sigma_1} \right) f_k + \frac{1}{2} \Psi_{ijk} \{f_j, f_k\}. \quad (29)$$

Since f is a periodic function with respect to $\sigma_{1,2}$, we can expand it in terms of an infinite number of strings, depending on the coordinate σ_1 :

$$f_i(\sigma_1, \sigma_2, t) = \sum_n X_i^n(\sigma_1, t) e^{in\sigma_2}. \quad (30)$$

Then, from the self-duality Eqs. (28), (29) we find that the winding number of the membrane is related to the center-of-mass momentum, which is transverse to the compactification directions A and B . Also, the infinite number of strings are coupled through the following equations

$$\dot{X}_i^n(\sigma_1, t) = \Psi_{ijk} \left(A_j n - B_j \frac{\partial}{\partial \sigma_1} \right) X_k^n + \frac{i}{2} \Psi_{ijk} \sum_{n_1+n_2=n} \left(n_2 \frac{\partial X_j^{n_1}}{\partial \sigma_1} X_k^{n_2} - n_1 X_j^{n_1} \frac{\partial X_k^{n_2}}{\partial \sigma_1} \right). \quad (31)$$

The string-like solution corresponds to the particular case $\partial f_i / \partial \sigma_2 = 0$, where we obtain

$$X_i^0 = X_i(\sigma_1, t) \rightarrow \dot{X}_i = \Psi_{ijk} B_k \frac{\partial X_j}{\partial \sigma_1}. \quad (32)$$

This equation is formally solved in vector form by

$$X(\sigma_1, t) = e^{tM} \frac{\partial}{\partial \sigma_1} X(\sigma_1, 0), \quad (33)$$

where we defined the 7×7 matrix $M_{ij} = \Psi_{ijk} B_k$. Explicit solutions are found by expanding X_i in terms of the eigenvectors of M . In fact, since M is real and antisymmetric, the real 7-dimensional vector space decomposes into three orthogonal two-dimensional subspaces, each corresponding to a pair of imaginary eigenvalues $\pm i\lambda$, and a one-dimensional subspace, in the direction of B_i , corresponding to the zero eigenvalue. Since, in addition, $(M^2)_{ij} = -B^2 \delta_{ij} + B_i B_j$ (as can be checked), we see that the imaginary eigenvalue pairs are all $\pm i|B|$. Therefore the problem decomposes into three 3-dimensional problems (one for each subspace) of the kind we solved before. The general solution is then

$$X_1^{(n)} + iX_2^{(n)} = F_n(\sigma_1 - iBt), \quad n = 1, 2, 3 \quad (34)$$

$$X^{(0)} = |B|t, \quad (35)$$

where $(X_1^{(n)}, X_2^{(n)})$ are the projections of the membrane coordinates on the n -th two-dimensional eigenspace and $X^{(0)} = X_i B_i / |B|$ is the projection on B_i . As an example, if we choose B_i in the third direction, $B_i = B\delta_{i3}$, we have

$$X_1 + iX_2 = F_1(\sigma_1 - iBt) \quad (36)$$

$$X_5 + iX_4 = F_2(\sigma_1 - iBt) \quad (37)$$

$$X_6 + iX_7 = F_3(\sigma_1 - iBt) \quad (38)$$

$$X_3 = Bt. \quad (39)$$

Considering, now, the case when at $t = 0$ we have a proper (not string-like) membrane configuration, with its periodic part dependent on both variables, we write

$$X_i = f_i^{\text{cl}} + f_i, \quad (40)$$

where $f_i^{\text{cl}} = A_i \sigma_1 + B_i \sigma_2$. The equation of the general case (24) can be written in a symbolic form, by defining the matrix differential operator

$$L_f^{ik} = \Psi_{ijk} \left(\frac{\partial f_j}{\partial \sigma_1} \frac{\partial}{\partial \sigma_2} - \frac{\partial f_j}{\partial \sigma_2} \frac{\partial}{\partial \sigma_1} \right) \quad (41)$$

as a vector equation

$$\dot{f} = \left(L_f^{\text{cl}} + \frac{1}{2} L_f \right) f. \quad (42)$$

It is possible to solve this non-linear matrix differential system by iteration of the solution of its linear part,

$$\dot{g} = L_f^{\text{cl}} g. \quad (43)$$

The above differential system can be written as a matrix integral equation as follows

$$f = g + \frac{1}{2} e^{tL_f^{\text{cl}}} \int_t^0 e^{-t'L_f^{\text{cl}}} L_f f dt'. \quad (44)$$

It is easy to show that the infinite iteration of the solution g solves the non-linear differential system and, moreover, when the initial configuration is a string, the second part of the integral equation is zero and the problem is reduced to the homogeneous case. The general solution of the homogeneous system (43) is

$$g(t) = e^{tL_f^{\text{cl}}} g(t=0), \quad (45)$$

where $f_i(t=0) = g_i(t=0)$.

At this point, we would like to note that in ref [5] for the case of zero-winding we have been able to separate the time and the parameter dependence of the coordinates of the octonionic self-dual membrane. The time equations are generalizations of Nahm matrix Eqs. (9), where in the place of the three $SU(2)$ - T_i matrices, a pair T_i, S_i appears. A generalization of the Euler Top equations using octonions has been proposed in ref [6], where it was shown that this system is an integrable one and the explicit set of seven conservation laws including their algebraic relation has been provided. This system of equations is a specific case of the generalized Nahm equations when T_i, S_i are proportional to the Pauli matrices. Thus, for every solution of the generalized Euler Top system, one can obtain the corresponding self-dual membrane.

We close our short analysis by pointing out the existence of a different kind of self-duality equations which satisfy also the second-order Euclidean equations which has been introduced for self-dual Yang-Mills fields in Ref. [15].⁴ This system of equations could be generalized to membranes embedded in dimensions $D = \dim(G)$ where G is any Lie algebra.

⁴ We thank T. Ivanova for bringing her work with A. Popov to our attention.

This system of self-duality equations is an integrable one (as it was pointed out to us by T. Ivanova) but the geometrical significance for the dynamics of the self-dual membrane is not obvious to us. On the other hand, it is interesting to see what type of world-volume membrane instantons are obtained by this method.

We would like to conclude with few remarks. A systematic approach for the solutions of the seven-dimensional equations has been proposed in the case of toroidal compactifications which turn out to provide world volume membrane instantons which play an important role in the understanding of the vacuum structure of supermembrane theory. The question of the surviving supersymmetries for various classes of solutions is an important problem for the determinations of the BPS states of the supermembrane. The richness of the self-duality equations concerning string excitations suggest that probably it is the right framework of examining the non-perturbative unification of string interactions. This goes along the lines of an old suggestion that supermembranes are string solitons or coherent states of interacting strings. It remains to be seen if the strong coupling problem of string interactions is tamed by the determination of the correct non-perturbative string vacuum.

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