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IRRELEVANT OPERATORS AND EQUIVALENT FIELD THEORIES *

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The model described by the Lagrangian

$$L = \bar{\psi}(i\gamma \cdot \partial + g_0\sigma)\,\psi + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\mu_0^2\sigma^2 + \sum_{n=4,6\ldots\infty} \frac{g_{0n}}{n!}\sigma^n$$

is studied in 1 + 1 dimensions. Wilson's renormalization prescription is adopted and a non-trivial infrared stable fixed point is obtained in the mean-field approximation. We demonstrate that at the fixed point the operators $\sigma\partial^2\sigma$ and $g_{0n}\sigma^n$ for $n = 4, 6, \ldots \infty$ become irrelevant and the theory becomes equivalent to the Gross-Neveu model

$$L = \bar{\psi}(i\gamma \cdot \partial)\,\psi + \tfrac{1}{2}f_0(\bar{\psi}\psi)^2\ .$$

The equivalence is independent of the values of the dimensionless bare couplings of the original model (universality). Similar results are obtained in the case of other models in 3 + 1 dimensions.

1. Introduction

In this paper we discuss in 1 + 1 dimensions the model described by the Lagrange density

$$L = \bar{\psi}(i\gamma \cdot \partial + g_0\sigma)\,\psi + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\mu_0^2\sigma^2 + \sum_{n=2}^{\infty} \frac{g_{02n}}{(2n)!}\sigma^{2n}\ .$$

The motivation for the study of this model was provided by previous work concerning the equivalence of the Yukawa and four-fermion theories expanded in a mean-field expansion [1] and by previous work of Wilson on field theoretic models in $4-\epsilon$ dimensions [2].

The model is expanded in a saddle-point expansion (mean-field expansion) after the fermion degrees of freedom have been exactly integrated out. The resulting

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theory is super-renormalizable in this expansion just as it is in ordinary perturbation theory. Following Wilson we give the dimensional bare couplings a "cut-off" dependence and keep the resultant dimensionless bare couplings fixed while this cut-off is sent to infinity. Thus, we define

$$g_0 = \lambda_0 \Lambda \qquad \text{and} \qquad g_{0n} = \lambda_{0n} \Lambda^2 \qquad \text{for} \qquad n = 4, 6, \ldots .$$

Conventionally the dimensional couplings are kept fixed. The renormalized theory that results from Wilson's limiting prescription is exactly equivalent to all orders in our expansion scheme to the Gross-Neveu model which is not a super-renormalizable theory. We shall show that the equivalence of the two theories occurs because they correspond to the same infrared fixed point of the renormalization group.

The mean-field expansion is being used here mainly for two reasons. First, it provides us, to lowest order, with a non-trivial fixed point as long as we adopt the already stated prescription. Second, in the case of the Gross-Neveu model it coincides with the $1/N$ expansion in which this model has been extensively studied [3].

When we approach the fixed point, certain operators appearing in the Lagrangian have vanishing contributions. This situation is the field theoretic analogue of a system that approaches the critical region. Near criticality, certain operators become irrelevant. As we shall show, the meson kinetic energy operator becomes irrelevant and consequently, the meson field ceases to be a fundamental field. This is to be expected since at the fixed point the renormalization factor of the meson field vanishes. The vanishing of a renormalization factor is a compositeness condition. The compositeness of the σ-operator converts the original theory of interacting fermions and mesons into a theory of self-interacting fermions.

With the prescription we use, the fixed point is reached whatever the values of the bare dimensionless couplings $\lambda_0, \lambda_{04}, \lambda_{06}$ .... Thus, the renormalized theory at the fixed point is independent of the strengths of all original interactions. This is universality [4].

The above phenomena are not solely characteristic of the Yukawa-type model. We also study a model with only scalar fields in 3 + 1 dimensions. This is described by the Lagrangian

$$L(\phi, \chi) = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 + \tfrac{1}{2} g_0 \chi \phi^2 + \tfrac{1}{2}(\partial_\mu \chi)^2 - \tfrac{1}{2}\mu_0^2 \chi^2 + \frac{g_{03}}{3!}\chi^3 + \frac{g_{04}}{4!}\chi^4 .$$

We demonstrate that in the mean-field approximation, this model is equivalent to the ordinary ϕ^4 theory

$$L(\phi) = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m_0^2 \phi^2 + \frac{\lambda_0}{4!}\phi^4 .$$

A possible physical interpretation of our approach is that the original Lagrangian contains a fundamental mass scale (the mass of some super heavy boson M_{B}). Making the assumption that all dimensional bare couplings have a dependence on this mass scale converts our theory into a critical theory since we are interested in momenta and masses in the infrared region q, $m \ll M_B$ [5]. The possibility that high-energy physics could be described as a critical phenomenon has received considerable attention. However, meaningful field theoretic models in four-dimensional space-time with non-trivial fixed points wait to be discovered.

2. The model

Let us consider in one space and one time dimensions the model described by

$$L = \bar{\psi}(i\gamma \cdot \partial + g_0\sigma)\,\psi + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\mu_0^2\sigma^2 + \sum_{n=2}^{\infty} \frac{g_{02n}}{(2n)!}\,\sigma^{2n}\,. \tag{2.1}$$

This model has been extensively studied elsewhere concurrently [6]. For completeness we repeat some of the results and in some cases reformulate them according to our present approach.

The generating functional of the field theory described by (2.1) is

$$Z(J, \bar{\eta}, \eta) = \int \mathrm{d}\sigma \mathrm{d}\psi \mathrm{d}\bar{\psi}\, \exp[i \int \mathrm{d}^2x(L + J\sigma + \bar{\eta}\psi + \bar{\psi}\eta)]\,.$$

J, $\bar{\eta}$ and η are c-number sources coupled to the fundamental fields appearing in (2.1). The fermions enter quadratically and can be integrated out exactly. The resulting functional in Euclidean space is

$$Z(J, \bar{\eta}, \eta) = \mathrm{e}^W = \int \mathrm{d}\sigma\, \exp\left\{\bar{\eta}S^{-1}\eta + \mathrm{tr}\ln S^{-1} + \tfrac{1}{2}\sigma(-\partial^2 - \mu_0^2)\,\sigma \right.$$
$$\left. + \sum_{n=4,6,\ldots\infty} \frac{g_{0n}}{n!}\,\sigma^n\right\}\,. \tag{2.2}$$

For convenience we have introduced

$$S^{-1}(x, y) = (i\gamma_\mu\partial_x^\mu + g_0\sigma(x))\,\delta^{(2)}(x - y)\,.$$

Up to this point we have made no approximation. The generating functional has been expressed as an integral over boson degrees of freedom only. Fermion Green functions can still be derived from it by taking variational derivatives with respect to the fermion sources.

Our next step is to introduce by hand a parameter ϵ in the generating functional.

$$Z_\epsilon = \mathrm{e}^{W/\epsilon} = \int \mathrm{d}\sigma\, \mathrm{e}^{-F\{\sigma, J, \bar{\eta}, \eta\}/\epsilon}\,.$$

Our approach will be to expand Z_ϵ in a Laplace expansion [7]. The connected functional $W = \epsilon \ln Z_\epsilon$ has the form

$$W \simeq W_0 + \epsilon W_1 + \epsilon^2 W_2 + \dots .$$

The order of the approximation is identified by the power of ϵ. If our integral were an ordinary integral this expansion would be an asymptotic expansion in ϵ as long as ϵ is small. In our case ϵ is just a book-keeping device and at the end of the computations it must be set equal to one. One of the renormalized parameters has to be identified as an alternative small parameter. The expanded functional is

$$Z_\epsilon \sim e^{-F\{\sigma_0\}/\epsilon} \exp[-\tfrac{1}{2}\mathrm{tr} \ln A] \left\{1 - \frac{\epsilon}{8}\iiiint C(xyzw) A^{-1}(xy) A^{-1}(zw) \right.$$
$$+ \frac{\epsilon}{24} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int B(xyz) B(abc)[2A^{-1}(xa) A^{-1}(yb) A^{-1}(zc)$$
$$\left. + 3A^{-1}(xy) A^{-1}(za) A^{-1}(bc)] + O(\epsilon^2)\right\} . \tag{2.3}$$

Here, we have introduced the mean field σ_0 defined by the minimum conditions

$$\left.\frac{\delta F}{\delta \sigma}\right|_{\sigma_0} = 0 , \quad \left.\frac{\delta^2 F}{\delta\sigma\delta\sigma}\right|_{\sigma_0} \equiv A(xy) > 0 . \tag{2.4}$$

The other functions appearing in (2.3) are defined as

$$C(x, y, z, w) \equiv \left.\frac{\delta^4 F}{\delta\sigma(x)\, \delta\sigma(y)\, \delta\sigma(z)\, \delta\sigma(w)}\right|_{\sigma_0} ,$$

$$B(x, y, w) \equiv \left.\frac{\delta^3 F}{\delta\sigma(x)\, \delta\sigma(y)\, \delta\sigma(z)}\right|_{\sigma_0} .$$

The lowest-order connected functional of our model according to (2.3) corresponds to the tree approximation of the effective Lagrangian $F\{\sigma_0, J, \bar{\eta}, \eta\}$.

$$W_0 = \bar{\eta} S^{-1} \eta + \mathrm{tr} \ln S^{-1} + \tfrac{1}{2}\sigma_0(-\partial^2 - \mu_0^2)\, \sigma_0 + \sum_{n=4,6\dots} \frac{g_{0n}}{n!} \sigma_0^n + J\sigma_0 .$$

The original Lagrangian (2.1) was invariant under the discrete transformation $\psi \to \gamma_5 \psi$, $\sigma \to -\sigma$. A bare mass for the fermion destroys this symmetry. We assume that the exact vacuum of the theory does not respect this symmetry and thus breaks it spontaneously. In the spontaneously broken theory the vacuum expectation value of σ is non-zero and a fermion mass proportional to it is generated. Of course, no Goldstone boson is associated with such discrete symmetry breaking. The lowest-order fermion propagator is *

$$S^{-1}(P) = \not{P} - m ,$$

* We could have started with a bare mass m_0. In that case there would be no reason to exclude odd self-couplings.

where

$$m \equiv -g_0\sigma_0 \ .$$

The set of all connected Green functions can be generated from the connected functional by taking variational derivatives with respect to the sources. The 1PI vertex functions can also be derived from the effective action by taking variational derivatives with respect to the classical fields.

The classical meson field to lowest order coincides with the mean field defined by (2.4). The lowest-order mean-field condition for non-zero mean field takes the form

$$\mu_0^2 = 2ig_0^2 \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (k^2 - (g_0\sigma_0)^2)^{-1} + \sum_{n=4,6,\dots} \frac{g_{0n}}{(n-1)!} \sigma_0^{n-2} \ . \tag{2.5}$$

The σ-propagator to zeroth order in the absence of sources is easily derived and is

$$\frac{\delta^2 \mathrm{T}}{\delta\sigma_c \delta\sigma_c} = \Delta_\sigma^{-1}(p^2) = p^2 - \mu_0^2 + g_0^2 \Pi(p^2) + \sum_{n=4,6,\dots} \frac{g_{0n}}{(n-2)!} \sigma_0^{n-2} \ .$$

The function $\Pi(p^2)$ stands for the fermion bubble and is

$$\Pi(p^2) \equiv i \ \mathrm{tr} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} S(k)\, S(k+p) = i \ \mathrm{tr} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} (\not{k} - m)^{-1} (\not{k} + \not{p} - m)^{-1}$$

$$= 2i \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{k^2 + k \cdot p + m^2}{[k^2 - m^2]\,[(k+p)^2 - m^2]} = 2i \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \int_0^1 \mathrm{d}x \frac{k^2 - p^2 x(1-x) + m^2}{k^2 + p^2 x(1-x) - m^2)^2}$$

The logarithmic divergence of the fermion bubble can be cancelled by the logarithmic divergence already contained in the bare mass μ_0^2 according to (2.5). In other words, one subtraction suffices. Instead we will subtract twice and introduce wave-function and mass renormalization according to the following conditions at zero momentum

$$\Delta_\sigma^{-1}(0) = -\frac{\mu^2}{Z_\sigma} \ , \qquad \left(\frac{\partial \Delta_\sigma^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\sigma} \ .$$

To lowest order we have $g^2 = g_0^2 Z_\sigma$. Thus, the renormalized propagator is

$$\bar{\Delta}_\sigma^{-1}(p^2) = Z_\sigma \Delta_\sigma^{-1}(p^2) = p^2 - \mu^2 + g^2 \ \mathrm{sub}_0^2 \Pi(p^2) \ .$$

The subtraction symbol stands for $\Pi(p^2) - \Pi(0) - p^2 \,[\partial\Pi/\partial p^2]_0$. The wave-function renormalization Z_σ is easily calculated from its definition. It is

$$Z_\sigma = \left(1 + \frac{g_0^2}{12\pi m^2}\right)^{-1} = 1 - \frac{g^2}{12\pi m^2} \ . \tag{2.6}$$

Consequently, the renormalized coupling is related to the bare coupling through

$$\frac{1}{g^2} = \frac{1}{g_0^2} + \frac{1}{12\pi m^2} \ . \tag{2.7}$$

The higher boson vertex functions are obtained from the effective action by taking variational derivatives with respect to the classical meson field. We find *

$$\Gamma^{(\kappa\sigma)}(p_1 \dots p_{\kappa-1}) = i(-g_0)^\kappa \operatorname{tr} \int \frac{d^2q}{(2\pi)^2} (S(q)\, S(q+p_1) \dots S(q+p_1+\dots+p_{\kappa-1}))$$

$$+ ((\kappa-1)!\ \mathrm{XT}) + \sum_{n \geqslant \kappa} \frac{g_{0n}}{(n-\kappa)!}\, \sigma_0^{n-\kappa}\,. \tag{2.8}$$

Here the κ-fermion polygons,

$$\Pi^{(\kappa)}(p_1 \dots p_{\kappa-1}) \equiv i(-1)^\kappa \operatorname{tr} \int \frac{d^2q}{(2\pi)^2} S(q) \dots S(q+p_1+\dots p_{\kappa-1})\,,$$

are finite and consequently the $\kappa\sigma$ vertex functions do not require any renormalization. Although no subtraction is required, we subtract once at zero momentum and define renormalized σ self-couplings through the normalization conditions

$$\Gamma^{(\kappa\sigma)}(0, \dots, 0) = \frac{g_\kappa}{Z_\sigma^{\kappa/2}}\,, \qquad \kappa = 3, 3, 5, \dots \infty\,.$$

The even boson self-couplings can be expressed in terms of the bare self-couplings through the introduction of new renormalization factors Z_κ.

$$g_\kappa = g_{0\kappa} Z_\sigma^{\kappa/2}/Z_\kappa\,, \qquad \kappa = 4, 6, \dots, \infty\,.$$

Our oversubtraction scheme has been designed to parametrize the renormalized theory in a form that will bring to the surface the limiting equivalence we have in mind. Going back to the $\kappa\sigma$ vertex functions we define the renormalized functions as

$$\overline{\Gamma}^{(\kappa\sigma)}(p_1 \dots p_{\kappa-1}) = Z_\sigma^{\kappa/2} \Gamma^{(\kappa\sigma)}(p_1 \dots p_{\kappa-1})\,.$$

In agreement with our normalization conditions the renormalized vertex functions are

$$\overline{\Gamma}^{(\kappa\sigma)}(p_1 \dots p_{\kappa-1}) = g_\kappa + (g)^\kappa\, \mathrm{sub}_0^1\, \{\Pi^{(\kappa)}(p_1 \dots p_{\kappa-1})\} \quad ((\kappa-1)!\ \mathrm{XT}) \tag{2.9}$$

The fermion polygons with all momentum zero can be easily calculated. They are given by the formula

$$\Pi^{(\kappa)}(0, \dots, 0) = \frac{m^{2-\kappa}}{\pi(\kappa-2)(\kappa-1)}\,, \qquad \kappa = 3, 4, 5, 6 \dots \infty$$

The renormalization factors Z_κ are given by the following expression.

$$\frac{1}{Z_\kappa} = \frac{(g_0)^\kappa (\kappa-3)!\, m^{2-\kappa}}{\pi g_0} + \frac{(g_0)^\kappa}{g_{0\kappa}} \sum_{n=4,6\dots} \frac{g_{0n}/(g_0)^n}{(n-\kappa)!} (-m)^{n-\kappa}\,, \tag{2.10}$$

* The symbol (XT) means: crossed terms.

κ takes even values. Substituting in the above expression the renormalized couplings, we obtain the sum rule

$$1 = \frac{(g)^\kappa}{g_\kappa} \frac{(\kappa - 3)!}{\pi} m^{2-\kappa} + \frac{(g)^\kappa}{g_\kappa} \sum_{n=4,6..} \frac{g_n/(g)^n}{(n-\kappa)!} Z_n (-m)^{n-\kappa} . \tag{2.11}$$

The renormalized σ-propagator is defined by the equation

$$\frac{\mu^2}{Z_\sigma} = \mu_0^2 - g_0^2 \Pi(0) - \sum_{n=4,6...} \frac{g_{0n}}{(n-2)!} \sigma_0^{n-2}$$

In terms of the gap equation (2.5) the above is reduced to

$$\mu^2 = \frac{g^2}{\pi} + g^2 \sum_{n=4,6...} \frac{g_n}{(g)^n} Z_n \frac{(-m)^{n-2}(n-2)}{(n-1)!} . \tag{2.12}$$

The renormalized theory has been parametrized in terms of the renormalized couplings g, g_3, g_4, g_5, and the fermion mass m. The renormalized meson mass is not an independent parameter but is a function of the couplings and m . The renormalization factors Z_σ and Z_n are finite functions of these parameters. Thus (2.12) can serve for the computation of this mass. μ^2 is not the physical meson mass but is related to it in a simple way. The physical meson mass is a solution of

$$\Delta_\sigma^{-1}(\mu_\sigma^2) = 0 \qquad \text{or} \qquad \mu_\sigma^2 = \mu^2 - g^2 \,\mathrm{sub}_0^2 \Pi(\mu_\sigma^2) .$$

3. The fixed point

The bare theory was parametrized in terms of the dimensional couplings $g_0, g_{04}, \ldots$ and the mass μ_0^2. The divergence associated with μ_0^2 suggests the presence of another mass in the bare theory, the cut-off Λ^2. In the renormalized theory the meson mass is exchanged for a new mass parameter, the fermion mass m which arose due to the asymmetric vacuum. We are going to make the assumption that the bare couplings are scaled in terms of the cut-off according to

$$g_0^2 = \lambda_0^2 \Lambda^2 \qquad \text{and} \qquad g_{0n} = \lambda_{0n} \Lambda^2 \qquad \text{for} \qquad n = 4, 6, \ldots \infty \tag{3.1}$$

Here $\lambda_0, \lambda_{04}, \lambda_{06}, \ldots$ are dimensionless. The renormalized theory is characterized by the couplings $g, g_3, g_4, g_5, \ldots$ and the mass m. We can define dimensionless renormalized parameters using the scale of the renormalized theory m;

$$g^2 = \lambda^2 m^2 \qquad \text{and} \qquad g_n = \lambda_n m^2 \qquad n = 3, 4, 5, 6, \ldots . \tag{3.2}$$

Here $\lambda, \lambda_3, \lambda_4, \ldots$ are dimensionless.

The renormalizability of the model can be summarized in the statement

$$\overline{\Gamma}^{(n\sigma)}(p_1 \ldots p_{n-1}; m, \lambda^2, \lambda_3, \ldots) = Z_\sigma^{n/2} \Gamma^{(n\sigma)}(p_1 \ldots p_{n-1}; \mu_0^2, g_0, \ldots, \Lambda^2) .$$

Dimensional analysis implies that

$$Z_\sigma = Z_\sigma\left(\lambda^2, ..., \lambda_n, ...; \frac{m}{\Lambda}\right),$$

$$Z_\kappa = Z_\kappa\left(\lambda^2, ..., \lambda_n, ...; \frac{m}{\Lambda}\right).$$

Considering the arbitrary vertex function $\Gamma^{(n\sigma)}$ and taking a derivative with respect to ln m while keeping g_0, g_{04}, ... and Λ^2 fixed, we obtain

$$\left(m\frac{\partial \Gamma^{(n\sigma)}}{\partial m}\right)_{g_0,...,\Lambda^2} = \overline{Z}_\sigma^{n/2}\left(m\frac{\partial}{\partial m} + \beta(\lambda^2, ...)\frac{\partial}{\partial \lambda^2} + \sum_{\kappa=3}^{\infty}\beta_\kappa(\lambda^2, ...)\frac{\partial}{\partial \lambda_\kappa}\right.$$

$$\left. - n\gamma_\sigma(\lambda^2, ...)\right)\overline{\Gamma}^{(n\sigma)} = \left(m\frac{\partial \mu_0^2}{\partial m}\right)_{g_0,...\Lambda^2}\frac{\partial \Gamma^{(n\sigma)}}{\partial \mu_0^2}.$$

We have introduced the well-known renormalization group β and γ functions defined in the usual manner as

$$\beta(\lambda^2, ...) \equiv \left(m\frac{\partial \lambda^2}{\partial m}\right)_{g_0^2,...\Lambda^2},$$

$$\beta_\kappa(\lambda^2, ...) \equiv \left(m\frac{\partial \lambda_\kappa}{\partial m}\right)_{g_0,...\Lambda^2}, \qquad \kappa = 3, 4, ...\infty,$$

$$\gamma_\sigma(\lambda^2, ...) \equiv \left(m\frac{\partial \ln Z_\sigma}{\partial m}\right)_{g_0,...\Lambda^2}.$$

Instead of expressing the right-hand side in terms of a derivative with respect to the bare mass μ_0^2 we can express it as a derivative with respect to the constant classical field σ which is a "bare-mass parameter" related to μ_0^2 through the gap equation. The Callan-Symanzik equations then become

$$\left(m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial \lambda^2} + \sum_{\kappa=3}^{\infty}\beta_\kappa\frac{\partial}{\partial \lambda_\kappa} - n\gamma_\sigma\right)\overline{\Gamma}^{(n\sigma)}(...) = Z_\sigma^{n/2} m\frac{\partial \sigma}{\partial m}\Gamma^{((n+1)\sigma)}(0; ...).$$

Considering the analogous equation for the inverse fermion propagator, we obtain

$$\left(\frac{\partial \sigma}{\partial m}\right)_{g_0,...\Lambda^2} = -Z_\sigma^{1/2}(1 - 2\gamma_\psi)/g.$$

Here γ_ψ stands for the anomalous dimension of the fermion field. Thus, the Callan-

Symanzik equations take the form

$$\left(m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda^2}+\sum_{\kappa=3}^{\infty}\beta_\kappa\frac{\partial}{\partial\lambda_\kappa}-n\gamma_\sigma\right)\overline{\Gamma}^{(n\sigma)}(\ldots)=-\frac{m}{g}(1-2\gamma_\psi)\,\overline{\Gamma}^{((n+1)\sigma)}(0;\ldots) \tag{3.3}$$

For $p_i^2/m^2 \to \infty$ (asymptotic region) the right-hand side becomes negligible due to Weinberg's theorem and the Callan-Symanzik equations reduce to the renormalization group equations. This is the region that is of interest in critical phenomena because that is where the correlation length of the renormalized theory $\xi^2 = 1/m^2$ becomes much larger than the length scale $x^2 = 1/p^2$ we are looking at

$$\xi^2/x^2 \to \infty\,.$$

The above equation becomes then

$$\left(m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial\lambda^2}+\sum_{\kappa=3}^{\infty}\beta_\kappa\frac{\partial}{\partial\lambda_\kappa}-n\gamma_\sigma\right)\overline{\Gamma}^{(n\sigma)}(\text{asym})\simeq 0\,.$$

Starting from the definition of the β and γ functions we can calculate them in the zeroth-order mean-field approximation. The Yukawa-type coupling leads to a β-function

$$\beta(\lambda^2)=-2\lambda^2\left(1-\frac{\lambda^2}{12\pi}\right).$$

The anomalous dimension of the meson field is

$$\gamma_\sigma(\lambda^2)=\frac{\lambda^2}{12\pi}\,.$$

The β-functions of the boson self-couplings are easily computed to be

$$\beta_\kappa(\lambda^2,\ldots)=\lambda_\kappa(-2+\kappa\gamma_\sigma)-\frac{\lambda_{\kappa+1}}{\lambda}(1-2\gamma_\psi)\,,$$

$$\kappa=3,4,5\,\ldots\infty\,.$$

The β-function of the Yukawa coupling (fig. 1) has an infrared stable fixed point at

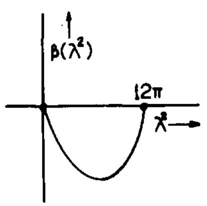

Fig. 1.

$\lambda^2 = 12\pi$. Examining the definition (2.7) of the renormalized dimensionless coupling λ^2 to this order

$$\frac{1}{\lambda^2} = \frac{m^2}{\Lambda^2}\frac{1}{\lambda_0^2} + \frac{1}{12\pi},$$

we see that the limiting value $\lambda^2 = 12\pi$ is obtained as soon as we take the cut-off to go to infinity, regardless of the value of the bare dimensionless coupling λ_0^2. Our renormalization prescription (3.1) has converted the theory into a theory sitting at the fixed point. The value of the renormalized coupling is independent of the value of the bare dimensionless coupling. The renormalization factor Z_σ to this order is according to (2.6)

$$Z_\sigma = 1 - \frac{\lambda^2}{12\pi},$$

which vanishes at the limiting value $\lambda^2 = 12\pi$. The vanishing of a renormalization factor is a compositeness condition. Later we shall see that at the fixed point the meson field ceases to be a fundamental canonical field and turns into a composite operator. The anomalous dimension of the meson field takes its maximum value at the fixed point

$$\gamma_\sigma(\lambda^2 = 12\pi) = 1 .$$

It is obvious from eq. (2.10) that our prescription (3.1) will make the renormalization factors Z_κ vanish as well. (2.10) can be written as

$$\frac{1}{Z_\kappa} = \frac{(\lambda_0)^\kappa}{\lambda_{0\kappa}}\left(\frac{\Lambda}{m}\right)^{\kappa-2}\left(\frac{(\kappa-3)!}{\pi} + \sum_{n=4,6\dots}\frac{\lambda_{0n}}{(\lambda_0)^n}(-m)^{2-n}\frac{(-)^{n-\kappa}}{(n-\kappa)!}\right),$$

when we take the cut-off to infinity we instantly have $Z_\kappa \to 0$, whatever the values of $\lambda_{0\kappa}$ are. From (2.11) it follows that the vanishing of the renormalization factors implies

$$\lambda_\kappa = \frac{(\lambda)^\kappa}{\pi}(\kappa-3)! , \qquad \kappa = 4, 6, \dots \infty . \tag{3.4}$$

The lowest order β-functions of the meson self-couplings are

$$\beta_\kappa = \lambda_\kappa\left(-2 + \frac{\kappa\lambda^2}{12\pi}\right) - \frac{\lambda_{\kappa+1}}{\lambda}.$$

At the fixed point $\lambda^2 = 12\pi$ they become

$$\beta_\kappa = \lambda_\kappa(\kappa-2) - \frac{\lambda_{\kappa+1}}{\sqrt{12\pi}} .$$

The choice $\lambda_\kappa = ((12\pi)^{\kappa/2}/\pi)(\kappa-3)!$, which agrees with (3.4) at the fixed point, makes the β_κ functions vanish, as can be easily seen, so

$$\beta_\kappa(\lambda^2 = 12\pi, ..., \lambda_n = \frac{(12\pi)^{n/2}}{\pi}(n-3)!, ...) = 0 , \qquad k = 3, 4, 5 \dots .$$

As we mentioned earlier, in the asymptotic region $p^2/m^2 \to \infty$, the Callan-Symanzik equations reduce to the renormalization group equations. The β-functions vanish for the critical theory, defined by the values $\lambda^* = \sqrt{12\pi}$ and $\lambda_n^* = ((12\pi)^{n/2}/\pi)(n-3)!$, and the anomalous dimension of the meson becomes unity. Consequently,

$$\left(m\frac{\partial}{\partial m} - n\right)\overline{\Gamma}^{(n\sigma)}(p; m, \lambda^*) \sim 0 .$$

A solution to the above equation is

$$\overline{\Gamma}^{(n\sigma)}(p; m, \lambda^*) = m^n \Phi(p) .$$

From dimensional considerations we must have

$$\overline{\Gamma}^{(n\sigma)}(\rho p; m, \lambda^*) = \rho^{2-n}\overline{\Gamma}^{(n\sigma)}(p; m, \lambda^*) .$$

For the two-point function the scaling law reads

$$\overline{\Gamma}^{(2\sigma)}(\rho^2 p^2) = \overline{\Gamma}^{(2\sigma)}(p^2) .$$

In order to verify this behavior let us re-examine the σ-propagator for the limiting theory. Since,

$$\overline{\Gamma}^{(2\sigma)}(p^2) = p^2 - \mu^2 + m^2\lambda^2 \,\mathrm{sub}_0^2\Pi(p^2) = p^2\left(1 - \frac{\lambda^2}{12\pi}\right) - \mu^2 + m^2\lambda^2 \,\mathrm{sub}_0^1\Pi(p^2)$$

when $\lambda^2 \to 12\pi$ the asymptotic behavior of the propagator ceases to be $\sim p^2$ and becomes

$$\overline{\Gamma}^{(2\sigma)}(p^2) \underset{p^2\to\infty}{\sim} m^2 \ln\left(\frac{p^2}{m^2}\right) .$$

Thus, the asymptotic behavior of the limiting theory is drastically different than the asymptotic behavior of the theory we started with.

The renormalized theory at the fixed point is characterized only by the fermion mass m. The σ-mass μ^2 is calculable. From (2.12) we obtain that $\mu^2 = 12m^2$. The physical σ-mass will be the solution of

$$\mu_\sigma^2 = 12m^2 - 12\pi m^2 \,\mathrm{sub}_0^2\Pi(\mu_\sigma^2)$$

which reduces to

$$\mathrm{sub}_0^1\Pi(\mu_\sigma^2) = \frac{1}{\pi}$$

and leads to $\mu_\sigma^2 = 4m^2$. Thus the physical σ-mass is at threshold to this order of approximation. The values of the renormalized couplings do not depend on the values of the bare couplings. That implies that the critical theory does not depend on the strength or the form of the bare interaction $\Sigma_{n=4,6,...}(g_{0n}/n!)\,\sigma^n$. This is reminiscent of the analogous situation in critical phenomena where near criticality the detailed microscopic structure of different systems becomes irrelevant (universality).

4. $(\bar{\psi}\psi)^2$ in 1 + 1 dimensions

In this section we shall consider the theory of a massless fermion field interaction *via* a quartic self-interaction

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial)\,\psi + \tfrac{1}{2}f_0(\bar{\psi}\psi)^2 \,. \tag{4.1}$$

We have examined this theory elsewhere [1,6] but for completeness we restate some of our previous results.

The bare theory has no dimensional parameter apart from the cut-off Λ^2. The bare coupling f_0 is dimensionless. The Lagrange density (4.1) can be rewritten as

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial + g_0\sigma)\,\psi - \tfrac{1}{2}\mu_0^2\sigma^2 \,.$$

We have made the identification

$$f_0 \equiv \left(\frac{g_0}{\mu_0}\right)^2 , \qquad \sigma = \frac{g_0}{\mu_0^2}(\bar{\psi}\psi)\,.$$

The reason that we rewrote the Lagrangian in terms of a dimensionless scalar operator σ is that we plan to make contact with the Yukawa-type model of the previous sections. There is no kinetic term for σ of course since σ is not a fundamental meson field but an auxiliary variable.

The fermions enter quadratically and can be integrated out exactly in the generating functional. The resulting functional is expressed as an integral over the boson variable only.

$$Z = \int \mathrm{d}\sigma\, \mathrm{e}^{-F\{\sigma,\bar{\eta},\eta,J\}} = \int \mathrm{d}\sigma \exp\{\bar{\eta}S\eta + \mathrm{tr}\ln S^{-1} - \tfrac{1}{2}\mu_0^2\sigma^2 + J\sigma\}\,.$$

Here, as before $S^{-1} \equiv i\not{\partial} + g_0\sigma$.

The functional can now be expanded in a mean-field expansion in exactly the same fashion as the theory of sects. 2 and 3. The zeroth-order connected functional is

$$W_0 = \bar{\eta}S(\sigma_0)\,\eta + \mathrm{tr}\ln S^{-1}(\sigma_0) - \tfrac{1}{2}\mu_0^2\sigma_0^2 + J\sigma_0 \,.$$

The mean field σ_0 is defined by the minimum conditions

$$\left.\frac{\delta F}{\delta\sigma}\right|_{\sigma_0} = 0\,, \qquad \left.\frac{\delta^2 F}{\delta\sigma\delta\sigma}\right|_{\sigma_0} > 0\,.$$

The first of them for a non-zero mean field σ_0 takes the form

$$\mu_0^2 = 2ig_0^2 \int \frac{d^2k}{(2\pi)^2} (k^2 - (g_0\sigma_0)^2)^{-1} .$$

To lowest order the classical σ-field, i.e. the vacuum expectation value of the operator σ, coincides with the mean field. A non-zero classical field breaks spontaneously the discrete γ_5 symmetry of the Lagrangian. As a consequence of a non-zero mean field the fermion acquires a mass $m = -g_0\sigma_0$, as can be seen from the lowest-order fermion propagator.

The zeroth-order σ-propagator is

$$\Delta_\sigma^{-1}(p^2) = -\mu_0^2 + g_0^2 \Pi(p^2) .$$

$\Pi(p^2)$ is the fermion bubble that appears in sect. 2. Having in mind the Yukawa-type model, we renormalize the propagator imposing the following normalization conditions at zero momentum:

$$\Delta_\sigma^{-1}(0) = \frac{-\mu^2}{Z_\sigma}, \qquad \left(\frac{\partial \Delta_\sigma^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\sigma} .$$

The resulting renormalized propagator

$$\overline{\Delta}_\sigma^{-1}(p^2) = p^2 - \mu^2 + g^2 \operatorname{sub}_0^2 \Pi(p^2)$$

is structurally identical with the analogous meson propagator of sect. 2. However, in the present case the renormalized coupling has a fixed value

$$\frac{1}{g^2} = \frac{Z_\sigma^{-1}}{g_0^2} = \frac{1}{g_0^2}\left(\frac{\partial \Delta_\sigma^{-1}}{\partial p^2}\right)_0 = \frac{1}{g_0^2} g_0^2 \left(\frac{\partial \Pi}{\partial p^2}\right)_0 = \frac{1}{12\pi m^2} .$$

Thus, the dimensionless renormalized coupling has the fixed value $\lambda^2 = 12\pi$. The σ-mass is not an independent parameter either. Its value can be computed using the mean-field condition (gap equation). The mass is $\mu^2 = 12m^2$. The physical mass can be found in relation to μ^2. It is

$$\mu_\sigma^2 = 4m^2 .$$

The higher σ-vertex are obtained from the effective action and to this order are just the fermion polygons

$$\Gamma^{(n\sigma)}(p_1 \ldots p_{k-1}) = i(-g_0)^k \operatorname{tr} \int \frac{d^2q}{(2\pi)^2} (S(q) \ldots S(q + p_1 + \ldots p_{k-1})) ((\kappa - 1)! \text{ XT}) .$$

Although the polygons are finite, we subtract once in accordance with the normalization condition

$$\Gamma^{(k)}(0, \ldots,) = g_k / Z^{k/2} , \qquad k = 3, 4, \ldots \infty .$$

The renormalized vertices then, are

$$\bar{\Gamma}^{(k\sigma)}(p_1 \dots p_{k-1}) = g_k + i(-g)^k \,\mathrm{sub}_0^1 \,\mathrm{tr} \int \frac{d^2q}{(2\pi)^2} (S(q) \dots S(q + p_1 + \dots p_{k-1})) \,.$$

The renormalized couplings are calculable. They are

$$g_k = m^2 \frac{\lambda^k}{\pi} (k-3)! \,, \qquad k = 3, 4, \dots \infty \,.$$

The dimensionless Yukawa-type coupling has a fixed value

$$\lambda = \sqrt{12\pi} \,.$$

Thus the couplings of our theory have the fixed values

$$\lambda^* = \sqrt{12\pi} \,, \qquad \lambda_k^* = \frac{(12\pi)^{k/2}}{\pi} (k-3)! \,, \qquad k = 3, 4 \dots \infty \,.$$

The appearance of the contact terms in the renormalized vertices is illusory since they have the value of the polygons at zero momentum and they exactly cancel. Writing the vertices in this way however, serves the purpose to make contact with the previous model. The vertex functions are structurally identical to the vertices of the Yukawa-like model.

The only independent parameter of the renormalized theory is the fermion mass m, while the bare theory was characterized by the dimensionless bare coupling f_0. This "dimensional transmutation" occurs because the massless theory is unstable and the instability is avoided with the spontaneous generation of mass by breaking the discrete γ_5 invariance of the Lagrangian.

From our derivation of the lowest order 1PI vertex functions it is already apparent that the renormalized Green functions of the self-coupled fermion model are identical functions of m, λ^2, λ_3, λ_4, ... and the momenta with the renormalized Green functions of the Yukawa-like model of the previous sections, when the dimensionless couplings are fixed to have the value

$$\lambda^{*2} = 12\pi \qquad \lambda_k^* = \frac{(12\pi)^{k/2}}{\pi} (k-3)!, \qquad k = 3, 4, 5, \dots \infty \,.$$

The lowest-order renormalized effective actions of both models become numerically identical when the dimensionless couplings are restricted to have the above values [6]. These values are exactly the values that make the β-functions of the Yukawa-type model vanish

$$\beta(\lambda^*, \lambda_3^*, \dots) = 0 \,,$$

$$\beta_\kappa(\lambda^*, \lambda_3^*, \dots) = 0 \,, \qquad \kappa = 3, 4, \dots \infty \,.$$

Thus, the Yukawa model (2.1) becomes entirely equivalent to the four-fermion model (4.1) at the fixed point.

This equivalence to lowest order in the mean-field expansion is good to all orders in this expansion. According to formula (2.3) if the lowest-order functionals F and their derivatives are identical, since higher orders are constructed by iterating the lowest order, the higher-order functionals will be the same. Thus, the Yukawa model at the fixed point will coincide with the self-coupled Fermi model to all orders in the mean-field expansion.

A striking difference between the original theory and the critical theory is that in the original theory we had a fundamental meson field interacting with the fermions. The critical theory is a theory of self-interacting fermions. The successor of the meson field is the composite operator $\overline{\psi}\psi$. The compositeness has as an immediate consequence the change in the asymptotic behavior of the σ-propagator.

5. Relevant and irrelevant operators

The meson field of the Yukawa-like model enters in the Lagrangian with an infinity of terms

$$\mathcal{L} = \overline{\psi}(i\gamma \cdot \partial + g_0\sigma)\,\psi + \tfrac{1}{2}\sigma(-\partial^2 - \mu_0^2)\,\sigma + \frac{g_{04}}{4!}\sigma^4 + \frac{g_{06}}{6!}\sigma^6 + \dots .$$

The critical theory on the other hand is characterized by

$$\mathcal{L} = \overline{\psi}(i\gamma \cdot \partial + g_0\sigma)\,\psi - \tfrac{1}{2}\mu_0^2\sigma^2 ,$$

in which the successor of the meson field σ enters in only two terms. The vanishing contribution of the operators $\sigma\partial^2\sigma$ and σ^4, σ^6, ... is a consequence of the fact that the theory occupies an infrared fixed point. In the language of critical phenomena these operators become irrelevant at criticality. How this happens can be sketched from the lowest-order effective action. The only needed counter-term for the Yukawa-like model in lowest order is the mass counter-term. In fact, since this theory is superrenormalizable for finite g_0 this is the only counter-term needed to all orders. However in the critical limit, the theory ceases to be super-renormalizable and becomes only renormalizable because of the change in the asymptotic behavior of the meson propagator. Additional counter-terms are needed for calculations at the critical point beyond lowest order. Our "over subtraction" procedure outlined in sect. 2 handles this limiting case smoothly and straightforwardly for higher order calculations [6,7].

The effective action in terms of the translated meson field $s = \sigma - \sigma_0$ is, to zeroth order,

$$\Gamma = \int \left\{ s\left(\frac{-\partial^2}{2}\right)s - \tfrac{1}{2}\mu_0^2 s^2 + \sum_{n=4,6\dots} \frac{g_{0n}}{n}(s+\sigma_0)^n \right.$$

$$- \sum_{n=2} \frac{(-)^n}{n} \operatorname{tr}\left(\frac{g_0 s}{i\gamma \cdot \partial + g_0 \sigma_0}\right)^n$$

$$\left. + \overline{\eta}(i\gamma \cdot \partial + g_0(s + \sigma_0))^{-1}\eta \right\} .$$

The linear in s term is not present because of the condition $\delta\Gamma/\delta s = 0$. Choosing

$$\delta\mu^2 = \mu_0^2 - g_0^2 \operatorname{tr}(i\gamma \cdot \partial + g_0 \sigma_0)^{-2} - \mu^2/Z_\sigma$$

we obtain

$$\Gamma = \int \left\{ Z_\sigma \overline{s} \left(\frac{-\partial^2}{2}\right) \overline{s} - \tfrac{1}{2}\mu^2 \overline{s}^2 - \sum_{n=3}^{\infty} \frac{(-)^n}{n} \operatorname{tr}\left(\frac{g\overline{s}}{i\gamma \cdot \partial + g\overline{\sigma}_0}\right)^n \right.$$

$$- \tfrac{1}{2}(g\overline{s})^2 \operatorname{sub}_0^1 \operatorname{tr}\left(\frac{1}{i\gamma \cdot \partial + g\overline{\sigma}_0}\right)^2$$

$$\left. + \sum_{n=4,6} \frac{g_n}{n!} Z_n (\overline{s} + \overline{\sigma}_0)^n + \overline{n}(i\gamma \cdot \partial + g(\overline{s} + \overline{\sigma}_0))^{-1}\eta \right\} .$$

Taking the limit $Z_\sigma \to 0$ and $Z_\eta \to 0$, the effective action reduces to the effective action of the self-coupled Fermi field

$$\Gamma = \int \left\{ \overline{\eta}(i\gamma \cdot \partial + g(\overline{s} + \overline{\sigma}_0))\, \eta^{-1} - \tfrac{1}{2}\mu^2 \overline{s}^2 - \sum_{n=3}^{\infty} \frac{(-)^n}{n} \operatorname{tr}\left(\frac{g\overline{s}}{i\gamma \cdot \partial + g\overline{\sigma}_0}\right)^n \right.$$

$$\left. - \tfrac{1}{2}(g\overline{s})^2 \operatorname{sub}_0^1 \operatorname{tr}(i\gamma \cdot \partial + g\overline{\sigma}_0)^{-2} \right\} .$$

The relevance or irrelevance [4] of a certain operator is a statement concerning its growth rate as we approach the fixed point. With our renormalization prescription the theory is converted to a critical theory as soon as we take the limit $\Lambda^2 \to \infty$.

Let us consider the individual operators and see how they grow with the cut-off. For example the boson self-coupling terms are

$$O_n \equiv g_{0n}\sigma^n = \lambda_{0n}\Lambda^2 Z_\sigma^{n/2}\overline{\sigma}^n = \frac{\lambda_{0n}\overline{\sigma}^n\Lambda^2}{(1 + \lambda_0^2\Lambda^2/12m^2)^{n/2}}$$

$$\underset{\Lambda^2 \to \infty}{\sim} \left(\frac{12\pi m^2}{\lambda_0^2}\right)^{n/2} \frac{\lambda_{0n}\overline{\sigma}^n}{\Lambda^{n-2}} \sim 0 .$$

Similarly, the kinetic energy operator is

$$O_k \equiv \sigma\partial^2\sigma = Z_\sigma(\overline{\sigma}\partial^2\overline{\sigma}) = \frac{(\overline{\sigma}\partial^2\overline{\sigma})}{(1 + \lambda_0^2\Lambda^2/12\pi m^2)^{1/2}} \underset{\Lambda^2 \to \infty}{\sim} 0 .$$

On the other hand the σ-mass term is relevant.

$$O_m \equiv \mu_0^2\sigma^2 = \mu_0^2\overline{\sigma}^2 Z_\sigma \underset{\Lambda^2 \to \infty}{\sim} \frac{\tfrac{1}{2}\lambda_0^2\Lambda^2 \ln(\Lambda^2/n^2)}{1 + \lambda_0^2\Lambda^2/12\pi m^2} ;$$

O_m does not go to zero at criticality. The logarithmic infinity is cancelled by the fermion bubble diagram and its limit is finite and non-zero.

6. Other models

Similar phenomena occur in other models. As an example we shall consider the following scalar model in 3+1 dimensions.

$$\mathcal{L}(\phi, \chi) = \tfrac{1}{2}\phi(-\partial^2 - m_0^2 + g_0\chi)\,\phi + \tfrac{1}{2}\chi(-\partial^2 - \mu_0^2)\,\chi + \frac{g_{03}}{3!}\chi^3 + \frac{g_{04}}{4!}\chi^4 \,. \quad (6.1)$$

The ϕ-field enters quadratically and can be integrated out. The resulting functional is an integral over χ only and can be expanded in a mean field expansion following sect. 2.

The lowest-order ϕ-propagator is

$$G_0^{-1}(p^2) = p^2 - m_0^2 + g_0\chi_0 = p^2 - m^2 \,.$$

χ_0 stands for the mean field.

The only divergence that shows up in the zeroth-order mean-field approximation is associated with the ϕ-bubble

$$\Pi(p^2) \equiv -\tfrac{1}{2}i\int\frac{\mathrm{d}^4k}{(2\pi)^4}G_0(k)\,G_0(k+p) = \frac{1}{2(4\pi)^2}\int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}r\, r(r - p^2x(1-x)$$
$$+ m^2)^{-2} \,.$$

The lowest-order χ-propagator is

$$\Delta_\chi^{-1}(p^2) = p^2 - \mu_0^2 + g_{03}\chi_0 + \tfrac{1}{2}g_{04}\chi_0^2 + g_0^2\Pi(p^2) \,.$$

A renormalized mass μ^2 and a renormalized coupling $g^2 = g_0^2 Z_\chi$ are introduced through the following normalization conditions at zero momentum

$$\Delta_\chi^{-1}(0) = \frac{-\mu^2}{Z_\chi}, \qquad \left(\frac{\partial\Delta_\chi^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\chi}.$$

The renormalized propagator takes the form

$$\overline{\Delta}_\chi^{-1}(p^2) = p^2 - \mu^2 \,\mathrm{sub}_0^2\Pi(p^2) \,.$$

The lowest-order dimensionless renormalized coupling is

$$\frac{1}{\lambda^2} \equiv \frac{m^2}{g^2} = \frac{m^2}{g_0^2} + \frac{1}{12(4\pi)^2} \,.$$

Fig. 2.

In terms of λ^2 the renormalization factor,

$$Z_\chi^{-1} = 1 + \frac{g_0^2}{12(4\pi)^2 m^2}$$

becomes

$$Z_\chi = 1 - \frac{\lambda^2}{12(4\pi)^2} .$$

Higher χ functions can be obtained from the effective action by taking variational derivatives with respect to the classical χ-field. The 3χ and 4χ vertices are

$$\Gamma^{(3\chi)}(p_1 p_2) = g_{03} + g_{04}\chi_0 + \tfrac{1}{2} i g_0^3 \int \frac{d^4 k}{(2\pi)^4} G_0(k)\, G_0(k+p_1)\, G_0(k+p_1+p_2)$$
$$+ (1\mathrm{XT}) ,$$

$$\Gamma^{(4\chi)}(p_1 p_2 p_3) = g_{04} - \tfrac{1}{2} i g_0^4 \int \frac{d^4 k}{(2\pi)^4} G_0(k)\, G_0(k+p_1)\, G_0(k+p_1+p_2)$$
$$\times G_0(k+p_1+p_3+p_2) + (5\mathrm{XT}) .$$

The lowest-order 3χ and 4χ vertices do not contain any divergences. Higher orders of course will contribute logarithmically divergent diagrams since we have a quartic coupling present (fig. 2). Although there is no divergence present we can still define renormalized couplings subtracting once in accordance with the normalization conditions:

$$\Gamma^{(3\chi)}(0,0) = \frac{g_3}{Z_\chi^{3/2}} , \qquad \Gamma^{(4\chi)}(0,0,0) = \frac{g_4}{Z_\chi^2} .$$

The renormalized χ self-couplings are defined as

$$g_3 = g_{03} \frac{Z_\chi^{3/2}}{Z_3} , \qquad g_4 = g_{04} \frac{Z_\chi^2}{Z_4} ,$$

while the corresponding dimensionless χ self-couplings are

$$\lambda_3 = g_3/m , \qquad \lambda_4 = g_4 .$$

The renormalized 3χ and 4χ vertex functions are

$$\overline{\Gamma}^{(3\chi)}(p_1 p_2) = \lambda_3 m + \tfrac{1}{2} i(\lambda)^3 m^3 \,\mathrm{sub}_0^1 \int \frac{d^4 k}{(2\pi)^4} G_0(k)\, G_0(k+p_1)\, G_0(k+p_1+p_2)$$

$$+ (1\mathrm{XT}) ,$$

$$\bar{\Gamma}^{(4\chi)}(p_1 p_2 p_3) = \lambda_4 - \tfrac{1}{2} i(\lambda)^4 m^4 \, \mathrm{sub}_0^1 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} G_0(k)\, G_0(k+p_1)\, G_0(k+p_1+p_2)$$

$$\times\, G_0(k+p_1+p_2+p_3) + (5\mathrm{XT}) .$$

The parameters of the bare theory are the masses m_0^2 and μ_0^2 and the couplings g_0, g_{03}, and g_{04}. The renormalized theory is expressed in terms of m^2, μ^2 and λ^2, λ_3 and λ_4. The β-functions associated with the dimensionless renormalized couplings can be defined in the usual way. In the lowest-order mean-field approximation we obtain

$$\beta(\lambda^2) = -2\lambda^2 \left(1 - \frac{\lambda^2}{12(4\pi)^2}\right),$$

$$\beta_3(\lambda^2, \lambda_3, \lambda_4) = -\lambda_3 \left(1 - \frac{\lambda^2}{4(4\pi)^2}\right) - \frac{2\lambda_4}{\lambda} ,$$

$$\beta_4(\lambda^2, \lambda_4) = \frac{\lambda^2}{3(4\pi)^2}(\lambda_4 - 6\lambda^2) .$$

The renormalization factors can be calculated in the mean-field approximation. We find

$$Z_\chi = 1 - \frac{\lambda^2}{12(4\pi)^2} ,$$

$$Z_3 = 1 - \frac{\lambda_4}{\lambda_3}\left(1 - \frac{\lambda^4}{2(4\pi)^2 \lambda_4}\right)\frac{(m_0^2 - m^2)}{\lambda} - \frac{\lambda^3}{2(4\pi)^2 \lambda_3},$$

$$Z_4 = 1 - \frac{\lambda^4}{2(4\pi)^2 \lambda_4} .$$

The β-functions vanish for the values

$$\lambda^* = 8\pi\sqrt{3} , \qquad \lambda_3^* = 48\pi\sqrt{3} \quad \text{and} \quad \lambda_4^* = 72(4\pi)^2 .$$

For the same values the renormalization factors vanish as well. The model possesses an infrared fixed point at $\lambda^* = 8\pi\sqrt{3}$. Examining the definition of the renormalized coupling

$$\frac{1}{\lambda^2} = \frac{m^2}{g_0^2} + \frac{1}{12(4\pi)^2} ,$$

we can see that Wilson's renormalization prescription $g_0^2 = \lambda_0^2 \Lambda^2$ carries the theory to the fixed point as soon as the cut-off is taken to infinity

$$\frac{1}{\lambda^2} = \frac{m^2}{\Lambda^2}\frac{1}{\lambda_0^2} + \frac{1}{12(4\pi)^2} \xrightarrow[\Lambda^2\to\infty]{} \frac{1}{12(4\pi)^2} = \frac{1}{\lambda^{*2}} \, .$$

This happens regardless of the value of λ_0^2.

Let us consider next the theory described by

$$\mathcal{L} = \tfrac{1}{2}\phi(-\partial^2 - m_0^2)\,\phi + \frac{f_0}{4!}\phi^4 \, .$$

We can rewrite it in terms of an auxiliary field χ as

$$\mathcal{L} = \tfrac{1}{2}\phi(-\partial^2 - m_0^2 + g_0\chi)\,\phi - \tfrac{1}{2}\mu_0^2\chi^2 \, .$$

Expanding this theory in a mean-field approximation we obtain to lowest order the following χ-propagator

$$\Delta_\chi^{-1}(p^2) = -\mu_0^2 + g_0^2\Pi(p^2) \, .$$

$\Pi(p^2)$ is the same scalar ϕ-bubble we encountered earlier. Employing the same normalization conditions as for the previous model:

$$\Delta_\chi^{-1}(0) = \frac{-\mu^2}{Z_\chi} \, , \qquad \left(\frac{\partial\Delta_\chi^{-1}}{\partial p^2}\right)_0 = \frac{1}{Z_\chi} \, ,$$

we obtain the renormalized χ-propagator in a structurally identical form with the previous model

$$\bar{\Delta}_\chi^{-1}(p^2) = p^2 - \mu^2 + g^2\,\mathrm{sub}_0^2\Pi(p^2) \, .$$

In this case however the renormalized coupling g^2 is fixed

$$\frac{1}{g^2} = \frac{Z_\chi^{-1}}{g_0^2} = \frac{1}{g_0^2}\left(\frac{\partial\Pi}{\partial p^2}\right)_0 g_0^2 = \frac{1}{12(4\pi)^2 m^2} \, .$$

The dimensionless renormalized coupling was the value

$$\lambda^2 = 12(4\pi)^2 \, .$$

The 3χ and 4λ renormalized vertices can be derived in a straightforward fashion. They are structurally identical with the vertices of the previous model if the dimensionless couplings are fixed to have the values $\lambda_3^* = 48\pi\sqrt{3}$ and $\lambda_4^* = 72(4\pi)^2$.

The only parameters of this model are m^2 and μ^2. The χ-field is not fundamental but represents a collective excitation. There are regions in the parameter space m^2, μ^2 for which the model is stable whatever the sign of the bare coupling f_0. The original model becomes entirely equivalent to ϕ^4 theory * at the fixed point. This

* Interesting equivalences of a ϕ^6 interaction in a $1/N$ approximation to the ϕ^4 interaction in lowest order in a different limit than ours have been observed by the Brandeis group. See, for example, ref. [8].

happens independently of the values of the dimensionless bare couplings. The operators that become irrelevant at criticality are $\chi\partial^2\chi$, χ^3 and χ^4.

This picture refers to the lowest order. To higher orders the 4χ coupling introduces divergences that give to the renormalization factors an explicit dependence on $\ln(\Lambda^2/m^2)$ apart from the dependence on Λ^2 through the dimensional couplings. The limit $Z_\chi \to \infty$ will still be attainable. For example if Z_χ (to order k) $\to 0$,

$$Z_\chi(\text{to order } k+1) = Z_\chi(\text{to order } k)(1 + \epsilon\alpha_1 \ldots + \epsilon^{k+1}\alpha_{k+1})^{-1} .$$

The functions $\alpha_1 \ldots \alpha_{k+1}$ contain divergent terms $(\ln \Lambda^2)$ with $n \leqslant k$. Thus, the limit $Z_\chi(k+1) \to 0$ is feasible. The lowest-order factor vanishes like $\sim 1/\Lambda^2$.

The vanishing of the renormalization factors corresponds to the vanishing of the β-functions for some values of the renormalized couplings *. We have no guarantee however that the fixed point will continue to be non-trivial. If the fixed point is trivial, the operator most likely to become relevant to higher orders is χ^4. The approach to the fixed point is guaranteed if we take its bare coupling equal to zero.

Another model that displays the same phenomena is the Yukawa theory in $4-\epsilon$ dimensions

$$\mathcal{L} = \bar{\psi}(i\gamma\cdot\partial + g_0\chi)\,\psi + \tfrac{1}{2}\chi(-\partial^2 - \mu_0^2)\,\chi + g_{03}\chi^3 + g_{04}\chi^4 .$$

Its equivalence to the self-coupled fermion field has been studied by the authors and Wilson [1,2]. In the mean-field approximation the dimensionless renormalized coupling is

$$\frac{1}{\lambda^2} = \frac{m^\epsilon}{g^2} = \frac{m^\epsilon}{g_0^2} + f(\epsilon) = \left(\frac{m}{\Lambda}\right)^\epsilon \frac{1}{\lambda_0^2} + f(\epsilon) .$$

$f(\epsilon)$ is logarithmically divergent as $\epsilon \to 0$. The renormalization factor of the χ-field is

$$Z_\chi = 1 - f(\epsilon)\,\lambda^2 .$$

The β-function of the coupling λ^2 is

$$\beta(\lambda^2) = -2\lambda^2(1 - f(\epsilon)\,\lambda^2) .$$

An infrared stable fixed point occurs at

$$\lambda^* = \frac{1}{\sqrt{f(\epsilon)}} .$$

This value is achieved, according to our prescription, when $\Lambda^2 \to \infty$. In the four-dimensional case $(\epsilon \to 0)$ the fixed point becomes trivial $(\lambda^* = 0)$.

* The renormalization factors satisfy the renormalization group equations. For example

$$\left(m\frac{\partial}{\partial m} + \beta\frac{\partial}{\partial\lambda} - 2\gamma_\sigma\right) Z_\sigma = 0 .$$

At the fixed point, obviously $Z_\sigma \sim [m/\Lambda]^{2\gamma_\sigma} \underset{\Lambda\to\infty}{\sim} 0$.

One can repeat the steps we undertook previously and prove the equivalence of this theory with the self-coupled fermion field theory

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial)\, \psi + \tfrac{1}{2} f_0 (\bar{\psi}\psi)^2 \,,$$

to all orders in the mean-field expansion. As long as the dimensionality is less than 4 and the theory is super-renormalizable the operators $\chi\partial^2\chi$, χ^3 and χ^4 are irrelevant.

In four dimensions, although in the lowest-order mean-field approximation the fixed point is trivial [9], one can still demonstrate the formal equivalence between the Yukawa theory and the four-fermion theory introducing the renormalized Yukawa coupling as an extra renormalized parameter independent of the bare Yukawa coupling [1].

7. Conclusions

(a) We have considered in 1 + 1 dimensions the model described by

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial + g_0 \sigma) + \tfrac{1}{2}\sigma(-\partial^2 - \mu_0^2)\, \sigma + \sum_{n=2}^{\infty} \frac{g_{02n}}{(2n)!}\, \sigma^{2n} \,.$$

(b) We assumed that the bare couplings have a cut-off dependence

$$g_0 = \lambda_0 \Lambda \,, \qquad \text{and} \qquad g_{0n} = \lambda_{0n} \Lambda^2 \,, \qquad n = 2, 4, \ldots \infty \,.$$

λ_0 and λ_{0n} are dimensionless.

(c) When the cut-off is taken to infinity the theory is converted into a critical theory occupying an infrared stable fixed point.

(d) The critical theory is exactly equivalent to the Gross-Neveu model

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial)\, \psi + \tfrac{1}{2} f_0 (\bar{\psi}\psi)^2 \,.$$

(e) The operators $\sigma\partial^2\sigma$, σ^4, σ^6, ..., σ^n, ... become irrelevant at criticality.

(f) The approach to the fixed point is entirely independent of the values of the dimensionless bare couplings (universality).

(g) The critical theory is a massless theory in the sense that it contains no bare dimensional parameters. Masses however are present due to dimensional transmutation.

(h) The fundamental meson of the original theory disappeares at the fixed point, leaving as a remnant a fermion-antifermion bound state, excited by the composite operator $\bar{\psi}\psi$. The conversion of the meson into a composite particle is accompanied by a change in the asymptotic behavior of its propagator. This has as a consequence the appearance of new primitive divergences that convert the theory from superrenormalizable to renormalizable.

(i) Higher orders in the mean-field expansion are not expected to alter the above picture. The equivalence is true also in the ordinary renormalized perturbation theory.

(j) Similar phenomena occur for other models, like the Yukawa theory in $4-\epsilon$ dimensions

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial + g_0 \sigma)\, \psi + \tfrac{1}{2}\sigma(-\partial^2 - \mu_0^2)\, \sigma + g_{03}\sigma^3 + g_{04}\sigma^4 \ .$$

This theory is equivalent, in the mean-field expansion to the four-fermion theory

$$\mathcal{L} = \bar{\psi}(i\gamma \cdot \partial)\, \psi + \tfrac{1}{2} f_0 (\bar{\psi}\psi)^2$$

provided we are at a fixed point. This suggests the renormalizability of the four-fermion theory in the mean-field expansion. The four-fermion theory is indeed renormalizable in the mean-field expansion as it has demonstrated elsewhere [1]. In four dimensions the fixed point becomes trivial but the above results still hold in a formal sense [1].

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