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Is the CMB shift parameter connected with the growth of cosmological perturbations?

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ABSTRACT

We verify numerically that in the context of general relativity (GR), flat models which have the same Ω_{m} and cosmic microwave background (CMB) shift parameter R but different $H(a)$ and $w(a)$ also have very similar (within less than 8 per cent) growth of perturbations even though the dark energy density evolution is quite different. This provides a direct connection between geometrical and dynamical tests of dark energy and may be used as a cosmological test of GR.

Key words: cosmic microwave background – large-scale structure of Universe.

1 INTRODUCTION

There is by now convincing evidence that the available high-quality cosmological data [Type Ia supernova (SNIa), cosmic microwave background (CMB) etc.] are well fitted by an emerging 'standard model'. In the context of general relativity (GR) this 'standard model', assuming flatness, is described by the Friedman equation:

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_{\mathrm{m}}(a) + \Omega_{\mathrm{DE}}(a)\right], \tag{1}$$

where $a(t)$ is the scale factor of the universe, $\Omega_{\mathrm{m}}(a)$ is the density parameter corresponding to the sum of baryonic and cold dark matter, with the latter needed to explain clustering, and an extra component $\Omega_{\mathrm{DE}}(a)$ with negative pressure called dark energy needed to explain the observed accelerated cosmic expansion (Riess et al. 1998; Perlmutter et al. 1999; Efstathiou et al. 2002; Tegmark et al. 2004; Nesseris & Perivolaropoulos 2005, 2007b; Spergel et al. 2007). During the last decade there have been many theoretical speculations regarding the nature of the exotic 'dark energy'. Various candidates have been proposed in the literature, among which a dynamical scalar field acting as vacuum energy (Ozer & Taha 1987; Caldwell, Dave & Steinhardt 1998; Peebles & Ratra 2003). Under this framework, high-energy field theories generically indicate that the equation of state of such a dark energy is a function of the cosmic time. To identify this type of evolution of the equation of state, a detailed form of the observed $H(z)$ is required which may be obtained by a combination of multiple dark energy probes. Such probes may be divided in two classes according to the methods used to obtain $H(z)$.

(i) Geometric methods probe the large-scale geometry of space–time directly through the redshift dependence of cosmological distances [$d_{\mathrm{L}}(z)$ or $d_{\mathrm{A}}(z)$]. They thus determine $H(z)$ independent of the validity of Einstein equations.

(ii) Dynamical methods determine $H(z)$ by measuring the evolution of energy density (background or perturbations) and using a gravity theory to relate them with geometry, i.e. with $H(z)$. These methods rely on knowledge of the dynamical equations that connect geometry with energy and may therefore be used in combination with geometric methods to test these dynamical equations.

A very accurate and deep geometrical probe of dark energy is the angular scale of the sound horizon at the last scattering surface as encoded in the location l_1^{TT} of the first peak of the CMB temperature perturbation spectrum. This probe is described by the so-called CMB shift parameter (cf. Bond, Efstathiou & Tegmark 1997; Trotta 2004; Nesseris & Perivolaropoulos 2007a) which is defined as

$$R = \frac{l_1'^{TT}}{l_1^{TT}}, \tag{2}$$

where l_1^{TT} is the temperature perturbation CMB spectrum multipole of the first acoustic peak. In the definition of R, l_1^{TT} corresponds to the model (with fixed Ω_{m}, Ω_{b} and h) characterized by the shift parameter and $l_1'^{T}$ to a *reference* flat standard cold dark matter (SCDM) model ($\Omega_{\mathrm{m}} = 1$) with the same $\omega_{\mathrm{m}} = \Omega_{\mathrm{m}}h^2$ and $\omega_{\mathrm{b}} = \Omega_{\mathrm{b}}h^2$ as the original model. Recently, Nesseris & Perivolaropoulos (2007a) have found that models based on GR that have identical shift parameter R and matter density Ω_{m} also lead to almost identical integrated Sachs–Wolfe (ISW) effect despite of their possible differences in the cosmic expansion histories.

The aim of this work is to investigate our suspicion that the (geometrical) CMB shift parameter is somehow associated with the (dynamical) fluctuation growth rate in the context of GR. Note that a possible violation of this connection may thus be viewed as a hint for modifications of GR. The structure of the paper is as follows. The basic theoretical elements are presented in Section 2. The results are presented in Section 3 by solving numerically the time evolution

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equation for the mass density contrast for various flat dark energy models that share the same value of shift parameter and value of Ω_m. In Section 4 we draw our conclusions. Finally, in Appendix A we have treated analytically, up to a certain point, the differential equation for the mass density contrast considering different dark energy models with a time varying equation of state.

2 THEORETICAL ELEMENTS

The location l_1^{TT} of the first acoustic peak in the CMB temperature spectrum can be connected with the angular diameter distance d_A and with the sound horizon r_s both at the last scattering surface ($z = z_{ls}$) and then the shift parameter (see Nesseris & Perivolaropoulos 2007a and references therein for details) can be brought to the form $R' = 2/\{\sqrt{\Omega_m}\int_0^{z_{ls}}[dz/E(z)]\}$. The expression usually used for the shift parameter is

$$R = \sqrt{\Omega_m}\int_{a_{ls}}^{1}\frac{da}{a^2 H(a)/H_0} = \sqrt{\Omega_m}\int_0^{z_{ls}}\frac{dz}{E(z)}, \qquad (3)$$

where $E(z) \equiv H(z)/H_0$.

In order to define $E(z)$ we use the Chevalier–Polarski–Linder (CPL; Chevallier & Polarski 2001; Linder 2003) parametrization for which

$$w(a) = w_0 + w_1(1-a) \qquad (4)$$

and

$$E^2(a) = \Omega_m a^{-3} + \frac{(1-\Omega_m)a^{-3}}{f(a)}, \qquad (5)$$

where

$$f(a) = \exp\left[-3\int_a^1 \frac{w(u)}{u}\,du\right] = a^{3(w_0+w_1)}e^{-3w_1(a-1)}. \qquad (6)$$

On the other hand, a dynamical probe of geometry is the measured linear growth factor of the matter density perturbations $\delta(a)$. The evolution equation of the growth factor for models where the dark energy fluid has a vanishing anisotropic stress and the matter fluid is not coupled to other matter species is given by (Peebles 1993; Stabenau & Jain 2006; Uzan 2007)

$$\ddot{\delta} + 2H(t)\dot{\delta} - 4\pi G\rho_m\delta = 0, \qquad (7)$$

where dots denote derivatives with respect to time. Useful expressions of the growth factor can be found for the Λ cold dark matter (ΛCDM) cosmology in Peebles (1993), for the quintessence scenario (w = const) in Silveira & Waga (1994), Wang & Steinhardt (1998), Basilakos (2003) and Nesseris & Perivolaropoulos (2008) and for the scalar tensor models in Gannouji & Polarski (2008). As an example, in the case of ΛCDM cosmology the growth factor is of the form

$$\delta(a) = \frac{5\Omega_m}{2}\frac{H(a)}{H_0}\int_0^a \frac{da'}{[a'H(a')/H_0]^3} \qquad (8)$$

which for $a = 1$ has some similarity with the form of the shift parameter (see equation 3). Finally, it is interesting to mention here that for dark energy models with a time varying equation of state, an analytical solution for $\delta(a)$ has yet to be found because the basic differential equation (7) becomes more complicated than in models with constant w. However, in a recent paper (Linder & Cahn 2007) a growth index γ was used to parametrize the linear growing mode including models with a time varying equation of state (see next section).

3 THE CONNECTION BETWEEN THE CMB SHIFT PARAMETER AND THE GROWTH FACTOR

In order to explore the above-mentioned connection of the CMB shift parameter R of equation (3) with the growth factor $\delta(a)$, we perform a Monte Carlo analysis in the parameter space of w_0– w_1 of the parametrization (4). In particular we fix Ω_m (for example 0.25) and compare the variation of the growth factor for w_0– w_1 pairs corresponding to fixed R with the corresponding variation when other combinations of w_0– w_1 are fixed. Specifically, we consider and compare four cases.

(i) Case 1: w_0 and w_1 are constrained by a fixed value of the CMB shift parameter, $R(w_0, w_1) = 1.7$.

(ii) Case 2: w_0 and w_1 are constrained by a linear relation of the form $w_1 = -2.9$–$3.1\ w_0$ approximating the locus of the w_0 and w_1 that satisfy the relation $R(w_0, w_1) = 1.7$ (see Fig. 1). This approximation is accurate to within about 5 per cent and provides

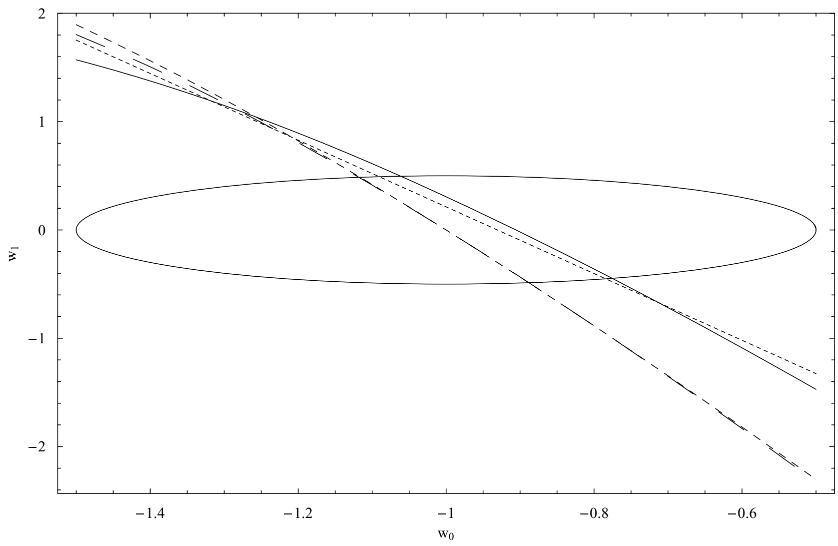

Figure 1. The locus of w_0, w_1 for case 1 (solid line), case 2 (dotted line), case 3 (circle, see text), case 4 ($m = 2, n = 0$, short dashed line; $m = 1, n = 1$, long dashed line).

Table 1. The ratio of the values of the range of the growth factor at $a = 1$ for various m, n to that for $m = 1, n = 0$. The case $m = 1, n = 0$ corresponds to the CMB shift parameter. Notice that for these values of m, n the ratio becomes minimal.

m/n	0	1	2	3	4
0	9.64	9.74	9.75	9.16	9.79
1	1.00	5.92	10.71	11.72	54.24
2	7.95	10.84	11.58	56.81	376.47
3	10.89	11.89	69.62	428.33	2 969.39
4	11.61	73.53	247.44	1422.99	20 633.90

an estimate of the uncertainties introduced in the predicted value of the growth factor if the shift parameter is not accurately measured.

(iii) Case 3: w_0 and w_1 are constrained to be on a circle of radius 0.5 and centre the ΛCDM point $(-1, 0)$.

(iv) Case 4: w_0 and w_1 are constrained by a fixed value of an integral ansatz of a form similar to the CMB shift parameter

$$A(w_0, w_1)_{m,n} = \sqrt{\Omega_{\mathrm{m}}} \int_0^{z_{\mathrm{ls}}} \frac{\mathrm{d}z}{(1+z)^n E^m(z)} \tag{9}$$

for various values of the parameters m, n (see Table 1). Note that for $(m, n) = (1, 0)$ we get the usual CMB shift parameter.

In the first two cases w_0 is a random variable in the range $[-1.5, -0.5]$ and for each value of w_0 we use the constraint equation to solve for w_1. In case 3, w_0 and w_1 are on a circle of radius 0.5 and centre the ΛCDM point $(-1, 0)$. In Fig. 1 we show the locus of the points w_0 and w_1 that satisfy the constraints of cases 1–4. Notice that in Fig. 1, case 3 is shown as an ellipse and not a circle due to the fact that the aspect ratio is chosen so that the loci for the other cases are shown optimally. Furthermore, in Fig. 2 we present the growth factor evolution which is derived by solving numerically equation (7) for the first three cases and all Monte Carlo values of w_0 and w_1 (100 pairs each time). In case 1 the values of $\delta(a = 1)$ are much more constrained than the other two cases as the range (dispersion) of the growth factor at $a = 1$, i.e. $\max[\delta(a = 1)] - \min[\delta(a = 1)]$ for the first case is 0.055 [corresponding to a variation of the mean value of $\delta(a = 1)$ of less than 8 per cent] while in the second and third cases it is 3.1 (33 per cent *variation*) and 5.9 (49 per cent *variation*) times that value. We have also investigated the sensitivity of our analysis to the matter density parameter. In particular, we confirmed that in the range $\Omega_{\mathrm{m}} \in [0.2, 0.3]$ our results depend weakly on the value of Ω_{m}. In fact, the present time dispersion of the growth factor for fixed shift parameter varies from 3.5 per cent for $\Omega_{\mathrm{m}} = 0.2$ to 9 per cent for $\Omega_{\mathrm{m}} = 0.3$ (8 per cent for $\Omega_{\mathrm{m}} = 0.25$). Thus our main result persists for all physical values of Ω_{m} and it strongly indicates that the CMB shift parameter is somehow *associated* with the growth factor in the context of GR.

The linear relation of the form $w_1 = -2.9$–$3.1w_0$ corresponding to Fig. 2(b) provides a rough approximation (good to about 5 per cent) of the locus of points that satisfy the relation $R(w_0, w_1) = 1.7$. This introduces significant additional dispersion to the present day growth factor (the dispersion goes to 33 per cent in Fig. 2b from the 8 per cent obtained with the exact locus of fixed R in Fig. 2a). Once we improve the w_1 expression with an appropriate quadratic term in w_0, the approximation improves from about 5 to about 0.3 per cent and the growth factor dispersion drops back to 8.5 per cent (almost the same as with the exact locus of fixed R). This is an interesting result as it means that the shift parameter should be measured with 1σ errors better than 1 per cent for a determination of the growth factor to an accuracy better than 10 per cent. This accuracy of measurement of the shift parameter however is not far from present day measurements which have determined R to within 1.5 per cent ($R = 1.7 \pm 0.03$ from Wang & Mukherjee 2007).

In case 4 the analysis is similar to case 1, i.e. w_0 is a random variable but now w_1 is found from the generalized constraint

$$A(w_0, w_1)_{m,n} = A_{\mathrm{fixed}\ m,n}, \tag{10}$$

where $A_{\mathrm{fixed}\ m,n}$ is the value of $A(w_0, w_1)_{m,n}$ for a ΛCDM cosmology ($w_0 = -1, w_1 = 0$) for the respective values of m, n. We computed the range (dispersion) of the growth factor at $a = 1$, i.e. $\max[\delta(a = 1)] - \min[\delta(a = 1)]$ for the values w_0 and w_1 derived from the constraint equation (10). In Table 1 we show the ratio of these values for various m, n to the value of the range of the growth factor for $m = 1, n = 0$ [when $A(w_0, w_1)_{m,n}$ goes over to the CMB shift parameter R], i.e.

$$\frac{\{\max[\delta(a = 1)] - \min[\delta(a = 1)]\}_{(m,n)}}{\{\max[\delta(a = 1)] - \min[\delta(a = 1)]\}_{(1,0)}}. \tag{11}$$

Finally, we checked that random values for both w_0 and w_1 give a much larger dispersion on the values of $\delta(a = 1)$, as exactly they should. Notice that the CMB shift parameter R ($m = 1, n = 0$) seems to constrain the growth factor by about an order of magnitude or more compared to other forms of the integral ansatz (different values of m and n). From a theoretical point of view a possible relation between the CMB shift parameter and the growth factor can be used as a viable test for the GR. As expected (Bertschinger 2006), changing the validity of Einstein's field equations (the so-called theory of modified gravity), we change accordingly the growth factor (see Gannouji & Polarski 2008 for a detailed investigation of the growth factor in scalar–tensor theories). In contrast the behaviour of the CMB shift parameter remains unaltered, simply because the latter is a geometrical function (see e.g. Nesseris & Perivolaropoulos 2007a). Thus, a mismatch between the measured value of the shift parameter and the measured value of the linear growth factor would be a hint towards modified gravity.

Verifying this connection between the growth factor and the CMB shift parameter R analytically requires an exact or approximate solution to the differential equation for the evolution of density perturbations when evolving dark energy is taken into account. In what

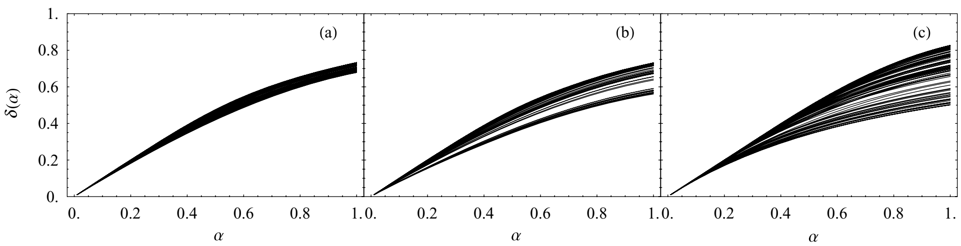

Figure 2. The growth factor for the first three cases and all Monte Carlo values of w_0 and w_1($\Omega_{\mathrm{m}} = 0.25$). Case 1 corresponds to (a) and cases 2 and 3 to (b) and (c), respectively. Note that for this figure we do not normalize the growth factor with its value at $a = 1$ as we are interested in its range of values at $z = 0$.

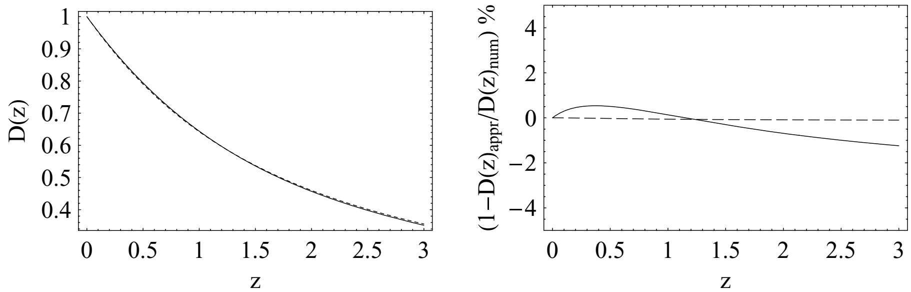

Figure 3. Left panel: the growth factor as a function of redshift. The solid line represents our analytical approximation (see equation 14), while the dashed line represents the parametrized growth factor (see equation 12 derived by Linder & Cahn 2007). Right panel: the per cent accuracy $(1 - D_{\rm appr}/D_{\rm num})$ per cent of the two approximations, equation (12) (dashed line) and equation (14) (solid line). Note that we use $\Omega_{\rm m} = 0.25$ and $(w_0, w_1) = (-0.95, 0.43)$.

follows we discuss some recent attempts towards the construction of such approximate solutions.

A well-known approximate solution to equation (7) is found by Linder & Cahn (2007), where a growth index γ was used to parametrize the linear growing mode for models with a time varying equation of state. Specifically, the growth index γ was defined through

$$D(a) = \exp\left[\int_1^a \frac{\Omega_{\rm m}^{\gamma}(u)}{u}\,{\rm d}u\right], \qquad (12)$$

where $D(a)$ is the growth factor normalized to unity and it was found that γ can be approximated by

$$\gamma = \frac{6 - 3(1 + w_{\infty})}{11 - 6(1 + w_{\infty})}, \qquad (13)$$

where $w_{\infty} \equiv w(z \gg 1)$.

The previous approach was based on the approximation that the universe is not too far from being matter dominated. However, by utilizing just some basic elements from the differential equation theory we have solved analytically up to a certain point equation (7) (see Appendix A for details), assuming that the equation of state parameter is a function of time. In this approach the growing $D_+(a)$ and decaying $D_-(a)$ modes are given by

$$D_{\pm}(a) \simeq a^{-3/2} E^{-1/2}(a) \exp\left[\mp\frac{\sqrt{21}}{3}\int_1^a |g(u)|^{1/2}\,{\rm d}u\right], \qquad (14)$$

where $g(a)$ is defined in Appendix A.

Utilizing the best-fitting cosmological parameters $\Omega_{\rm m} = 0.25$ and $(w_0, w_1) = (-0.95, 0.43)$ obtained from the Gold06 SNIa data set and the CMB shift parameter, in the right-hand panel of Fig. 3, we present the approximated growth factor (left-hand panel) as a function of redshift by utilizing equation (14) (solid line) and equation (12) (dashed line), respectively . It is obvious that our analytical approximation is indeed close to that found by Linder & Cahn (2007). In the right-hand panel of Fig. 3 we show the per cent accuracy $(1 - D_{\rm appr}/D_{\rm num})$ per cent of the two approximations D appr, compared to the numerical solution. Equation (12) deviates from the numerical solution by 0.17 per cent while our $D_+(z)$ approximation deviates by 0.5–1.1 per cent.

4 CONCLUSIONS

In this work, we found that flat models which have the same $\Omega_{\rm m}$ and CMB shift parameter R but different $H(a)$ and $w(a)$ also have very similar growth of perturbations even though the dark energy density evolution is quite different. This was done by comparing various forms of constraints for w_0 and w_1, besides the CMB shift parameter, by a Monte Carlo simulation. In all cases considered, models constrained by the CMB shift parameter had also a very similar growth factor.

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APPENDIX A

In this appendix we try to treat analytically, as much as possible, the problem of the growth factor evolution for dark energy models with a time varying equation of state.

The time evolution equation for the mass density contrast, modelled as a pressureless fluid, is obtained from the Euler and matter stress energy conservation equations as

$$\ddot{\delta} + 2H(t)\dot{\delta} - 4\pi G\rho_{\rm m}\delta = 0, \tag{A1}$$

where dots denote derivatives with respect to time. This differential equation is valid for models where the dark energy fluid has a vanishing anisotropic stress and the matter fluid is not coupled to other matter species; however, see Uzan (2007) for a detailed discussion of the modifications that appear on the right-hand side of the above equation when such terms are present. Changing variables from t to a the above equation becomes (see also Linder 2003)

$$\delta'' + A(a)\delta' + B(a)\delta = 0, \tag{A2}$$

where $A(a) = (3/2a)\{1 - [(1 - \Omega_{\rm m})w(a)]/[1 - \Omega_{\rm m} + \Omega_{\rm m}f(a)]\}$ and $B(a) = -(3/2a^2)\,\{[\Omega_{\rm m}f(a)]/[1 - \Omega_{\rm m} + \Omega_{\rm m}f(a)]\}$.

Performing now the following transformation:

$$\delta(a) = y(a)\exp\left[-\frac{1}{2}\int_1^a A(u)\,{\rm d}u\right], \tag{A3}$$

the linear mass density fluctuations is written as

$$\delta(a) = y(a)\,a^{-3/2}E^{-1/2}(a), \tag{A4}$$

and the unknown function y satisfies the following differential equation:

$$y'' - g(a)y = 0 \tag{A5}$$

with a relevant factor of

$$g(a) = \frac{1}{2}A'(a) + \frac{1}{4}A^2(a) - B(a). \tag{A6}$$

It becomes evident that a major part of the pure solution is described by the expression $a^{-3/2}\,E^{-1/2}\,(a)$.

Of course, in order to solve fully the problem we have to derive the functional form of y. In particular, we write equation (A5) as follows:

$$\frac{y'^2}{2} - \frac{g(a)y^2}{2} + \int_1^a \frac{y^2(u)}{2}g'(u)\,{\rm d}u = -c_1 \tag{A7}$$

or

$$|g(a)y^2| = \left|c^2 + \int_1^a y^2(u)g'(u)\,{\rm d}u + y'^2\right| \qquad \left(c^2 = 2c_1\right). \tag{A8}$$

From a mathematical point of view we can select the integration constant c to be large enough such as

$$n^2|g(a)|y^2 \le n^2c^2 + \int_1^a n^2|g(u)|y^2(u)\frac{|g'(u)|}{|g(u)|}\,{\rm d}u + y'^2, \tag{A9}$$

where n^2 is the normalization constant of the problem. Now we can use Gronwall's theorem (see Gronwall 1919), which is a well-known theorem from the differential equation theory.

Theorem: let us assume that $\mu : [a,\beta] \to [0,\infty)$ and $y : [a,\beta] \to [0,\infty)$ continuous functions and $\lambda \in \mathcal{R}$. If

$$y \le \lambda + \left|\int_{t_0}^t \mu(x)y(x)\,{\rm d}x\right| \qquad \forall t \in [a,\beta], \tag{A10}$$

then

$$y \le \lambda\exp\left(\left|\int_{t_0}^t \mu(x)\,{\rm d}x\right|\right) \qquad \forall t \in [a,\beta], \tag{A11}$$

and using it on (A9) we get the following useful formula:

$$n^2|g(a)|y^2 \le n^2c^2\exp\left(\int_1^a \frac{|g'(u)|}{|g(u)|}\,{\rm d}u\right) + y'^2. \tag{A12}$$

Doing so it turns out that a possible approximation could be

$$n^2|g(a)|y^2 \simeq n^2c^2|g(a)| + y'^2 \tag{A13}$$

from which we get that

$$y(a) \simeq c\cosh\left(\pm n\int_1^a |g(u)|^{1/2}\,{\rm d}u\right). \tag{A14}$$

Taking into account equation (A4), we can obtain the following approximation:

$$\delta(a) \simeq c\;a^{-3/2}E^{-1/2}(a)\cosh\left(\pm n\int_1^a |g(u)|^{1/2}\,{\rm d}u\right). \tag{A15}$$

In order to normalize our analytical expression we use as a limiting case the Einstein–de Sitter model ($\Omega_{\rm m} = 1, g(a) = 21/16a^2$) in which the behaviour of the corresponding growing mode is well known $D_+(a) = a$. Indeed, doing so we get $n = \sqrt{21}/3$ and thus the following normalized growing D_+ and decaying D_- modes, respectively, become

$$D_\pm(a) \simeq a^{-3/2}E^{-1/2}(a)\exp\left(\mp\frac{\sqrt{21}}{3}\int_1^a |g(u)|^{1/2}\,{\rm d}u\right). \tag{A16}$$

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