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Knitting neutrino mass textures with or without Tri–Bi maximal mixing

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ABSTRACT

The solar and baseline neutrino oscillation data suggest bimaximal neutrino mixing among the first two generations, and trimaximal mixing between all three neutrino flavors. It has been conjectured that this indicates the existence of an underlying symmetry for the leptonic fermion mass textures. The experimentally measured quantities, however, are associated to the latter indirectly and in a rather complicated way through the mixing matrices of the charged leptons and neutrinos. Motivated by these facts, we derive exact analytical expressions which directly link the charged lepton and neutrino mass and mixing parameters to measured quantities and obtain constraints on the parameter space. We discuss deviations from Tri–Bi mixing matrices and present minimal extensions of the Harrison, Perkins and Scott matrices capable of interpreting all neutrino data.

1. Introduction

Since the experimental confirmation of neutrino oscillations, there have been assiduous efforts to measure the exact mixing angles and their tiny masses.[1] The last few years we are in a position to know up to a high accuracy the three neutrino mixing angles and the mass squared differences. Several phenomenological explorations have led to the conclusion that the so-called Tri–Bi (TB) maximal mixing [6] is in remarkable agreement with the solar and atmospheric neutrino data. Indeed, the experimental range of the three angles lie in the range

$$\sin^2\theta_{12} \approx 0.312^{+0.019}_{-0.018}, \qquad \sin^2\theta_{23} \approx 0.466^{+0.073}_{-0.058}, \qquad \sin^2\theta_{13} \approx 0.126^{+0.053}_{-0.049} \tag{1}$$

while the values of TB-prediction are $\frac{1}{3}, \frac{1}{2}, 0$ respectively.

Another interesting aspect is the fact that in the basis where the charged lepton mass matrix is diagonal, TB-mixing is independent of the neutrino mass eigenvalues, the symmetric neutrino mass elements need only to satisfy three simple relations [7]

$$m_{e\mu} = m_{e\tau}, \qquad m_{\mu\mu} = m_{\tau\tau}, \qquad m_{ee} + m_{e\mu} = m_{\mu\mu} + m_{\mu\tau}.$$

One may attempt to attribute the regularity of the data in the leptonic sector to the existence of some particular symmetry of a suitable theoretical model. However, the above picture is definitely different from the corresponding one in the quark sector thus, in view of accumulating experimental evidence during the last decade, the reconciliation of neutrino data with simple $U(1)$ family symmetry models [8] is rather unlikely. It has been suggested for example, that this specific structure might originate from a discrete non-Abelian symmetry [9,10]. A different point of view is taken however in [7] where the authors claim that TB-mixing might be completely accidental as they found that significant violations from TB-mixing may occur within the present experimental bounds, with 1–3 mixing in particular leading to substantial deviations. Several other suggestions including the introduction of discrete and unified theories have appeared in the literature [4–20].

The modifications on the TB-mixing suggested by several of these proposals are based on perturbative considerations of the original TB-mass textures and the mixing matrix. However, in order to consistently study the effects on the mixing and the tiny mass differences

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in the neutrino sector, a rather accurate approach is required. Further, a major issue the present days is the exact measurement of the θ_{13} angle which in the original TB-model was assumed to be exactly zero, while data do allow for a small deviation. In addition, several measurable effects depend crucially on the value of the θ_{13} angle. For example, the survival probability of the reactor neutrinos involves both θ_{13} and Δm^2_{31} [21], and current bounds allow a small value for θ_{13}. If this angle is non-zero indeed, its value is sensitive to any modification of the mass matrix; in this case, approximate results and perturbative expansions may not be adequate to reliably determine the measured parameters in terms of mass textures eventually dictated by some symmetry. The aim of this Letter is to fill this gap. Assuming only general Hermitian mass squared matrices for the charge leptons and neutrinos, we will derive analytical results for the mixing and mass-squared differences. To this end, using well-known theorems from the spectral theory of matrices, we express a general 3×3 Hermitian fermion mass matrix as a second degree polynomial of a unitary matrix. This way, we are able to disentangle the mass eigenstates which appear only in the coefficients of this expansion. We then express the neutrino mixing angles as functions of variables that parametrize the unitary matrices which diagonalize the charged lepton and neutrino mass matrices[2] respectively. This procedure gives enough flexibility to determine the experimentally allowed range of the parameters, and, at the same time seek for mass textures eventually dictated by some underlying symmetry. As an immediate benefit of this approach we get a non-zero θ_{13} angle emerging even in the minimal TB-scenario, provided that at least one non-zero phase in the neutrino texture is assumed. In addition, we will present a second example where compatible neutrino textures arise, otherwise not accessible by perturbative treatment around the TB-solution.

2. Formulation

As explained in the introduction, the TB-maximal mixing is compatible with the observed neutrino data. However, the very specific form of the mass matrices postulated in this approach can only be embedded in particular classes of unified theories and even less string derived models. Early and present endeavors in this direction for example involve the rather promising A_4 symmetry as far as the neutrino sector is concerned. Thus, A_4 can be generated by elements S, T satisfying $S^2 = T^3 = (ST)^3 = 1$ and these can be viewed [15] as a subgroup of the modular group which plays a fundamental role in string theory. Nevertheless, a straightforward application to the quark sector is not very satisfactory since it predicts unacceptably small quark mixing, while only contrived variants can possibly reconcile data from both the quark and lepton sectors [12,16–19]. Other attempts to generate TB-mixing relying on the non-Abelian family symmetry $\Delta(27)$ give results which are also compatible with quark mixing and can in principle be embedded in a unified gauge theory [20]. However, the usefulness of parametrizations dictated by such symmetries is limited within the prescribed scenario and might not capture cases exhibiting other possible interesting properties beyond TB-mixing. Moreover, if we seriously wish to exploit the idea that some other underlying symmetry is found hidden behind the regularity of the neutrino data, perturbative investigations around the TB-solution are highly unlikely to have a chance. We should not also ignore the variety of unified or string models where the existing vacua along flat directions usually break symmetries in a hard way and lead to complicated mass textures. Renormalization group effects, as well as instanton contributions [22–25] may further obscure the original symmetry.

Taking into account the above considerations, we infer that the complicated picture of the model building landscape as well as the sensitivity of the neutrino data on TB-departures, call for a detailed and *exact* treatment of the neutrino sector. To accomplish this, in this section, we will develop a new formalism for describing 3×3 mass matrices and their corresponding diagonalizing transformations. In doing this, we will generalize a formalism which appears to be a special property of the original TB-construction, namely the independence of the mixing angles from the eigenvalues. This will be a built-in property of our suggested formalism and will facilitate the analysis of the complicated structure of the leptonic sector.

We wish to analyze general models based on GUTs, SUSY-GUTs and strings which predict a variety of fermion mass textures m_f not necessarily symmetric or Hermitian. In the present analysis we will consider the Hermitian squares $m_f m_f^\dagger$ of the 3×3 fermion mass matrices which capture the physical properties of a whole class of fermion mass textures m_f.

A general Hermitian 3×3 matrix contains 9 independent elements and can be written as

$$H = i \ln U \tag{2}$$

where U a unitary matrix. Using the Caley–Hamilton theorem we can write

$$H = b_1 I + b_2 U + b_3 U^2 \tag{3}$$

where b_1, b_2, b_3 are complex in general. Reversing the argument, we propose to write a Hermitian mass matrix M in the form

$$M = b_1 I + b_2 U + b_3 U^2 \tag{4}$$

where U is a unitary matrix and, without loss of generality, we assume that $\det U = 1$. Now, the standard CKM form for a unitary matrix contains four independent elements and has determinant one. Adding six degrees of freedom (d.o.f.) from the complex b_i coefficients we have a total of ten, so one d.o.f. is redundant, and can be removed by requiring one eigenvalue of U to be one as described in the next paragraph.

Since U and M obviously commute, the above expression can be diagonalized by means of a similarity transformation. Thus, once we have expressed a given M in terms of U, we can find its diagonalizing matrix simply by diagonalizing U. A diagonal unitary matrix is uniquely defined by

$$U_d = \begin{bmatrix} e^{ia_1} & 0 & 0 \\ 0 & e^{ia_2} & 0 \\ 0 & 0 & e^{ia_3} \end{bmatrix}. \tag{5}$$

[2] We note that the method developed here is general and can be applied equally well to the quark sector.

One phase can be absorbed into a redefinition of the coefficients b_2 and b_3 and taking into account the determinant condition, we end up with

$$U_d=\begin{bmatrix} e^{ia} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-ia}\end{bmatrix} \tag{6}$$

where the ordering of the diagonal elements may vary. Thus, our unitary matrix can always be chosen to have one eigenvalue equal to one. Denoting by m_1, m_2, m_3 the (real) eigenvalues of M, we have the equations

$$m_1=b_1+b_2e^{i\alpha}+b_3e^{2i\alpha},\qquad m_2=b_1+b_2+b_3,\qquad m_3=b_1+b_2e^{-i\alpha}+b_3e^{-2i\alpha}. \tag{7}$$

The solution of the above system for b_1, b_2, b_3 gives

$$b_1=-\frac{1}{4}\csc^2\frac{\alpha}{2}\left(\frac{e^{-\frac{3}{2}i\alpha}m_1+e^{\frac{3}{2}i\alpha}m_3}{2\cos\frac{a}{2}}-m_2\right), \tag{8}$$

$$b_2=+\frac{1}{4}\csc^2\frac{\alpha}{2}\left(e^{-i\alpha}(m_1-m_2)-e^{i\alpha}(m_2-m_3)\right), \tag{9}$$

$$b_3=-\frac{1}{4}\csc^2\frac{\alpha}{2}\frac{e^{-\frac{i}{2}\alpha}(m_1-m_2)-e^{\frac{i}{2}\alpha}(m_2-m_3)}{2\cos\frac{a}{2}}. \tag{10}$$

Therefore, using this parametrization, we have succeeded to disentangle the mass eigenvalues of M from the diagonalizing matrix. The eigenmasses m_i are given as functions of the coefficients b_i and the phase α only. Consequently, for a given mass spectrum we may reconstruct the fermion mass texture by simply computing the coefficients b_i from relations (8)–(10) and a suitably chosen unitary matrix U. If for example, the mixing effects are accurately described by the experimental data, the mixing angles can be specified and U can be readily determined from the mixing matrix and the phase α. Next, we concentrate on the unitary matrix U assuming the standard parametrization in terms of three angles θ_{12}, θ_{23}, θ_{13} and a phase δ. We have

$$U=\begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13}\\ -c_{23}s_{12}-c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23}-e^{i\delta}s_{12}s_{13}s_{23} & c_{13}s_{23}\\ -c_{12}c_{23}s_{13}e^{i\delta}+s_{12}s_{23} & -c_{23}s_{12}s_{13}e^{i\delta}-c_{12}s_{23} & c_{13}c_{23}\end{bmatrix}. \tag{11}$$

The requirement to have one eigenvalue equal to one leads to the constraint

$$\sin\delta\sin\theta_{12}\sin\theta_{13}\sin\theta_{23}=0. \tag{12}$$

This condition is satisfied if one of the parameters vanishes generating four distinct structures for U:

$$U_1=\begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13}\\ -c_{23}s_{12}-c_{12}s_{13}s_{23} & c_{12}c_{23}-s_{12}s_{13}s_{23} & c_{13}s_{23}\\ -c_{12}c_{23}s_{13}+s_{12}s_{23} & -c_{23}s_{12}s_{13}-c_{12}s_{23} & c_{13}c_{23}\end{bmatrix},\quad \delta=0, \tag{13}$$

$$U_2=\begin{bmatrix} c_{13} & 0 & e^{-i\delta}s_{13}\\ -s_{13}s_{23}e^{i\delta} & c_{23} & c_{13}s_{23}\\ -c_{23}s_{13}e^{i\delta} & -s_{23} & c_{13}c_{23}\end{bmatrix},\quad \theta_{12}=0, \tag{14}$$

$$U_3=\begin{bmatrix} c_{12} & s_{12} & 0\\ -c_{23}s_{12} & c_{12}c_{23} & s_{23}\\ s_{12}s_{23} & -c_{12}s_{23} & c_{23}\end{bmatrix},\quad \theta_{13}=0, \tag{15}$$

$$U_4=\begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13}\\ -s_{12} & c_{12} & 0\\ -c_{12}s_{13}e^{i\delta} & -s_{12}s_{13}e^{i\delta} & c_{13}\end{bmatrix},\quad \theta_{23}=0. \tag{16}$$

In the present analysis, out of the four possible forms for U we choose the most appropriate cases and work out its implications on the leptonic sector. Since the mixing effects depend on the combined charged lepton and neutrino diagonalizing matrices, we treat them separately.

2.1. The neutrinos

We start with the neutrino sector. Here we do not assume a specific embedding of the matrices in a given (GUT) model, thus the absolute scale of the neutrino eigenmasses is arbitrary and can be easily chosen in consistency with the $\beta\beta$-decay constraints.

We will analyze the case where the unitary matrix associated to the neutrino mass texture M_ν is expressed as in (4) with U given by the specific form (14), i.e.

$$U=\begin{bmatrix} c_{13} & 0 & e^{-i\delta}s_{13}\\ -s_{13}s_{23}e^{i\delta} & c_{23} & c_{13}s_{23}\\ -c_{23}s_{13}e^{i\delta} & -s_{23} & c_{13}c_{23}\end{bmatrix}. \tag{17}$$

By construction, the above matrix admits one eigenvalue equal to one, thus the eigenvalues of U are 1 and $e^{\pm i\alpha}$. The diagonalizing matrix for U is difficult to find in simple form, so we introduce the following parametrization.

$$\tan\theta_{13} = \frac{2z_1}{1-z_1^2}, \tag{18}$$

$$\tan\theta_{23} = \frac{2z_1 z_2\sqrt{(1+z_1^2)(1-z_2^2)}}{z_2^2 - z_1^2 + 2z_1^2 z_2^2}, \tag{19}$$

$$\delta = \theta + \frac{\pi}{2}. \tag{20}$$

Then, the diagonalizing matrix for U is

$$V_\nu(z_1, z_2, \theta) = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\theta}\frac{1}{p}(z_2 - \imath q^2 z_1) & \sqrt{2}q\imath e^{i\theta} & -e^{i\theta}\frac{1}{p}(z_2 + \imath q^2 z_1) \\ \frac{q}{p}(z_1 z_2 - \imath) & -\sqrt{2}z_2 & \frac{q}{p}(z_1 z_2 + \imath) \\ p & \sqrt{2}q z_1 & p \end{bmatrix} \tag{21}$$

where p, q are functions of $z_{1,2}$ given by

$$p = \sqrt{\frac{1+z_1^2 z_2^2}{(1+z_1^2)}}, \tag{22}$$

$$q = \sqrt{\frac{1-z_2^2}{1+z_1^2}}. \tag{23}$$

We can easily check that

$$V_\nu^\dagger U V_\nu = \text{diagonal}\left[e^{i\alpha}, 1, e^{-i\alpha}\right]. \tag{24}$$

The eigenvalue α depends only on z_1, z_2

$$e^{i\alpha} = -\frac{z_1 - iz_2}{z_1 + iz_2}, \quad \text{or} \quad \alpha = \tan^{-1}\frac{z_1}{z_2}.$$

The following relations are also useful:

$$z_1 = \tan\frac{\theta_{13}}{2}, \qquad z_2 = \frac{z_1\cos\frac{\theta_{23}}{2}}{\sqrt{z_1^2 + \sin^2\frac{\theta_{23}}{2}}}, \qquad p = \sqrt{1 - z_2^2\tan^2\frac{\theta_{23}}{2}}, \qquad q = \frac{z_1}{z_2}\cot\frac{\theta_{23}}{2}.$$

Using these equations, all the elements of (21) can be expressed in terms of the trigonometric entries of the unitary matrix U.

2.2. The charged leptons

We now derive similar formulae for the charged leptons. As in the case of neutrinos, we choose to write the mass matrix in terms of a unitary matrix U in accordance to formula (4). For reasons that will become clear later, we choose the ordering of the U matrix eigenvalues to be as follows:

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{-i\alpha} \end{bmatrix}. \tag{25}$$

If the eigenvalues of M are m_1, m_2, m_3 the coefficients b_i in (4) are given by equations analogous to (8)–(10).

For the case of charged leptons, we confine ourselves to orthogonal matrices. Therefore, out of the four possible forms for U we choose (13)

$$U = \begin{bmatrix} c_{12}c_{13} & c_{13}s_{12} & s_{13} \\ -c_{23}s_{12} - c_{12}s_{13}s_{23} & c_{12}c_{23} - s_{12}s_{13}s_{23} & c_{13}s_{23} \\ -c_{12}c_{23}s_{13} + s_{12}s_{23} & -c_{23}s_{12}s_{13} - c_{12}s_{23} & c_{13}c_{23} \end{bmatrix} \tag{26}$$

which corresponds to the structure U_1. Next, we use the fact that an orthogonal matrix can be written as[3]

$$U = e^{\alpha\hat{n}\cdot\vec{s}} = 1 + \sin\alpha\,\hat{n}\cdot\vec{s} + (1-\cos\alpha)(\hat{n}\cdot\vec{s})^2$$

where $\hat{n} = (n_1, n_2, n_3)$ is a unit vector and the 3×3 matrices s_i satisfy the conditions

$$[s_i, s_j] = \varepsilon_{ijk}s_k$$

[3] For a detailed mathematical analysis of the subsequent formalism in the context of the fermion mass matrices see [14].

and are explicitly given by

$$s_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \tag{27}$$

$$s_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{28}$$

$$s_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

The matrix U is diagonalized by means of the matrix

$$V_l(n_1, n_2, n_3) = \frac{1}{\sqrt{2}\sqrt{n_1^2 + n_3^2}} \begin{bmatrix} \sqrt{2}\sqrt{n_1^2 + n_3^2}n_1 & n_1 n_2 - i n_3 & n_1 n_2 + i n_3 \\ -\sqrt{2}\sqrt{n_1^2 + n_3^2}n_2 & n_1^2 + n_3^2 & n_1^2 + n_3^2 \\ \sqrt{2}\sqrt{n_1^2 + n_3^2}n_3 & n_2 n_3 + i n_1 & n_2 n_3 - i n_1 \end{bmatrix} \tag{30}$$

and we may check that

$$V_l^\dagger U V_l = \text{diagonal}\left[1, e^{i\alpha}, e^{-i\alpha}\right]. \tag{31}$$

2.3. The leptonic mixing matrix

In the previous sections we managed to obtain the diagonalizing matrices for the charged lepton and neutrino mass textures V_l and V_ν in closed form. In accordance to standard notation, the leptonic mixing matrix is defined to be

$$V_M = e^{i\psi} V_l^\dagger V_\nu. \tag{32}$$

The phase factor is introduced in order to have the matrix determinant equal to one. A closer inspection reveals that all the elements of so derived V_M are complex. It can be shown [26] however, that (32) can be rendered equivalent to the standard form given in (11). The proof goes as follows. A Hermitian matrix M can by diagonalized by means of a unitary transformation

$$U^\dagger M U = \text{Diag}[m_1, m_2, m_3] = D \cdot \text{Diag}[m_1, m_2, m_3] \cdot D^\dagger \tag{33}$$

where D is a unitary matrix of the form

$$D = \text{Diag}\left[e^{id_1}, e^{id_2}, e^{id_3}\right]. \tag{34}$$

This way, if U is a diagonalizing matrix so is UD. The lepton mixing matrix is given by

$$V_M = V_l^\dagger V_\nu \tag{35}$$

and taking the above into account, V_M can be equivalently written as

$$V_M = D^\dagger V_l^\dagger V_\nu C \tag{36}$$

where

$$C = \text{Diag}\left[e^{ic_1}, e^{ic_2}, e^{ic_3}\right], \qquad D = \text{Diag}\left[e^{id_1}, e^{id_2}, e^{id_3}\right].$$

If we require (36) be reduced to the standard form, the six phases can be uniquely determined.

3. Analysis

Using the above results, we will now proceed to determine possible deviations from the TB-mixing which fit the experimental data and determine the allowed range for the parameters $z_1, z_2, \theta, \hat{n}$ in the neutrino and charged lepton sectors respectively. Before further pursuing the general case, we will first present the simplest and possibly the most elegant way of extending the Tri–Bi maximal mixing.

3.1. Example: The minimal case

We start with the neutrino sector and introduce values for the parameters $z_{1,2}$ which are in accordance with the TB-scenario. We put $z_2 = -1$ and we get $\tan\theta_{23} = 0$ whilst for the eigenvalue of the unitary matrix U we get

$$e^{i\alpha} = \frac{i + z_1}{i - z_1}$$

so that $\tan\alpha = -\frac{2z_1}{1 - z_1^2}$ and thus, $\alpha = -\theta_{13}$. This way the U matrix becomes

$$U=\begin{bmatrix}\cos\alpha & 0 & -ie^{i\theta}\sin\alpha\\ 0 & 1 & 0\\ -ie^{-i\theta}\sin\alpha & 0 & \cos\alpha\end{bmatrix}. \tag{37}$$

The neutrino mass matrix takes the simple form

$$\begin{bmatrix}\frac{1}{2}(m_1+m_3) & 0 & \frac{1}{2}e^{i\theta}(m_3-m_1)\\ 0 & m_2 & 0\\ \frac{1}{2}e^{-i\theta}(m_3-m_1) & 0 & \frac{1}{2}(m_1+m_3)\end{bmatrix}. \tag{38}$$

For $\theta=\pi$ it is exactly the texture of the neutrino mass matrix introduced in the case of TB-mixing [6]. The matrix (38) is written in terms of its real eigenmass values with the $\{13\},\{31\}$ entries multiplied by a phase factor.

Next, we proceed with the charged lepton sector. The TB-matrix [6] corresponds to

$$n_1=\frac{1}{\sqrt{3}},\qquad n_2=-\frac{1}{\sqrt{3}},\qquad n_3=\frac{1}{\sqrt{3}}. \tag{39}$$

We know however, that the TB-case does not reproduce the experimental data since it predicts a zero θ_{13} angle. A minimal extension arises if we introduce in the neutrino mixing matrix the parameters

$$z_2=-1,\qquad \theta=\pi+\varphi$$

while keeping the charged lepton diagonalizing matrix as above. Then the mixing matrix is given by

$$V_M=e^{-\frac{i\pi}{6}}e^{-\frac{i\varphi}{3}}V_l^\dagger\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)V_\nu(z_1,-1,\varphi+\pi). \tag{40}$$

After some algebra and the removal of the redundant phase factors [26], the matrix can be brought into canonical form given by

$$V_M=\begin{bmatrix}\sqrt{\frac{2}{3}}\cos\frac{\varphi}{2} & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}\sin\frac{\varphi}{2}\\ -\sqrt{\frac{2}{3}}\sin(\frac{\varphi}{2}+\frac{\pi}{6}) & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}}\cos(\frac{\varphi}{2}+\frac{\pi}{6})\\ \sqrt{\frac{2}{3}}\sin(\frac{\varphi}{2}-\frac{\pi}{6}) & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}}\cos(\frac{\varphi}{2}-\frac{\pi}{6})\end{bmatrix}. \tag{41}$$

The experimental bounds are:

$$0.0871557<|\sin\theta_{13}|<0.224931, \tag{42}$$

$$0.68728<|\tan\theta_{12}|<0.713293, \tag{43}$$

$$0.213895<|\tan\theta_{23}|<1.09131 \tag{44}$$

and we have

$$(V_M)_{11}=\sin\theta_{13}=-\sqrt{\frac{2}{3}}\sin\frac{\varphi}{2}, \tag{45}$$

$$\frac{(V_M)_{23}}{(V_M)_{33}}=\tan\theta_{23}=-\frac{\cos(\frac{\varphi}{2}+\frac{\pi}{6})}{\cos(\frac{\varphi}{2}-\frac{\pi}{6})}, \tag{46}$$

$$\frac{(V_M)_{11}}{(V_M)_{12}}=\tan\theta_{12}=\frac{1}{\sqrt{2}\cos\frac{\varphi}{2}}. \tag{47}$$

Combining the above, we find that all the constraints are satisfied for

$$\frac{\pi}{15}\lesssim\varphi\lesssim\frac{\pi}{12}. \tag{48}$$

This is a rather interesting result: it states that TB-mixing can reconcile the neutrino data by a suitable choice of the phase parameter θ parametrizing the neutrino diagonalizing matrix. In the minimal case we are here dealing with, this phase coincides with the phase in the $\{13\},\{31\}$ elements of the neutrino mass matrix. In the original TB-model this phase is simply taken to be $\theta=\pi$. When shifted by a value φ lying in the range (48), neutrino data are exactly predicted.

The advantage of this solution as compared to any perturbative approach around the TB-solution aiming to fit a non-zero θ_{13} angle is rather obvious: indeed, it is shown that a non-zero θ_{13} angle that preserves the symmetric and zero form texture of the M_ℓ and M_ν matrices can be naturally incorporated into the minimal TB-scheme. This is of crucial importance if we really wish to attribute their simple structure to some kind of discrete or other symmetry of the theory [9–20].

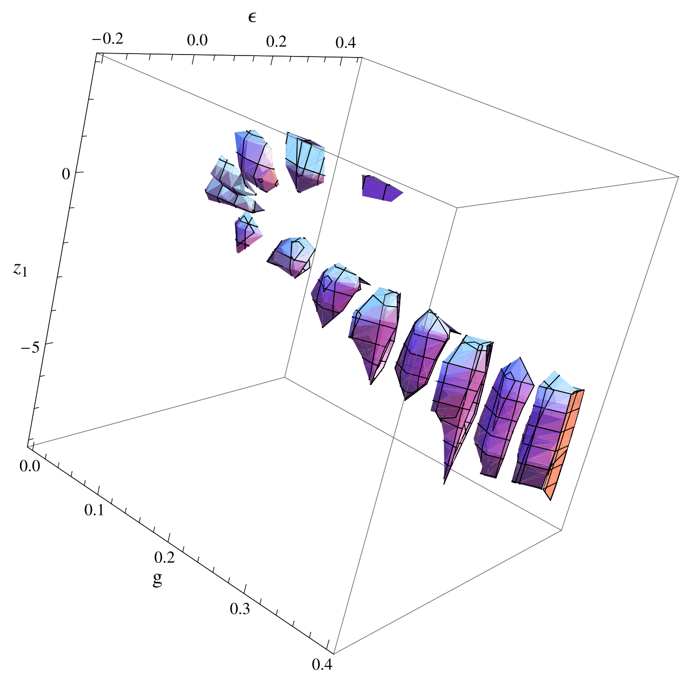

Fig. 1. In this plot, deviations from TB-mixing are parametrized in terms of z_1, $g^2 = 1 + z_2$ of the neutrino and ϵ of the corresponding charged lepton diagonalizing matrices. (See (21), (49).)

3.2. The general case

We explore now regions of the parameter space which signal departures from the TB-case. Deviations can be easily obtained by assuming for example that

$$n_1 = \frac{1}{\sqrt{3}}, \qquad n_3 = \frac{1}{\sqrt{3}} - \varepsilon, \qquad n_2 = -\sqrt{1 - n_1^2 - n_3^2} \tag{49}$$

in the charged leptons sector. Similarly, we choose to write $z_2 = -1 + g^2$ in the neutrino diagonalizing matrix (21). In Fig. 1 we plot the ranges of these parameters subject to the well-known constraints of the mixing angles, while we keep $\theta = \pi$.

We observe that the allowed values of g lie in the range $0.05 \lesssim g \lesssim 0.6$, those of ε in the range $-0.2 \lesssim \varepsilon \lesssim 1$ while acceptable values for z_1 cover a wider range lying from $z_1 \sim 2.4$ to large negative values $z_1 \sim -10$. It appears that there are wide regions in the parameter space consistent with data which significantly deviate from the TB-mixing picture.

Next, in order to determine mass textures related to possible exact symmetries, we scan the g, ε ranges for fixed z_1 values. Here, we will concentrate in the subregions $g \sim [0.2\text{–}0.6]$, $\varepsilon \sim [0.1\text{–}0.6]$ and search for values corresponding to exactly known trigonometric quantities.

Let us choose $\varepsilon = 1/\sqrt{3}$. This eliminates one entry in V_l which assumes the form

$$V_\ell = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

Upon inspection, we observe that the remaining two parameters of the neutrino diagonalizing matrix can be taken to be $z_1 = \tan\frac{7\pi}{12}$, $z_2 = -\frac{1}{\sqrt{2}}$.

With this choice, we can readily check that the matrix formed by the moduli of the elements of the leptonic mixing matrix is given by

$$\begin{pmatrix} 0.806056 & 0.586939 & 0.0759986 \\ 0.420639 & 0.655601 & 0.627096 \\ 0.416337 & 0.475067 & 0.775225 \end{pmatrix}$$

pretty much close to the experimental data shown below

$$\begin{bmatrix} \cdots & 0.546431\text{–}0.580416 & 0.03141\text{–}0.14091 \\ \cdots & \cdots & 0.63505\text{–}0.736914 \\ \cdots & \cdots & \cdots \end{bmatrix}$$

where the missing elements are determined by unitarity. For the above choice of parameters, the charged lepton mass matrix obtained by substituting (26) into the general form (4), is found to be

$$M_l = \begin{pmatrix} \frac{m_1+m_2+m_3}{3} & -\frac{2m_1-m_2-m_3}{3\sqrt{2}} & -i\frac{m_2-m_3}{\sqrt{6}} \\ -\frac{2m_1-m_2-m_3}{3\sqrt{2}} & -\frac{4m_1+m_2+m_3}{6} & -i\frac{m_2-m_3}{2\sqrt{3}} \\ i\frac{m_2-m_3}{\sqrt{6}} & i\frac{m_2-m_3}{2\sqrt{3}} & \frac{m_2+m_3}{2} \end{pmatrix}. \tag{50}$$

The matrix (50) exhibits an interesting structure and one could think of several ways to link it to a possible existence of underlying symmetries. For example, in cases of string derived models with several singlet fields $\phi, \phi', \ldots$ acquiring vevs, one defines expansion parameters $\epsilon = \langle\phi\rangle/M$, $\epsilon' = \langle\phi'\rangle/M, \ldots$ with M being the cutoff scale of the higher theory. Then, to leading order we get the hierarchy $m_3 \gg m_2 \gg m_1$ and we can approximate this matrix by

$$M_l \approx \begin{pmatrix} |\epsilon|^2 & \epsilon\bar{\epsilon}' & \epsilon \\ \bar{\epsilon}\epsilon' & -|\epsilon'|^2 & \epsilon' \\ \bar{\epsilon} & \bar{\epsilon}' & 1 \end{pmatrix} m_\ell \tag{51}$$

with $\epsilon = i\sqrt{\frac{2}{3}}$, $\epsilon' = \frac{i}{\sqrt{3}}$ and m_ℓ a mass parameter related to charged lepton mass scale.

The corresponding neutrino mass matrix is given by

$$M_\nu = \begin{pmatrix} \frac{\alpha_+(m_1+\xi_- m_2+m_3)}{4} & \frac{\beta_+(m_1-m_3)+i\gamma_-(m_1-2m_2+m_3)}{4} & \frac{4\sqrt{2}(m_1-m_3)-i(m_1-2m_2+m_3)}{16} \\ \ldots & \frac{m_1+2m_2+m_3}{4} & \frac{i\beta_-(m_1-m_3)+\gamma_+(m_1-2m_2+m_3)}{4} \\ \ldots & \ldots & \frac{\alpha_-(m_1+\xi_+ m_2+m_3)}{4} \end{pmatrix}$$

where the dots stand for the corresponding complex conjugate entries and the various coefficients are

$$\alpha_\pm = \frac{6 \pm \sqrt{3}}{4}, \qquad \beta_\pm = \frac{1 \pm \sqrt{3}}{2}, \qquad \gamma_\pm = \frac{1}{2}\sqrt{\tan(\pi/4 \pm \pi/6)} = \frac{\sqrt{2 \pm \sqrt{3}}}{2}, \qquad \xi_\pm = \frac{32}{33}\alpha_\pm\gamma_\pm^2.$$

It is to be noted that all the off-diagonal entries of the neutrino mass matrix are expressed only in terms of the squared neutrino mass differences (note that we have assumed Hermitian squared mass matrices thus we have the correspondence $m_i \leftrightarrow m_{\nu_i}^2$). The resulting structure is now more complicated than the corresponding charged lepton one. This is of course to be anticipated in models employing the see-saw mechanism, since the effective neutrino mass matrix is a product of the Dirac and the heavy right handed Majorana neutrino mass matrices $M_\nu \propto m_D M_N^{-1} m_D^T$. Depending on the specific structure of the hypothetical original theory, there are even more options to attribute this matrix to symmetry properties [27], the analysis of this issue however goes beyond the scope of this Letter.

4. Conclusions

In this Letter we have investigated possible forms for the charged lepton and neutrino mass textures which can reconcile the experimental data on neutrino oscillations. In our analysis we have considered the Hermitian squares of either mass matrix and used standard techniques to express each one of them as a second degree polynomial of a suitably chosen unitary matrix. Since the eigenmass dependence is encapsulated in the coefficients of this expansion only, we can express the neutrino mixing angles analytically, as functions of the parameters which define the unitary matrices that generate the charged lepton and neutrino mass textures respectively. Next, we may use the available neutrino data on the mixing angles to constrain this parameter space. In particular, taking into account that the mass matrices suggested by Harrison et al. are in good agreement with the Tri–Bi maximal neutrino mixing, we explored the parameter space for allowed deviations. We have found that the actual data including a non-vanishing θ_{13} angle can be nicely captured, by only introducing a single phase in the {13} and {31} entries of the neutrino mass texture in the original TB-scheme. Furthermore, upon varying the free parameters of our model in a wider range, we have found that neutrino data can be accommodated even for large deviations from the TB-matrices too.

References

[1] R.N. Mohapatra, et al., Rep. Prog. Phys. 70 (2007) 1757, arXiv:hep-ph/0510213.
[2] M. Maltoni, T. Schwetz, M.A. Tortola, J.W.F. Valle, New J. Phys. 6 (2004) 122, arXiv:hep-ph/0405172.
[3] J.R. Ellis, G.K. Leontaris, S. Lola, D.V. Nanopoulos, Eur. Phys. J. C 9 (1999) 389, arXiv:hep-ph/9808251.
[4] Z.-z. Xing, Phys. Lett. B 533 (2002) 85, hep-ph/0204049.
[5] G. Altarelli, F. Feruglio, Rev. Mod. Phys. 82 (2010) 2701, arXiv:1002.0211 [hep-ph].
[6] P.F. Harrison, D.H. Perkins, W.G. Scott, Phys. Lett. B 530 (2002) 167, hep-ph/0202074.
[7] M. Abbas, A.Y. Smirnov, Phys. Rev. D 82 (2010) 013008, arXiv:1004.0099 [hep-ph].
[8] G.K. Leontaris, J. Rizos, Nucl. Phys. B 567 (2000) 32, arXiv:hep-ph/9909206.
[9] C.S. Lam, Phys. Rev. D 78 (2008) 073015, arXiv:0809.1185 [hep-ph].
[10] I. de Medeiros Varzielas, S.F. King, G.G. Ross, Phys. Lett. B 644 (2007) 153, arXiv:hep-ph/0512313.
[11] K.S. Babu, E. Ma, J.W.F. Valle, Phys. Lett. B 552 (2003) 207, arXiv:hep-ph/0206292.
[12] X.G. He, Y.Y. Keum, R.R. Volkas, JHEP 0604 (2006) 039, arXiv:hep-ph/0601001.
[13] S. Antusch, S.F. King, C. Luhn, M. Spinrath, Nucl. Phys. B 850 (2011) 477, arXiv:1103.5930 [hep-ph].
[14] G.K. Leontaris, N.D. Vlachos, JHEP 1001 (2010) 016, arXiv:0909.4701 [hep-th].
[15] G. Altarelli, F. Feruglio, Nucl. Phys. B 741 (2006) 215, arXiv:hep-ph/0512103.
[16] E. Ma, G. Rajasekaran, Phys. Rev. D 64 (2001) 113012, arXiv:hep-ph/0106291.
[17] F. Feruglio, C. Hagedorn, Y. Lin, L. Merlo, Nucl. Phys. B 775 (2007) 120, arXiv:hep-ph/0702194;
F. Feruglio, C. Hagedorn, Y. Lin, L. Merlo, Nucl. Phys. B 836 (2010) 127 (Erratum).
[18] F. Bazzocchi, M. Frigerio, S. Morisi, Phys. Rev. D 78 (2008) 116018, arXiv:0809.3573 [hep-ph].
[19] M. Honda, M. Tanimoto, Prog. Theor. Phys. 119 (2008) 583, arXiv:0801.0181 [hep-ph].

[20] I. de Medeiros Varzielas, S.F. King, G.G. Ross, Phys. Lett. B 648 (2007) 201, arXiv:hep-ph/0607045.
[21] S.M. Bilenky, D. Nicolo, S.T. Petcov, Phys. Lett. B 538 (2002) 77, arXiv:hep-ph/0112216;
J. Bernabeu, S. Palomares Ruiz, S.T. Petcov, Nucl. Phys. B 669 (2003) 255, arXiv:hep-ph/0305152;
M.C. Gonzalez-Garcia, M. Maltoni, Eur. Phys. J. C 26 (2003) 417, arXiv:hep-ph/0202218.
[22] Ralf Blumenhagen, Mirjam Cvetič, Timo Weigand, Nucl. Phys. B 771 (2007) 113, arXiv:hep-th/0609191.
[23] L.E. Ibanez, A.M. Uranga, JHEP 0703 (2007) 052, arXiv:hep-th/0609213.
[24] G.K. Leontaris, Int. J. Mod. Phys. A 24 (2009) 6035, arXiv:0903.3691 [hep-ph].
[25] M. Cvetič, J. Halverson, P. Langacker, R. Richter, JHEP 1010 (2010) 094, arXiv:1001.3148 [hep-th].
[26] G.K. Leontaris, N.D. Vlachos, in preparation.
[27] H.K. Dreiner, G.K. Leontaris, S. Lola, G.G. Ross, C. Scheich, Nucl. Phys. B 436 (1995) 461, arXiv:hep-ph/9409369.