Solitons in normally dispersive mode-locked lasers

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Soliton pulses in normally dispersive mode-locked lasers are considered using a nonlinear Schrödinger equation, appropriately modified to model power (intensity) and energy saturations. Strongly chirped, localized pulses are obtained when the effects of nonlinearity, dispersion, saturated gain, filtering, and loss form an appropriate balance. In the case of constant dispersion, perturbation theory yields a set of uncoupled equations for the amplitude and the phase of the soliton pulse. In dispersion-managed (DM) systems, an asymptotic multiple-scale theory is used to analyze the dynamics. This equation, which describes solitons in the anomalous regime, also admits higher-order solitons, the so-called antisymmetric soliton or bisoliton, in both constant dispersion and DM systems. Such pulses have been observed in recent experiments.

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I. INTRODUCTION

Fiber amplifiers are key components in optical telecommunication systems and are often used as high-power ultrafast sources. They are generally configured to operate in a range where nonlinear effects are negligible. However, recent results have demonstrated an interesting operating regime where nonlinear propagation is exploited to generate an ultrashort nearly parabolic pulse that grows self-similarly as it is amplified [1]. Such self-similar dynamical effects have attracted considerable recent interest in the study of nonlinear optical pulse propagation [2].

Self-similar pulses have been asymptotically derived from the nonlinear Schrödinger (NLS) equation in the normal regime with linear gain [3]. This pulse shape represents a type of nonlinear attractor toward which a rather general-shaped input pulse tends to after sufficient distance. Such self-similar pulses are often termed "similaritons" [4].

Femtosecond solid-state lasers, such as those based on the Ti:sapphire gain medium, have received considerable attention in the field of ultrafast science. In the past decade, following the discovery of Kerr-lens mode locking, the performance of these lasers has led to their widespread use [5]. More recently, ultrashort pulse mode locking has been demonstrated in fiber lasers operating in the normal regime. Wave-breaking-free operation has been achieved with pulse energies much greater than those attained by stretched-pulse lasers [6–8].

Pulse propagation in a laser cavity is governed by the interaction of chromatic dispersion, self-phase modulation, saturable gain and filtering, and intensity discrimination. In the anomalous regime, various models have been used to describe this propagation, including Ginzburg-Landau (GL) systems [9–13] and the so-called "master equation" [14,15]. The master equation is a generalization of the classical NLS equation modified to contain gain, filtering, and loss terms. Gain and filtering are saturated by energy (i.e., the time integral of the pulse power), while loss is represented by a cubic nonlinearity. If the pulse energy is taken to be constant the master equation reduces to a GL-type system. These equations exhibit a variety of solutions ranging from unstable, chaotic to quasiperiodic, strong amplitude growth and mode locking over a small region of parameter space [16].

Here we use the power energy saturation (PES) equation discussed in Refs. [17,18] to obtain localized pulses that propagate in a normally dispersive laser, consistent with recent experimental observations [6,19]. In the normal dispersion regime, a different type of pulse shaping occurs which is qualitatively distinct from the well-known soliton [18] and the dispersion-managed (DM) soliton [17] of the anomalous dispersion regime both of which are described by this equation. Pulse formation in an ultrashort pulse laser in the anomalous regime is typically dominated by the interplay between dispersion and nonlinearity. Suitable gain media and an effective saturable absorber are required for initiation of pulsed operation from intracavity noise and subsequent stabilization of the pulse. The pulses found here are positively chirped throughout the cavity, consistent with experimental observations [6]. Detailed aspects include (i) the approximately parabolic temporal amplitude profile near the peak of the pulse with a transition to steep decay [6] and (ii) the large and significant spectral profile of the pulse. This is unlike the standard hyperbolic secant constant dispersion soliton solution and the Gaussian DM soliton of the anomalous regime.

The dimensionless distributed constant dispersion model describing the propagation of pulses in a laser cavity is given by

$$i\psi_z - \frac{d_0}{2}\psi_{tt} + |\psi|^2\psi = \frac{ig}{1+\epsilon E}\psi + \frac{i\tau}{1+\epsilon E}\psi_{tt} - \frac{il}{1+\delta P}\psi, \quad (1)$$

where $\psi(z,t)$ is the slowly varying electromagnetic pulse envelope, $E(z)=\int_{-\infty}^{+\infty}|\psi|^2 dt$ is the pulse energy, and $P(z,t)=|\psi|^2$ is the instantaneous pulse power (intensity). The first term on the right-hand side represents saturable gain, the second is spectral filtering, and the third the saturable loss; the parameters g (dimensionless gain), $\tau>0$ (filtering), $l>0$ (loss), and ϵ, δ (dimensionless inverses of the saturation energy and

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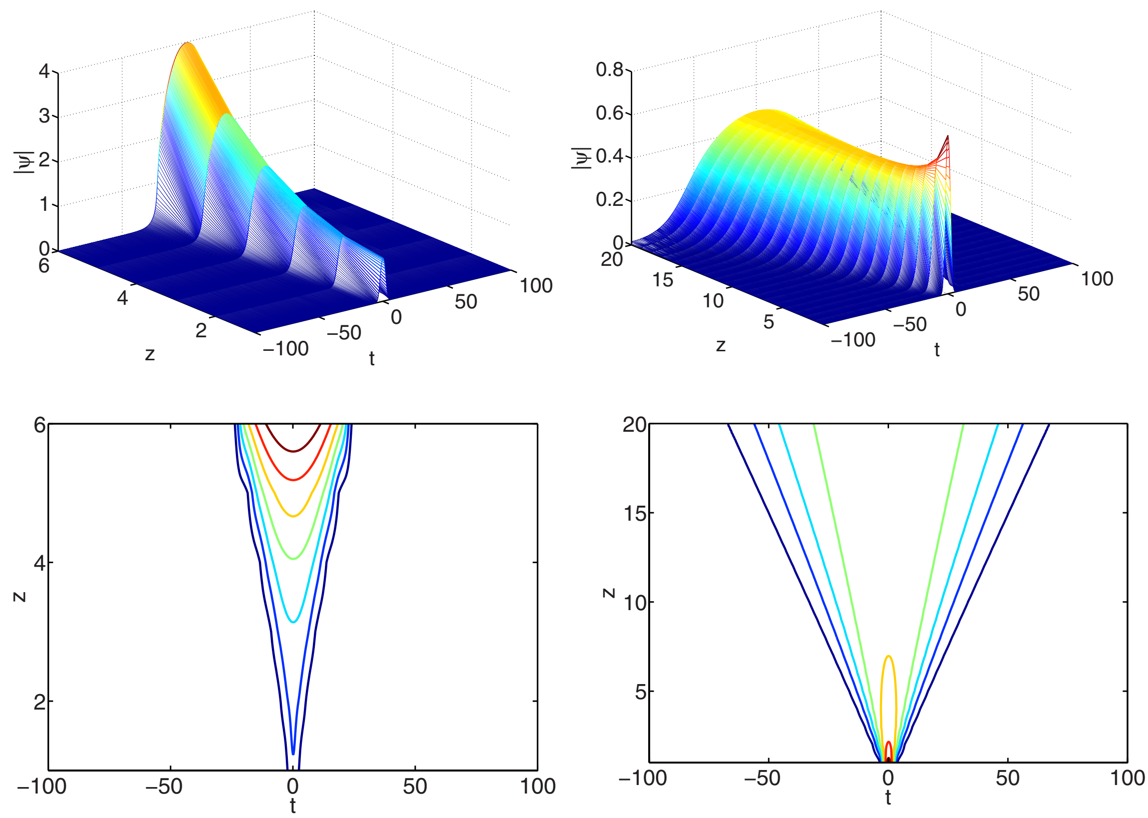

FIG. 1. (Color online) Self-similar evolution of a pulse under linear (left) and saturated (right) gains.

power, respectively) are all positive, real constants. The dimensionless constant dispersion is represented by $d_0 > 0$. DM systems are considered later (Sec. IV). Notice that gain and filtering are saturated with energy while loss is saturated with power (intensity). The gain and filtering mechanisms are related to the energy of the pulse while the loss is related to the power (intensity) of the pulse. The filtering mechanism takes into account the frequency widow in the laser crystal. The loss terms are often referred to as fast saturable absorbers. Absorbers of the type introduced in the above model are often used. In fact, Haus [20] introduced such a saturable term and then expanded the power saturation term in a Taylor series, keeping only the first two terms (up to cubic nonlinearity). This resulted in the master equation. In other models, loss is introduced in the form of fast saturable power absorbers which are placed periodically [6,19]. This type of lumped model has also been studied in dispersion-managed systems. We refer to the above equation as the PES equation. All parameters appear in nondimensional form; later we relate these parameters to typical physical values obtained from experimental data. We find that the essential features of the lumped model are included in this distributive PES equation [21] in lasers operating in both the anomalous and normal regimes and in dispersion-managed systems as discussed below. Hereafter, the right-hand side of Eq. (1) will be denoted by $Q[\psi]$ and will represent the perturbing effect of the system.

Additional terms such as higher-order linear and nonlinear dispersions, delayed nonlinear response, and Raman type terms can be important in some mode-locked laser applications [22–24]. The study of the dynamics of normal solitons in the PES model with additional higher-order perturbing terms is an interesting topic for future study. More recently it was shown [25] that dispersion-managed models with power (intensity)-energy saturation are in good agreement with experimental results in mode-locked Ti:sapphire lasers. It is therefore important to study the above distributed model.

II. SIMILARITONS AND SELF-SIMILAR EVOLUTION

The effects of energy and power saturations in this model are crucial. If we set $\tau = l = 0$ and $\epsilon = 0$, we recover the similariton supporting equation, namely, the NLS with linear gain [3]. In this system, pulses evolve in a self-similar way in both amplitude and width. Since energy saturation was present in the original formulation of the master equation, it is useful to compare the difference. To illustrate, we evolve Eq. (1) including only the gain term in the right-hand side with and without the saturation. The resulting evolutions are shown in Fig. 1. Here, $g = 0.5$ and $\epsilon = 0$ (left) $\epsilon = 1$ (right). Without energy saturation and starting from a unit Gaussian under linear gain, the pulse undergoes a rapid increase in both its amplitude and width. As indicated in the corresponding contour plot, a self-similar evolution results; this is the well-known similariton. However, when the gain is saturated with energy, the pulse undergoes a decrease in its peak amplitude to a constant value for the duration of the evolution (locks) but still continues to evolve and grow linearly in its width. This suggests that the energy saturation term provides

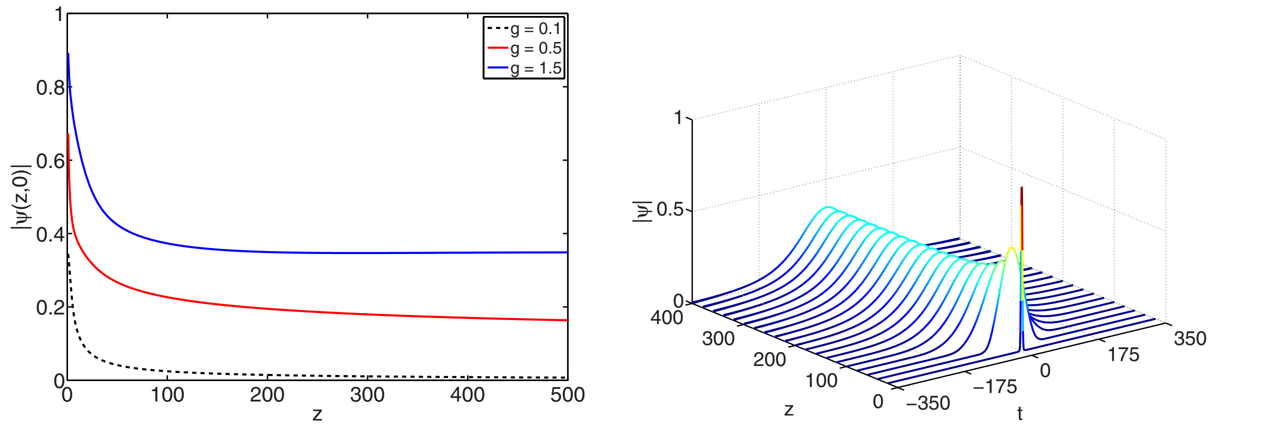

FIG. 2. (Color online) Evolution of the pulse peak of an arbitrary initial profile under the PES equation with different values of gain. The damped pulse-peak evolution is shown with a dashed line. In the right figure, the complete evolution is given for $g=1.5$.

a mechanism to control the growth in the amplitude of pulse.

The spectral filtering plays a crucial role in maintaining a short pulse duration with high energy. This phenomenon is generic to mode locking with normal dispersion as also discussed in Ref. [26]. With spectral filtering in the cavity, robust, localized, high-energy, ultrashort pulses can be generated. It has been observed that spectral filtering of a chirped pulse in the cavity is a major component of the pulse shaping in these lasers [8]. When $\tau \neq 0$, the effects of self-similar evolution are still present but not as prominent. This is to be expected since the filtering effects acts as an overall loss to the propagation of the pulse.

Furthermore, a laser can only operate when a sufficient amount of both gain *and* loss is present. Passive mode locking generally utilizes saturable absorbers, such as the type introduced in the PES model.

III. PULSES IN THE CONSTANT DISPERSION REGIME

The dynamics of pulses evolving under the PES equation are studied next. All terms are kept constant and only the gain parameter g is changed. More precisely, typical values are taken: $d_0=\epsilon=\delta=1$, $\tau=l=0.1$. The evolution of the pulse peak for different values of the gain parameter g is shown in Fig. 2. When $g=0.1$, the pulse decays quickly due to excessive loss with no noticeable oscillatory or chaotic behavior; the pulse exhibits damped evolution. When $g=0.5$, due to the loss in the system, the pulse undergoes an initial decrease and then continues to slowly dissipate. When $g=1.5$, a localized evolution is obtained. With sufficient gain in the system, the pulse amplitude initially decreases but then a steady state is reached. The features of self-similar evolution are initially present since the sharp and narrow initial Gaussian rapidly evolves into a wide but finite-energy pulse. Three regimes are observed: (a) when the loss is much grater than the gain, the pulse decays to zero, (b) when the loss is again the prominent effect but sufficient gain exists in the system to sustain a very slowly decaying evolution resulting in a “quasisoliton” state, and (c) the soliton regime above a certain value of gain.

Localized modes of Eq. (1) are examined next. The spectral renormalization (SPRZ) method of Ref. [27] is employed to calculate these solutions with spectral accuracy. With SPRZ, in each iteration, the ratio between the dispersive and nonlinear parts of the equation is modified until convergence is achieved. Assuming localized solutions of the form $\psi(z,t)=u(t)\exp(i\mu z)$, $u(\pm\infty)\rightarrow 0$, we obtain from Eq. (1)

$$-\mu u-\frac{d_0}{2}u_{tt}+|u|^2u=Q[u],$$

which is a nonlinear eigenvalue problem with respect to the propagation constant μ. To find these solutions, we must determine the appropriate value(s) of the propagation constant μ for which a solution actually exists. The criterion for determining μ is that the renormalization constant in SPRZ is real. With this additional requirement, we obtain only one value of μ, for a specific set of parameters, that the iteration converges. When $g<l$ in Eq. (1), we do not find a solution, i.e., we do not find a value of μ for which the above iteration will converge. This is consistent with the observation that when the effect of loss is stronger than the gain, the only acceptable solution is the trivial or quasisoliton solution. Solutions of the PES for various values of g and the corresponding propagation constant are depicted in Fig. 3. Notice the change in the pulse width and amplitude. As the gain

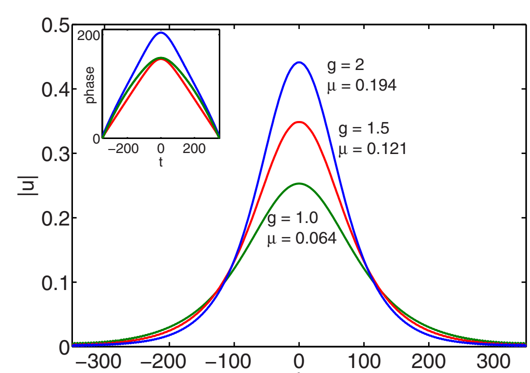

FIG. 3. (Color online) Solutions of Eq. (1) for different values of the gain parameter g and the corresponding propagation constants. In the inset, the corresponding phases are plotted.

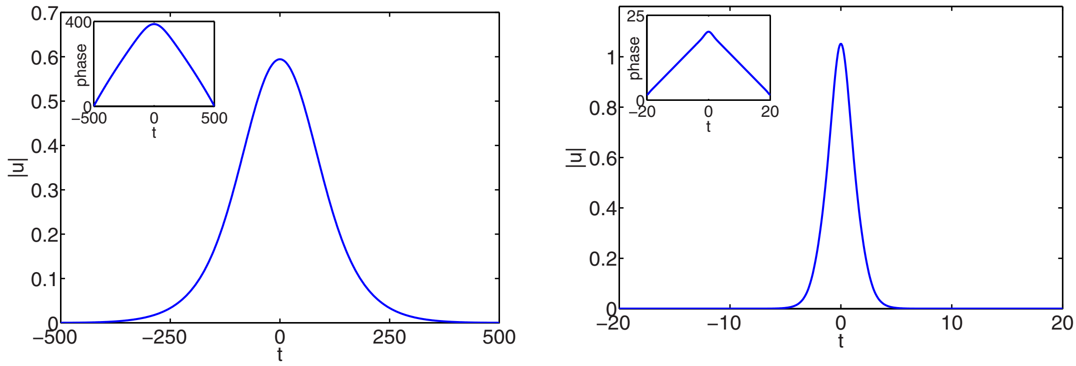

FIG. 4. (Color online) The soliton solution and its phase corresponding to the experimental values of the parameters given in Refs. [7] (left) and [8] (right).

parameter increases, so does the amplitude and the pulse become narrower.

As indicated above, the values for the gain, loss, and filtering parameters provide operating regions of the laser system which depend critically on the gain parameter. The parameters ϵ,δ were conveniently chosen to have unit value. Changing these values corresponding to experimental data confirms the above observations. If we begin with a dimensional system and rescale, we find $d_0=\beta z_*/t_*^2$, $g=g_* z_*$, $\tau=g_* z_*/\Delta\omega^2 t_*^2$, $\epsilon=E_*/E_{\text{sat}}$, and $\delta=P_*/P_{\text{sat}}$. For a Ti:sapphire laser [7], $\beta=60\ \text{fs}^2/\text{mm}$ is the group-velocity dispersion, $z_*=2$ mm the characteristic length, $t_*=10$ fs the characteristic time, $P_*=5$ MW the characteristic power, $E_*=P_* t_*$ the characteristic energy, $g_*=20$ dB the dimensional gain over the crystal, $\Delta\omega^2=100\ \text{fs}^{-1}$ the frequency cutoff, and $E_{\text{sat}}=10$ nJ, $P_{\text{sat}}=2$ MW are the saturated energy and power, respectively. The loss is taken as a distributive saturable absorber and its nondimensional value is $l=0.3$. The corresponding nondimensional values are $d_0=1.2$, $g=5$, $\tau=0.02$, $l=0.3$, $\epsilon=5.5$, and $\delta=2.5$. For the fiber laser of Ref. [8], $\beta=230\ \text{fs}^2/\text{mm}$, $z_*=1.6$ mm, $t_*=170$ fs, $P_*=0.13$ MW, $E_*=20$ nJ, $g_*=15$ dB, $\Delta\omega^2=6\ \text{fs}^{-1}$, $0.25\ \text{nJ}\leq E_{\text{sat}}\leq 6$ nJ, and $0.1\ \text{kW}\leq P_{\text{sat}}\leq 2.4$ kW, so that $d_0=0.0013$, $g=3.5$, $\tau=3.25$, $l=0.7$, $2.5\leq\epsilon\leq 60$ (here taken 5.5 as before), and $40\leq\delta\leq 1000$ (here taken 50). In Fig. 4, we show the resulting soliton for these parameters which agrees with the observations in Refs. [7,8].

A. Asymptotic theory of solitons in the normal regime

Using the spectral renormalization method, we have found localized pulses in the normal dispersive regime for a broad range of parameters, provided that sufficient gain is present in the system. These pulses are slowly varying in t, with large phase. This suggests that by assuming a slow time scale in the equation and using perturbation theory, i.e., a WKB-type expansion, the soliton system can be reduced to simpler ordinary differential equations for the amplitude and the phase of the pulse. The first step before performing perturbation theory is to write the solution of Eq. (1) in the form $\psi(z,t)=R\exp(i\mu z+i\theta)$, where $R=R(t)$ and $\theta=\theta(t)$ are the pulse amplitude and phase, respectively. Substituting into the equation and equating real and imaginary parts, we get

$$-\mu R-\frac{d_0}{2}(R_{tt}-R\theta_t^2)+R^3=-\frac{\tau}{1+\epsilon E}(2R_t\theta_t+R\theta_{tt}), \tag{2a}$$

$$-\frac{d_0}{2}(2R_t\theta_t+R\theta_{tt})=\frac{g}{1+\epsilon E}R+\frac{\tau}{1+\epsilon E}-\frac{l}{1+\delta R^2}R. \tag{2b}$$

We then take the characteristic time length of the pulse to be such that we can define a scaling in the independent variable of the form $\nu=T/t$ or $T=\nu t$ so that $R=R(\nu t)$, $\theta_t=O(1)$ (i.e., θ is large), and $\theta_{tt}=O(\nu)$, where $\nu\ll 1$. Then Eq. (2a) becomes

$$-\mu R+\frac{d_0}{2}R\theta_t^2+R^3=\frac{d_0}{2}\nu^2 R_{TT}-\frac{\tau}{1+\epsilon E}(2\nu R_T\theta_t+R\theta_{tt}).$$

The leading-order equation is

$$\theta_t^2=\frac{2}{d_0}(\mu-R^2). \tag{3}$$

Using the same argument and this newly derived Eq. (3), we then find that Eq. (2b) to leading order reads

$$R_t=-\operatorname{sgn}(t)\frac{\sqrt{2(\mu-R^2)/d_0}}{3R^2-2\mu}\times\left(\frac{g}{1+\epsilon E}-\frac{4\tau(\mu-R^2)/d_0}{1+\epsilon E}-\frac{l}{1+\delta R^2}\right). \tag{4}$$

This is now a nonlocal first-order differential equation for $R=R(t)$, since $E=\int_{-\infty}^{+\infty}R^2dt$. We also note that imposing $\theta_t(t=0)=0\Rightarrow\mu\approx R^2(0)$.

To remove the nonlocality, another condition is needed and it is based on the singular points of Eq. (4). Recall that $R(t)$ is a decaying function in $t\in[0,+\infty)$ and $\mu\approx R^2(0)\geq R(t)^2$. Thus, there exists a point in t such that the denominator in the equation becomes zero. To remove the singularity, we require that the numerator of the equation is also zero at the same point which leads to

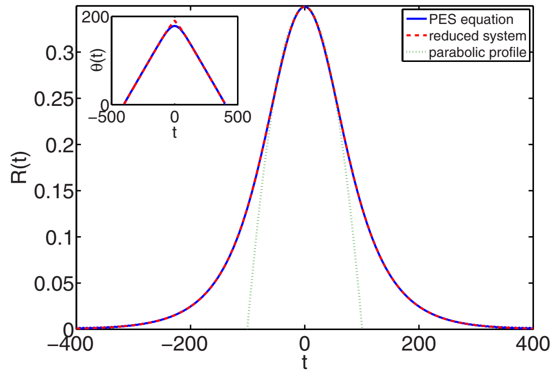

FIG. 5. (Color online) Solutions of the complete PES equation and the reduced system. The relative parabolic profile is also shown. Here, $g=1.5$.

$$1+\epsilon E=\frac{1}{l}\left(g-\frac{2\tau}{3d_0}\mu\right)\left(1+\frac{2\delta}{3}\mu\right).$$

Thus Eq. (4) is now a first-order equation which can be solved by standard numerical methods and analyzed by phase plane methods. The resulting solutions from the PES equation and the reduced equations are compared in Fig. 5. Notice that here $\mu=0.1216$, while $R^2(0)=0.1218$.

B. Higher-order modes

Finally, we mention the intriguing observation of higher-order solutions: i.e., antisymmetric solitons or bisolitons in mode-locked lasers operating in the normal regime. Recent experiments [19] demonstrate that higher-order antisymmetric or bisolitons can propagate in these lasers. These solitons differ significantly from the higher-order solutions of the classical NLS and the dispersion-managed solitons since they do not exhibit any oscillatory or breathing behavior as they propagate in the cavity. These solutions are different from the so-called bistable solitons [28], a class of solutions which NLS systems with generalized nonlinearities exhibit, and are not observed in a Kerr-type medium.

The bisolitons are pairs of regular solitons whose peak amplitudes have a difference in phase $\Delta\theta=\pi$ and an appropriate separation [19]. In the PES equation, we find these solutions by evolving the initial condition in the constant dispersive normal regime: $\psi(0,t)=t\exp(-t^2)$; again $g=1.5$, $\tau=l=0.1$, and $d_0=\epsilon=\delta=1$. A typical bisoliton is shown in Fig. 6. Interestingly, away from the peaks, the pulse is well approximated by a single soliton.

IV. DISPERSION-MANAGED SYSTEMS

The introduction of dispersion and nonlinear management induces rapidly varying dynamics which often obscure the main features. To overcome this difficulty we employ a multiscale method and work with the underlying averaged equation. Solitons as solutions of equations with constant dispersion are localized modes whose amplitude is constant in time. On the other hand, dispersion-managed solitons exhibit rapid breathing behavior, namely, their amplitude changes

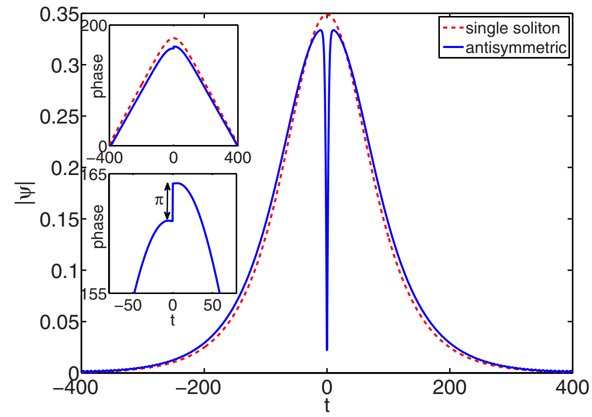

FIG. 6. (Color online) Antisymmetric soliton of the PES equation. In the inset, the phase of the pulse is shown and in the blow up the jump at $t=0$ is calculated to be exactly π.

according to the dispersion map. A convenient feature of the averaged model is that the modes represent pulses averaged over one cavity round trip, as though the pulses were propagating in a system with the (constant) average cavity dispersion. In the averaged equation, the DM solitons have constant amplitude. For a pulse with amplitude $u(z,t)$, power $P(z,t)=|u|^2$, and energy $E(z)=\int_{-\infty}^{+\infty}|u|^2dt$, which is propagating along the z direction, our model equation takes the form [17]

$$i\frac{\partial u}{\partial z}+\frac{d(z)}{2}\frac{\partial^2 u}{\partial t^2}+n(z)|u|^2u=\frac{ig}{1+\epsilon E}u+\frac{i\tau}{1+\epsilon E}u_{tt}-\frac{il}{1+\delta P}u. \qquad (5)$$

Note that $u(z,t)$ will be used for the description of the pulse envelope to distinguish this case from the constant dispersion case where $\psi(z,t)$ was used. We model the effect of dispersion management by splitting the dispersion $d(z)$ into two components [29] $d(z)=d_0+\Delta(z/z_a)/z_a$, where d_0 is the constant-averaged dispersion. The variable z_a is the dispersion-map period, which measures the ratio of the characteristic nonlinear distance to the characteristic dispersion length. Typically, z_a is small, i.e., $z_a\ll 1$. The function $d(\zeta)=d(z/z_a)$ is large and periodic and $n(z/z_a)=n(\zeta)$ is $O(1)$ and periodic. The path-averaged dispersion is d_0 and $\Delta(z)$ is rapidly varying and has zero average. Within each map period, the dispersion flips its sign as follows: $\Delta(\zeta)=\{-\Delta_1,0<\zeta<1/2,\Delta_1,1/2<\zeta<1\}$, whereas the propagation is periodically linear-nonlinear-linear, i.e., $n(\zeta)=\{0,0<\zeta<1/2,n_0,1/2<\zeta<1\}$. A key parameter is the map strength $s=\Delta_1/4$, which models the variability of the dispersion around the average. To illustrate the features of this system, we set $d_0=-1,n_0=1$ and allow s to vary.

To obtain the averaged equation, we introduce multiple scales and apply perturbation theory [29]. Define the new variables for distance $\zeta=z/z_a$ and $Z=z$ representing the short- and long-scale dynamics, respectively. Next, expand u in powers of $z_a\ll 1$ as

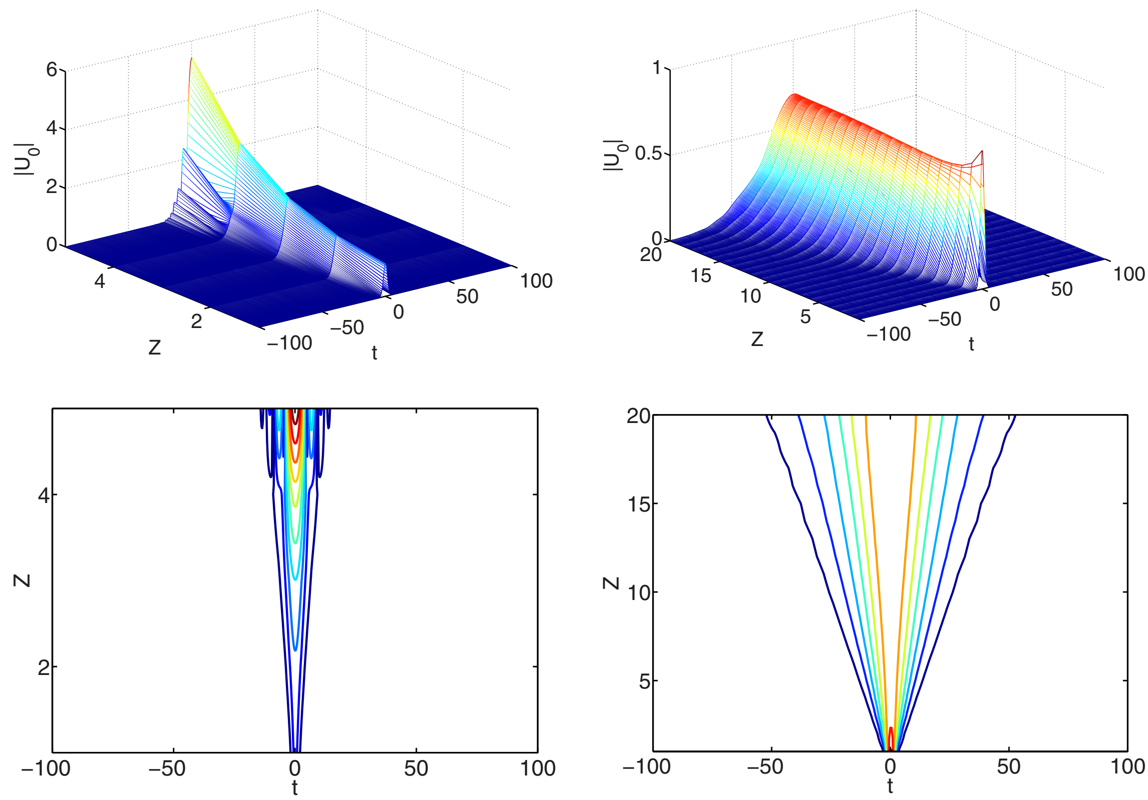

FIG. 7. (Color online) Self-similar evolution of the dispersion-managed system under linear (left) and saturated (right) gains. Here $g=0.5$ and $s=10$.

$$u(\zeta,Z,t) = u^{(0)}(\zeta,Z,t) + z_a u^{(1)}(\zeta,Z,t) + O(z_a^2).$$

In this way, Eq. (5) breaks into a series of equations corresponding to the different powers of z_a. At $O(z_a^{-1})$,

$$i\frac{\partial u^{(0)}}{\partial \zeta} + \frac{\Delta(\zeta)}{2}\frac{\partial^2 u^{(0)}}{\partial t^2} = 0. \quad (6)$$

To leading order, the evolution of the pulse is determined by the large variations of $d(z)$ about the mean and nonlinearity and residual dispersion represent only a small perturbation to the linear equation. The linear equation, Eq. (6), can be solved using Fourier transforms, namely,

$$\hat{u}^{(0)}(\zeta,Z,\omega) = \exp\left[-i\frac{\omega^2}{2}C(\zeta)\right]\hat{U}_0(Z,\omega), \quad (7)$$

where $C(\zeta)=\int_0^\zeta \Delta(\zeta')d\zeta'$ and $\hat{U}_0(Z,\omega)=\hat{u}^{(0)}(\zeta=0,Z,\omega)$. The Fourier transform of any function, $f(t)$, is denoted

$$\hat{f}(\omega) = \mathcal{F}\{f(t)\} \equiv \int_{-\infty}^{+\infty} f(t)\exp(i\omega t)dt.$$

The function $\hat{U}_0$ represents the slowly evolving amplitude of $\hat{u}$, whereas the fast oscillations induced by the local values of the dispersion are included in the exponential term. The function $\hat{U}_0$ is arbitrary at this stage and is determined by removing secular terms at the next order of perturbation.

This procedure determines an equation for $\hat{U}_0(Z,\omega)$ which is given by

$$i\frac{\partial \hat{U}_0}{\partial Z} - \frac{d_0}{2}\omega^2\hat{U}_0 + \int_0^1 \exp\left[-i\frac{\omega^2}{2}C(\zeta)\right]\left[n(\zeta)\mathcal{F}\{|u^{(0)}|^2u^{(0)}\} - \mathcal{F}\{Q[u^{(0)}]\}\right]d\zeta = 0. \quad (8)$$

This is a nonlocal equation for $U_0(Z,\omega)$ and describes the averaged dynamics of the pulse envelope. We refer to Eq. (8) as the dispersion-managed power energy saturation equation (DMPES).

The effects of self-similar evolution in this DM system are shown in Fig. 7. As in the case of constant dispersion, without energy saturation and starting from a unit Gaussian under linear gain, the pulse undergoes a rapid increase in both its amplitude and width resulting in a quasi-self-similar evolution. However, when the gain is saturated with energy, the pulse undergoes a decrease in its peak amplitude to a constant value for the duration of the evolution (locks) but still continues to evolve and grow linearly in its width.

The mode-locking mechanism of more general pulses inserted in the cavity at $z=0$ is addressed next. To study such an evolution, we integrate Eq. (8) with Eq. (7) (using fourth order Runge-Kutta) with a given initial profile $U_0(0,t)=\exp(-t^2)$. In order to lock onto localized solutions, the gain parameter g needs to be sufficiently large to counter the two lossy terms (filtering and loss). In what follows, $d_0=-1$ and we fix $\tau=l=0.1$ and $\epsilon=\delta=1$. As in the case of constant dispersion, enough gain must be present in the system for the pulse not to dissipate to zero. As seen in Fig. 8, three cases are again observed: dumped evolution, a slowly decaying

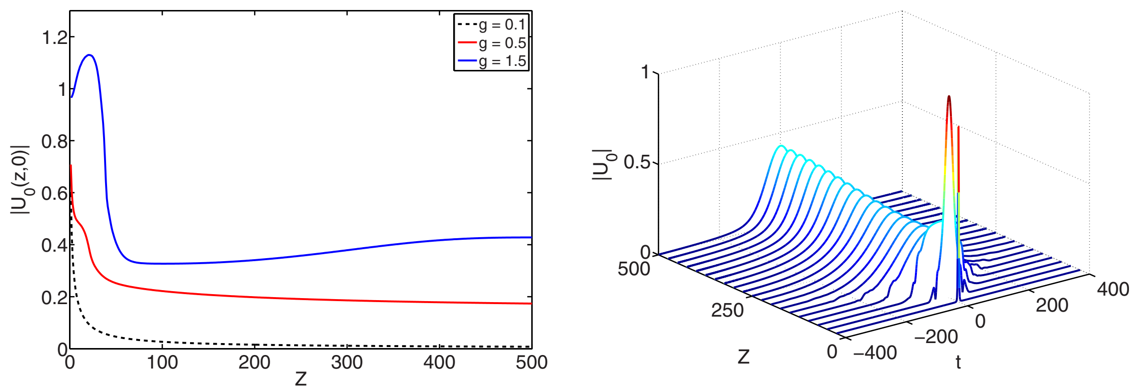

FIG. 8. (Color online) Evolution of the pulse peak of an arbitrary initial profile under the DMPES equation with different values of gain. The damped pulse-peak evolution is shown with a dashed line. In the right figure, the complete evolution is given for $g=1.5$. Here, $s=10$.

evolution resulting to quasisolitons, and the mode-locking regime. Notice that the transient state, before mode-locking occurs, is now more pronounced than before. The resulting pulses can also be found with the SPRZ method. Here we fix the gain parameter, $g=1.5$, and vary the dispersion strength. The results for $s=1$ and $s=10$ are shown in Fig. 9. Recall that the case $s=0.1$ is close to the constant dispersion case and is not repeated here. Unlike the anomalous case [17] the map strength seems to be overwhelmed by the size of the soliton and its chirp. In fact, the cases of weak ($s=0.1$) and moderate ($s=1$) dispersion strengths are essentially the same. We note that the pulses become somewhat broader as the map strength s increases.

We conclude with the higher state or a bisoliton in DM systems, shown in Fig. 10. We find these solutions by evolving an initial condition $u(t,0)=t\exp(-t^2)$, with $g=1.5$ and all other parameters are the same as in the previous DM figures. Numerical studies indicate that these bisoliton states are stable. Further research needs to be done to define the stability characteristics.

V. CONCLUSIONS

Soliton propagation in mode-locked lasers as described by the power energy saturation equation was analyzed. This model captures the qualitative features of different types of mode-locked lasers operating in the normal regime. The PES model describes mode locking to steady solitons in both the anomalous and normal regimes. Depending on the size of the gain parameter, pulses are either damped, i.e., decay to zero, or are asymptotically attracted to a localized solitary wave. Instabilities and blow up are not observed for a wide range of the parameters, even when the dissipative mechanisms are strong. The energy-saturated gain and filtering and power-saturated loss, which are typically small perturbation effects in lasers, are crucial in the mode-locking mechanism. The same results are obtained for both constant and dispersion-managed systems.

In the normally dispersive regime, pulses are wide and strongly chirped; unlike solitons, their anomalous regime counterparts (which also satisfy the PES equation), they are not well approximated by the unperturbed equations without gain, filtering, and loss. These pulses agree with those observed recently in experiments. To further analyze their properties, an asymptotically reduced first-order equation was obtained. Finally, it was shown that the model also supports higher-order soliton solutions: the so-called antisymmetric or bisoliton in both constant as well as DM systems.

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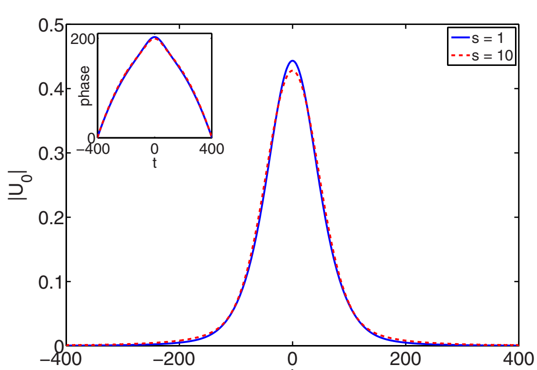

FIG. 9. (Color online) Soliton profiles of the DMPES equation corresponding to different map strengths. In the inset, we plot the corresponding phases. The phases of $s=1,10$ are nearly indistinguishable.

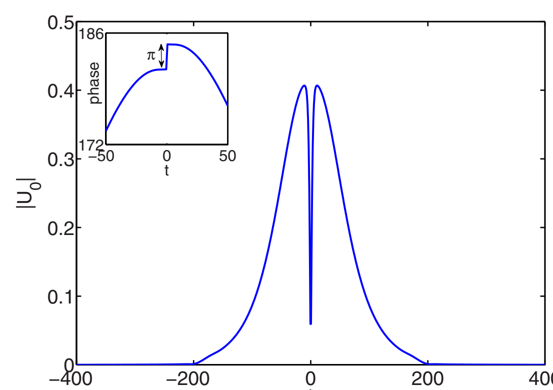

FIG. 10. (Color online) Antisymmetric soliton of the DMPES for $s=10$. In the inset, the phase jump at $t=0$ is calculated to be exactly π.

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