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SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH STATE-DEPENDENT DELAY

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Abstract. In this paper we study the existence of solutions for the initial value problem for semilinear functional differential equations of fractional order with state-dependent delay. The nonlinear alternative of Leray-Schauder type is the main tool in our analysis.

1. Introduction

Recently in [7], existence results were proved for an initial value problem for functional differential equations of fractional order with state-dependent delay

$$D^\beta y(t) = f(t, y_{\rho(t,y_t)}), \quad t \in J = [0, b],\ 0 < \beta < 1, \tag{1.1}$$

$$y(t) = \varphi(t), \quad t \in (-\infty, 0] \tag{1.2}$$

as well as for neutral functional differential equations of fractional order with state-dependent delay

$$D^\beta [y(t) - g(t, y_{\rho(t,y_t)})] = f(t, y_{\rho(t,y_t)}), \quad \text{for } t \in J, \tag{1.3}$$

$$y(t) = \varphi(t), \quad t \in (-\infty, 0], \tag{1.4}$$

where D^β is the standard Riemman-Liouville fractional derivative, $f : J \times \mathcal{B} \to \mathbb{R}$, $g : J \times \mathcal{B} \to \mathbb{R}$ and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are appropriate given functions, $\varphi \in \mathcal{B}$, $\varphi(0) = 0$, $g(0, \varphi) = 0$ and $\mathcal{B}$ is called a *phase space*.

The purpose of this paper is to extend the results of [7] by studying the existence of solutions for initial value problems for a functional semilinear differential equations of fractional order with state-dependent delay, as well as, for a neutral functional semilinear differential equations of fractional order with state-dependent delay. In particular, in Section 3, we consider the following initial value problem for a functional semilinear differential equations of fractional order with state-dependent delay

$$D^\beta y(t) = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad t \in J = [0, b], \quad 0 < \beta < 1, \tag{1.5}$$

$$y(t) = \varphi(t), \quad t \in (-\infty, 0], \tag{1.6}$$

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while in Section 4, we consider the following initial value problem for a neutral functional semilinear differential equations of fractional order with state-dependent delay,

$$D^{\beta}[y(t)-g(t,y_{\rho(t,y_t)})]=A[y(t)-g(t,y_{\rho(t,y_t)})]+f(t,y_{\rho(t,y_t)}),\quad t\in J, \tag{1.7}$$

$$y(t)=\varphi(t),\quad t\in(-\infty,0], \tag{1.8}$$

where D^{β} is the standard Riemman-Liouville fractional derivative.

Here, $f : J\times\mathcal{B}\to E$, $g : J\times\mathcal{B}\to E$ and $\rho : J\times\mathcal{B}\to(-\infty,b]$ are appropriate given functions, $\varphi\in\mathcal{B}$, $\varphi(0)=0$, $g(0,\varphi)=0$, $A : D(A)\subseteq E\to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, and $\mathcal{B}$ is called a *phase space* that will be defined later (see Section 2).

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [14] (see also Kappel and Schappacher [21] and Schumacher [36]). For a detailed discussion on this topic we refer the reader to the book by Hino *et al* [20].

While functional differential equations have been used in modelling a panorama of natural phenomena as discussed in the books by Kolmanovskii and Myshkis [23] and Hale and Lunel [15], it has been only recently that fractional differential equations have begun to see extensive utilization in modelling problems that arise in engineering and other sciences, including viscoelasticity, electrochemistry, control, porous media flow, physics, mechanics and others [11, 19, 22, 30, 32, 33, 37]. On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equation has received a significant amount of attention in the last years, we refer to [2, 3, 5, 10, 16, 17, 18] and the references therein.

In part, differential equations of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics and other fields can be described with the help of fractional differential equations, see [11, 12, 19, 29, 33, 34, 35] and references therein. The theory of differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations, for example see [1, 4, 6, 8, 9, 22, 24, 25, 26, 27, 28, 32, 37].

Our approach is based on the nonlinear alternative of Leray-Schauder type [13]. These results can be considered as a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout this paper.

By $C(J,E)$ we denote the Banach space of continuous functions from J into E with the norm

$$\|y\|_{\infty}:=\sup\{|y(t)| : t\in J\}.$$

Now, we recall some definitions and facts about fractional derivatives and fractional integrals of arbitrary orders, see [22, 30, 32, 33].

Definition 2.1. The fractional primitive of order $\beta > 0$ of a function $h : (0, b] \to E$ is defined by
$$I_0^\beta h(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)ds,$$
provided the right hand side exists pointwise on $(0, b]$, where Γ is the gamma function.

For instance, $I^\beta h$ exists for all $\beta > 0$, when $h \in C((0, b], E) \cap L^1((0, b], E)$; note also that when $h \in C([0, b], E)$ then $I^\beta h \in C([0, b], E)$ and moreover $I^\beta h(0) = 0$.

Definition 2.2. The fractional derivative of order $\beta > 0$ of a continuous function $h : (o, b] \to E$ is given by
$$\frac{d^\beta h(t)}{dt^\beta} = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t (t-s)^{-\beta} h(s)ds = \frac{d}{dt} I_a^{1-\beta} h(t).$$

In this paper, we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to those introduced in [20]. More precisely, $\mathcal{B}$ will be a linear space of all functions from $(-\infty, 0]$ to E endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ satisfying the following axioms:

(A) If $y : (-\infty, b] \to E$, $b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
  (i) $y_t \in \mathcal{B}$,
  (ii) $\|y_t\|_{\mathcal{B}} \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}$,
  (iii) $|y(t)| \le H\|y_t\|_{\mathcal{B}}$,
  where $H > 0$ is a constant, $K : [0, \infty) \to [1, \infty)$ is continuous, $M : [0, \infty) \to [1, \infty)$ is locally bounded and $H,\ K,\ M$ are independent of $y(\cdot)$.

(A1) For the function $y(\cdot)$ in (A), y_t is a $\mathcal{B}$-valued continuous function on $[0, b]$.

(A2) The space $\mathcal{B}$ is complete.

The next lemma is a consequence of the phase space axioms and is proved in [16].

Lemma 2.3. *Let $\varphi \in \mathcal{B}$ and $I = (\gamma, 0]$ be such that $\varphi_t \in \mathcal{B}$ for every $t \in I$. Assume that there exists a locally bounded function $J^\varphi : I \to [0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in I$. If $y : (\infty, b] \to \mathbb{R}$ is continuous on J and $y_0 = \varphi$, then*
$$\|y_t\|_{\mathcal{B}} \le (M_b + J^\varphi(\max\{\gamma, -|s|\})\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)| : \theta \in [0, \max\{0, s\}]\},$$
for $s \in (\gamma, b]$, where we denoted $K_b = \sup_{t\in J} K(t)$ and $M_b = \sup_{t\in J} M(t)$.

3. Main Result

In this section, the nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions of problem (1.5)-(1.6).

Let us start by defining what we mean by a solution of problem (1.5)-(1.6).

Definition 3.1. A function $y : (-\infty, b] \to E$ is said to be a solution of (1.5)-(1.6) if $y_0 = \varphi, y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and
$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s, y_{\rho(s,y_s)})\, ds, \quad t \in J.$$

In what follows we assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous and $\varphi \in \mathcal{B}$ and the following hypotheses are satisfied

(H1) A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ in E, which is compact for $t > 0$, and there exist constant $M \geq 1$ such that $\|T(t)\|_{B(E)} \leq M, t \geq 0$;
(H2) $f : J \times \mathcal{B} \to E$ is a continuous function;
(H3) there exists $p \in C([0, b], \mathbb{R}^+)$ and $\Omega : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|f(t, u)| \leq p(t)\Omega(\|u\|_{\mathcal{B}})$$

for $t \in [0, b]$ and each $u \in \mathcal{B}$;
(H4) there exists a number $K_0 > 0$ such that

$$\frac{K_0}{(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + MK_b\Omega(K_0)\|I^{\beta}p\|_{\infty}} > 1;$$

(H5) the function $t \to \varphi_t$ is well defined and continuous from the set $\mathcal{R}(\rho^-) = \{\rho(s, \psi) : (s, \psi) \in J \times B, \rho(s, \psi) \leq 0\}$ into $\mathcal{B}$. Moreover, there exists a continuous and bounded function $J^{\varphi} : \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \leq J^{\varphi}(t)\|\varphi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.

Remark 3.2. The hypothesis (H5) is adapted from [16], where we refer for remarks concerning this hypothesis.

Theorem 3.3. *Assume that the hypotheses* (H1)–(H5) *hold. If* $\rho(t, \psi) \leq t$ *for every* $(t, \psi) \in J \times \mathcal{B}$*, then the* (1.5)-(1.6) *has at least one solution on* $(-\infty, b]$.

Proof. Let $Y = \{u \in C(J, E) : u(0) = \varphi(0) = 0\}$ endowed with the uniform convergence topology and $N : Y \to Y$ be the operator defined by

$$Ny(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} T(t - s) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds, \quad t \in J,$$

where $\bar{y} : (-\infty, b] \to E$ is such that $\bar{y}_0 = \varphi$ and $\bar{y} = y$ on J. From axiom (A) and our assumption on φ, we infer that $Ny(\cdot)$ is well defined and continuous.

Let $\bar{\varphi} : (-\infty, b] \to E$ be the extension of φ to $(-\infty, b]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on J and $\tilde{J}^{\varphi} = \sup\{J^{\varphi} : s \in \mathcal{R}(\rho^-)\}$.

We will prove that $N(\cdot)$ is completely continuous from $B_r(\bar{\varphi}|_J, Y)$ to $B_r(\bar{\varphi}|_J, Y)$.
Step 1: N is continuous on $B_r(\bar{\varphi}|_J, Y)$. This was proved in [16, p. 515, Step 3].
Step 2: N maps bounded sets into bounded sets. If $y \in B_r(\bar{\varphi}|_J, Y)$, from Lemma 2.3 follows that

$$\|\bar{y}_{\rho(t,\bar{y}_t)}\|_{\mathcal{B}} \leq r^* := (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b r$$

and so

$$\begin{aligned}
|(Ny)(t)| &= \frac{M}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} T(t - s) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds \\
&\leq \frac{M}{\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} p(s)\Omega(\|\bar{y}_{\rho(s,\bar{y}_s)}\|_{\mathcal{B}})\, ds \\
&\leq \frac{M}{\Gamma(\beta)} \|p\|_{\infty} \Omega(r^*) \int_0^t (t - s)^{\beta-1}\, ds \\
&\leq \frac{Mb^{\beta}}{\Gamma(\beta + 1)} \|p\|_{\infty} \Omega(r^*).
\end{aligned}$$

Thus

$$\|Ny\|_\infty \le \frac{Mb^\beta}{\Gamma(\beta+1)}\|p\|_\infty \Omega(r^*) := \ell.$$

Step 3: N maps bounded sets into equicontinuous sets of B. Let $t_1,\ t_2 \in (0,b]$ with $t_1 < t_2$ and B_α be a bounded set as in Step 2. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \le \epsilon$.
Case 1. If $\tau_1 > \epsilon$ then

$$\begin{aligned}
&|(Ny)(t_2) - (Ny)(t_1)| \\
&\le \frac{1}{\Gamma(\beta)}\int_0^{t_1-\epsilon} [(t_2-s)^{\beta-1}T(t_2-s) - (t_1-s)^{\beta-1}T(t_1-s)]|f(s,\bar y_{\rho(s,\bar y_s)})|\,ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_{t_1-\epsilon}^{t_1} \big[(t_2-s)^{\beta-1}T(t_2-s) - (t_1-s)^{\beta-1}T(t_1-s)\big]\,|f(s,\bar y_{\rho(s,\bar y_s)})|\,ds \\
&\quad + \frac{1}{\Gamma(\beta)}\int_{t_1}^{t_2} (t_2-s)^{\beta-1}T(t_2-s)|f(s,\bar y_{\rho(s,\bar y_s)})|\,ds \\
&\le \frac{\|p\|_\infty\Omega(r^*)}{\Gamma(\beta)}\Big(\Big|\int_0^{t_1-\epsilon}[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]T(t_2-s)\,ds\Big| \\
&\quad + \Big|\int_0^{t_1-\epsilon}(t_1-s)^{\beta-1}T(t_1-\epsilon-s)[T(t_2-t_1-\epsilon) - T(\epsilon)]ds\Big| \\
&\quad + \Big|\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]T(t_2-s)\,ds\Big| \\
&\quad + \Big|\int_{t_1-\epsilon}^{t_1}(t_1-s)^{\beta-1}T(t_1-\epsilon-s)[T(t_2-t_1-\epsilon) - T(\epsilon)]\,ds\Big| + \int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\Big) \\
&\le \frac{\|p\|_\infty\Omega(r^*)}{\Gamma(\beta)}\Big(M\int_0^{t_1-\epsilon}[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]\,ds \\
&\quad + M\|T(t_2-t_1-\epsilon) - T(\epsilon)\|_{B(E)}\int_0^{t_1-\epsilon}(t_2-s)^{\beta-1}\,ds \\
&\quad + M\int_{t_1-\epsilon}^{t_1}[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]ds \\
&\quad + M\|T(t_2-t_1-\epsilon) - T(\epsilon)\|_{B(E)}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\beta-1}\,ds + M\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\,ds\Big),
\end{aligned}$$

where we have used the semigroup identities

$$T(\tau_2 - s) = T(\tau_2 - \tau_1 + \epsilon)T(\tau_1 - s - \epsilon), \quad T(\tau_1 - s) = T(\tau_1 - s - \epsilon)T(\epsilon).$$

Case 2. Let $\tau_1 \le \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$\begin{aligned}
|(Ny)(t_2) - (Ny)(t_1)| &\le \frac{1}{\Gamma(\beta)}\Big|\int_0^{t_2}(t_2-s)^{\beta-1}T(t_2-s)f(s,\bar y_{\rho(s,\bar y_s)})\,ds \\
&\quad - \int_0^{t_1}(t_2-s)^{\beta-1}T(t_2-s)f(s,\bar y_{\rho(s,\bar y_s)})\,ds\Big| \\
&\le M\frac{\|p\|_\infty\Omega(r^*)}{\Gamma(\beta)}\Big(\int_0^{2\epsilon}(t_2-s)^{\beta-1}\,ds + \int_0^{\epsilon}(t_1-s)^{\beta-1}\,ds\Big).
\end{aligned}$$

Note equicontinuity follows since (i). $T(t), t \geq 0$ is a strongly continuous semigroup and (ii). $T(t)$ is compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$) [31].

From the steps 1 to 3, together with Arzelá-Ascoli theorem, it suffices to show that N maps B_α into a precompact set in E.

Let $0 < t < b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_\alpha$ we define

$$N_\epsilon(y)(t) = \frac{T(\epsilon)}{\Gamma(\beta)} \int_0^{t-\epsilon} (t-s-\epsilon)^{\beta-1} T(t-s-\epsilon) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds.$$

Since $T(t)$ is a compact operator for $t > 0$, the set $Y_\epsilon(t) = \{N_\epsilon(y)(t) : y \in B_\alpha\}$ is precompact in E for every ϵ, $0 < \epsilon < t$. Moreover

$$\begin{aligned} &|N(y)(t) - N_\epsilon(y)(t)| \\ &\leq M \frac{\|p\|_\infty \Omega(r^*)}{\Gamma(\beta)} \Big(\int_0^{t-\epsilon} [(t-s)^{\beta-1} - (t-s-\epsilon)^{\beta-1}]\, ds + \int_{t-\epsilon}^t (t-s)^{\beta-1}\, ds \Big). \end{aligned}$$

Therefore, the set $Y(t) = \{N(y)(t) : y \in B_\alpha\}$ is precompact in E. Hence the operator N is completely continuous.

Step 4: (A priori bounds). We now show there exists an open set $U \subseteq Y$ with $y \neq \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $y \in Y$ and $y = \lambda N(y)$ for some $0 < \lambda < 1$. Then for each $t \in [0, b]$ we have

$$y(t) = \lambda \Big[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds \Big].$$

This implies by (H3) and lemma 2.3 that

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ||T(t-s)| f(s, \bar{y}_{\rho(s,\bar{y}_s)})|\, ds \\ &\leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega((M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \sup\{|\bar{y}(s)| : s \in [0,t]\})\, ds, \end{aligned}$$

since $\rho(s, \bar{y}_s) \leq s$ for every $s \in J$. Here $\bar{J}^\phi = \sup\{J^\phi(s) : s \in \mathcal{R}(\rho^-)\}$.

Set $\mu(t) = \sup\{|y(s)| : 0 \leq s \leq t\}$ $t \in [0, b]$. Then we have

$$\mu(t) \leq \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega((M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \mu(s))\, ds.$$

If $\xi(t) = (M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \mu(t)$ then we obtain

$$\begin{aligned} \xi(t) &\leq (M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + \frac{MK_b}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega(\xi(s))\, ds \\ &\leq (M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + MK_b \Omega(\|\xi\|_\infty) \|I^\beta p\|_\infty. \end{aligned}$$

Then

$$\frac{\|\xi\|_\infty}{(M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + MK_b \Omega(\|\xi\|_\infty) \|I^\beta p\|_\infty} \leq 1\,.$$

By (H4), there exists M_* such that $\|y\|_\infty \neq M_*$. Set

$$U = \{y \in Y : \|y\|_\infty < M^* + 1\}.$$

Then $N : \overline{U} \to Y$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N(y)$, for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [13], we deduce that N has a fixed point y in U, which is a solution of (1.5)-(1.6). $\square$

4. NFDEs of Fractional Order

In this section we give an existence result for (1.7)-(1.8).

Definition 4.1. A function $y : (-\infty, b] \to E$ is said to be a solution of (1.7)-(1.8) if $y_0 = \varphi, y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and

$$y(t) = g(s, y_{\rho(s,y_s)}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s, y_{\rho(s,y_s)}) ds, \quad t \in J.$$

Theorem 4.2. *Assume* (H1)-(H3), (H5) *are satisfied. In addition we suppose that the following two conditions hold:*

(H6) *the function g is continuous and completely continuous, and for any bounded set Q in $\mathcal{B} \cap C([0,b], E)$, the set $\{t \to g(t, y_t) : y \in Q\}$ is equicontinuous in $C([0,b], E)$, and there exist constants $0 \le d_1 < 1/K_b$, $d_2 \ge 0$ such that*

$$|g(t,u)| \le d_1 \|u\|_{\mathcal{B}} + d_2, \quad t \in [0,b], \; u \in \mathcal{B};$$

(H7) *there exists a number $K_0 > 0$ such that*

$$\frac{\|\xi\|_\infty}{(M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + \frac{K_b}{1-K_b d_1}\big\{d_1(M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + d_2 + M\Omega(\|\xi\|_\infty)\|I^\beta p\|_\infty\big\}} > 1.$$

If $\rho(t,\psi) \le t$ for every $(t,\psi) \in J \times \mathcal{B}$, then the (1.7)-(1.8) *has at least one solution on $(-\infty, b]$.*

Proof. Consider the operator $N_0 : C((-\infty,b], E) \to C((-\infty,b], E)$ defined by,

$$N_0(y)(t) = \begin{cases} \varphi(t), & \text{if } t \in (-\infty, 0], \\ g(t, y_{\rho(t,y_t)}) & \\ +\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} T(t-s) f(s, y_{\rho(s,y_s)}) ds, & \text{if } t \in [0,b]. \end{cases}$$

In analogy to Theorem 3.3, we consider the operator $N_1 : Y \to Y$ defined by

$$(N_1 y)(t) = \begin{cases} 0, & t \le 0 \\ g(t, \bar{y}_{\rho(s,\bar{y}_s)}) + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} T(t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)}) ds, & t \in [0,b]. \end{cases}$$

We shall show that the operator N_1 is continuous and completely continuous. Using (H6) it suffices to show that the operator $N_2 : Y \to Y$, defined by

$$N_2(y)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds, \quad t \in [0,b],$$

is continuous and completely continuous. This was proved in Theorem 3.3.

We now show there exists an open set $U \subseteq Y$ with $y \ne \lambda N_1(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $y \in Y$ and $y = \lambda N_1(y)$ for some $0 < \lambda < 1$. Then

$$y(t) = \lambda\Big[g(s, \bar{y}_{\rho(s,\bar{y}_s)}) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} T(t-s) f(s, \bar{y}_{\rho(s,\bar{y}_s)})\, ds\Big], \quad t \in [0,b],$$

and

$$\begin{aligned} |y(t)| &\le d_1\Big[(M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \sup\{|y(s)| : s \in [0,t]\}\Big] + d_2 \\ &\quad + \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega((M_b + \tilde{J}^\varphi)\|\varphi\|_{\mathcal{B}} + K_b \sup\{|\bar{y}(s)| : s \in [0,t]\}) ds, \end{aligned}$$

for $t \in (0, b]$. If $\mu(t) = \sup\{|y(s)| : s \in [0, t]\}$ then

$$\begin{aligned}\mu(t) &\leq d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_1 K_b \mu(t) + d_2 \\ &\quad + \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \mu(s))\, ds,\end{aligned}$$

or

$$\begin{aligned}\mu(t) &\leq \frac{1}{1 - K_b d_1} \big[d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_2 \big] \\ &\quad + \frac{1}{1 - K_b d_1} \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega((M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \mu(s))\, ds,\end{aligned}$$

for $t \in (0, b]$. If $\xi(t) = (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b \mu(s)$ then we have

$$\begin{aligned}\xi(t) &\leq (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} \\ &\quad + \frac{K_b}{1 - K_b d_1} \Big\{ d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_2 + \frac{M}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} p(s) \Omega(\xi(s))\, ds \Big\} \\ &\leq (M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} \\ &\quad + \frac{K_b}{1 - K_b d_1} \Big\{ d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_2 + M\Omega(\|\xi\|_\infty)\|I^\beta p\|_\infty \Big\}.\end{aligned}$$

Consequently,

$$\frac{\|\xi\|_\infty}{(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + \frac{K_b}{1 - K_b d_1} \big\{ d_1(M_b + \tilde{J}^{\varphi})\|\varphi\|_{\mathcal{B}} + d_2 + M\Omega(\|\xi\|_\infty)\|I^\beta p\|_\infty \big\}} \leq 1.$$

By (H7), there exists L^* such that $\|y\|_\infty \neq L^*$. Set

$$U_1 = \{ y \in Y : \|y\|_\infty < L^* + 1 \}.$$

From the choice of U_1 there is no $y \in \partial U_1$ such that $y = \lambda N_1(y)$ for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [13], we deduce that N_1 has a fixed point y in U_1. Then N_1 has a fixed point, which is a solution of (1.7)-(1.8). $\square$

References

[1] O. P. Agrawal; Analytical schemes for a new class of fractional differential equations, *J. Phys. A* **40 (21)** (2007), 5469-5477.

[2] W. Aiello, H. I. Freedman and J. Wu; Analysis of a model representing stage-structured population growth with state-dependent time delay, *SIAM J. Appl. Math.* **52(3)** (1992), 855-869.

[3] M. Bartha; Periodic solutions for differential equations with state-dependent delay and positive feedback, *Nonlinear Anal.* **53(6)** (2003), 839-857.

[4] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab; Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[5] Y. Cao, J. Fan and T. C. Gard; The effects of state-dependent time delay on a stage-structured population growth model, *Nonlinear Anal.* **19(2)** (1992), 95-105.

[6] M. A. Darwich and S. K. Ntouyas; Existence results for a fractional functional differential equation of mixed type, *Commun. Appl. Nonlinear Anal.* **15** (2008), 47-55.

[7] M. A. Darwish and S. K. Ntouyas; Functional differential equations of fractional order with state-dependent delay, *Dynam. Systems Appl.* (to appear).

[8] M. A. Darwich and S. K. Ntouyas; Existence results for perturbed functional differential equations of mixed type with fractional order, *Nonlinear Studies* (to appear).

[9] D. Delboso and L. Rodino; Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609-625.
[10] A. Domoshnitsky, M. Drakhlin and E. Litsyn; On equations with delay depending on solution, *Nonlinear Anal.* **49(5)** (2002), 689-701.
[11] M. Garh, A. Rao and S. L. Kalla; Fractional generalization of temperature fields problems in oil strata, *Mat. Bilten* **30** (2006), 71-84.
[12] W. G. Glockle and T. F. Nonnenmacher; A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68** (1995), 46-53.
[13] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[14] J. Hale and J. Kato; Phase space for retarded equations with infinite delay, *Funkcial. Ekvac.* **21** (1978), 11-41.
[15] J. Hale and S. M. Verduyn Lunel; *Introduction to Functional Differential Equations*, Applied Mathematical Sciences **99**, Springer-Verlag, New York, 1993.
[16] E. Hernández, A. Prokopczyk and L. Ladeira; A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal. Real World Appl.* **7(4)** (2006), 510-519.
[17] E. Hernández, M. Pierri and G. Goncalves; Existence results for an impulsive abstract partial differential equation with state-dependent delay, *Comput. Math. Appl.* **52(3-4)** (2006), 411-420.
[18] E. Hernández, R. Sakthivel, Rathinasamy and A. Tanaka; Existence results for impulsive evolution differential equations with state-dependent delay, *Electron. J. Differential Equations* **2008**(2008) no. 28, 1-11.
[19] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[20] Y. Hino, S. Murakami and T. Naito; *Functional Differential Equations with Infinite Delay*, in: Lecture Notes in Mathematics, **1473**, Springer-Verlag, Berlin, 1991.
[21] F. Kappel and W. Schappacher; Some considerations to the fundamental theory of infinite delay equations, *J. Differential Equations* **37** (1980), 141-183.
[22] A. A. Kilbas, H. M. Srivastava and Juan J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[23] V. Kolmanovskii and A. Myshkis; *Introduction to the Theory and Applications of Functional-Differential Equations*, Mathematics and its Applications **463**, Kluwer Academic Publishers, Dordrecht, 1999.
[24] V. Lakshmikantham; Theory of fractional functional differential equation, *Nonlinear Anal.* **69** (2008), 3337-3343.
[25] V. Lakshmikantham and J. V. Devi; Theory of fractional differential equations in a Banach space, *Eur. J. Pure Appl. Math.* **1** (1) (2008), 38-45.
[26] V. Lakshmikantham and A. S. Vatsala; Basic theory of fractional differential inequalities and applications, *Commun. Appl. Anal.* **11** (2007), no. 3-4, 395-402.
[27] V. Lakshmikantham and A. S. Vatsala; Basic theory of fractional differential equations, *Nonlinear Anal.* **69** (2008), 2677-2682.
[28] V. Lakshmikantham and A. S. Vatsala; General uniqueness and monotone iteration technique in fractional differential equations, *Appl. Math. Lett.* **21** (2008), 828-834.
[29] F. Mainardi; Fractional calculus: Some basic problems in continuum and statistical mechanis, in "Fractals and Fractional Calculus in Continuum Mechanics" (A. Carpinteri and F. Mainardi, Eds), pp. 291-348, Springer-Verlag, Wien, 1997.
[30] K. S. Miller and B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[31] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
[32] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[33] S. G. Samko, A. A. Kilbas and O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[34] R. K. Saxena and S. L. Kalla; On a fractional generalization of free electron laser equation, *Appl. Math. Comput.* **143** (2003), 89-97.
[35] R. K. Saxena, A. M. Mathai and H. L. Haubold; On generalized fractional kinetic equations, *Phys. A* **344** (2004), 657-664.
[36] K. Schumacher; Existence and continuous dependence for differential equations with unbounded delay, *Arch. Rational Mech. Anal.* **64** (1978), 315-335.

[37] C. Yu and G. Gao; Existence of fractional differential equations, *J. Math. Anal. Appl.* **310** (2005), 26-29.

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