SIAM J. MATRIX ANAL. APPL.
Vol. 30, No. 2, pp. 700–712

REACHABILITY AND HOLDABILITY OF NONNEGATIVE STATES*

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Abstract. Linear differential systems $\dot{x}(t) = Ax(t)$ ($A \in \mathbb{R}^{n\times n}$, $x_0 = x(0) \in \mathbb{R}^n$, $t \geq 0$) whose solutions become and remain nonnegative are studied. It is shown that the eigenvalue of A furthest to the right must be real and must possess nonnegative right and left eigenvectors. Moreover, for some $a \geq 0$, $A + aI$ must be eventually nonnegative, that is, its powers must become and remain entrywise nonnegative. Initial conditions x_0 that result in nonnegative states $x(t)$ in finite time are shown to form a convex cone that is related to the matrix exponential e^{tA} and its eventual nonnegativity.

Key words. eventually nonnegative matrix, exponentially nonnegative matrix, point of nonnegative potential, Perron–Frobenius, Metzler matrix, convex cone

AMS subject classifications. 15A48, 93B03, 65F10

DOI. 10.1137/070693850

1. Introduction.

In dynamical systems theory, one is frequently interested in qualitative information regarding state evolution. In particular, due to physical and modeling constraints arising in engineering, biological, medical, behavioral, and economic applications, it is commonly of interest to impose or consider conditions for nonnegativity of the states; see, e.g., [2, 6]. Such applications typically draw on the theory, or directly take the form, of a linear differential system,

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n\times n}, \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0, \tag{1.1}$$

whose solution is given by $x(t) = e^{tA}x_0$. We shall refer here to the set

$$\{x(t) = e^{tA}x_0 \mid t \in [0,\infty)\}$$

as the *trajectory emanating from* x_0 and say that x_0 gives rise to this trajectory. In this paper we will consider conditions for the entrywise nonnegativity of the trajectories associated with (1.1). Our main concern is the following "hit and hold" problem:

When does the trajectory emanating from an initial point x_0 become (entrywise) nonnegative and remain nonnegative for all time thereafter?

More specifically, we will seek characterizations of system parameters that lead to a trajectory becoming nonnegative at a finite time (*reachability of* $\mathbb{R}^n_+$) and remaining nonnegative for all time thereafter (*holdability of* $\mathbb{R}^n_+$). This endeavor will comprise two related efforts:

(1) Study matrices $A \in \mathbb{R}^{n\times n}$ for which there exists $t_0 \in [0,\infty)$ such that $e^{tA} \geq 0$ for all $t \geq t_0$. We shall term such matrices *eventually exponentially nonnegative*.

(2) Given an eventually exponentially nonnegative matrix A, study initial points $x_0 \in \mathbb{R}^n$ for which there exists $\hat{t} \in [0,\infty)$ such that $e^{tA}x_0 \geq 0$ for all $t \geq \hat{t}$. We shall refer to such initial points as *points of nonnegative potential*.

Some comments regarding these two goals and the structure of this paper are in order.

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*Received by the editors June 6, 2007; accepted for publication (in revised form) by D. Szyld February 22, 2008; published electronically June 18, 2008.
http://www.siam.org/journals/simax/30-2/69385.html
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First, matrices all of whose off-diagonal entries are nonnegative (known as essentially nonnegative or Metzler matrices) are eventually exponentially nonnegative (with $t_0 = 0$). However, as we shall see in section 3, the eventually exponentially nonnegative matrices form a larger matrix class. They are closely related to the eventually nonnegative matrices, namely, matrices whose powers become and remain nonnegative. It is this latter fact that provides further motivation for our study, as eventually nonnegative matrices arise in the theory of positive control systems; see e.g., [16].

Second, it is clear that $\mathbb{R}^n_+$ (the nonnegative orthant) comprises points of nonnegative potential but as we shall see, in the general case, the totality of such points forms a convex cone that strictly contains $\mathbb{R}^n_+$. Our relevant analysis is in section 4, where points of nonnegative potential and the asymptotic behavior of solutions are connected to the matrix exponential e^{tA} and its eventual nonnegativity. We note that even in applications where initial points and states are de facto nonnegative, points of nonnegative potential outside $\mathbb{R}^n_+$ can be of practical interest. For example, suppose that for some $x_0 \in \mathbb{R}^n$, $\hat{x}_0 = Ax_0$ is a point of nonnegative potential. Then there exists $\hat{t} \geq 0$ such that for all $t \geq \hat{t}$,

$$\dot{x}(t) \ = \ \frac{d}{dt}(e^{tA}x_0) \ = \ Ae^{tA}x_0 \ = \ e^{tA}Ax_0 \ = \ e^{tA}\hat{x}_0 \geq 0;$$

that is, the trajectory emanating from x_0 becomes (at $t = \hat{t}$) and remains entrywise nondecreasing. This situation occurs, e.g., when (1.1) models species that reach a symbiotic state after which none of the populations decreases; see [9].

2. Notation, definitions, and preliminaries. Given an $n \times n$ matrix A, the spectrum of A is denoted by $\sigma(A)$ and its spectral radius by $\rho(A) = \max\{|\lambda| \mid \lambda \in \sigma(A)\}$. An eigenvalue λ of A is said to be *dominant* if $|\lambda| = \rho(A)$. The *spectral abscissa* of A is defined and denoted by $\lambda(A) := \max\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$. By $\operatorname{index}_0(A)$ we denote the degree of 0 as a root of the minimal polynomial of A. Consequently, when we say $\operatorname{index}_0(A) \leq 1$, we mean that either A is invertible or that the size of the largest nilpotent Jordan block in the Jordan canonical form of A is 1×1.

The *nonnegative orthant* in $\mathbb{R}^n$, that is, the set of all nonnegative vectors in $\mathbb{R}^n$, is denoted by $\mathbb{R}^n_+$. For $x \in \mathbb{R}^n$, we use the notation $x \geq 0$ interchangeably with $x \in \mathbb{R}^n_+$.

An $n \times n$ matrix A is called *reducible* if there exists a permutation matrix P such that

$$PAP^T \ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square, nonvacuous matrices. Otherwise, A is called *irreducible*. Recall that irreducibility of A is equivalent to the directed graph of A, $G(A)$, being strongly connected, namely, the existence of a path of edges leading from any vertex i to any other vertex j. For details and further terminology regarding directed graphs, see [1].

Every reducible matrix A can be symmetrically permuted to its *Frobenius normal form*; namely, for every reducible matrix $A \in \mathbb{R}^{n\times n}$, there exists a permutation matrix P such that

$$PAP^T \ = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_{pp} \end{bmatrix},$$

where each diagonal block A_{jj} $(j = 1, 2, \ldots, p)$ is square and either irreducible or the 1×1 zero matrix. Note that the diagonal blocks in the Frobenius normal form correspond to a partition of the vertices of $G(A)$ into classes of strongly connected vertex subsets (a singleton is considered strongly connected).

To the Frobenius normal form of A above we associate the *reduced graph* $R(A)$ of A defined as follows: $R(A)$ has p vertices, each one of them corresponding to a strongly connected set of vertices in the directed graph of A. $R(A)$ has a directed edge from i to j if and only if $A_{ij} \neq 0$. In section 3 we will consider the *transitive closure* of $R(A)$, $\overline{R(A)}$, which is the directed graph obtained from $R(A)$ having an edge from i to j if and only if there is a path from i to j in $R(A)$.

DEFINITION 2.1. *An $n \times n$ matrix $A = [a_{ij}]$ is called*

- nonnegative (positive), *denoted by $A \geq 0$ $(A > 0)$, if $a_{ij} \geq 0$ (> 0) for all i and j;*
- essentially nonnegative (positive), *denoted by $A \overset{\mathrm{s}}{\geq} 0$ $(A \overset{\mathrm{s}}{>} 0)$, if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for all $i \neq j$;*
- eventually nonnegative (positive), *denoted by $A \overset{\mathrm{v}}{\geq} 0$ $(A \overset{\mathrm{v}}{>} 0)$, if there exists positive integer k_0 such that $A^k \geq 0$ $(A^k > 0)$ for all $k \geq k_0$; we denote the smallest such positive integer by $k_0 = k_0(A)$ and refer to it as the* power index *of A;*
- exponentially nonnegative (positive) *if for all $t \geq 0$, $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0$ $(e^{tA} > 0)$;*
- eventually exponentially nonnegative (positive) *if there exists $t_0 \in [0, \infty)$ such that for all $t \geq t_0$, $e^{tA} \geq 0$ $(e^{tA} > 0)$. We denote the smallest such nonnegative number by $t_0 = t_0(A)$ and refer to it as the* exponential index *of A.*

LEMMA 2.2. *Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:*
(i) *A is eventually exponentially nonnegative.*
(ii) *There exists $a \in \mathbb{R}$ such that $A + aI$ is eventually exponentially nonnegative.*
(iii) *For all $a \in \mathbb{R}$, $A + aI$ is eventually exponentially nonnegative.*

Proof. The equivalences follow readily from the fact that as aI and A commute,

$$e^{t(A+aI)} = e^{atI} e^{tA} = e^{at} e^{tA}. \qquad \square$$

We conclude this section with some notions crucial to the analysis in section 3.

DEFINITION 2.3. *We say that $A \in \mathbb{R}^{n \times n}$ has*

- *the* Perron–Frobenius property *if $\rho(A) > 0$, $\rho(A) \in \sigma(A)$, and there exists a nonnegative eigenvector corresponding to $\rho(A)$;*
- *the* strong Perron–Frobenius property *if, in addition to having the Perron–Frobenius property, $\rho(A)$ is a simple eigenvalue such that*

$$\rho(A) > |\lambda| \quad \text{for all} \quad \lambda \in \sigma(A), \;\; \lambda \neq \rho(A),$$

and if there is a strictly positive eigenvector corresponding to $\rho(A)$.

By the Perron–Frobenius theorem, every nonnilpotent $A \geq 0$ has the Perron–Frobenius property and every primitive $A \geq 0$ has the strong Perron–Frobenius property; see [1].

3. Eventually exponentially nonnegative matrices. There is a well-known equivalence between the notions of exponential nonnegativity and essential nonnegativity; see [1, Chapter 6, Theorem (3.12)]. We include a proof of this result next for completeness.

LEMMA 3.1. *$A \in \mathbb{R}^{n\times n}$ is exponentially nonnegative if and only if $A \overset{s}{\geq} 0$.*

Proof. If $A \overset{s}{\geq} 0$, then there exists large enough $\alpha \geq 0$ such that $A + \alpha I \geq 0$. Hence, as A and αI commute, we have that for all $t \geq 0$,

$$e^{tA} = e^{-t\alpha I} e^{t(A+\alpha I)} = e^{-t\alpha} e^{t(A+\alpha I)} \geq 0.$$

Conversely, let $e^{tA} \geq 0$ for all $t \geq 0$ and by way of contradiction suppose that $a_{ij} < 0$ for some $i \neq j$. Then, denoting the entries of A^k by $a_{ij}^{(k)}$, we have

$$(e^{tA})_{ij} = t a_{ij} + \frac{t^2}{2!} a_{ij}^{(2)} + \frac{t^3}{3!} a_{ij}^{(3)} + \cdots .$$

Thus, letting $t \to 0^+$ we have that for some $t > 0$, $(e^{tA})_{ij} < 0$, a contradiction. □

As a consequence of the above lemma, every essentially nonnegative matrix A is eventually exponentially nonnegative with exponential index $t_0 = 0$. We proceed with a characterization of eventually exponentially positive matrices based on some recent results proven in [11].

THEOREM 3.2 (see [11, Theorem 2.2]). *For a matrix $A \in \mathbb{R}^{n\times n}$ the following are equivalent:*

(i) *Both matrices A and A^T have the strong Perron–Frobenius property.*

(ii) *A is eventually positive.*

(iii) *A^T is eventually positive.*

Our main result thus far is the following extension of Theorem 3.2.

THEOREM 3.3. *For a matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:*

(i) *There exists $a \geq 0$ such that both matrices $A + aI$ and $A^T + aI$ have the strong Perron–Frobenius property.*

(ii) *$A + aI$ is eventually positive for some $a \geq 0$.*

(iii) *$A^T + aI$ is eventually positive for some $a \geq 0$.*

(iv) *A is eventually exponentially positive.*

(v) *A^T is eventually exponentially positive.*

Proof. The equivalence of (i)–(iii) is the content of Theorem 3.2 applied to $A + aI$. We will argue the equivalence of (ii) and (iv), with the equivalence of (iii) and (v) being analogous:

Let $A + aI$ be eventually positive and let k_0 be a positive integer such that $(A + aI)^k > 0$ for all $k \geq k_0$. Then there exists large enough $t_0 > 0$ so that the first $k_0 - 1$ terms of the series

$$e^{t(A+aI)} = \sum_{m=0}^{\infty} \frac{t^m (A + aI)^m}{m!}$$

are dominated by the remaining terms, rendering every entry of $e^{t(A+aI)}$ positive for all $t \geq t_0$. It follows that $e^{tA} = e^{-ta} e^{t(A+aI)}$ is positive for all $t \geq t_0$. That is, A is eventually exponentially positive. Conversely, suppose A is eventually exponentially positive. As $(e^A)^k = e^{kA}$, it follows that e^A is eventually positive. Thus, by Theorem 3.2, e^A has the strong Perron–Frobenius property. Recall that $\sigma(e^A) = \{e^\lambda : \lambda \in \sigma(A)\}$ and so $\rho(e^A) = e^\lambda$ for some $\lambda \in \sigma(A)$. Then for each $\mu \in \sigma(A)$ with $\mu \neq \lambda$ we have

$$e^\lambda > |e^\mu| = |e^{\mathrm{Re}\,\mu + i\mathrm{Im}\,\mu}| = e^{\mathrm{Re}\,\mu}.$$

Hence λ is the spectral abscissa of A, namely, $\lambda > \operatorname{Re}\mu$ for all $\mu \in \sigma(A)$ with $\mu \neq \lambda$. In turn, this means that there exists large enough $a > 0$ such that

$$\lambda + a > |\mu + a| \quad \text{for all } \mu \in \sigma(A), \;\; \mu \neq \lambda.$$

As $A + aI$ shares its eigenspaces with e^A, it follows that $A + aI$ has the strong Perron–Frobenius property. Invoking Theorem 3.2 once more, we have that $A + aI$ is eventually positive. ∎

Remark 3.4. Note that the equivalence of (ii) and (iv) in Theorem 3.3 represents a generalization of the fact that $A \overset{s}{>} 0$ is equivalent to A being exponentially positive.

Example 3.5. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

and observe that

$$A^2 = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 5 & 9 & 13 & 13 \\ 5 & 9 & 13 & 13 \\ 1 & 3 & 5 & 5 \\ 3 & 6 & 9 & 9 \end{bmatrix}.$$

It is easily checked that A is an eventually positive matrix with power index $k_0 = 3$, so by Theorem 3.3, A is an eventually exponentially positive matrix. Computing e^{tA} for $t = 1, 2$ we obtain, respectively,

$$\begin{bmatrix} 5.0401 & 6.3618 & 8.6836 & 8.6836 \\ 4.0401 & 7.3618 & 8.6836 & 8.6836 \\ -0.4655 & 2.7873 & 5.0401 & 4.0401 \\ 2.7873 & 3.5746 & 6.3618 & 7.3618 \end{bmatrix}, \begin{bmatrix} 71.2660 & 134.1429 & 198.0199 & 198.0199 \\ 70.2660 & 135.1429 & 198.0199 & 198.0199 \\ 18.4960 & 45.3810 & 71.2660 & 70.2660 \\ 45.3810 & 88.7620 & 134.1429 & 135.1429 \end{bmatrix}.$$

Taking into consideration the location of the nonpositive entries of A and A^2, we infer that the exponential index of A is $t_0 \in (1, 2)$.

Next we focus on eventually exponentially *nonnegative* matrices and connect them to eventually *nonnegative* matrices. In what follows we state and prove conditions that are sufficient for eventual exponential nonnegativity and investigate necessary conditions. To do so, we first need to discuss the relationship among the Frobenius normal forms of the powers of an eventually nonnegative matrix. This topic and its relation to the spectrum are studied extensively in [3, 4]. Below we summarize and paraphrase some of these results.

THEOREM 3.6 (see [3, Theorems 3.4 and 3.5]). *Let $A \in \mathbb{R}^{n\times n}$ be eventually nonnegative with* $\operatorname{index}_0(A) \leq 1$. *Then there exists a positive integer q and a permutation matrix P such that*

(i) $A^k \geq 0$ *for all* $k \geq q$;

(ii) PAP^T *and* PA^qP^T *are simultaneously in Frobenius normal form;*

(iii) $\overline{R(A)} = \overline{R(A^q)}$.

THEOREM 3.7. *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative with* $\operatorname{index}_0(A) \leq 1$. *Then A is an eventually exponentially nonnegative matrix.*

Proof. To avoid trivialities, suppose $n \geq 2$ and recall Theorem 3.6(ii). Without loss of generality, assume $P = I$; otherwise our considerations apply to a permutational similarity of A. Thus A and A^q are assumed to be in Frobenius normal form as follows:

$$(3.1)\quad A = \begin{bmatrix} A_{11} & \cdots & \cdots & A_{1p} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{pp} \end{bmatrix} \quad \text{and} \quad A^q = \begin{bmatrix} A_{11}^{(q)} & \cdots & \cdots & A_{1p}^{(q)} \\ 0 & \ddots & \ddots & A_{2k}^{(q)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & A_{pp}^{(q)} \end{bmatrix}.$$

Consider the power series $e^{tA} = \sum_{k=0}^{\infty} \frac{t^m A^k}{k!}$ partitioned in blocks conformably to the matrices in (3.1) and let $c_{ij}(t)$ be the (i,j)th entry of e^{tA}. Abusing slightly the notation, let $\{1, 2, \ldots, p\}$ denote the p strongly connected classes in $G(A)$ implied by (3.1). Let i belong to class u and j to class v, where $u, v \in \{1, 2, \ldots, p\}$. The following cases ensue:

Suppose that $p = 1$. As $n \geq 2$, A is irreducible. Thus for all powers $k \geq q$, the (i,j)th entry of A^k is nonnegative and is indeed positive for at least some powers $\geq q$. As a consequence, as $t \geq 0$ increases, $c_{ij}(t)$ is dominated in the power series by positive terms. That is, $c_{ij}(t)$ becomes and remains positive for all large enough $t \geq 0$.

Suppose next that $p > 1$. The blocks in the lower triangular part of the block partition of each A^k implied by (3.1) must be zero; namely, if $u > v$, then $c_{ij}(t) = 0$ for all $t \geq 0$.

If $u = v$, that is, if i, j belong to the same equivalence class, then either A_{uu} and $A_{uu}^{(q)}$ are both equal to the 1×1 zero matrix or they are both irreducible. In the former case, $c_{ij}(t) = 0$ for all $t \geq 0$, and in the latter case, $c_{ij}(t)$ becomes and remains positive for all large enough $t \geq 0$ analogously to the $p = 1$ case above.

Finally, let us consider the sign of $c_{ij}(t)$ when $u < v$. Let $a_{ij}^{(k)}$ denote the (i,j)th entry of A^k. If $a_{ij}^{(k)} = 0$ for all $k < q$, then by Theorem 3.6(i) we have that $c_{ij}(t) \geq 0$ for all $t \geq 0$. If $a_{ij}^{(k)} \neq 0$ for some $k < q$, then there must be a path form i to j in $G(A)$. Thus there is a path from u to v in $R(A)$. By Theorem 3.6(iii), $\overline{R(A)} = \overline{R(A^q)}$ and so there must be a path from u to v in $R(A^q)$. It follows that there is a path from i to j in $G(A^q)$. In turn, this implies that there is a power $m \geq q$ such that $a_{ij}^{(m)} > 0$. As $a_{ij}^{(k)} \geq 0$ for all $k \geq q$, we have once again that $c_{ij}(t)$ is dominated in the power series by positive terms and so it becomes and remains positive for all large enough $t \geq 0$.

To conclude, we have shown that each entry $c_{ij}(t)$ of e^{tA} becomes and remains nonnegative for all large enough $t \geq 0$, namely, that A is eventually exponentially positive. □

COROLLARY 3.8. *Let $A \in \mathbb{R}^{n \times n}$ such that $A + aI$ is eventually nonnegative for all $a \in [a_1, a_2]$ $(a_1 < a_2)$. Then A is an eventually exponentially nonnegative matrix.*

Proof. Since $\sigma(A)$ is a finite set, there exists $a \in [a_1, a_2]$ such that $A + aI$ is invertible. Hence $\text{index}_0(A + aI) = 0$ and so by Theorem 3.7, $A + aI$ is eventually exponentially nonnegative. By Lemma 2.2, it follows that A is eventually exponentially nonnegative. □

We illustrate the above results on eventual nonnegativity with the following examples.

Example 3.9. Consider

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

for which

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 5 & 5 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & 11 & 9 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 8 & 8 \end{bmatrix}, \quad A^5 = \begin{bmatrix} 0 & 1 & 11 & 9 \\ 1 & 0 & 21 & 21 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 16 & 16 \end{bmatrix}.$$

Notice that A is reducible, eventually nonnegative, and, referring to Theorem 3.6, $q = k_0 = 2$. Since $\text{index}_0(A) = 1$, Theorem 3.7 implies that A is an eventually exponentially nonnegative matrix. For illustration, we compute e^{tA} for $t = 1, 2$ to be, respectively,

$$\begin{bmatrix} 1.5431 & 1.1752 & 2.3404 & -0.0100 \\ 1.1752 & 1.5431 & 4.0487 & 2.9625 \\ 0 & 0 & 4.1945 & 3.1945 \\ 0 & 0 & 3.1945 & 4.1945 \end{bmatrix}, \quad \begin{bmatrix} 3.7622 & 3.6269 & 18.1543 & 10.9006 \\ 3.6269 & 3.7622 & 35.4439 & 29.9195 \\ 0 & 0 & 27.7991 & 26.7991 \\ 0 & 0 & 26.7991 & 27.7991 \end{bmatrix}.$$

This confirms A is an eventually exponentially nonnegative matrix with $1 < t_0 < 2$.

Example 3.10. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and its sequence of powers

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 0 & 0 \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots).$$

The matrix A is eventually nonnegative with $k_0 = 2$. As the $(1, 2)$ block of A^k is 0 for all $k \geq 2$, while the one of A is not and contains negative entries, A is not eventually exponentially nonnegative. In agreement, the assumptions of Theorem 3.7 do not hold since $\text{index}_0(A) = 2$.

The failure of eventual nonnegativity to force eventual exponential nonnegativity observed in the above example can occur even if A is irreducible, as the following example shows.

Example 3.11. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

and its sequence of powers

$$A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix} \quad (k = 2, 3, \ldots).$$

The matrix A is an eventually nonnegative matrix with $k_0 = 2$ and $\mathrm{index}_0(A) = 2$. As the assumptions of Theorem 3.7 do not hold, we may not conclude that A is eventually exponentially nonnegative. The $(2, 1)$ block of A^k is 0 for all $k \geq 2$, while the one of A is not and contains negative entries. Thus A is not eventually exponentially nonnegative. Indeed,

$$e^A = \begin{bmatrix} 4.1945 & 3.1945 & 7.3891 & 7.3891 \\ 3.1945 & 4.1945 & 7.3891 & 7.3891 \\ -1 & 1 & 4.1945 & 3.1945 \\ 1 & -1 & 3.1945 & 4.1945 \end{bmatrix}, \quad e^{3A} = \begin{bmatrix} 202.2 & 201.2 & 1210.3 & 1210.3 \\ 201.2 & 202.2 & 1210.3 & 1210.3 \\ -3 & 3 & 202.2 & 201.2 \\ 3 & -3 & 201.2 & 202.2 \end{bmatrix}.$$

We now turn our attention to necessary conditions for eventual exponential nonnegativity for which we need to quote some results from [11]. Note that in the first theorem below from [11], we have added the assumption that A is not nilpotent; the need for this assumption is observed in [5].

THEOREM 3.12 (see [11, Theorem 2.3]). *Let $A \in \mathbb{R}^{n\times n}$ be an eventually nonnegative matrix which is not nilpotent. Then both A and A^T have the Perron–Frobenius property.*

THEOREM 3.13 (see [11, Theorem 2.4]). *Let both $A \in \mathbb{R}^{n\times n}$ and A^T have the Perron–Frobenius property. If $\rho(A)$ is a simple and the only dominant eigenvalue of A, then*

$$\lim_{k\to\infty} \left(\frac{A}{\rho(A)}\right)^k = xy^T,$$

where x and y are, respectively, right and left nonnegative eigenvectors of A corresponding to $\rho(A)$, satisfying $x^T y = 1$.

THEOREM 3.14. *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix. Then the following hold:*

(i) *e^A and e^{A^T} have the Perron–Frobenius property.*

(ii) *If $\rho(e^A)$ is a simple eigenvalue of e^A and $\rho(e^A) = e^{\rho(A)}$, then there exists $a_0 \geq 0$ such that $\lim_{k\to\infty} ((A + aI)/(\rho(A + aI))^k = xy^T$ for all $a > a_0$, where x and y are, respectively, right and left nonnegative eigenvectors of A corresponding to $\rho(A)$, satisfying $x^T y = 1$.*

Proof. (i) Let A be eventually exponentially nonnegative. As $(e^A)^k = e^{kA}$, it follows that e^A is eventually nonnegative. Thus, by Theorem 3.12 and since e^A and e^{A^T} are not nilpotent, they have the Perron–Frobenius property.

(ii) From (i) we specifically have that $\rho(e^A) \in \sigma(e^A)$. Let x, y be right and left nonnegative eigenvectors, respectively, corresponding to $\rho(e^A)$ and normalized so that $x^T y = 1$. As in the proof of Theorem 3.3, $\rho(e^A) = e^\lambda$ for some $\lambda \in \sigma(A)$ with $\lambda > \operatorname{Re}\mu$ for all $\mu \in \sigma(A) \setminus \{\lambda\}$. This means that there exists large enough $a_0 > 0$, such that for all $a \geq a_0$,

$$\rho(A + aI) = \lambda + a > |\mu + a| \quad \text{for all } \mu \in \sigma(A), \;\; \mu \neq \lambda.$$

As $A + aI$ and e^A share eigenvectors, we obtain that for all $a > a_0$, $A + aI$ and $A^T + aI$ both have the Perron–Frobenius property, with $\lambda + a$ being simple and their only dominant eigenvalue. Applying Theorem 3.13 to $A + aI$, we thus obtain

$$\lim_{k\to\infty} \frac{1}{\rho(A + aI)^k}(A + aI)^k = xy^T \geq 0. \qquad (3.2)$$

∎

Remark 3.15. Referring to the proof of Theorem 3.14, by (3.2) we have that if $(xy^T)_{ij} > 0$, then $\left((A + aI)^k\right)_{ij} > 0$ for all k sufficiently large. In particular, if $xy^T > 0$, then $A + aI$ is eventually nonnegative for all $a > a_0$. If, however, xy^T is nonnegative but not strictly positive, $A+aI$ can fail to be eventually nonnegative for all $a \in \mathbb{R}$. This situation is illustrated by the matrix A in Example 3.11.

4. Points of nonnegative potential. In this section $A \in \mathbb{R}^{n\times n}$ denotes an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. We will study points of nonnegative potential, that is, the set

$$X_A(\mathbb{R}^n_+) \;=\; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\, (\forall t \geq \hat{t})\, [e^{tA}x_0 \geq 0]\}. \qquad (4.1)$$

$X_A(\mathbb{R}^n_+)$ comprises all initial points giving rise to trajectories of (1.1) that reach $\mathbb{R}^n_+$ at some finite time and stay in $\mathbb{R}^n_+$ for all time thereafter.

First, let us recall some basic facts and terminology on convex cones in $\mathbb{R}^n$. Our references are [1, Chapter 1] and [12]. A convex set $K \subseteq \mathbb{R}^n$ is called a *convex cone* if $aK \subseteq K$ for all $a \geq 0$. A convex cone is called *polyhedral* if it consists of all finite nonnegative linear combinations of the elements of a finite set. A convex cone K is *pointed* if $K \cap (-K) = \{0\}$ and *solid* if its topological interior is nonempty. A pointed, solid convex cone is called a *proper cone*. The nonnegative orthant $\mathbb{R}^n_+$ is indeed a proper cone; it is also a polyhedral cone, comprising all finite nonnegative combinations of the standard basis vectors. Any subset of $\mathbb{R}^n$ of the form $K = S\mathbb{R}^n_+$, where S is an invertible matrix, is a proper polyhedral cone and referred to as a *simplicial cone*.

Given an eventually exponentially nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with exponential index $t_0 = t_0(A) \geq 0$, define the simplicial cone

$$K = e^{t_0 A}\mathbb{R}^n_+ = \{x_0 \in \mathbb{R}^n \mid (\exists y \geq 0)\, [x_0 = e^{t_0 A} y]\}$$

and consider the sets

$$Y_A(K) \;=\; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\, [e^{\hat{t}A}x_0 \in K]\} \qquad (4.2)$$

and

$$X_A(K) \;=\; \{x_0 \in \mathbb{R}^n \mid (\exists \hat{t} = \hat{t}(x_0) \geq 0)\, (\forall t \geq \hat{t})\, [e^{tA}x_0 \in K]\}. \qquad (4.3)$$

Lemma 4.1. *Let K, $Y_A(K)$ as defined above. Then $K \subseteq \mathbb{R}^n_+ \subseteq Y_A(K)$.*

Proof. We have that $K \subseteq \mathbb{R}^n_+$ since $e^{t_0 A} \geq 0$. If $x_0 \in \mathbb{R}^n_+$, then for $\hat{t} = 2t_0$, $e^{\hat{t}A}x_0 = e^{t_0 A}\,(e^{t_0 A}x_0) \in K$. Hence, $\mathbb{R}^n_+ \subseteq Y_A(K)$. $\square$

Note that the sets $Y_A(K)$, $X_A(K)$, and $X_A(\mathbb{R}^n_+)$ are convex cones. They are not necessarily closed sets, however. For example, when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

it can be shown that $X_A(\mathbb{R}^2_+)$ consists of the whole upper plane excluding the negative x-axis.

The set $Y_A(K)$ comprises initial points for which the trajectories enter K at some time. The set $X_A(K)$ comprises initial points for which the trajectories enter K at some time and remain in K for all time thereafter. The set of points of nonnegative potential, $X_A(\mathbb{R}^n_+)$, comprises initial points for which the trajectories at some time become nonnegative and remain nonnegative for all time thereafter. Next we shall argue that $Y_A(K)$, $X_A(K)$, and $X_A(\mathbb{R}^n_+)$ coincide and interpret this result subsequently.

PROPOSITION 4.2. *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$ and let $K = e^{t_0 A}\mathbb{R}^n_+$. Then*

$$Y_A(K) \;=\; X_A(\mathbb{R}^n_+) \;=\; X_A(K).$$

Proof. We begin by proving the first equality. If $x_0 \in Y_A(K)$, then there exists $\hat{t} \geq 0$ and $y \geq 0$ such that $e^{\hat{t}A}x_0 = e^{t_0 A}y$. Thus, $x_0 = e^{(t_0-\hat{t})A}y$ and so $e^{tA}x_0 = e^{(t+t_0-\hat{t})A}y \geq 0$ if $t + t_0 - \hat{t} \geq t_0$, i.e., for all $t \geq \hat{t}$. It follows that $x_0 \in X_A(\mathbb{R}^n_+)$, i.e., $Y_A(K) \subseteq X_A(\mathbb{R}^n_+)$. For the opposite containment, let $x_0 \in X_A(\mathbb{R}^n_+)$; that is, there exists $\hat{t} \geq 0$ such that $e^{tA}x_0 \geq 0$ for all $t \geq \hat{t}$. Let $\tilde{t} = \hat{t} + t_0$. Then $e^{\tilde{t}A}x_0 = e^{t_0 A}(e^{\hat{t}Ax_0}) \in K$, proving that $X_A(\mathbb{R}^n_+) \subseteq Y_A(K)$ and thus equality holds.

For the second equality, clearly $X_A(K) \subseteq X_A(\mathbb{R}^n_+)$ since $K \subseteq \mathbb{R}^n_+$. To show the opposite containment, let $x_0 \in X_A(\mathbb{R}^n_+)$. Then there exists $\hat{t} \geq 0$ such that $e^{t_0 A}e^{sA}x_0 \in K$ for all $s \geq \hat{t}$. That is, $e^{tA}x_0 \in K$ for all $t \geq t_0 + \hat{t}$ and thus $x_0 \in X_A(K)$. $\square$

Remark 4.3. Referring to Proposition 4.2, we must make the following observations:

(i) If $t_0 = 0$ (i.e., if $A \overset{s}{\geq} 0$, or equivalently if $e^{tA} \geq 0$ for all $t \geq 0$), then $K = \mathbb{R}^n_+$. In this case, $X_A(\mathbb{R}^n_+)$ coincides with the *reachability cone* of the nonnegative orthant for an essentially nonnegative matrix, which is studied in detail in [10, 9].

(ii) The equality $X_A(\mathbb{R}^n_+) = X_A(K)$, in conjunction with Lemma 4.1, can be interpreted as saying that the simplicial cone $K = e^{t_0 A}\mathbb{R}^n_+$ serves as an attractor set for trajectories emanating at points of nonnegative potential; in other words, trajectories emanating in $X_A(\mathbb{R}^n_+)$ always reach and remain in $K \subseteq \mathbb{R}^n_+$ after a finite time.

(iii) Our observations so far imply that the trajectory emanating from a point of nonnegative potential will enter cone K; however, it may subsequently exit K while it remains nonnegative, and it will eventually re-enter K and remain in K for all finite time thereafter. This situation is illustrated by the following example.

Example 4.4. Consider the matrix

$$A = \begin{bmatrix} 0.3929 & -0.8393 & 1.1071 & 1.3393 \\ 1.0357 & 0.6964 & -0.5357 & 0.8036 \\ 1.0357 & -0.3036 & 0.4643 & 0.8036 \\ 1.4643 & 1.0536 & -0.9643 & 0.4464 \end{bmatrix}.$$

It can be checked that A and A^T have the strong Perron–Frobenius property and so, by Theorems 3.2 and 3.3, A is an eventually exponentially positive matrix. Using MATLAB and a bisection method, we estimated (within five decimals) the exponential index to be $t_0 = t_0(A) = 2.64378$. The matrices e^A and $e^{t_0 A}$ are

$$e^A = \begin{bmatrix} 3.6277 & -0.7991 & 1.4260 & 3.1345 \\ 3.0341 & 2.2579 & -0.6987 & 2.7958 \\ 3.0341 & -0.4604 & 2.0196 & 2.7958 \\ 3.3050 & 1.4836 & -0.9696 & 3.5701 \end{bmatrix}$$

and

$$e^{t_0 A} = \begin{bmatrix} 91.902 & 3.5982 & 14.0615 & 88.299 \\ 91.499 & 18.162 & 0.3981 & 87.801 \\ 91.499 & 4.0959 & 14.4643 & 87.801 \\ 91.897 & 17.494 & 0 & 88.469 \end{bmatrix}.$$

Hence the cone $K = e^{t_0 A}\mathbb{R}^n_+$ is the cone generated by the columns of the matrix $e^{t_0 A}$ above. Consider now the following trajectory points $x(t) = e^{tA}x(0)$:

$$x_0 = x(0) = \begin{bmatrix} -1.1617 \\ 0.6014 \\ 0.9693 \\ 1.0887 \end{bmatrix}, \quad x(1) = e^A x_0 = \begin{bmatrix} 0.1 \\ 0.2 \\ 1.2 \\ 0 \end{bmatrix}, \quad x(2) = e^{2A} x_0 = \begin{bmatrix} 1.9141 \\ -0.0834 \\ 2.6348 \\ -0.5363 \end{bmatrix},$$

$$e^{(t_0+1)A} x_0 = \begin{bmatrix} 26.7836 \\ 13.2600 \\ 27.3263 \\ 12.6884 \end{bmatrix}, \quad e^{(t_0+2)A} x_0 = \begin{bmatrix} 165.3049 \\ 127.5845 \\ 165.8206 \\ 126.9949 \end{bmatrix}, \quad e^{(2t_0+1)A} x_0 = \begin{bmatrix} 4013.8 \\ 3816.4 \\ 4014.3 \\ 3815.8 \end{bmatrix}.$$

Observe the following: $e^{(t_0+1)A}x_0 \in K$ since $e^A x_0 \in \mathbb{R}^n_+$; $e^{(t_0+2)A}x_0 \notin K$ since $e^{2A}x_0 \notin \mathbb{R}^n_+$; $e^{(2t_0+1)A}x_0 \in K$ since $e^{(t_0+1)A}x_0 \in \mathbb{R}^n_+$; finally, trajectory points $x(t)$ are in K for all $t \geq 2t_0 + 1$. In other words, the trajectory emanating at x_0 enters K, exits K, and eventually re-enters and remains in K for all time thereafter.

In view of the above example, a natural question arises: When is it possible that all trajectories emanating in $X_A(\mathbb{R}^n_+)$ reach and never exit K? This is equivalent to asking whether or not $e^{tA}K \subseteq K$ for all $t \geq 0$. To resolve this question, we will invoke the following extension of Lemma 3.1 from $\mathbb{R}^n_+$ to simplicial cones, which can be found in [13, 14].

LEMMA 4.5. *Let $A \in \mathbb{R}^{n\times n}$ and $K = S\mathbb{R}^n_+$, where $S \in \mathbb{R}^{n\times n}$ is nonsingular. Then there exists $a \geq 0$ such that $(A + aI)K \subseteq K$ if and only if $e^{tA}K \subseteq K$ for all $t \geq 0$.*

Proof. Consider the similarity transformation $A \to B = S^{-1}AS$. We claim that there exists $a \geq 0$ such that $(A + aI)K \subseteq K$ if and only if $B \overset{s}{\geq} 0$. Indeed, if $(A + aI)K \subseteq K$, then

$$(B + aI)\mathbb{R}^n_+ = S^{-1}(A + aI)S\mathbb{R}^n_+ = S^{-1}(A + aI)K \subseteq S^{-1}K = \mathbb{R}^n_+.$$

Conversely, if $B \overset{s}{\geq} 0$, then there exists $a \geq 0$ such $B + aI = S^{-1}(A + aI)S \geq 0$. Hence for each $x \in K$, there exists $y \geq 0$ such that

$$S^{-1}(A + aI)x = S^{-1}(A + aI)Sy = z \geq 0.$$

That is, $(A + aI)Sy = Sz \in K$. Similarly, one can show that $e^{tA}K \subseteq K$ for all $t \geq 0$ if and only if $e^{tB} \geq 0$ for all $t \geq 0$. ∎

We note in passing that Lemma 4.5 holds more generally for every polyhedral cone K; see [13, 14].

COROLLARY 4.6. *Let $A \in \mathbb{R}^{n\times n}$ be an eventually exponentially nonnegative matrix with exponential index $t_0 = t_0(A) \geq 0$. Let $K = e^{t_0 A}\mathbb{R}^n_+$. Then $e^{tA}K \subseteq K$ for all $t \geq 0$ if and only if $t_0 = 0$ (or equivalently, if and only if $A \overset{s}{\geq} 0$).*

Proof. If $t_0 = 0$, then $K = \mathbb{R}^n_+$ and $e^{tA} \geq 0$ for all $t \geq 0$. For the converse, suppose $e^{tA}K \subseteq K$ for all $t \geq 0$. We must show that $t_0 = 0$. Let $y \geq 0$ and consider $x_0 = e^{t_0 A}y \in K$. As $e^{tA}x_0 \in K$ for all $t \geq 0$, there must exist $z \geq 0$ such that

$$e^{(t+t_0)A}\, y \;=\; e^{t_0 A}\, z \quad \text{for all } t \geq 0.$$

But this means $e^{tA}y = z \geq 0$ for all $t \geq 0$. Since y was taken arbitrary in $\mathbb{R}^n_+$, we have $e^{tA}\mathbb{R}^n_+ \subseteq \mathbb{R}^n_+$ for all $t \geq 0$; that is, $t_0 = 0$. ∎

We conclude this section with a discussion on a possible numerical test for points of nonnegative potential. When $A = [a_{ij}] \overset{s}{\geq} 0$, $X_A(\mathbb{R}^n_+)$ admits a numerical characterization reported in [10] and briefly described in the following. Consider the sequence $\{x_k\}$ generated from x_0 by the Cauchy–Euler finite differences scheme

$$x_k = (I + hA)^k x_0, \quad k = 0, 1, \ldots,$$

which we refer to as *the discrete trajectory* (*associated with the time-step* h) *emanating from* x_0. Define the quantity

$$h(A) = \sup\left\{ h \mid \min_{1\leq i\leq n}(1 + h a_{ii}) > 0 \right\}$$

and notice that $h(A) = \sup\{h \mid (I + hA) \geq 0\} > 0$, as well as that $h(A) = \infty$ when $A \geq 0$.

For any $h \in (0, h(A))$, denote by $X_{A,h}(\mathbb{R}^n_+)$ the set of all initial states $x_0 \in \mathbb{R}^n$ that give rise to discrete trajectories $\{x_k\}$ which become and remain (due to nonnegativity of $I + hA$) nonnegative; that is,

$$X_{A,h}(\mathbb{R}^n_+) = \{x_0 \in \mathbb{R}^n \mid (\exists k_0 = k_0(x_0) \geq 0)\, (\forall k \geq k_0)\, [(I + hA)^k x_0 \in \mathbb{R}^n_+]\}.$$

We refer to $X_{A,h}(\mathbb{R}^n)$ as the *discrete reachability cone* (*of* $\mathbb{R}^n_+$ *under* A *with respect to* h). The geometric and algebraic properties of the discrete reachability cone are studied extensively in [8, 10].

THEOREM 4.7 (see [10]). *Let $A \in \mathbb{R}^{n\times n}$ be an essentially nonnegative matrix and let $h \in (0, h(A))$ such that $(I + hA)$ is invertible. Then $X_A(\mathbb{R}^n_+) = X_{A,h}(\mathbb{R}^n_+)$.*

When $A \overset{s}{\geq} 0$, Theorem 4.7 suggests a simple test to find out whether or not a given initial point x_0 belongs to $X_A(\mathbb{R}^n_+)$: 1. Choose a positive $h < h(A)$ such that the iteration matrix $I + hA$ is invertible. 2. Check whether for some nonnegative integer k, $x_k = (I + hA)^k x_0$ is nonnegative (in which case $x_0 \in X_A(\mathbb{R}^n_+)$) or decide that x_k will never be nonnegative (in which case $x_0 \notin X_A(\mathbb{R}^n_+)$).

As noted in [15], Theorem 4.7 can be generalized from $\mathbb{R}^n_+$ to any simplicial cone K such that $e^{tA}K \subseteq K$ for all $t \geq 0$. Thus, in view of Proposition 4.2, the question arising is whether the above test can be extended to $X_A(\mathbb{R}^n_+) = X_A(K)$, when A is eventually exponentially nonnegative with exponential index $t_0 \geq 0$ and $K = e^{t_0 A}\mathbb{R}^n_+$. By Corollary 4.6, however, it follows that the answer is in the negative when

$t_0 > 0$. The development of a characterization of points of nonnegative potential in terms of discrete trajectories will likely require a close examination of the generalized eigenspaces of A as in the proof of Theorem 4.7. We plan to undertake this task in future work, as well as perform a numerical analysis of the associated test.

5. Conclusions. We considered the problem of when a trajectory $x(t) = e^{tA}x_0$ $(t \geq 0)$ becomes and remains nonnegative. Naturally, we needed to study (1) matrices A for which e^{tA} becomes and remains nonnegative and (2) initial points x_0 giving rise to nonnegative trajectories, which we called points of nonnegative potential. The combination of such matrices and initial points results in trajectories that reach and stay in the nonnegative orthant. We discovered that eventual nonnegativity of the exponential matrix is intimately related to eventual nonnegativity of the powers of A (section 3). We also found that the collection of points of nonnegative potential coincides with the collection of initial points that reach and stay in a certain simplicial cone K associated with e^{tA}. Interestingly, trajectories emanating at points of nonnegative potential may enter and subsequently exit this cone K; however, K eventually attracts such trajectories permanently (section 4). Our results generalize and parallel well-known facts in nonnegative systems theory and are illustrated with several examples.

Acknowledgment. The authors would like to sincerely thank the anonymous referees and the editor, Daniel Szyld, for several comments that helped us correct and improve the original manuscript.

REFERENCES

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, SIAM, Philadelphia, 1994.
[2] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative Matrices in Dynamic Systems.* Wiley-Interscience, New York, 1989.
[3] S. Carnochan Naqvi and J. J. McDonald, *The combinatorial structure of eventually nonnegative matrices*, Electron. J. Linear Algebra, 9 (2002), pp. 255–269.
[4] S. Carnochan Naqvi and J. J. McDonald, *Eventually nonnegative matrices are similar to semi-nonnegative matrices*, Linear Algebra Appl., 381 (2004), pp. 245–258.
[5] A. Elhashash and D. B. Szyld, *Perron-Frobenius Properties of General Matrices*, preprint, 2007.
[6] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, Wiley-Interscience, New York, 2000.
[7] M. Neumann and R. J. Stern, *Boundary results for positively invariant cones and their reachability cones*, Linear Multilinear Algebra, 17 (1985), pp. 143–154.
[8] M. Neumann and R. J. Stern, *Cone reachability for linear differential systems*, Appl. Anal., 20 (1986), pp. 57–71.
[9] M. Neumann and M. Tsatsomeros, *Symbiosis points for linear differential systems*, Linear Multilinear Algebra, 30 (1991), pp. 49–59.
[10] M. Neumann, R. J. Stern, and M. Tsatsomeros, *The reachability cones of essentially nonnegative matrices*, Linear Multilinear Algebra, 28 (1991), pp. 213–224.
[11] D. Noutsos, *On Perron-Frobenius property of matrices having some negative entries*, Linear Algebra Appl., 412 (2005), pp. 132–153.
[12] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1997.
[13] H. Schneider and M. Vidyasagar, *Cross-positive matrices*, SIAM J. Numer. Anal., 7 (1970), pp. 508–519.
[14] R. J. Stern and M. J. Tsatsomeros, *Extended M-matrices and subtangentiality*, Linear Algebra Appl., 97 (1987), pp. 1–11.
[15] M. J. Tsatsomeros, *Reachability of Nonnegative and Symbiotic States*, Ph.D. thesis, University of Connecticut, Storrs, CT, 1990.
[16] B. G. Zaslavsky, *Eventually nonnegative realization of difference control systems*, in Dynamical Systems and Related Topics, Adv. Ser. Dynam. Systems 9, World Scientific, River Edge, NJ, 1991, pp. 573–602.