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Pulse dynamics and solitons in mode-locked lasers

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A model system is presented that describes the physical properties of mode-locked lasers. This distributed system incorporates gain and filtering saturated with energy while loss is saturated with power. It is found that general initial pulses evolve to stable localized solutions which exist for wide choices of the parameters, the only requirement being sufficient gain. Moreover, these pulses are essentially solitons of the classical nonlinear Schrödinger (NLS) equation. In the anomalous regime, the additional terms present in the system serve to provide the mode locking mechanism. Consequently, these pulses are approximated by the classical NLS soliton, given by hyperbolic secant functions, in agreement with recent experiments.

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Ultrashort pulses have been the subject of extensive research and have many important applications, including communications [1], optical clock technology [2], high-order harmonic generation [3], extreme physics [4], and measuring of the fundamental constants of nature [5]. One of the techniques used to produce ultrashort pulses is mode locking in laser systems. Mode-locked (ML) lasers have been studied for many years, with origins dating back more than four decades. However, due to their complicated dynamics it is only in recent years that researchers have begun to better understand and utilize their true potential. ML laser pulses can be generated actively by use of an external element or passively via the Kerr-lens mechanism. The latter produces shorter pulses and will be the focus of this paper.

Unlike a typical CW laser, a ML laser emits a series of short intense pulses at a steady repetition rate. In practice, ML laser pulses are produced in a laser gain medium with sufficiently broad gain bandwidth. The effect is to generate pulsed operation and phase lock different longitudinal modes [6]. In the frequency domain, the pulses from the ML laser correspond to a frequency comb, i.e., a sequence of spectral lines.

A common technique developed for passive mode locking is Kerr lens mode locking (KLM). KLM is a method of mode locking lasers via the optical nonlinear Kerr effect. This method allows the generation of light pulses with a duration of a few femtoseconds. Intensity changes with lengths of nanoseconds are amplified by the Kerr-lensing process and the pulse length further shrinks to achieve higher field strengths in the center of the pulse. This sharpening process is only limited by the bandwidth achievable with the laser material and the cavity mirrors as well as the dispersion of the cavity.

A laser can only operate if a sufficient amount of gain and loss is present. Passive mode locking generally utilizes saturable absorbers. KLM can simulate an effective saturable absorber action that is extremely fast and is useful for short pulse generation over a wide range of wavelengths. However, generation of ultrashort pulses also requires the action of gain saturation.

In order to model the effects of nonlinearity, dispersion, bandwidth limited gain, energy saturation and intensity discrimination in a laser cavity the so-called master-equation was introduced [7,8]. The master equation is a generalization of the classical nonlinear Schrödinger equation (NLS) modified to contain gain, filtering, and loss terms. Gain and filtering are saturated by energy (i.e., the time integral of the pulse power), while loss is represented by a cubic nonlinearity. For certain values of the parameters this equation exhibits a range of phenomena including mode locking pulse evolution, pulses which disperse into radiation, pulses which evolve to a nonlocalized quasiperiodic state, and pulses whose amplitude grows rapidly. In the latter case, if the nonlinear gain is too high, the linear attenuation terms are unable to prevent the pulse from blowing up, suggesting the breakdown of the master mode locking model [9]. Unfortunately, there is only a small window of parameter space which allows for the generation of stable mode-locked pulses. In particular, the model is highly sensitive to the nonlinear loss and/or gain parameter. To overcome this sensitivity, other types of terms, such as quintic terms, can be added to the master equation in order to stabilize the solutions. This, however, only slightly increases the parameter range for mode locking (instabilities may still occur) and also adds another parameter to the model.

Another commonly used model is based on Ginzburg-Landau (GL) type equations [10]. In fact, if the pulse energy is taken to be constant the master-equation reduces to a GL type system. These models do not include saturable terms and chirp-free pulses are only found for specific values of the parameters. In general, these equations exhibit a wide spectrum of pulses ranging from pulses with complex and chaotic dynamics to pulses whose amplitude grows rapidly. These systems are also employed to model "soliton explosions" [11] and make use of a stretched cavity in the laser and the lack of spectral filtering. Such pulses occur when the laser is close to unstable operation.

Interestingly, recent experiments in the constant anomalous regime [12] indicate that the normalized intensity of a pulse in a mode-locked laser fits a hyperbolic secant spectrum well. Other experiments indicate that pulses should be chirp-free [13]. In addition, it has been shown [14], in dispersion managed systems, with net anomalous dispersion, that when saturable absorption and gain are excluded from

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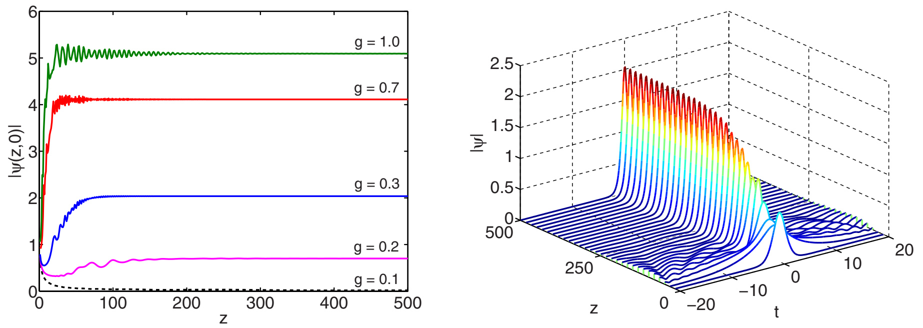

FIG. 1. (Color online) Evolution of the pulse peak of an arbitrary initial profile under NLSGL with different values of gain. The damped pulse-peak evolution is shown with a dashed line. In the second figure, the complete evolution is given for $g=0.3$.

the theoretical model the underlying model's predictions are in good agreement with experiment. Namely, while the gain and loss mechanisms are necessary to generate the pulse, it can be expected that the long time pulse dynamics should be governed by the simpler unperturbed equations, when the pulse's behavior should be dominated by the interplay of the Kerr nonlinearity and the linear dispersion.

The distributed model describing the propagation of pulses in a laser cavity is given in dimensionless form by

$$i\psi_z + \frac{d_0}{2}\psi_{tt} + |\psi|^2\psi = \frac{ig}{1+E/E_{\rm sat}}\psi + \frac{i\tau}{1+E/E_{\rm sat}}\psi_{tt} - \frac{il}{1+P/P_{\rm sat}}\psi, \tag{1}$$

where $E=E(z)=\int_{-\infty}^{+\infty}|\psi|^2dt$ is the pulse energy, $E_{\rm sat}$ is the saturation energy, $P=P(z,t)=|\psi|^2$ is the instantaneous pulse power, $P_{\rm sat}$ is the saturation power, d_0 is the dispersion (which is constant in this study), and the parameters g, τ, l are all positive real constants. The first term on the right-hand side represents saturable gain, the second is spectral filtering, and the third saturable loss. The gain filtering mechanisms are related to the energy of the pulse while the loss is related to the power (intensity) of the pulse. Saturation terms prevent the pulse from reaching a singular state; i.e., "infinite" energy or a blowup in amplitude. Indeed, if blowup were to occur that would mean that both the amplitude and the energy of the pulse are large, hence, the perturbing effects are very small thus reducing the equation to the unperturbed NLS, which admits a stable finite solution.

Further inspection of Eq. (1) reveals that a first-order Taylor approximation results in the master equation where the cubic term accounts for the saturable loss, while a higher approximation (second order) results in higher-order GL equations with quintic terms. Moreover, we find that this model has stable soliton solutions over a wide parameter range and does not exhibit blowup solutions. We refer to Eq. (1) as the perturbed NLS with gain-loss or NLSGL and the right-hand side of Eq. (1), i.e., what we shall refer to as the perturbing contribution, is denoted hereafter by $Q[\psi]$.

In some models loss is introduced in the form of fast saturable power absorbers which are placed periodically. This type of lumped model [15] has been studied in dispersion-managed systems operating in the normal regime. We find that the essential features of the lumped model are included in this distributive equation, even in dispersion-managed systems [16]. In fact, Eq. (1) also exhibits pulse solutions in the normal regime ($d_0<0$) [17]. We briefly discuss this at the end of this article.

Power saturation models also arise in other important problems in nonlinear optics. For example, in the study of the dynamics of localized lattice modes (solitons, vortices, etc.) propagating in photorefractive nonlinear crystals [18,19]. If the nonlinear term in these equations was simply a cubic nonlinearity, without saturation, two-dimensional fundamental lattice solitons would be vulnerable to blow up singularity formation, which is not observed. Thus saturable terms are crucial in these problems. Our model is based on the Kerr effect and hence Kerr type nonlinearity is used. The effects of gain and loss in photorefractive materials will be studied in another article.

Our analysis begins with the dynamics of pulses evolving under the NLSGL equation. In practice, the mode locking mechanism is often not self-starting; rather it requires a crucial misalignment to occur. Such misalignments can be regarded as forming a general initial state of an evolution equation, not as the initial state of an exact soliton solution. This observation leads us to consider an arbitrary initial profile, e.g., $\psi(0,t)=\exp(-At^2)$, and to employ a fourth-order Runge-Kutta method to evolve Eq. (1) in z. Furthermore, all terms are kept constant and only the gain parameter g changes. More precisely, $E_{\rm sat}=P_{\rm sat}=A=1$, $\tau=l=0.1$, and assuming constant anomalous dispersion, $d_0=1$ ($d_0>0$). The filtering in the equation also acts as an additional loss term for the system. For stable soliton solutions to exist the gain parameter g needs only to be sufficiently large to counter the two loss terms.

The evolution of the pulse peak for different values of the gain parameter g is shown in Fig. 1. When $g=0.1$ the pulse vanishes quickly due to excessive loss with no noticeable oscillatory behavior; the pulse simply collapses, making this a damped evolution. When $g=0.2, 0.3$, due to the loss in the system the pulse initially undergoes a relative to its amplitude sharp decrease. However, it rapidly recovers and evolves into a stable solution. Similar to the damped evolution, the amplitude is initially decreased but the resulting evolution is stable. Interestingly enough, when $g=0.7, 1$, and the perturbations can no longer be considered small, a stable evolution is again obtained, although somewhat different from the case above. Now with excessive gain in the system, the pulse amplitude increases and the steady state is rapidly reached. The only major difference between the modes is the resulting amplitude and the width of the pulse, i.e., for larger

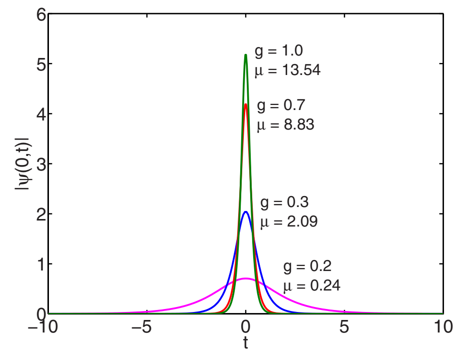

FIG. 2. (Color online) Solutions of Eq. (1) for different values of the gain parameter g and the corresponding propagation constants.

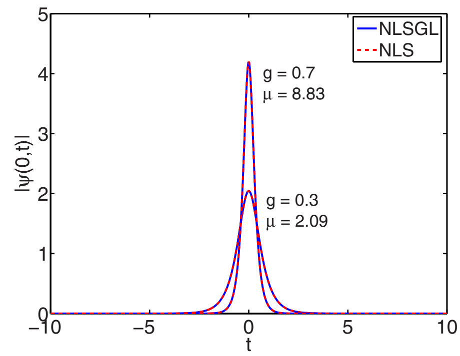

FIG. 3. (Color online) Solitons of the perturbed and unperturbed equations.

g the pulse is larger and narrower (see Fig. 2).

The above suggest that in the NLSGL model, the mode locking effect is always present for $g > g^*$, a critical gain value. Without enough gain, i.e., $g \leqslant g^*$, pulses dissipate to the trivial zero state. Furthermore, there is no complex radiation states or states whose amplitudes grow without bound for any choice of parameters studied. In terms of solutions, Eq. (1) admits soliton states for all values of $g > g^* \geqslant l$ (recall, here $l=0.1$). This has also been verified by analytical methods (soliton perturbation theory), but due to the lack of space the details will be presented elsewhere.

The existence of modes, namely, solutions of Eq. (1), is examined next. We employ the spectral renormalization method introduced in Ref. [20]. This is a spectrally accurate iterative method that in each iteration modifies the ratio between the dispersive and nonlinear parts of the equation until convergence is achieved. Assuming localized solutions of the form $\psi(z,t)=u(t)\exp(i\mu z)$, $u(\pm\infty)\to 0$, we obtain

$$-\mu u + \frac{1}{2}u_{tt} + |u|^2 u = Q[u],$$

which is a nonlinear eigenvalue problem with respect to the propagation constant μ. Taking the Fourier transform (FT) of the equation and setting $u=\lambda v \Leftrightarrow \hat{u}=\lambda\hat{v}$ results in

$$-(\mu+\omega^2/2)\hat{v} = \mathcal{F}\{Q[\lambda_n v_n]/\lambda_n - |\lambda_n v_n|^2 v_n\}.$$

The usual definition of the FT is used, namely,

$$\hat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{+\infty} f(t)\exp(i\omega t)dt.$$

The iterative scheme is

$$\hat{v}_{n+1} = -\frac{\mathcal{F}\{Q[\lambda_n v_n]/\lambda_n - |\lambda_n v_n|^2 v_n\}}{\mu+\omega^2/2} \tag{2}$$

and the renormalization parameter λ_n, in each iteration, is defined by the roots of the algebraic equation

$$\int_{-\infty}^{+\infty}(\mu+\omega^2/2)|\hat{v}_n|^2 d\omega + \int_{-\infty}^{+\infty}(\mathcal{F}\{Q[\lambda_n v_n]/\lambda_n - |\lambda_n v_n|^2 v_n\})\hat{v}_n^* d\omega = 0. \tag{3}$$

Starting with a general initial guess for $v(t)$, say, $v_0 = \exp(-At^2)$, we can construct solutions using Eqs. (2) and (3). This initial guess can be important in the convergence of the method. Equation (3) cannot be solved analytically for λ, and thus a numerical method, say Newton iteration, must be used. The convergence of the Newton iteration depends on the initial guess. If the method seems not to be converging at the first iteration for n and Newton's method cannot find a reasonable λ, then an exponential with different width should be considered; e.g., take $A<1$ at $n=0$.

Further, to find these solutions, we must first determine the appropriate value(s) of the propagation constant μ for which a solution actually exists. The criterion for determining μ is that $\mathrm{Im}\{\lambda_n\}=0$. With this additional requirement we obtain only one value of μ, for a specific set of parameters, that a solution exists. Also, when $g \leqslant l$, in Eq. (1), we do not find a solution, i.e., we do not find a value of μ for which the above iteration will converge. This is consistent with the observation that when the effect of loss is stronger than the gain the only acceptable solution is the trivial solution. Solutions of the NLSGL equation for various values of g and the corresponding propagation constant are depicted in Fig. 2. Notice the change in the pulse width and amplitude. As the gain parameter increases so does the amplitude and the pulse becomes narrower. The energy and the amplitude of the pulse increases with g. In fact, the energy changes according to $E \sim \sqrt{\mu}$. Indeed, from soliton theory of the classical NLS equation this is exactly the way a classical soliton's energy changes. The key difference being that in the pure NLS a semi-infinite set of μ exists, whereas now μ is unique for the given set of parameters. Hence one expects that the solutions of the two equations, NLSGL and NLS, are comparable. In Fig. 3 we plot the two solutions for different values of g. In each case the same value of μ is used. The amplitudes match so closely that they are indistinguishable in the figure, meaning the perturbing effect is strictly the mode locking mechanism, i.e., its effect is to mode lock to a soliton of the pure NLS with the appropriate propagation constant. The solitons of the unperturbed NLS system are well known in closed analytical form, i.e., they are expressed in terms of the hyperbolic secant function, $\psi = \sqrt{2\mu}\,\mathrm{sech}(\sqrt{2\mu}t)\exp(-i\mu z)$, and therefore describe solitons of the NLSGL to a good approximation.

This constraint on the propagation constant seems to relax for large values of μ. Indeed, as μ grows larger the energy and power of the pulse also increase resulting in a better approximation to a Hamiltonian system, the unperturbed NLS equation. In this regime approximate pulse solutions which are obtained by weakening the accuracy requirement in the mode finding algorithm exist for wide ranges of μ, we refer to these solutions as quasisolitons.

Another type of pulse that has recently attracted much attention are the pulses propagating in a ML laser in the normal dispersion regime [15] (the pulses sometimes grow self-similarly depending on the gain and/or loss mechanism [21]). Indeed we find mode-locked pulse solutions of Eq. (1) with $d_0 < 0$. The main difference these pulses exhibit from those in the anomalous regime is that they are now highly chirped, very wide (almost parabolic) pulses that *do not* correspond to the solution of the unperturbed equation. A typical mode locking evolution for these pulses is shown in Fig. 4. In the second part of the figure we compare two typical solitons corresponding to the same parameters with $g=1$ and $d_0=-1$. We will discuss the pulse dynamics and properties in the normal regime in a future communication [17].

To conclude, we have analyzed a distributive model system which can be used in the study of pulse propagation in mode-locked lasers. The results of the pulse dynamics in the anomalous regime are simple: pulses are either damped, i.e., decay to zero, or their evolution leads to mode locking. Complicated evolution (chaotic, radiation, or strong growth) is not observed for a wide range of the parameters even when the perturbations cannot be considered small at the initial instant. The saturated (energy) gain and filtering, and saturated (power) loss, though crucial to the mode-locking mechanism, after evolution they are only found to be perturbative effects. The resulting modes are essentially the modes of the unperturbed NLS system, i.e., hyperbolic secants, for the same corresponding propagation constant.

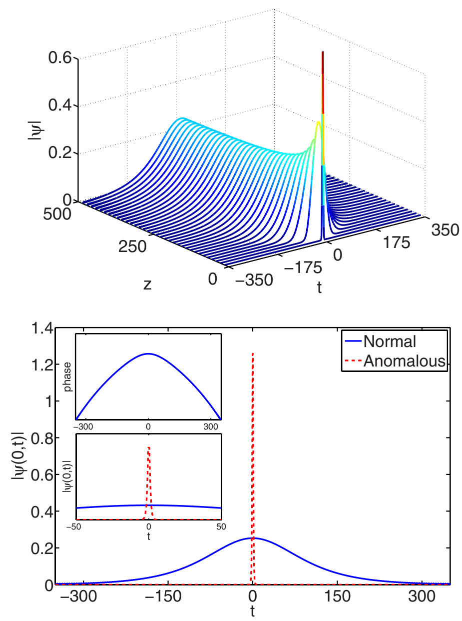

FIG. 4. (Color online) Top: Evolution of an arbitrary unit Gaussian under the NLSGL in the normal regime ($d_0=-1$). Bottom: Comparison with the corresponding mode of the anomalous equation. In the insets the phase of the chirped pulse and a zoom-in of the two solutions are shown. The soliton of the NLSGL equation in the anomalous regime is essentially chirp-free.

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