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Christos G. Philos; Yiannis G. Sficas

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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SECOND AND THIRD ORDER RETARDED DIFFERENTIAL EQUATIONS

CH. G. PHILOS and Y. G. SFICAS, Ioannina

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0. INTRODUCTION

In this paper we deal with second and third order retarded differential equations and give some results on their oscillatory and asymptotic behavior which are analogous to the "Kneser-type" ones for second and third order ordinary differential equations. More precisely, we consider the linear retarded differential equations

$$(E_1)\qquad [r(t)\,x'(t)]' + p(t)\,x[g(t)] = 0\,,\quad t \geqq t_0\,,$$

$$(E_2)\qquad [r(t)\,[r(t)\,x'(t)]']' + p(t)\,x[g(t)] = 0\,,\quad t \geqq t_0\,,$$

where r *is a positive continuous function on the interval* $[t_0, \infty)$ *with*

$$\int^{\infty} \frac{dt}{r(t)} = \infty\,,$$

p *is a nonnegative continuous function on* $[t_0, \infty)$ *and* g *is a continuously differentiable and increasing function on* $[t_0, \infty)$ *such that*

$$\lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g(t) \leqq t \quad \textit{for every} \quad t \geqq t_0\,.$$

We also consider the (not necessarily linear) retarded differential equations

$$(E_1')\qquad [r(t)\,x'(t)]' + p(t)\,\Phi(x[g(t)]) = 0\,,$$

$$(E_2')\qquad [r(t)\,[r(t)\,x'(t)]']' + p(t)\,\Phi(x[g(t)]) = 0\,,$$

where Φ *is a continuous function which is defined at least on* $\mathbb{R} - \{0\}$ ($\mathbb{R}$ is the real line) *and has the sign property*

$$y \neq 0 \Rightarrow y\,\Phi(y) > 0\,.$$

Sufficient smoothness for the existence of solutions of the above differential equations which are defined for all large t will be assumed without mention. In what follows, we consider only such solutions $x(t)$ which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function on an interval $[T, \infty)$ is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*.

For the sake of brevity, we use the notations

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}, \quad t \geqq t_0, \quad P(t) = \int_{t}^{\infty} p(s)\, ds, \quad t \geqq t_0,$$

in the case where $\int^{\infty} p(t)\, dt < \infty$, and

$$S_{\Phi} = \max \left\{ \limsup_{y \to \infty} \frac{y}{\Phi(y)}, \ \limsup_{y \to -\infty} \frac{y}{\Phi(y)} \right\}.$$

1. OSCILLATION OF SECOND ORDER RETARDED DIFFERENTIAL EQUATIONS

In this section we deal with the oscillation of the solutions of the second order retarded differential equations (E_1) and (E_1'). It is known (cf. [3, 4]) that *the condition*

$$(H_1) \qquad \int^{\infty} \frac{1}{r(t)} P(t)\, dt = \infty$$

is sufficient for all bounded solutions of (E_1) [*or, more generally, of* (E_1')] *to be oscillatory.* Moreover (cf. [6]), *under the condition*

$$\int^{\infty} p(t)\, R[g(t)]\, dt = \infty,$$

for every nonoscillatory solution x of (E_1) *we have* $\lim_{t \to \infty} r(t)\, x'(t) = 0$. In the present, sufficient conditions for all solutions of (E_1) or (E_1') to be oscillatory are established.

Theorem 1. *The condition*

$$(C_1) \qquad \liminf_{t \to \infty} P(t)\, R[g(t)] > \tfrac{1}{4}$$

is sufficient for all solutions of (E_1) *to be oscillatory.*

Proof. We observe that (C_1) implies (H_1) and hence it is enough to prove that (E_1) does not have any unbounded nonoscillatory solutions. Moreover, because of the linearity of the equation (E_1), with respect to the nonoscillatory solutions of this equation we can confine our discussion only to the positive ones.

Let x be a positive unbounded solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of the equation (E_1) and let $T \geqq T_0$ be chosen so that

$$g(t) \geqq T_0 \quad \text{for every} \quad t \geqq T.$$

Then from (E_1) it follows that $(rx')'$ is nonpositive on $[T, \infty)$. Moreover, $(rx')'$ is not identically zero on any interval of the form $[T', \infty)$, $T' \geqq T$, since (C_1) ensures that the same holds for the function p. Thus, by the fact that $\int^\infty [1/r(t)]\, dt = \infty$, we can easily verify that x' is positive on $[T, \infty)$. Furthermore, we observe that (C_1) implies $\int^\infty p(t)\, R[g(t)]\, dt = \infty$ and consequently we always have $\lim_{t\to\infty} r(t)\, x'(t) = 0$. So, (E_1) gives

$$r(t)\, x'(t) = \int_t^\infty p(s)\, x[g(s)]\, ds \quad \text{for all} \quad t \geqq T.$$

Now, let K be the set of all $k > 0$ for which there exists a $T_k \geqq T$ such that the function $X_k = x/R^k$ is increasing on $[T_k, \infty)$. Since

$$\lim_{t\to\infty} \frac{x(t)}{R(t)} = \lim_{t\to\infty} r(t)\, x'(t) = 0,$$

we always have $k < 1$ for any $k \in K$. Moreover, K is not empty. Indeed, by (C_1), we choose a $T^* \geqq T$ such that for every $t \geqq T^*$

$$g(t) \geqq T \quad \text{and} \quad P(t)\, R[g(t)] \geqq \tfrac{1}{4}.$$

Then for $t \geqq T^*$ we obtain

$$r[g(t)]\, x'[g(t)] \geqq r(t)\, x'(t) = \int_t^\infty p(s)\, x[g(s)]\, ds$$

$$\geqq P(t)\, x[g(t)] \geqq \frac{1}{4} \frac{x[g(t)]}{R[g(t)]}.$$

Thus, because of the assumptions on g, we have

$$r(t)\, x'(t) \geqq \frac{1}{4} \frac{x(t)}{R(t)} \quad \text{for every} \quad t \geqq T^*.$$

By using this inequality, it is easy to see that the function $X_{1/4}$ has a nonnegative derivative on $[T^*, \infty)$, which means that $\frac{1}{4} \in K$.

Next, let k be an arbitrary number in K and let $T_k \geqq T$ be such that the function X_k is increasing on $[T_k, \infty)$. By (C_1), we choose a $T_k^* \geqq T_k$ so that for every $t \geqq T_k^*$

$$g(t) \geqq T_k \quad \text{and} \quad P(t)\, R[g(t)] \geqq c,$$

where c is a number with $c > \frac{1}{4}$. Then for $t \geqq T_k^*$ we get

$$r[g(t)]\, X_k'[g(t)]\, R^k[g(t)] + kX_k[g(t)]\, R^{k-1}[g(t)] = r[g(t)]\, x'[g(t)] \geqq$$

$$\geqq r(t)\, x'(t) = \int_t^\infty p(s)\, x[g(s)]\, ds = \int_t^\infty p(s)\, X_k[g(s)]\, R^k[g(s)]\, ds \geqq$$

$$\geqq X_k[g(t)] \int_t^\infty p(s)\, R^k[g(s)]\, ds \geqq c^k X_k[g(t)] \int_t^\infty p(s)\, P^{-k}(s)\, ds =$$

$$= -c^k X_k[g(t)] \int_t^\infty P^{-k}(s)\, \mathrm{d}P(s) = \frac{c^k}{1-k} X_k[g(t)]\, P^{1-k}(t) \geqq$$

$$\geqq \frac{c}{1-k} X_k[g(t)]\, R^{k-1}[g(t)] .$$

So, for every $t \geqq T_k^*$ we have

$$r[g(t)]\, X_k'[g(t)] \geqq \left(\frac{c}{1-k} - k\right) \frac{X_k[g(t)]}{R[g(t)]},$$

which implies that

$$r(t)\, X_k'(t) \geqq \left(\frac{c}{1-k} - k\right) \frac{X_k(t)}{R(t)} \quad \text{for} \quad t \geqq T_k^* .$$

Since $c > \frac{1}{4}$, the minimum $2\sqrt{(c)} - 1 = m$ of the function

$$\sigma(\theta) = \frac{c}{1-\theta} - \theta, \quad 0 < \theta < 1$$

is positive and we have

$$r(t)\, X_k'(t) \geqq m \frac{X_k(t)}{R(t)} \quad \text{for every} \quad t \geqq T_k^* .$$

By this inequality, we can easily verify that the function $X_{k+m} = X_k/R^m$ has a nonnegative derivative on $[T_k^*, \infty)$, which means that $k + m$ belongs in K. But, as k can be chosen arbitrarily close to $\sup K$ and m is positive and independent of the choise of k, this is a contradiction.

Remark 1. Theorem 1 generalizes recent results in [2] and [7] concerning the special case where $r = 1$. Also, in the case of ordinary differential equations this theorem leads to a well-known classical oscillation result due to Hille [1] (cf. also Swanson [8, p. 45]). The method which we have used in the proof of Theorem 1 patterns after that in [7].

Theorem 1'. *The condition*

(C_1') $$\liminf_{t\to\infty} P(t)\, R[g(t)] > \tfrac{1}{4} S_\Phi$$

is sufficient for all solutions of (E_1') *to be oscillatory.*

Proof. Since (C_1') implies (H_1), it suffices to prove that (E_1') has not unbounded nonoscillatory solutions. Furthermore, the substitution $z = -x$ transforms (E_1') into the equation

$$[r(t)\, z'(t)]' + p(t)\, \hat{\Phi}(z[g(t)]) = 0 ,$$

where $\hat{\Phi}(y) = -\Phi(-y)$ for all y in the domain of Φ. The transformed equation inherits from (E_1') all conditions possed. Hence, with respect to the nonoscillatory solutions of the equation (E_1') we can restrict our attention only to the positive ones.

Let x be a positive unbounded solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the equation (E_1') and let $T \geqq T_0$ be chosen so that

$$g(t) \geqq T_0 \quad \text{for every} \quad t \geqq T.$$

Then it is obvious that the restriction of x on the interval $[T, \infty)$ is a (positive) solution (on this interval) of the linear equation

$$[r(t)\, w'(t)]' + \tilde{p}(t)\, w[g(t)] = 0\,, \quad t \geqq t_0\,,$$

where

$$\tilde{p}(t) = \begin{cases} p(t)\, \dfrac{\Phi(x[g(t)])}{x[g(t)]}\,, & \text{if} \quad t \geqq T \\ \tilde{p}(T)\,, \quad \text{if} \quad t_0 \leqq t \leqq T\,. \end{cases}$$

Thus, by Theorem 1, we must have

$$\liminf_{t\to\infty} R[g(t)] \int_t^\infty \tilde{p}(s)\, ds \leqq \tfrac{1}{4}\,.$$

From (E_1') it follows that x is increasing on $[T, \infty)$. So, if we choose a $T^* \geqq T$ so that

$$g(t) \geqq T \quad \text{for every} \quad t \geqq T^*\,,$$

then for $t \geqq T^*$ we obtain

$$P(t) = \int_t^\infty \tilde{p}(s) \frac{x[g(s)]}{\Phi(x[g(s)])}\, ds \leqq \left[\sup_{s \geqq t} \frac{x[g(s)]}{\Phi(x[g(s)])}\right] \int_t^\infty \tilde{p}(s)\, ds \leqq$$

$$\leqq \left[\sup_{y \geqq x[g(t)]} \frac{y}{\Phi(y)}\right] \int_t^\infty \tilde{p}(s)\, ds\,.$$

Hence, because of $\lim_{t\to\infty} x(t) = \infty$, we get

$$\liminf_{t\to\infty} P(t)\, R[g(t)] \leqq \left[\limsup_{y\to\infty} \frac{y}{\Phi(y)}\right] \left[\liminf_{t\to\infty} R[g(t)] \int_t^\infty \tilde{p}(s)\, ds\right] \leqq$$

$$\leqq \frac{1}{4} \limsup_{y\to\infty} \frac{y}{\Phi(y)} \leqq \frac{1}{4} S_\Phi\,,$$

which contradicts (C_1').

Remark 2. Suppose that $S_\Phi < \infty$. Then (cf. [5]) the condition $\int^\infty p(t)\, dt = \infty$ is also sufficient for all solutions of (E_1') to be oscillatory.

Remark 3. It is known (cf. [5]) that the condition

$$(\hat{C}_1') \qquad \limsup_{t\to\infty} P(t)\, R[g(t)] > S_\Phi$$

is also sufficient for all solutions of (E_1') to be oscillatory. We note here that it is possible to have the condition (C_1') valid while $(\hat{C}_1')$ fails. For example, in the case of the equation

$$x''(t) + \frac{1}{t^2} x\left(\frac{t}{2}\right) = 0, \quad t \geqq 1$$

the condition (C_1') holds while $(\hat{C}_1')$ is not satisfied.

2. OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF THIRD ORDER RETARDED DIFFERENTIAL EQUATIONS

Here, we are concerned with the oscillatory and asymptotic behavior of the solutions of the third order retarded differential equations (E_2) and (E_2'). It is known (cf. [3, 4]) that, *under the condition*

$$(H_2) \quad \textit{either} \quad \int^{\infty} \frac{1}{r(t)} P(t)\, dt = \infty \quad \textit{or} \quad \int^{\infty} \frac{1}{r(t)} \int_t^{\infty} \frac{1}{r(s)} P(s)\, ds\, dt = \infty,$$

every bounded solution x of (E_2) [or, more generally, of (E_2')] is oscillatory or such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} r(t)\, x'(t) = \lim_{t\to\infty} r(t)\, [r(t)\, x'(t)]' = 0 \quad \textit{monotonically}.$$

Moreover (cf. [6]), *if*

$$(H_3) \quad \textit{either} \quad \int^{\infty} R[g(t)]\, p(t)\, dt = \infty \quad \textit{or} \quad \int^{\infty} \frac{1}{r(t)} \int_t^{\infty} R[g(s)]\, p(s)\, ds\, dt = \infty,$$

then for every unbounded nonoscillatory solution x of (E_2) we have

$$\lim_{t\to\infty} [x(t)/R(t)] = \pm\infty.$$

In this section, we shall give conditions, under which every solution of (E_2) or (E_2') is oscillatory or tending to zero at ∞. For this purpose, we need the following lemma.

Lemma. *Let u be a nonnegative function on an interval $[\tau, \infty)$, $\tau \geqq t_0$, such that $(r(ru')')'$ exists on $[\tau, \infty)$. Suppose that*

$$u' \geqq 0 \quad \textit{and} \quad (r(ru')')' \leqq 0 \quad \textit{on} \quad [\tau, \infty).$$

Then the function

$$U(t) = u(t)\left[\int_\tau^t \frac{ds}{r(s)}\right]^{-2}, \quad t > \tau$$

is decreasing.

Proof. By our assumptions on u and the fact that $\int^{\infty} [1/r(t)]\, dt = \infty$, it is easy to see that $(ru')'$ is nonnegative on $[\tau, \infty)$. Let

$$G(t) = u(t) \int_\tau^t \frac{ds}{r(s)} - \frac{1}{2} r(t)\, u'(t) \left[\int_\tau^t \frac{ds}{r(s)}\right]^2, \quad t \geqq \tau .$$

For every $t \geqq \tau$ we obtain

$$r(t)\, G'(t) = u(t) - \frac{1}{2} r(t) \left[r(t)\, u'(t)\right]' \left[\int_\tau^t \frac{ds}{r(s)}\right]^2 =$$

$$= u(\tau) + r(\tau)\, u'(\tau) \int_\tau^t \frac{ds}{r(s)} + \int_\tau^t \frac{1}{r(s)} \int_\tau^s \frac{1}{r(w)} \left[r(w) \left[r(w)\, u'(w)\right]'\right] dw\, ds -$$

$$- \frac{1}{2} r(t) \left[r(t)\, u'(t)\right]' \left[\int_\tau^t \frac{ds}{r(s)}\right]^2 \geqq$$

$$\geqq r(t) \left[r(t)\, u'(t)\right]' \left\{\int_\tau^t \frac{1}{r(s)} \int_\tau^s \frac{1}{r(w)}\, dw\, ds - \frac{1}{2} \left[\int_\tau^t \frac{ds}{r(s)}\right]^2\right\} = 0 .$$

Thus, G is increasing on $[\tau, \infty)$. Since $G(\tau) = 0$, we have that G is nonnegative on $[\tau, \infty)$ and consequently

$$u(t) \geqq \frac{1}{2} r(t)\, u'(t) \int_\tau^t \frac{ds}{r(s)} \quad \text{for every} \quad t > \tau .$$

By using this inequality, we can easily see that the function U has a nonpositive derivative on (τ, ∞), which proves the lemma.

Theorem 2. *Under the condition*

$$(\mathrm{C}_2) \qquad \min \left\{\liminf_{t\to\infty} P(t)\, R^2[g(t)],\ \frac{1}{2} \liminf_{t\to\infty} r(t)\, p(t)\, R(t)\, R^2[g(t)]\right\} > \frac{1}{3\sqrt{3}},$$

every solution x of (E_2) *is oscillatory or such that*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} r(t)\, x'(t) = \lim_{t\to\infty} r(t) \left[r(t)\, x'(t)\right]' = 0 \quad \textit{monotonically} .$$

Proof. Condition (C_2) implies (H_2) and so it is enough to prove the nonexistence of unbounded nonoscillatory solutions of (E_2). Moreover, for the study of the non-oscillatory solutions of (E_2) it suffices to deal only with the positive ones.

Let x be a positive unbounded solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of the equation (E_2) and let $\tau \geqq T_0$ be chosen so that

$$g(t) \geqq T_0 \quad \text{for every} \quad t \geqq \tau .$$

Then the function $(r(rx')')'$ is nonpositive on $[\tau, \infty)$. Furthermore, this function is not identically zero on any interval of the form $[\tau', \infty)$, $\tau' \geqq \tau$, since, because of (C_2), the same is valid for p. Thus, by $\int^{\infty} [1/r(t)]\, dt = \infty$, $(rx')'$ is positive on $[\tau, \infty)$.

Moreover, x' is eventually positive. We suppose, without loss of generality, that x' is positive on the whole interval $[\tau, \infty)$. We remark that (E_2) gives

$$r(t)\,[r(t)\,x'(t)]' \geqq \int_t^\infty p(s)\,x[g(s)]\,\mathrm{d}s \quad \text{for every} \quad t \geqq \tau\,.$$

We put

$$v(t) = R(t)\,r(t)\,x'(t) - x(t)\,, \quad t \geqq \tau\,.$$

Then

$$v'(t) = R(t)\,[r(t)\,x'(t)]' > 0 \quad \text{for} \quad t \geqq \tau$$

and consequently v is either negative on $[\tau, \infty)$ or eventually positive. The case where $v < 0$ on $[\tau, \infty)$ is impossible. Indeed, in this case

$$\left[\frac{x(t)}{R(t)}\right]' = \frac{v(t)}{r(t)\,R^2(t)} < 0 \quad \text{for all} \quad t \geqq \tau$$

and hence

$$\lim_{t\to\infty} [x(t)/R(t)] < \infty\,.$$

This is a contradiction, since condition (C_2) implies (H_3). Thus, v is eventually positive, i.e.

$$R(t)\,r(t)\,x'(t) - x(t) > 0 \quad \text{for all large } t\,.$$

Now, by (C_2), we consider two constants c_1, c with $2/3\,\sqrt{3} < c_1 < c < 2$ and

$$\min\{2 \liminf_{t\to\infty} P(t)\,R^2[g(t)],\ \liminf_{t\to\infty} r(t)\,p(t)\,R(t)\,R^2[g(t)]\} > c\,.$$

Furthermore, we choose a $T > \tau$ so that for every $t \geqq T$

$$g(t) > \tau\,,$$

$$\left[\int_\tau^{g(t)} \frac{\mathrm{d}s}{r(s)}\right]^2 R^{-2}[g(t)] \geqq \frac{c_1}{c}\,,$$

$$\min\{2\,P(t)\,R^2[g(t)],\ r(t)\,p(t)\,R(t)\,R^2[g(t)]\} \geqq c\,,$$

$$R(T)\,r(T)\,x'(T) - x(T) > 0\,.$$

Let k be a number with

$$1 < k < \frac{1}{2} + \frac{c}{4} + \frac{1}{2}\sqrt{\left(1 + \frac{c^2}{4} + 3c\right)}$$

and such that

$$R(T)\,r(T)\,x'(T) - k\,x(T) > 0\,.$$

We shall prove that the function $X_k = x/R^k$ is increasing on $[T, \infty)$. To do this, we first remark that

$$\lim_{t\to\infty} x[g(t)]\,P(t) = 0\,,$$

since for every $t \geqq T$

$$x[g(t)]\, P(t) = x[g(t)] \int_t^\infty p(s)\, ds \leqq \int_t^\infty p(s)\, x[g(s)]\, ds\,.$$

So, for every $t \geqq T$ we obtain

$$r[g(t)]\,(rX_k')'\,[g(t)]\, R^k[g(t)] + 2kr[g(t)]\, X_k'[g(t)] R^{k-1}[g(t)] +$$
$$+ k(k-1)\, X_k[g(t)]\, R^{k-2}[g(t)] = r[g(t)]\,(rx')'\,[g(t)] \geqq r(t)\,[r(t)\, x'(t)]' \geqq$$
$$\geqq \int_t^\infty p(s)\, x[g(s)]\, ds = -\int_t^\infty x[g(s)]\, dP(s) =$$
$$= -x[g(s)]\, P(s)|_t^\infty + \int_t^\infty P(s)\, x'[g(s)]\, g'(s)\, ds =$$
$$= -\lim_{s\to\infty} x[g(s)]\, P(s) + P(t)\, x[g(t)] + \int_t^\infty P(s)\, x'[g(s)]\, g'(s)\, ds =$$
$$= P(t)\, x[g(t)] + \int_t^\infty P(s)\, x'[g(s)]\, g'(s)\, ds \geqq$$
$$\geqq P(t)\, x[g(t)] + r[g(t)]\, x'[g(t)] \int_t^\infty \frac{P(s)}{r[g(s)]}\, g'(s)\, ds \geqq$$
$$\geqq \frac{c}{2} R^{-2}[g(t)]\, x[g(t)] + \frac{c}{2} r[g(t)]\, x'[g(t)] \int_t^\infty \frac{R^{-2}[g(s)]}{r[g(s)]}\, g'(s)\, ds =$$
$$= \frac{c}{2} r[g(t)]\, X_k'[g(t)]\, R^{k-1}[g(t)] + \frac{c}{2}(k+1)\, X_k[g(t)]\, R^{k-2}[g(t)]\,.$$

Thus, for $t \geqq T$ we have

$$r(t)\,[r(t)\, X_k'(t)]' + \left(2k - \frac{c}{2}\right)\frac{r(t)\, X_k'(t)}{R(t)} + \left[k^2 - \left(1 + \frac{c}{2}\right)k - \frac{c}{2}\right] X_k(t)\, R^{-2}(t) \geqq 0\,.$$

It is easy to see that $k^2 - (1 + \frac{1}{2}c)\, k - \frac{1}{2}c \leqq 0$ and therefore

(1) $$[r(t)\, X_k'(t)]' + \left(2k - \frac{c}{2}\right)\frac{X_k'(t)}{R(t)} \geqq 0 \quad \text{for all} \quad t \geqq T.$$

If $\tilde{t} \geqq T$ is such that the function rX_k' takes a local minimum at $\tilde{t}$, then $(rX_k')'\,(\tilde{t}) = 0$ and so, since $2k - \frac{1}{2}c > 0$, the above inequality gives $X_k'(\tilde{t}) \geqq 0$. Moreover,

$$X_k'(T) = [R(T)\, r(T)\, x'(T) - k\, x(T)]/r(T)\, R^{k+1}(T) > 0\,.$$

Thus, X_k' is either nonnegative on $[T, \infty)$ or such that for some $\tilde{T} > T$

$$X_k'(\tilde{T}) = 0 \quad \text{and} \quad X_k'(t) < 0 \quad \text{for} \quad t > \tilde{T}.$$

But, the latter case is impossible, since then (1) gives

$$[r(t)\,X_k'(t)]' > 0 \quad \text{for all} \quad t > \tilde{T},$$

which is a contradiction. We have thus proved that $X_k' \geqq 0$ on $[T, \infty)$, which means that X_k is increasing on $[T, \infty)$.

Next, we consider the set K of all numbers k, $1 < k < 2$, for which the function $X_k = x/R^k$ is increasing on $[T_k, \infty)$ for some $T_k \geqq T$. We observe that the set K is nonempty and we put $k_0 = \sup K$.

Let k be an arbitrary number in K and $T_k \geqq T$ be such that $X_k' \geqq 0$ on $[T_k, \infty)$.

We shall prove that the function rX_k' is decreasing on $[T_k', \infty)$ for some $T_k' \geqq T_k$. To this end, by using the lemma, for every $t \geqq T_k$ we get

$$[r(t)\,[r(t)\,X_k'(t)]']'\,R^k(t) + 3k[r(t)\,X_k'(t)]'\,R^{k-1}(t) + 3k(k-1)\,X_k'(t)\,R^{k-2}(t) +$$

$$+ k(k-1)(k-2)\frac{X_k(t)}{r(t)}\,R^{k-3}(t) = [r(t)\,[r(t)\,x'(t)]']' = -p(t)\,x[g(t)] \leqq$$

$$\leqq -c\,\frac{1}{r(t)\,R(t)}\,x[g(t)]\,R^{-2}[g(t)] =$$

$$= -c\,\frac{1}{r(t)\,R(t)}\left\{\left[\int_\tau^{g(t)} \frac{ds}{r(s)}\right]^2 R^{-2}[g(t)]\right\}\left\{x[g(t)]\left[\int_\tau^{g(t)} \frac{ds}{r(s)}\right]^{-2}\right\} \leqq$$

$$\leqq -c_1\,\frac{1}{r(t)\,R(t)}\,x(t)\left[\int_\tau^t \frac{ds}{r(s)}\right]^{-2} \leqq -c_1\,\frac{X_k(t)}{r(t)}\,R^{k-3}(t)\,.$$

That is, for all $t \geqq T_k$ it holds

$$[r(t)\,[r(t)\,X_k'(t)]']'\,R^3(t) + 3k[r(t)\,X_k'(t)]'\,R^2(t) + 3k(k-1)\,X_k'(t)\,R(t) +$$

$$+ [k(k-1)(k-2) + c_1]\,\frac{X_k(t)}{r(t)} \leqq 0\,.$$

We remark that the maximum of the function

$$\sigma(\theta) = -\theta(\theta-1)(\theta-2)\,, \quad 1 \leqq \theta \leqq 2$$

is $2/3\,\sqrt{3}$ which is less than c_1. Hence we have

$$k(k-1)(k-2) + c_1 \geqq 0\,.$$

Moreover, we observe that $k(k-1) > 0$. Thus,

$$(2) \qquad [r(t)\,[r(t)\,X_k'(t)]']'\,R(t) + 3k[r(t)\,X_k'(t)]' \leqq 0\,, \quad t \geqq T_k\,.$$

This inequality implies that $(rX_k')'$ is eventually nonpositive.

To prove this assertion, we first assume that $(rX_k')' > 0$ on some interval $[\tau_k, \infty)$,

$\tau_k \geqq T_k$. Then

$$r(t)\,X_k'(t) > r(\tau_k)\,X_k'(\tau_k) \quad \text{for} \quad t > \tau_k$$

and consequently there exist a $\tau_k^* > \tau_k$ and a positive constant α such that

$$r(t)\,X_k'(t) \geqq \alpha \quad \text{for all} \quad t \geqq \tau_k^* .$$

So,

$$X_k(t) - X_k(\tau_k^*) \geqq \alpha \int_{\tau_k^*}^{t} \frac{ds}{r(s)}, \quad t \geqq \tau_k^* ,$$

which gives

$$\liminf_{t\to\infty} \frac{x(t)}{R^{k+1}(t)} \equiv \liminf_{t\to\infty} \frac{X_k(t)}{R(t)} \geqq \alpha > 0 .$$

But, by the lemma and the fact that $k > 1$, we have

$$\lim_{t\to\infty} \frac{x(t)}{R^{k+1}(t)} = 0 ,$$

i.e. a contradiction. Next, if the function $r(rX_k')'$ takes a local maximum at $\bar{t} \geqq T_k$, then $(r(rX_k')')'(\bar{t}) = 0$, and hence (2) gives $(rX_k')'(\bar{t}) \leqq 0$. We have thus proved that the only possible case is that $(rX_k')' \leqq 0$ on $[T_k', \infty)$ for some $T_k' \geqq T_k$.

Now, we choose a $T_k^* \geqq T_k'$ so that

$$g(t) \geqq T_k' \quad \text{for all} \quad t \geqq T_k^* .$$

Then for every $t \geqq T_k^*$ we obtain

$$r[g(t)]\,(rX_k')'\,[g(t)]\,R^k[g(t)] + 2kr[g(t)]\,X_k'[g(t)]\,R^{k-1}[g(t)] +$$
$$+ k(k-1)\,X_k[g(t)]\,R^{k-2}[g(t)] = r[g(t)]\,(rx')'\,[g(t)] \geqq r(t)\,[r(t)\,x'(t)]' \geqq$$
$$\geqq \int_t^\infty p(s)\,x[g(s)]\,ds = P(t)\,x[g(t)] + \int_t^\infty P(s)\,x'[g(s)]\,g'(s)\,ds =$$
$$= P(t)\,X_k[g(t)]\,R^k[g(t)] + \int_t^\infty P(s)\,X_k'[g(s)]\,R^k[g(s)]\,g'(s)\,ds +$$
$$+ k\int_t^\infty P(s)\,\frac{X_k[g(s)]}{r[g(s)]}\,R^{k-1}[g(s)]\,g'(s)\,ds \geqq \frac{c}{2}\,X_k[g(t)]\,R^{k-2}[g(t)] +$$
$$+ \frac{kc}{2}\,X_k[g(t)]\int_t^\infty \frac{R^{k-3}[g(s)]}{r[g(s)]}\,g'(s)\,ds = \frac{c}{2}\left(1 - \frac{k}{k-2}\right)X_k[g(t)]\,R^{k-2}[g(t)].$$

Therefore we derive that

$$r(t)\,X_k'(t) \geqq \frac{k^3 - 3k^2 + 2k + c}{2k(2-k)}\,\frac{X_k(t)}{R(t)} \quad \text{for} \quad t \geqq T_k^* .$$

Since the minimum of the function

$$\sigma_1(\theta) = \theta^3 - 3\theta^2 + 2\theta + c\,, \quad 1 \leqq \theta \leqq 2$$

is $c - 2/3\sqrt{3} > 0$, we get

$$r(t)\,X_k'(t) \geqq m\,\frac{X_k(t)}{R(t)} \quad \text{for every} \quad t \geqq T^*\,,$$

where

$$m = \frac{1}{2}\left(c - \frac{2}{3\sqrt{3}}\right).$$

By the last inequality, it is easy to see that the function $X_{k+m} = X_k/R^m$ is increasing on $[T_k^*, \infty)$. This, as k can be chosen arbitrarily close to k_0, is a contradiction.

Remark 4. Theorem 2 generalizes a recent result in [7] concerning the particular case where $r = 1$. The technique used here in proving Theorem 2 patterns after that in [7].

Theorem 2'. *Under the condition*

$$(\mathrm{C}_2') \qquad \min\{\liminf_{t\to\infty} P(t)\,R^2[g(t)],\ \tfrac{1}{2}\liminf_{t\to\infty} r(t)\,p(t)\,R(t)\,R^2[g(t)]\} > \frac{1}{3\sqrt{3}}\,S_\Phi\,,$$

every solution x of (E_2') is oscillatory or such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} r(t)\,x'(t) = \lim_{t\to\infty} r(t)\,[r(t)\,x'(t)]' = 0 \quad \textit{monotonically}\,.$$

Proof. Condition (C_2') ensures that (H_2) holds and hence we can restrict our attention only to the unbounded solutions of (E_2'). Furthermore, the substitution $z = -x$ transforms (E_2') into an equation of the same form satisfying the conditions possed for (E_2'). Thus, in order to study the existence or not of nonoscillatory solutions of (E_2') we can concentrate our interest only to the positive ones.

Let x be a positive unbounded solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the equation (E_2'). Moreover, let $T \geqq T_0$ be such that

$$g(t) \geqq T_0 \quad \text{for every} \quad t \geqq T\,.$$

Then the restriction of x on $[T, \infty)$ is a solution of the (linear) equation

$$[r(t)\,[r(t)\,w'(t)]']' + \tilde{p}(t)\,w[g(t)] = 0\,, \quad t \geqq t_0\,,$$

where

$$\tilde{p}(t) = \begin{cases} p(t)\,\dfrac{\Phi(x[g(t)])}{x[g(t)]}\,, & \text{if} \quad t \geqq T \\ \tilde{p}(T)\,, & \text{if} \quad t_0 \leqq t \leqq T\,. \end{cases}$$

By Theorem 2, for the function $\tilde{p}$ we always have

$$\min\left\{\liminf_{t\to\infty} R^2[g(t)] \int_t^\infty \tilde{p}(s)\,\mathrm{d}s,\ \tfrac{1}{2}\liminf_{t\to\infty} r(t)\,\tilde{p}(t)\,R(t)\,R^2[g(t)]\right\} \leqq \frac{1}{3\sqrt{3}}\,.$$

From (E_2') it follows that x is eventually increasing. We suppose, without loss of generality, that x is increasing at least on the interval $[T, \infty)$ and we consider a $T^* \geqq T$ such that

$$g(t) \geqq T \quad \text{for every} \quad t \geqq T^* .$$

Then for $t \geqq T^*$ we obtain

$$P(t) = \int_t^\infty \tilde{p}(s) \frac{x[g(s)]}{\Phi(x[g(s)])} \mathrm{d}s \leqq \left[\sup_{s \geqq t} \frac{x[g(s)]}{\Phi(x[g(s)])}\right] \int_t^\infty \tilde{p}(s) \mathrm{d}s \leqq$$

$$\leqq \left[\sup_{y \geqq x[g(t)]} \frac{y}{\Phi(y)}\right] \int_t^\infty \tilde{p}(s) \mathrm{d}s$$

and

$$p(t) = \tilde{p}(t) \frac{x[g(t)]}{\Phi(x[g(t)])} \leqq \tilde{p}(t) \left[\sup_{y \geqq x[g(t)]} \frac{y}{\Phi(y)}\right].$$

Thus, by $\lim_{t \to \infty} x(t) = \infty$, we derive

$$\min \{\liminf_{t \to \infty} P(t) R^2[g(t)], \tfrac{1}{2} \liminf_{t \to \infty} r(t) p(t) R(t) R^2[g(t)]\} \leqq$$

$$\leqq \left[\limsup_{y \to \infty} \frac{y}{\Phi(y)}\right] \min \left\{\liminf_{t \to \infty} R^2[g(t)] \int_t^\infty \tilde{p}(s) \mathrm{d}s, \right.$$

$$\left. \tfrac{1}{2} \liminf_{t \to \infty} r(t) \tilde{p}(t) R(t) R^2[g(t)]\right\} \leqq$$

$$\leqq \frac{1}{3\sqrt{3}} \limsup_{y \to \infty} \frac{y}{\Phi(y)} \leqq \frac{1}{3\sqrt{3}} S_\Phi ,$$

which contradicts (C_2').

Remark 5. Suppose that $S_\Phi < \infty$. Then (cf. [5]) we have also the conclusion of Theorem 2′, provided that $\int^\infty p(t) \mathrm{d}t = \infty$.

Remark 6. It is known (cf. [5]) that, under the condition

$$(\hat{C}_2') \qquad \limsup_{t \to \infty} R[g(t)] \int_t^\infty p(s) \left[\int_{g(t)}^{g(s)} \frac{\mathrm{d}w}{r(w)}\right] \mathrm{d}s > S_\Phi ,$$

we have also the conclusion of Theorem 2′ for the solutions of (E_2'). We note here that it is possible to have the condition (C_2') valid while $(\hat{C}_2')$ fails as, for example, in the case of the equation

$$x'''(t) + \frac{4}{t^3} x\left(\frac{t}{2}\right) = 0 , \quad t \geqq 1 .$$

Remark 7. It remains an open question to the authors if Theorem 2′ can be extended for the differential equation

$$[r_2(t)\,[r_1(t)\,x'(t)]']' + p(t)\,\Phi(x[g(t)]) = 0\,,$$

where r_1, r_2 are positive continuous functions on $[t_0, \infty)$ such that

$$\int^\infty \frac{dt}{r_1(t)} = \int^\infty \frac{dt}{r_2(t)} = \infty$$

and, in general, $r_1 \neq r_2$.

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Authors' address: Department of Mathematics, University of Ioannina, Ioannina, Greece.