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Review Article

Oscillation Criteria for Second-Order Delay, Difference, and Functional Equations

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Consider the second-order linear delay differential equation $x''(t) + p(t)x(\tau(t)) = 0$, $t \geq t_0$, where $p \in C([t_0, \infty), \mathbb{R}^+)$, $\tau \in C([t_0, \infty), \mathbb{R})$, $\tau(t)$ is nondecreasing, $\tau(t) \leq t$ for $t \geq t_0$ and $\lim_{t\to\infty}\tau(t) = \infty$, the (discrete analogue) second-order difference equation $\Delta^2 x(n) + p(n)x(\tau(n)) = 0$, where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to \mathbb{R}^+$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) \leq n-1$, and $\lim_{n\to\infty}\tau(n) = +\infty$, and the second-order functional equation $x(g(t)) = P(t)x(t) + Q(t)x(g^2(t))$, $t \geq t_0$, where the functions $P, Q \in C([t_0, \infty), \mathbb{R}^+)$, $g \in C([t_0, \infty), \mathbb{R})$, $g(t) \not\equiv t$ for $t \geq t_0$, $\lim_{t\to\infty} g(t) = \infty$, and g^2 denotes the 2th iterate of the function g, that is, $g^0(t) = t$, $g^2(t) = g(g(t))$, $t \geq t_0$. The most interesting oscillation criteria for the second-order linear delay differential equation, the second-order difference equation and the second-order functional equation, especially in the case where $\liminf_{t\to\infty}\int_{\tau(t)}^{t}\tau(s)p(s)ds \leq 1/e$ and $\limsup_{t\to\infty}\int_{\tau(t)}^{t}\tau(s)p(s)ds < 1$ for the second-order linear delay differential equation, and $0 < \liminf_{t\to\infty}\{Q(t)P(g(t))\} \leq 1/4$ and $\limsup_{t\to\infty}\{Q(t)P(g(t))\} < 1$, for the second-order functional equation, are presented.

1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the second-order delay differential equation

$$x''(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $p \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau \in C([t_0, \infty), \mathbb{R})$, $\tau(t)$ is nondecreasing, $\tau(t) \leq t$ for $t \geq t_0$, and $\lim_{t\to\infty}\tau(t) = \infty$, has been the subject of many investigations; see, for example, [1–21] and the references cited therein.

By a solution of (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) second-order difference equation

$$\Delta^2 x(n) + p(n)x(\tau(n)) = 0, \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to \mathbb{R}^+$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) \leq n - 1$, and $\lim_{n\to\infty}\tau(n) = +\infty$, has also attracted growing attention in the recent few years. The reader is referred to [22–26] and the references cited therein.

By a solution of (1) we mean a sequence $x(n)$ which is defined for $n \geq \min\{\tau(n) : n \geq 0\}$ and which satisfies $(1)'$ for all $n \geq 0$. A solution x of $(1)'$ is said to be *oscillatory* if the terms x of the solution are neither eventually positive nor eventually negative. Otherwise the solution is called *nonoscillatory*.

The oscillation theory of second-order functional equations of the form

$$x(g(t)) = P(t)x(t) + Q(t)x\left(g^2(t)\right), \quad t \geq t_0, \tag{1$''$}$$

where $P, Q \in C([t_0, \infty), \mathbb{R}^+)$, $g \in C([t_0, \infty), \mathbb{R})$ are given real-valued functions, x is an unknown real-valued function, $g(t) \not\equiv t$ for $t \geq t_0$, $\lim_{t\to\infty} g(t) = \infty$, and g^2 denotes the 2nd iterate of the function g, that is,

$$g^0(t) = t, \quad g^2(t) = g(g(t)), \quad t \geq t_0, \tag{1.1}$$

has also been developed during the last decade. We refer to [27–35] and the references cited therein.

By a solution of $(1)''$ we mean a real-valued function $x : [t_0, \infty) \to \mathbb{R}$ such that $\sup\{|x(s)| : s \geq t^*\} > 0$ for any $t^* \geq t_0$ and x satisfies $(1)''$ on $[t_0, \infty)$.

In this paper our purpose is to present the state of the art on the oscillation of all solutions to (1), $(1)'$, and $(1)''$, especially in the case where

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds \leq \frac{1}{e}, \qquad \limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds < 1 \tag{1.2}$$

for (1), and

$$0 < \liminf_{t\to\infty} \{Q(t)P(g(t))\} \leq \frac{1}{4}, \qquad \limsup_{t\to\infty} \{Q(t)P(g(t))\} < 1 \tag{1.3}$$

for $(1)''$.

2. Oscillation Criteria for (1)

In this section we study the second-order delay equation (1). For the case of ordinary differential equations, that is, when $\tau(t) \equiv t$, the history of the problem began as early as in 1836 by the work of Sturm [16] and was continued in 1893 by Kneser [8]. Essential contribution to the subject was made by E. Hille, A. Wintner, Ph. Hartman, W. Leighton, Z. Nehari, and others (see the monograph by Swanson [17] and the references cited therein). In particular, in 1948 Hille [7] obtained the following well-known oscillation criteria. Let

$$\limsup_{t\to\infty} t\int_t^{+\infty} p(s)ds > 1 \tag{2.1}$$

or

$$\liminf_{t\to\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4}, \tag{2.2}$$

the conditions being assumed to be satisfied if the integral diverges. Then (1) with $\tau(t) \equiv t$ is oscillatory.

For the delay differential equation (1), earlier oscillation results can be found in the monographs by Myshkis [14] and Norkin [15]. In 1968 Waltman [18] and in 1970 Bradley [1] proved that (1) is oscillatory if

$$\int^{+\infty} p(t)dt = +\infty. \tag{2.3}$$

Proceeding in the direction of generalization of Hille's criteria, in 1971 Wong [20] showed that if $\tau(t) \geq \alpha t$ for $t \geq 0$ with $0 < \alpha \leq 1$, then the condition

$$\liminf_{t\to\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4\alpha} \tag{2.4}$$

is sufficient for the oscillation of (1). In 1973 Erbe [2] generalized this condition to

$$\liminf_{t\to\infty} t\int_t^{+\infty} \frac{\tau(s)}{s} p(s)ds > \frac{1}{4} \tag{2.5}$$

without any additional restriction on τ. In 1987 Yan [21] obtained some general criteria improving the previous ones.

An oscillation criterion of different type is given in 1986 by Koplatadze [9] and in 1988 by Wei [19], where it is proved that (1) is oscillatory if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau(s)p(s)ds > 1 \tag{C_1}$$

or

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}\tau(s)p(s)ds > \frac{1}{e}. \tag{C_2}$$

The conditions (C_1) and (C_2) are analogous to the oscillation conditions (see [36])

$$A := \limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds > 1, \tag{H_1}$$

$$\alpha := \liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds > \frac{1}{e}, \tag{H_2}$$

respectively, for the first-order delay equation

$$x'(t) + p(t)x(\tau(t)) = 0. \tag{2.6}$$

The essential difference between (2.1), (2.2), and (C_1), (C_2) is that the first two can guarantee oscillation for ordinary differential equations as well, while the last two work only for delay equations. Unlike first-order differential equations, where the oscillatory character is due to the delay only (see [36]), equation (1) can be oscillatory without any delay at all, that is, in the case $\tau(t) \equiv t$. Figuratively speaking, two factors contribute to the oscillatory character of (1): the presence of the delay and the second-order nature of the equation. The conditions (2.1), (2.2), and (C_1), (C_2) illustrate the role of these factors taken separately.

In 1999 Koplatadze et al. [11] derived the following.

Theorem 2.1 (see [11]). *Let*

$$\limsup_{t\to\infty}\left\{\tau(t)\int_{t}^{+\infty}p(s)ds + \int_{\tau(t)}^{t}\tau(s)p(s)ds + [\tau(t)]^{-1}\int_{0}^{\tau(t)}s\tau(s)p(s)ds\right\} > 1. \tag{2.7}$$

Then (1) *is oscillatory.*

The following corollaries being more convenient for applications can be deduced from this theorem.

Corollary 2.2 (see [11]). *Let*

$$\limsup_{t\to\infty}\tau(t)\int_{t}^{+\infty}p(s)ds + \liminf_{t\to\infty}t^{-1}\int_{0}^{t}s\tau(s)p(s)ds > 1 \tag{2.8}$$

or

$$\liminf_{t\to\infty}\tau(t)\int_{t}^{+\infty}p(s)ds + \limsup_{t\to\infty}t^{-1}\int_{0}^{t}s\tau(s)p(s)ds > 1. \tag{2.9}$$

Then (1) *is oscillatory.*

Corollary 2.3 (see [11]). *Let*

$$\limsup_{t\to\infty} \tau(t)\int_t^{+\infty} p(s)ds > 1 \tag{2.10}$$

or

$$\limsup_{t\to\infty} t^{-1}\int_0^t s\tau(s)p(s)ds > 1. \tag{2.11}$$

Then (1) *is oscillatory.*

In the case of ordinary differential equations ($\tau(t) \equiv t$) the following theorem was given in [11].

Theorem 2.4 (see [11]). *Let* $\tau(t) \equiv t$ *and*

$$\limsup_{t\to\infty} \left\{ t\int_t^{+\infty} p(s)ds + t^{-1}\int_0^t s^2 p(s)ds \right\} > 1. \tag{2.12}$$

Then (1) *is oscillatory.*

In what follows it will be assumed that the condition

$$\int^{+\infty} \tau(s)p(s)ds = +\infty \tag{2.13}$$

is fulfilled. As it follows from Lemma 4.1 in [10], this condition is necessary for (1) to be oscillatory. The study being devoted to the problem of oscillation of (1), the condition (2.13) does not affect the generality.

Here oscillation results are obtained for (1) by reducing it to a first-order equation. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay equations only and do not work in the ordinary case.

Theorem 2.5 (see [12]). *Let* (2.13) *be fulfilled and let the differential inequality*

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi \right) p(t)x(\tau(t)) \le 0 \tag{2.14}$$

have no eventually positive solution. Then (1) *is oscillatory.*

Theorem 2.5 reduces the question of oscillation of (1) to that of the absence of eventually positive solutions of the differential inequality

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi \right) p(t)x(\tau(t)) \le 0. \tag{2.15}$$

So oscillation results for first-order delay differential equations can be applied since the oscillation of the equation

$$u'(t) + g(t)u(\delta(t)) = 0 \tag{2.16}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$u'(t) + g(t)u(\delta(t)) \leq 0. \tag{2.17}$$

This fact is a simple consequence of the following comparison theorem deriving the oscillation of (2.16) from the oscillation of the equation

$$v'(t) + h(t)v(\sigma(t)) = 0. \tag{2.18}$$

We assume that $g, h : R_+ \to R_+$ are locally integrable, $\delta, \sigma : R_+ \to R$ are continuous, $\delta(t) \leq t$, $\sigma(t) \leq t$ for $t \in R_+$, and $\delta(t) \to +\infty$, $\sigma(t) \to +\infty$ as $t \to +\infty$.

Theorem 2.6 (see [12]). *Let*

$$g(t) \geq h(t), \quad \delta(t) \leq \sigma(t), \quad \textit{for } t \in R_+, \tag{2.19}$$

and let (2.18) *be oscillatory. Then* (2.16) *is also oscillatory.*

Corollary 2.7 (see [12]). *Let* (2.16) *be oscillatory. Then the inequality* (2.17) *has no eventually positive solution.*

Turning to applications of Theorem 2.5, we will use it together with the criteria (H_1) and (H_2) to get the following.

Theorem 2.8 (see [12]). *Let*

$$K := \limsup_{t\to\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right)p(s)ds > 1 \tag{C_3}$$

or

$$k := \liminf_{t\to\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right)p(s)ds > \frac{1}{e}. \tag{C_4}$$

Then (1) *is oscillatory.*

To apply Theorem 2.5 it suffices to note that (i) equation (2.13) is fulfilled since otherwise $k = K = 0$; (ii) since $\tau(t) \to +\infty$ as $t \to +\infty$, the relations (C_4) and (C_4) imply the same relations with 0 changed by any $T \geq 0$.

Remark 2.9 (see [12]). Theorem 2.8 improves the criteria (C_1), (C_2) by Koplatadze [9] and Wei [19] mentioned above. This is directly seen from (C_3), (C_4) and can be easily checked if we take $\tau(t) \equiv t-\tau_0$ and $p(t) \equiv p_0/(t-\tau_0)$ for $t \geq 2\tau_0$, where the constants $\tau_0 > 0$ and $p_0 > 0$ satisfy

$$\tau_0 p_0 < \frac{1}{e}. \tag{2.20}$$

In this case neither of (C_1), (C_2) is applicable for (1) while both (C_3), (C_4) give the positive conclusion about its oscillation. Note also that this is exactly the case where the oscillation is due to the delay since the corresponding equation without delay is nonoscillatory.

Remark 2.10 (see [12]). The criteria (C_3), (C_4) look like (H_1), (H_2) but there is an essential difference between them pointed out in the introduction. The condition (H_2) is close to the necessary one, since according to [9] if $A \leq 1/e$, then (2.16) is nonoscillatory. On the other hand, for an oscillatory equation (1) without delay we have $k = K = 0$. Nevertheless, the constant $1/e$ in Theorem 2.8 is also the best possible in the sense that for any $\varepsilon \in (0, 1/e]$ it cannot be replaced by $1/e - \varepsilon$ without affecting the validity of the theorem. This is illustrated by the following.

Example 2.11 (see [12]). Let $\varepsilon \in (0, 1/e]$, $1 - e\varepsilon < \beta < 1$, $\tau(t) \equiv \alpha t$, and $p(t) \equiv \beta(1-\beta)\alpha^{-\beta}t^{-2}$, where $\alpha = e^{1/(\beta-1)}$. Then (C_4) is fulfilled with 1/e replaced by $1/e - \varepsilon$. Nevertheless (1) has a nonoscillatory solution, namely, $u(t) \equiv t^{\beta}$. Indeed, denoting $c = \beta(1-\beta)\alpha^{-\beta}$, we see that the expression under the limit sign in (C_4) is constant and equals $\alpha c|\ln\alpha|(1+\alpha c) = (\beta/e)(1 + (\beta(1-\beta))/e) > \beta/e > 1/e - \varepsilon$.

Note that there is a gap between the conditions (C_3), (C_4) when $0 \leq k \leq 1/e$, $k < K$. In the case of first-order equations (cf., [36–48]), using results in this direction, one can derive various sufficient conditions for the oscillation of (1). According to Remark 2.9, neither of them can be optimal in the above sense but, nevertheless, they are of interest since they cannot be derived from other known results in the literature. We combine Theorem 2.5 with Corollary 1 [40] to obtain the following theorem.

Theorem 2.12 (see [12]). *Let K and k be defined by (C_3), (C_4), $0 \leq k \leq 1/e$, and*

$$K > k + \frac{1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}, \tag{C$_9$}$$

where $\lambda(k)$ is the smaller root of the equation $\lambda = \exp(k\lambda)$. Then (1) is oscillatory.

Note that the condition (C_9) is analogous to the condition (C_9) in [40].

3. Oscillation Criteria for $(1)'$

In this section we study the second-order difference equation $(1)'$, where $\Delta x(n) = x(n+1) - x(n)$, $\Delta^2 = \Delta \circ \Delta$, $p : \mathbb{N} \to R_+$, $\tau : \mathbb{N} \to \mathbb{N}$, $\tau(n) \leq n-1$, and $\lim_{n\to\infty}\tau(n) = +\infty$.

In 1994, Wyrwińska [26] proved that all solutions of $(1)'$ are oscillatory if

$$\limsup_{n\to\infty}\left\{\sum_{i=\tau(n)}^{n}[\tau(i)-2]p(i)+[\tau(n)-2]\sum_{i=n+1}^{\infty}p(i)\right\}>1, \qquad (C_1)'$$

while, in 1997, Agarwal et al. [22] proved that, in the special case of the second-order difference equation with constant delay

$$\Delta^2 x(n)+p(n)x(n-k)=0, \quad k\geq 1, \qquad (1)'_c$$

all solutions are oscillatory if

$$\liminf_{n\to\infty}\sum_{i=n-k}^{n-1}(i-k)p(i)>2\left(\frac{k}{k+1}\right)^{k+1}. \qquad (3.1)$$

In 2001, Grzegorczyk and Werbowski [23] studied $(1)'_c$ and proved that under the following conditions

$$\limsup_{n\to\infty}\left\{\sum_{i=n-k}^{n}(i-n+k+1)p(i)+\left[(n-k-2)+\sum_{i=n_1}^{n-k-1}(i-k)^2p(i)\right]\times\sum_{i=n+1}^{\infty}p(i)\right\}>1, \quad \text{for some } n_1>n_0, \qquad (3.2)$$

or

$$\liminf_{n\to\infty}\sum_{i=n-k}^{n-1}(i-k-1)p(i)>\left(\frac{k}{k+1}\right)^{k+1} \qquad (C_2)'$$

all solutions of $(1)'_c$ are oscillatory. Observe that the last condition $(C_2)'$ may be seen as the discrete analogue of the condition (C_2).

In 2001 Koplatadze [24] studied the oscillatory behaviour of all solutions to $(1)'$ with variable delay and established the following.

Theorem 3.1 (see [24]). *Assume that*

$$\inf\left\{\frac{1}{1-\lambda}\liminf_{n\to\infty}n^{-\lambda}\sum_{i=1}^{n}ip(i)\tau^{\lambda}(i):\lambda\in(0,1)\right\}>1,$$
$$\liminf_{n\to\infty}n^{-1}\sum_{i=1}^{n}ip(i)\tau(i)>0. \qquad (3.3)$$

Then all solutions of $(1)'$ *oscillate.*

Corollary 3.2 (see [24]). *Let $\alpha > 0$ and*

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} i^2 p(i) > \max\left\{\alpha^{-\lambda}\lambda(1-\lambda) : \lambda \in [0,1]\right\}. \tag{3.4}$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0, \quad n \geq \max\left\{1, \frac{1}{\alpha}\right\}, \quad n \in N \tag{3.5}$$

oscillate.

Corollary 3.3 (see [24]). *Let n_0 be an integer and*

$$\liminf_{n\to\infty} n^{-1}\sum_{i=1}^{n} i^2 p(i) > \frac{1}{4}. \tag{3.6}$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n-n_0) = 0, \quad n \geq \max\{1, n_0+1\}, \quad n \in N \tag{3.7}$$

oscillate.

In 2002 Koplatadze et al. [25] studied $(1)'$ and established the following.

Theorem 3.4 (see [25]). *Assume that*

$$\liminf_{n\to\infty} \frac{\tau(n)}{n} = \alpha \in (0,\infty), \tag{3.8}$$

$$\liminf_{n\to\infty} n\sum_{i=n}^{\infty} p(i) > \max\left\{\alpha^{-\lambda}\lambda(1-\lambda) : \lambda \in [0,1]\right\}. \tag{3.9}$$

Then all solutions of $(1)'$ oscillate.

In the case where $\alpha = 1$, the following discrete analogue of Hille's well-known oscillation theorem for ordinary differential equations (see (2.2)) is derived.

Theorem 3.5 (see [25]). *Let n_0 be an integer and*

$$\liminf_{n\to\infty} n\sum_{i=n}^{\infty} p(i) > \frac{1}{4}. \tag{3.10}$$

Then all solutions of the equation

$$\Delta^2 x(n) + p(n)x(n - n_0) = 0, \quad n \geq n_0, \tag{3.11}$$

oscillate.

Remark 3.6 (see [25]). As in case of ordinary differential equations, the constant 1/4 in (3.10) is optimal in the sense that the strict inequality cannot be replaced by the nonstrict one. More than that, the same is true for the condition (3.9) as well. To ascertain this, denote by c the right-hand side of (3.9) and by λ_0 the point where the maximum is achieved. The sequence $x(n) = n^{\lambda_0}$ obviously is a nonoscillatory solution of the equation

$$\Delta^2 x(n) + p(n)x([\alpha n]) = 0, \tag{3.12}$$

where $p(n) = -\Delta^2(n^{\lambda_0})/[\alpha n]^{\lambda_0}$ and $[\alpha]$ denotes the integer part of α. It can be easily calculated that

$$p(n) = -\frac{c}{n^2} + o\left(\frac{1}{n^2}\right) \quad \text{as } n \longrightarrow \infty. \tag{3.13}$$

Hence for arbitrary $\varepsilon > 0$, $p(n) \geq (c - \varepsilon)/n^2$ for $n \in \mathbb{N}_{n_0}$ with $n_0 \in \mathbb{N}$ sufficiently large. Using the inequality $\sum_{i=n}^{\infty} i^2 \geq n^{-1}$ and the arbitrariness of ε, we obtain

$$\liminf_{n \to \infty} n \sum_{i=n}^{\infty} p(i) \geq c. \tag{3.14}$$

This limit cannot be greater than c by Theorem 3.4. Therefore it equals c and (3.9) is violated.

4. Oscillation Criteria for $(1)''$

In this section we study the functional equation $(1)''$.

In 1993 Domshlak [27] studied the oscillatory behaviour of equations of this type. In 1994, Golda and Werbowski [28] proved that all solutions of $(1)''$ oscillate if

$$\mathbf{A} := \limsup_{t \to \infty} \{Q(t)P(g(t))\} > 1 \tag*{$(C_1)''$}$$

or

$$\mathrm{a} := \liminf_{t \to \infty} \{Q(t)P(g(t))\} > \frac{1}{4}. \tag*{$(C_2)''$}$$

In the same paper they also improved condition $(C_1)''$ to

$$\limsup_{t\to\infty}\left\{Q(t)P(g(t))+\sum_{i=0}^{k}\prod_{j=0}^{i}Q\left(g^{j+1}(t)\right)P\left(g^{j+2}(t)\right)\right\}>1, \tag{4.1}$$

where $k \geq 0$ is some integer.

In 1995, Nowakowska and Werbowski [29] extended condition $(C_1)''$ to higher-order linear functional equations. In 1996, Shen [30], in 1997, Zhang et al. [35], and, in 1998, Zhang et al. [34] studied functional equations with variable coefficients and constant delay, while in 1999, Yan and Zhang [33] considered a system with constant coefficients.

It should be noted that conditions $(C_1)''$ and $(C_2)''$ may be seen as the analogues of the oscillation conditions (C_1) and (C_2) for (1) and $(C_1)'$ and $(C_2)'$ for $(1)'$.

As far as the lower bound $1/4$ in the condition $(C_2)''$ is concerned, as it was pointed out in [28], it cannot be replaced by a smaller number. Recently, in [32], this fact was generalized by proving that

$$Q(t)P(g(t)) \leq \frac{1}{4}, \quad \text{for large } t, \tag{$(N_1)'$}$$

implies that $(1)''$ has a nonoscillatory solution.

It is obvious that there is a gap between the conditions $(C_1)''$ and $(C_2)''$ when the limit $\lim_{t\to\infty}\{Q(t)P(g(t))\}$ does not exist. How to fill this gap is an interesting problem. Here we should mention that condition (4.1) is an attempt in this direction. In fact, from condition (4.1) we can obtain (see [31]) that all solutions of $(1)''$ oscillate if $0 \leq a \leq 1/4$ and

$$\mathbf{A} > \frac{1-2a}{1-a}. \tag{4.2}$$

In 2002, Shen and Stavroulakis [31] proved the following.

Theorem 4.1 (see [31]). *Assume that $0 \leq a \leq 1/4$ and that for some integer $k \geq 0$*

$$\limsup_{t\to\infty}\left\{\overline{a}Q(t)P(g(t))+\sum_{i=0}^{k}\overline{a}^{i}\prod_{j=0}^{i}Q\left(g^{j+1}(t)\right)P\left(g^{j+2}(t)\right)\right\}>1, \tag{4.3}$$

where $\overline{a} = \left((1+\sqrt{1-4a})/2\right)^{-1}$. Then all solutions of $(1)''$ oscillate.

Corollary 4.2 (see [31]). *Assume that $0 \leq a \leq 1/4$ and*

$$\mathbf{A} > \left(\frac{1+\sqrt{1-4a}}{2}\right)^{2}. \tag{4.4}$$

Then all solutions of $(1)''$ oscillate.

Remark 4.3 (see [31]). It is to be noted that as $a \to 0$ the condition (4.3) reduces to the condition (4.1) and the conditions (4.4) and (4.2) reduce to the condition $(C_1)''$. However the improvement is clear as $0 < a \leq 1/4$ because

$$1 > \frac{1-2a}{1-a} > \left(\frac{1+\sqrt{1-4a}}{2}\right)^2. \tag{4.5}$$

It is interesting to observe that when $a \to 1/4$ condition (4.4) reduces to

$$\mathbf{A} > 1/4, \tag{4.6}$$

which cannot be improved in the sense that the lower bound $1/4$ cannot be replaced by a smaller number.

Example 4.4 (see [31]). Consider the equation

$$x\left(t - 2\sin^2 t\right) = x(t) + \left(\frac{1}{4} + q\cos^2 t\right) x\left(t - 2\sin^2 t - 2\sin^2\left(t - 2\sin^2 t\right)\right), \tag{4.7}$$

where $g(t) = t - 2\sin^2 t$, $P(t) \equiv 1$, $Q(t) = 1/4 + q\cos^2 t$, and $q > 0$ is a constant. It is easy to see that

$$\begin{aligned} a &= \liminf_{t\to\infty}\left(\frac{1}{4} + q\cos^2 t\right) = \frac{1}{4}, \\ \mathbf{A} &= \limsup_{t\to\infty}\left(\frac{1}{4} + q\cos^2 t\right) = \frac{1}{4} + q > \frac{1}{4}. \end{aligned} \tag{4.8}$$

Thus, by Corollary 4.2 all solutions of (4.7) oscillate. However, the condition $(C_1)''$ is satisfied only for $q > 3/4$, while the condition (4.2) is satisfied for (much smaller) $q > 5/12$.

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