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A study of second-order q -difference equations with boundary conditions

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Abstract

This paper studies a boundary value problem of nonlinear second-order q -difference equations with non-separated boundary conditions. As a first step, the given boundary value problem is converted to an equivalent integral operator equation by using the q -difference calculus. Then the existence and uniqueness of solutions of the problem is proved via the resulting integral operator equation by means of Leray-Schauder nonlinear alternative and some standard fixed point theorems. Our approach is simpler than the one involving the typical series solution form of q -difference equations. The results corresponding to a second-order q -difference equation with anti-periodic boundary conditions appear as a special case. Furthermore, our results reduce to the corresponding results for classical second-order boundary value problems with non-separated boundary conditions in the limit $q \rightarrow 1$, which provides a useful check.

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1 Introduction

In this paper, we discuss the existence of solutions for the second-order q -difference equation with non-separated boundary conditions

$$\begin{cases} D_q^2 u(t) = f(t, u(t)), & t \in I, \\ u(0) = \eta u(T), \quad D_q u(0) = \eta D_q u(T), \end{cases} \quad (1.1)$$

where $f \in C(I \times \mathbb{R}, \mathbb{R})$, $I = [0, T] \cap q^{\mathbb{N}}$, $q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}\} \cup \{0\}$ and $T \in q^{\mathbb{N}}$ is a fixed constant, and $\eta \neq 1$ is a fixed real number.

The study of q -difference equations, initiated in the beginning of the 20th century ([1-4]), and, up to date, it has evolved into a multidisciplinary subject, (for example, see ([5-15]) and references therein). For some recent work on q -difference equations, we refer the reader to the papers ([16-23]). However, the theory of boundary value problems for nonlinear q -difference equations is still in the initial stage and many aspects of this theory need to be explored.

The main objective of this paper is to develop some existence and uniqueness results for the boundary value problem (1.1). Our results are based on a variety of fixed point theorems such as Banach's contraction principle, Leray-Schauder nonlinear alternative

and Krasnoselskii's fixed point theorem. Some illustrative examples and special cases are also discussed.

2 Preliminaries

Let us recall some basic concepts of q -calculus [14,15].

For $0 < q < 1$, we define the q -derivative of a real valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t \rightarrow 0} D_q f(t).$$

The higher order q -derivatives are given by

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

The q -integral of a function f defined in the interval $J_a = [a, b] \cap q^{\mathbb{N}}$ is given by

$$\int_a^x f(t) d_q t := \sum_{n=0}^{\infty} (1-q)q^n [xf(xq^n) - af(q^n a)], \quad x \in J_a,$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t) d_q t = \sum_{n=0}^{\infty} x(1-q)q^n f(xq^n),$$

provided the series converges. If $a \in J_0 = [0, b] \cap q^{\mathbb{N}}$ and f is defined in the interval J_0 , then

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly, we have

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.1}$$

and if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0).$$

In q -calculus, the product rule and integration by parts formula are

$$D_q(gh)(t) = (D_q g(t))h(t) + g(qt)D_q h(t), \tag{2.2}$$

$$\int_0^x f(t)D_q g(t)d_q t = [f(t)g(t)]_0^x - \int_0^x D_q f(t)g(qt)d_q t. \tag{2.3}$$

In the limit $q \rightarrow 1$ the above results correspond to their counterparts in standard calculus.

Lemma 2.1 *The unique solution of (1.1) is given by*

$$u(t) = \int_0^t (t - qs)f(s, u(s))d_qs + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]f(s, u(s))d_qs. \tag{2.4}$$

Proof. Integrating the equation $D_q^2 u = f(t, u)$, we get

$$D_q u(t) = \int_0^t f(s, u(s)) d_q s + a_1. \tag{2.5}$$

Integrating (2.5), we obtain

$$u(t) = \int_0^t \left(\int_0^v f(s, u(s)) d_q s \right) d_q v + a_1 t + a_2, \tag{2.6}$$

which, on changing the order of integration, takes the form

$$u(t) = \int_0^t \left(\int_{qs}^t f(s, u(s)) d_q v \right) d_q s + a_1 t + a_2. \tag{2.7}$$

Alternatively, (2.7) can be written as

$$u(t) = \int_0^t (t - qs) f(s, u(s)) d_q s + a_1 t + a_2. \tag{2.8}$$

Here a_1, a_2 are arbitrary constants. Conversely, differentiating (2.8) by applying the formulae (2.1) and (2.2) of q -calculus, one can easily obtain $D_q^2 u = f(t, u)$.

Now, using the boundary conditions of (1.1) in (2.8), we find that

$$\begin{aligned} a_1 &= \frac{-\eta}{(\eta - 1)} \int_0^T f(s, u(s)) d_q s, \\ a_2 &= \frac{\eta^2 T}{(\eta - 1)^2} \int_0^T f(s, u(s)) d_q s - \frac{\eta}{(\eta - 1)} \int_0^T (T - qs) f(s, u(s)) d_q s. \end{aligned}$$

Substituting the values of a_1 and a_2 in (2.8), we obtain (2.4). This completes the proof.

For the forthcoming analysis, let $\mathcal{C} = C(I, \mathbb{R})$ denotes the Banach space of all continuous functions from I to \mathbb{R} endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in I\}$.

Furthermore, we set

$$\begin{aligned} \Lambda_1 &= \sup_{t \in I} \left\{ \frac{t^2}{1+q} + \frac{|\eta(1+\eta q)| T^2}{(1+q)(\eta-1)^2} + \left| \frac{\eta}{\eta-1} \right| tT \right\} \\ &= \left(\frac{1}{1+q} + \frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \left| \frac{\eta}{\eta-1} \right| \right) T^2. \end{aligned} \tag{2.9}$$

Theorem 2.1 Let $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function satisfying the condition

$$|(H_0) \quad |f(t, u) - f(t, v)| \leq L |u - v|, \forall t \in I, u, v \in \mathbb{R},$$

where L is a Lipschitz constant. Then the boundary value problem (1.1) has a unique solution, provided $\Lambda = L \Lambda_1 < 1$, where Λ_1 is given by (2.9).

Proof. We define $F : \mathcal{C} \rightarrow \mathcal{C}$ by

$$F u(t) = \int_0^t (t - qs) f(s, u(s)) d_q s + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] f(s, u(s)) d_q s \tag{2.10}$$

for $u \in \mathcal{C}$ and $t \in I$. Let us set $\sup_{t \in I} |f(t, 0)| = M_0$ and choose

$$r \geq \frac{M_0 \Lambda_1}{1 - \delta}, \tag{2.11}$$

where δ is such that $\Lambda \leq \delta < 1$. Now we show that $FB_r \subset B_r$, where $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned} \|(Fu)(t)\| &= \sup_{t \in I} \left| \int_0^t (t - qs) f(s, u(s)) d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] f(s, u(s)) d_qs \right| \\ &\leq \sup_{t \in I} \left| \int_0^t (t - qs) (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] (|f(s, u(s)) - f(s, 0)| + |f(s, 0)|) d_qs \right| \\ &\leq \sup_{t \in I} \left| \int_0^t (t - qs) (L|u(s)| + |f(s, 0)|) d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] (L|u(s)| + |f(s, 0)|) d_qs \right| \\ &\leq (L\|u\| + M_0) \sup_{t \in I} \left| \int_0^t (t - qs) d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] d_qs \right| \\ &\leq (L\|u\| + M_0) \sup_{t \in I} \left\{ \frac{t^2}{1 + q} + \frac{|\eta(1 + \eta q)| T^2}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right| tT \right\} \\ &\leq \Lambda_1(Lr + M_0) \leq (\Lambda + 1 - \delta)r \leq r. \end{aligned}$$

Now, for $u, v \in \mathcal{C}$ and for each $t \in I$, we obtain

$$\begin{aligned} &\|(Fu)(t) - (Fv)(t)\| \\ &= \sup_{t \in I} |(Fu)(t) - (Fv)(t)| \\ &\leq \sup_{t \in I} \left| \int_0^t (t - qs) [f(s, u(s)) - f(s, v(s))] d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] [f(s, u(s)) - f(s, v(s))] d_qs \right| \\ &\leq L \sup_{t \in I} \left| \int_0^t (t - qs) d_qs + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)] d_qs \right| \|u - v\| \\ &\leq L \sup_{t \in I} \left\{ \frac{t^2}{1 + q} + \frac{|\eta(1 + \eta q)| T^2}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right| tT \right\} \|u - v\| \\ &\leq \Lambda \|u - v\|, \end{aligned}$$

where

$$\Lambda = L \left(\frac{1}{1 + q} + \frac{|\eta(1 + \eta q)|}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right| \right) T^2,$$

which depends only on the parameters involved in the problem. As $\Lambda < 1$, therefore \mathfrak{X} is a contraction. Thus, the conclusion of the theorem follows by Banach's contraction mapping principle. This completes the proof.

The next existence result is based on Leray-Schauder alternative.

Theorem 2.2 (Nonlinear alternative for single valued maps)[24]. *Let E be a Banach space, C a closed, convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F: \bar{U} \rightarrow C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either*

- (i) F has a fixed point in \bar{U} , or
- (ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Theorem 2.3 Assume that:

(H₁) there exists a continuous nondecreasing function $\psi: [0, \infty) \rightarrow (0, \infty)$ and a function $p \in L^1(I, \mathbb{R}_+)$ such that

$$|f(t, u)| \leq p(t)\psi(|u|) \text{ for each } (t, u) \in I \times \mathbb{R};$$

(H₂) there exists a number $M < 0$ such that

$$\frac{M}{T \left(1 + \frac{|\eta|(1+|1-\eta|)}{(\eta-1)^2} \right) \psi(M) \|p\|_{L^1}} > 1,$$

where

$$\|p\|_{L^1} = \int_0^T p(s) d_q s \neq 0.$$

Then the BVP (1.1) has at least one solution.

Proof. We define $F: \mathcal{C} \rightarrow \mathcal{C}$ as in (2.10). The proof consists of several steps.

(i) F maps bounded sets into bounded sets in $C(I, \mathbb{R})$.

Let $B_k = \{u \in C(I, \mathbb{R}) : \|u\| \leq k\}$ be a bounded set in $C(I, \mathbb{R})$ and $u \in B_k$. Then we have

$$\begin{aligned} |Fu(t)| &\leq \int_0^t |t-qs| |f(s, u(s))| d_q s + \frac{|\eta|}{(\eta-1)^2} \int_0^T |T+(1-\eta)(t-qs)| |f(s, u(s))| d_q s \\ &\leq T \left(1 + \frac{|\eta|(1+|1-\eta|)}{(\eta-1)^2} \right) \int_0^T p(s)\psi(\|u\|) d_q s \\ &\leq T \left(1 + \frac{|\eta|(1+|1-\eta|)}{(\eta-1)^2} \right) \psi(k) \int_0^T p(s) d_q s. \end{aligned}$$

Thus

$$\|Fu\| \leq T \left(1 + \frac{|\eta|(1+|1-\eta|)}{(\eta-1)^2} \right) \psi(k) \|p\|_{L^1}.$$

(ii) F maps bounded sets into equicontinuous sets of $C(I, \mathbb{R})$.

Let $r_1, r_2 \in I$, $r_1 < r_2$ and B_k be a bounded set of $C(I, \mathbb{R})$ as before. Then for $u \in B_k$ we have

$$\begin{aligned}
 |Fu(r_2) - Fu(r_1)| &= \left| \int_0^{r_2} (r_2 - qs)f(s, u(s))d_qs - \int_0^{r_1} (r_1 - qs)f(s, u(s))d_qs \right. \\
 &\quad \left. + \frac{\eta(1 - \eta)}{(\eta - 1)^2} \int_0^T (r_2 - r_1)f(s, u(s))d_qs \right| \\
 &= \left| \int_0^{r_1} (r_2 - r_1)f(s, u(s))d_qs + \int_{r_1}^{r_2} (r_2 - qs)f(s, u(s))d_qs \right. \\
 &\quad \left. + \frac{\eta(1 - \eta)}{(\eta - 1)^2} \int_0^T (r_2 - r_1)f(s, u(s))d_qs \right| \\
 &\leq \int_0^{r_1} |r_2 - r_1|p(s)\psi(k)d_qs + \int_{r_1}^{r_2} |r_2 - qs|p(s)\psi(k)d_qs \\
 &\quad + \frac{|\eta|}{|1 - \eta|} \int_0^T |r_2 - r_1|p(s)\psi(k)d_qs.
 \end{aligned}$$

The right hand side tends to zero as $r_2 - r_1 \rightarrow 0$. As a consequence of Arzelá-Ascoli theorem, we can conclude that $F : C(I, \mathbb{R}) \rightarrow C(I, \mathbb{R})$ is completely continuous.

(iii) Let $\lambda \in (0,1)$ and let $u = \lambda Fu$. Then, for $t \in I$, we have

$$\begin{aligned}
 |u(t)| &= |\lambda Fu(t)| \leq \int_0^t |t - qs| |f(s, u(s))| d_qs \\
 &\quad + \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)| |f(s, u(s))| d_qs \\
 &\leq T \left(1 + \frac{|\eta|(1 + |1 - \eta|)}{(\eta - 1)^2} \right) \psi(\|u\|) \int_0^T p(s) d,
 \end{aligned}$$

and consequently

$$\frac{\|u\|}{T \left(1 + \frac{|\eta|(1 + |1 - \eta|)}{(\eta - 1)^2} \right) \psi(\|u\|) \|p\|_{L^1}} \leq 1.$$

In view of (H_2) , there exists M such that $\|u\| \neq M$. Let us set

$$U = \{u \in C(I, \mathbb{R}) : \|u\| < M\}.$$

Note that the operator $F : \bar{U} \rightarrow C(I, \mathbb{R})$ is continuous and completely continuous (which is well known to be compact restricted to bounded sets). From the choice of U , there is no $u \in \partial U$ such that $u = \lambda F(u)$ for some $\lambda \in (0,1)$. Consequently, by Theorem 2.2, we deduce that \mathfrak{K} has a fixed point $u \in \bar{U}$ which is a solution of the problem (1.1). This completes the proof.

Theorem 2.4 Assume that there exist constants $0 \leq c < 1/\Delta_2$ and $N > 0$ such that $|f(t, u)| \leq \frac{c}{T^2} |u| + N$ for all $t \in I, u \in C(I, \mathbb{R})$, where

$$\Delta_2 = \frac{1}{1 + q} + \frac{|\eta(1 + \eta q)|}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right|. \tag{2.12}$$

Then the BVP (1.1) has at least one solution.

Proof. We define $F : \mathcal{C} \rightarrow \mathcal{C}$ as in (2.10). As in Theorem 2.3, using the assumption $|f(t, u)| \leq \frac{c}{T^2} |u| + N$ (a special form of the condition (H_1) with $p(t) = 1$, $\psi(|u|) = \frac{c}{T^2} |u| + N$), we can prove that \mathfrak{K} is completely continuous.

Assume that $u = \lambda Fu$ for some $\lambda \in [0, 1]$. For all $t \in I$ we have

$$\begin{aligned} |u(t)| &= |\lambda Fu(t)| \\ &= \left| \int_0^t (t - qs)f(s, u(s))d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]f(s, u(s))d_qs \right| \\ &\leq \left(\frac{c}{T^2} |u| + N \right) \Delta_2 T^2 \\ &\leq \Delta_2 (c|u| + NT^2), \end{aligned}$$

which implies that

$$\|u\| \leq \frac{N\Delta_2 T^2}{1 - c\Delta_2}.$$

Letting

$$R = \frac{N\Delta_2 T^2}{1 - c\Delta_2} + 1,$$

we have $\|u\| \leq R$. Consequently, by the nonlinear alternative of Leray-Schauder type [24], this completes the proof.

For next theorem, we need the following fixed point theorem [25].

Theorem 2.5 *Let Ω be an open bounded subset of a Banach space E with $0 \in \Omega$ and $B : \bar{\Omega} \rightarrow E$ be a compact operator. Then B has a fixed point in $\bar{\Omega}$ provided $\|Bu - u\|^2 \geq \|Bu\|^2 - \|u\|^2, u \in \partial \Omega$.*

Theorem 2.6 *Assume that there exists a positive constant M_1 such that*

$$|f(t, u)| \leq \frac{M_1}{\Lambda_1}, \quad \forall t \in I, \quad u \in [-M_1, M_1],$$

where Λ_1 is given by (2.9). Then the BVP (1.1) has at least one solution.

Proof. Let us define $\Gamma = \{u \in C(I, \mathbb{R}) : \|u\| < M_1\}$. As in Theorem 2.3, using the assumption on $f(t, u)$, it can be shown that \mathbb{F} is completely continuous. Thus, in view of Theorem 2.5, we just need to show that

$$\|Fu\| \leq \|u\|, \quad \forall u \in \partial \Gamma. \tag{2.13}$$

For all $t \in I, u \in \partial \Gamma$, we have

$$\begin{aligned} |Fu(t)| &= \left| \int_0^t (t - qs)f(s, u(s))d_qs \right. \\ &\quad \left. + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]f(s, u(s))d_qs \right| \\ &\leq \left| \int_0^t (t - qs)f(s, u(s))d_qs \right| \\ &\quad + \left| \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]f(s, u(s))d_qs \right| \\ &\leq \frac{M_1}{\Lambda_1} \left| \int_0^t (t - qs)d_qs + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]d_qs \right|, \end{aligned}$$

which yields $\|Fu\| \leq M_1$. Since (2.13) holds, therefore, we obtain the result.

Remark 2.1 In view of the assumption $|f(t, u)| \leq \frac{c}{T^2} \|u\| + N, u \in C(I, \mathbb{R}), t \in I$ of

Theorem 2.4, we see that it suffices to take $M_1 \geq \frac{N\Delta_2 T^2}{1 - c\Delta_2}$.

Before presenting the last result, we state a fixed point theorem due to Krasnoselskii [26] which is needed to prove the existence of solutions of (1.1).

Theorem 2.7 (Krasnoselskii) Let Ω be a closed convex, bounded and nonempty subset of a Banach space X . Let F_1, F_2 be the operators such that: (i) $F_1 x + F_2 y \in \Omega$ whenever $x, y \in \Omega$; (ii) F_1 is compact and continuous; (iii) F_2 is a contraction mapping. Then there exists $z \in \Omega$ such that $z = F_1 z + F_2 z$.

Theorem 2.8 Assume that $f: I \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying (H_0) and the following assumption holds:

(H_3) $|f(t, u)| \leq \mu(t), \forall (t, u) \in I \times \mathbb{R}$, and $\mu \in C(I, \mathbb{R}_+)$.

If

$$\left(\frac{|\eta(1 + \eta q)|}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right| \right) LT^2 < 1, \tag{2.14}$$

then the boundary value problem (1.1) has at least one solution on I .

Proof. With $\max_{t \in I} |\mu(t)| = \|\mu\|$, let us fix $r \geq \|\mu\| \Lambda_1$ (Λ_1 is given by (2.9)) and define $B_r = \{u \in C : \|u\| \leq r\}$.

Define the operators $F_1 u$ and $F_2 u$ on the set B_r as

$$F_1 u(t) = \int_0^t (t - qs)f(s, u(s))d_qs,$$

$$F_2 u(t) = \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t - qs)]f(s, u(s))d_qs.$$

For $u, v \in B_r$, we have

$$\|F_1 u + F_2 v\| \leq \|\mu\| \left(\frac{1}{1 + q} + \frac{|\eta(1 + \eta q)|}{(1 + q)(\eta - 1)^2} + \left| \frac{\eta}{\eta - 1} \right| \right) T^2 = \|\mu\| \Lambda_1 \leq r,$$

which implies that $F_1 u + F_2 v \in B_r$. In view of the condition (2.14), it follows that F_2 is a contraction mapping. Continuity of f together with the assumption (H_3) implies that the operator F_1 is continuous and uniformly bounded on B_r . Let us define $\sup_{(t,u) \in I \times B_r} |f(t,u)| = f_{\max} < \infty$. Then, for $t_1, t_2 \in I$ with $t_2 < t_1$ and $u \in B_r$, we have

$$\begin{aligned} \|(F_1 u)(t_1) - (F_1 u)(t_2)\| &= \sup_{(.,u) \in I \times B_r} \left| \int_0^{t_1} (t_1 - qs)f(s, u(s))d_qs \right. \\ &\quad \left. - \int_0^{t_2} (t_2 - qs)f(s, u(s))d_qs \right| \\ &= \sup_{(.,u) \in I \times B_r} \left| \int_0^{t_1} (t_1 - t_2)f(s, u(s))d_qs \right. \\ &\quad \left. + \int_{t_2}^{t_1} (t_2 - qs)f(s, u(s))d_qs \right| \\ &\leq f_{\max} \left(\int_0^{t_1} (t_1 - t_2)d_qs + \int_{t_2}^{t_1} |t_2 - qs|d_qs \right), \end{aligned}$$

which is independent of u and tends to zero as $t_2 - t_1 \rightarrow 0$. So F_1 is relatively compact on B_r . Hence, by the Arzelá-Ascoli Theorem, F_1 is compact on B_r . Thus all the assumptions of Theorem 2.7 are satisfied and the conclusion of Theorem 2.7 implies that the boundary value problem (1.1) has at least one solution on I . This completes the proof.

3 Examples

Example 3.1. Consider

$$\begin{cases} D_{\frac{1}{2}}^2 u(t) = L(t^2 + \cos t + 1 + \tan^{-1} u(t)), & 0 \leq t \leq 1, L > 0, \\ u(0) = \frac{1}{3}u(1), \quad D_{\frac{1}{2}} u(0) = \frac{1}{3}D_{\frac{1}{2}} u(1). \end{cases} \quad (3.1)$$

Here, $f(t, u) = L(t^2 + \cos t + 1 + \tan^{-1} u)$, L to be fixed later, $q = 1/2$, $\eta = 1/3$ and $T = 1$. Clearly

$$|f(t, u) - f(t, v)| \leq L |\tan^{-1} u - \tan^{-1} v| \leq L |u - v|$$

and

$$\Delta_1 = \left(\frac{1}{1+q} + \frac{|\eta(1+\eta q)|}{(1+q)(\eta-1)^2} + \left| \frac{\eta}{\eta-1} \right| \right) T^2 = \frac{21}{12}.$$

For $L < \frac{1}{\Delta_1} = \frac{12}{21}$, it follows by Theorem 2.1 that the problem (3.1) has a unique solution.

Example 3.2. Consider the following problem

$$\begin{cases} D_{\frac{1}{2}}^2 u(t) = \frac{c}{(2\pi)} \sin(2\pi u(t)) + \frac{|u(t)|}{2(1+|u(t)|)} + \frac{1}{2}, & 0 \leq t \leq 1, c \geq 0, \\ u(0) = \frac{1}{3}u(1), \quad D_{\frac{1}{2}} u(0) = \frac{1}{3}D_{\frac{1}{2}} u(1). \end{cases} \quad (3.2)$$

Here $q = 1/2$, $\eta = 1/3$, $T = 1$, and c will be fixed later. Observe that

$$|f(t, u)| = \left| \frac{c}{(2\pi)} \sin(2\pi u) + \frac{|u(t)|}{2(1+|u(t)|)} + \frac{1}{2} \right| \leq c|u| + 1,$$

and $\Delta_2 = \frac{21}{12}$. Obviously $N = 1$ and we can choose $c < \frac{1}{\Delta_2} = \frac{12}{21}$. Thus, the conclusion of Theorem 2.4 applies to the problem (3.2).

Remark. By fixing η in the results of this paper, we can obtain some special new results for q -difference equations. For instance, our results correspond to the ones for second-order q -difference equations with anti-periodic boundary conditions ($u(0) = -u(T)$, $D_q u(0) = -D_q u(T)$) for $\eta = -1$, and the results for an initial value problem of second-order q -difference equations follow for $\eta = 0$. Furthermore, the classical results for second order boundary value problems with non-separated boundary conditions follow in the limit $q \rightarrow 1^-$.

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Authors' contributions

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Competing interests

The authors declare that they have no competing interests.

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