Nonlinear optics in a birefringent optical fiber

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We extend the perturbation theory of the nonlinear Schrödinger equation for the study of perturbed (nonintegrable) forms of the vector equation. We derive a set of linear equations that describe the radiation field shed by the soliton as it propagates down a birefringent optical fiber. The formalism is applied to the case when strong birefringence and higher-order dispersion are present in the fiber and to the study of polarization mode dispersion. Finally we discuss an analytical treatment of the mechanism that generates the soliton shadow.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NSE) is the model equation best suited to describe the propagation of an optical pulse in weakly nonlinear and dispersive media. A basic property of this equation is complete integrability by means of the inverse scattering transform (IST) [1], with the soliton—a localized optical pulse—one the fundamental solutions. Refinements to the basic model equation—such as the addition of further dispersive terms—produces an equation which is generally not integrable, but is susceptible to analysis using a perturbative theory developed around the IST [2,3]. Studies to date have tended to concentrate on the scaler form of the perturbed NSE, implicitly ignoring the influence of the polarization of the optical pulse. When the optical fiber along which the soliton propagates is birefringent, the effects of polarization are important (for example, a change in the polarization state of the pulse results in a change in the speed of propagation of the soliton) and we must then introduce a model equation: the vector nonlinear Schrödinger equation (VNSE).

When an ultrashort pulse propagates down an anomalously dispersive birefringent optical fiber, complex features develop which require explanation. The object of this article is to describe a formalism, developed within the framework of inverse scattering theory based on the Manakov system [4], which admits useful application to the study of many of these observed features. One such feature is polarization mode dispersion (PMD) which is one of the most important considerations in transmission systems; this is discussed further below. The theory developed in this article is a direct extension of one developed previously for the scaler problem [2,4]. Several authors have addressed the problem of the development of a perturbation theory for application to the study of perturbed forms of the (integrable) VNSE. An earlier study of this problem utilizes a perturbation theory derived from a direct linearization of the VNSE [5]. This is a complementary approach to that developed here, but does not, we contend, use the best mathematical framework—that based on the IST. The present analysis extends that published by Midrio *et al.* [6] who computed the change in the vector soliton parameters in the perturbed birefringent system, but did not give full consideration to the generation of the radiation field. Further work is reported in Refs. [7–12], initiated primarily by a need to address the problem of PMD. PMD is discussed explicitly in Ref. [13]; it is shown first that a change in the polarization state of the soliton results in a change in the soliton velocity. A random change in the birefringent axes along the fiber necessarily results in a random variation in the polarization state of the soliton pulse propagation along the fiber, with a resulting stochastic fluctuation in the pulse velocity. This is the origin of PMD. Further work in Ref. [14] compares PMD jitter with that arising from other sources of stochasticity within the fiber.

The article is structured as follows: in Sec. II we introduce the perturbed VNSE while in Sec. III a formal perturbation theory centered around IST is developed. In particular, evolution equations for the scattering data associated with the Manakov system are derived for the general case when arbitrary perturbation terms are added to the VNSE, and we show how this data is linked with the radiation field in a manner which bears a close resemblance to the Fourier transform pair obtained for linear systems. The conserved quantities for the VNSE are then introduced and (briefly) discussed, and evolution equations for those are obtained for the general case of an arbitrary perturbation, which need not be small; this is an exact result. Some features of inverse scattering theory for the Manakov system are also included in the Appendixes. Different applications of the perturbation theory are then discussed in Sec. IV. An associate field formalism analogous to that introduced for the scalar problem [15] is first introduced and a connection is established between these (two) components and the components of the radiation field. Three problems by way of application are then discussed: we consider first the case when strong birefringence is the only perturbation in the fiber, and then when birefringent and third-order dispersion are both present. We conclude with a detailed discussion of the generation of the soliton shadow.

II. THE PERTURBED VECTOR SYSTEM

Ultrashort pulse propagation down an anomalously dispersive, birefringent optical fiber is described by VNSE

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$$i(q_{1,x}+\mu q_{1,t}+\beta q_2)+\kappa q_1-q_{1,tt}-q_1(|q_1|^2+A|q_2|^2)-Bq_2^2q_1^*=iF_1, \tag{1a}$$

$$i(q_{2,x}-\mu q_{2,t}-\beta q_1)-\kappa q_2-q_{2,tt}-q_2(|q_2|^2-A|q_1|^2)+Bq_1^2q_2^*=iF_2, \tag{1b}$$

with $A+B=1$. Here, κ is the weak birefringence parameter, corresponding to a difference in the phase velocities between the two polarization modes, and μ is the strong birefringence parameter representing one half of the modal group velocity difference. The function $\beta(t)$ describes the twisting of the birefringence axis with distance down the fiber. The parameter A is the normalized cross-phase modulation coefficient. The equations describe the coupling of two linearly polarized modes, with q_1 and q_2 the complex amplitudes in each mode, in which case $A=2/3$, $B=1/3$. A simple transformation to circularly polarized modes results in a similar set of equations where now the subscripts "1" and "2" correspond to differently circularly polarized modes and $A=2$, $B=0$. Finally, F_1 and F_2 represent the higher-order effects for each mode, which may include higher-order dispersion, Brillouin scattering, and so on. We rearrange Eqs. (1) so that the perturbing terms containing the parameters μ, β, and κ are taken over to the right hand side and thereafter considered as special choices of $\boldsymbol{F}=(F_1,F_2)^{\mathrm{T}}$, namely we take this system in the form

$$i\boldsymbol{q}_x-\boldsymbol{q}_{tt}-2\boldsymbol{q}^{\dagger}\boldsymbol{q}\boldsymbol{q}=i\boldsymbol{F}, \tag{2}$$

where $\boldsymbol{q}=(q_1,q_2)^{\mathrm{T}}$. Deviations of A from the value $A=1$, and the term with the parameter B, are also subsumed into F. When $\boldsymbol{F}$ is set to zero the VNSE equation is known to be integrable using the techniques of inverse scattering theory [1]. In particular, it has the single soliton solution

$$\boldsymbol{q}\equiv\boldsymbol{q}_s=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}q_s, \tag{3}$$

where (scalar) q_s is defined by

$$q_s=2\eta_1\exp[-2i\xi_1t+4i(\xi_1^2-\eta_1^2)x]\mathrm{sech}[2\eta_1(t-4\xi_1t)]. \tag{4}$$

The solution, Eq. (3) is hereafter denoted as $\boldsymbol{q}_s$, the vector soliton. The parameters ξ_1 and η_1 characterize the soliton, and θ is the projection angle of the pulse onto each polarization mode. Without loss of generality, we hereafter set $\xi_1=0$ and $2\eta_1=1$.

Comment. With $\boldsymbol{F}=-\mu\sigma_3\boldsymbol{q}_t$, the transformation

$$\boldsymbol{p}=\exp(-i\mu\sigma_3t/2-i\mu^2x/4)\boldsymbol{q} \tag{5}$$

removes the birefringent term, producing the Manakov evolution equation for $\boldsymbol{p}(x,t)$

$$i\boldsymbol{p}_x-\boldsymbol{p}_{tt}-2\boldsymbol{p}^{\dagger}\boldsymbol{p}\boldsymbol{p}=\boldsymbol{0}.$$

This, of course, has the soliton solution quoted on the right hand side of Eq. (3), which we temporarily denote as $\boldsymbol{p}_s$. The corresponding soliton solution for $\boldsymbol{q}(x,t)$, which we denote $\boldsymbol{q}_s^{(\mu)}$, is found by inverting the transformation (5), to yield

$$\boldsymbol{q}_s^{(\mu)}(x,t)=\exp(i\mu^2x/4)\begin{pmatrix}\exp(i\mu t/2)\cos\theta\\ \exp(-i\mu t/2)\sin\theta\end{pmatrix}q_s, \tag{6}$$

with scalar q_s defined in Eq. (4). This approach has been discussed in Refs. [10,11] and will not be repeated here. In any case, this transform is not applicable when other effects, such as higher order dispersion, are present in the fiber.

III. PERTURBATION THEORY

A. The evolution equations

A general evolution equation for the VNSE family can be expressed in the form

$$i\boldsymbol{q}_x-k(-i\mathcal{D})\boldsymbol{q}=\boldsymbol{0}, \tag{7}$$

where $k(\omega)$ is an arbitrary function of the operator $\mathcal{D}$, which is defined by the action

$$\mathcal{D}\boldsymbol{f}=\boldsymbol{f}_t-\int_t^{+\infty}\{\boldsymbol{q}^{\dagger},\boldsymbol{f}\}_{\mathrm{A}}dt'\boldsymbol{q}(t)$$

on any vector function $\boldsymbol{f}$. Here $\{,\}_{\mathrm{A}}$ denotes the anti-Hermitian anticommutator operation, so that $\{\boldsymbol{h}^{\dagger},\boldsymbol{f}\}_{\mathrm{A}}=\boldsymbol{h}^{\dagger}\boldsymbol{f}+\boldsymbol{f}h^{\dagger}-\boldsymbol{f}^{\dagger}\boldsymbol{h}-\boldsymbol{h}f^{\dagger}$ for any vector functions $\boldsymbol{f}$ and $\boldsymbol{h}$. Further, $k(\omega)$ is the dispersion function derived from the linearized form of the appropriate member of the VNSE family, with $\boldsymbol{q}\sim\exp(i\omega t-ikx)$. For example, the choice $k(\omega)=-\omega^2$ gives the VNSE equation. Note the simplicity of the operator $\mathcal{D}$: the prescription $\omega\rightarrow-i\partial_t\rightarrow-i\mathcal{D}$ takes us from dispersion function, to linearized form of the NSE equation, to the VNSE equation.

The spectral transform is a mapping from a potential $\boldsymbol{q}(x,t)$ into a set of scattering data $S_{ij}(x,\zeta)$, $i,j=1,2,3$, where ζ is the eigenparameter. The inverse transform permits construction of the "potential" $\boldsymbol{q}$ from a limited set of the data S_{ij}. Formally, we have [16]

$$S_{ij}=\int_{-\infty}^{+\infty}\boldsymbol{\phi}^{(j)}\wedge\boldsymbol{\psi}^{(i)}\begin{pmatrix}\boldsymbol{q}\\ -\boldsymbol{q}^*\end{pmatrix}dt, \tag{8}$$

where

$$\begin{pmatrix}\boldsymbol{q}\\ -\boldsymbol{q}^*\end{pmatrix}=\frac{1}{\pi}\int_C\left(\frac{S_{21}}{S_{11}}\boldsymbol{\psi}^{(2)}\vee\tilde{\boldsymbol{\psi}}^{(1)}+\frac{S_{31}}{S_{11}}\boldsymbol{\psi}^{(3)}\vee\tilde{\boldsymbol{\psi}}^{(1)}\right)d\zeta-\frac{1}{\pi}\int_{\bar{C}}\left(\frac{\Delta_{21}}{\Delta_{11}}\boldsymbol{\psi}^{(1)}\vee\tilde{\boldsymbol{\psi}}^{(2)}+\frac{\Delta_{31}}{\Delta_{11}}\boldsymbol{\psi}^{(1)}\vee\tilde{\boldsymbol{\psi}}^{(3)}\right)d\zeta. \tag{9}$$

Here, $\boldsymbol{\phi}^{(i)}\wedge\tilde{\boldsymbol{\psi}}^{(j)}$ and $\boldsymbol{\psi}^{(i)}\vee\tilde{\boldsymbol{\psi}}^{(j)}$ are four component row and column vectors, respectively, whose components are made of products between Jost function components for the forward and adjoint scattering problems. Namely,

$$\boldsymbol{\phi}^{(i)}\wedge\tilde{\boldsymbol{\psi}}^{(j)}=(\phi_2^{(i)}\tilde{\psi}_1^{(j)},\phi_3^{(i)}\tilde{\psi}_1^{(j)},\phi_1^{(i)}\tilde{\psi}_2^{(j)},\phi_1^{(i)}\tilde{\psi}_3^{(j)}),$$

$$\boldsymbol{\psi}^{(i)}\vee\tilde{\boldsymbol{\psi}}^{(j)}=(\psi_1^{(i)}\tilde{\psi}_2^{(j)},\psi_1^{(i)}\tilde{\psi}_3^{(j)},-\psi_2^{(i)}\tilde{\psi}_1^{(j)},-\psi_3^{(i)}\tilde{\psi}_1^{(j)})^T.$$

The quantities Δ_{ij} are cofactors of the matrix elements S_{ij}, while C ($\bar{C}$) is a contour running from $-\infty+i\epsilon$ ($-\infty-i\epsilon$) to $+\infty+i\epsilon$ ($+\infty-i\epsilon$) passing above (below) all zeros of S_{11} (Δ_{11}). A summary of the spectral transform and some remarks for these quantities can also be found in the Appendixes; see Ref. [16] for further details.

For a perturbed system, Eq. (7) is modified by adding $(i\boldsymbol{F},-i\boldsymbol{F}^*)$ to the right hand side, where $\boldsymbol{F}$ is the perturbation, e.g., strong birefringence in the fiber, with $\boldsymbol{F}=(\mu q_{1t},-\mu q_{2t})^T$, etc. This modified form for Eq. (7) can be substituted into

$$S_{ij,x}=\int_{-\infty}^{+\infty}\boldsymbol{\phi}^{(j)}\wedge\tilde{\boldsymbol{\psi}}^{(i)}\begin{pmatrix}\boldsymbol{q}_x\\ -\boldsymbol{q}_x^*\end{pmatrix}dt,$$

which expresses the evolution of the scattering data to give the final result

$$S_{ij,x}=S_{ij,x}^{(0)}+\int_{-\infty}^{+\infty}\boldsymbol{\phi}^{(j)}\wedge\tilde{\boldsymbol{\psi}}^{(i)}\begin{pmatrix}\boldsymbol{F}\\ -\boldsymbol{F}^*\end{pmatrix}dt, \tag{10}$$

where the term $S_{ij,x}^{(0)}$ represents the evolution for the unperturbed system. In particular, $S_{i1,x}^{(0)}=-4i\zeta^2S_{i1}, i=2,3,$ while $S_{11,x}^{(0)}=0$.

B. The radiation fields

We are interested in the case when a pulse comprising a soliton and a radiation field propagates down the fiber. Then, $\boldsymbol{q}(x,t)=\boldsymbol{q}_s(x,t)+\boldsymbol{\delta q}(x,t)$. Expressing the integrals in Eq. (9) in terms of their discrete and continuum contributions gives

$$\begin{aligned}\begin{pmatrix}q_1\\ q_2\end{pmatrix}=&\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{S_{21}}{S_{11}}\begin{pmatrix}\psi_1^{(2)}\tilde{\psi}_2^{(1)}\\ \psi_1^{(2)}\tilde{\psi}_3^{(1)}\end{pmatrix}+\frac{S_{31}}{S_{11}}\begin{pmatrix}\psi_1^{(3)}\tilde{\psi}_2^{(1)}\\ \psi_1^{(3)}\tilde{\psi}_3^{(1)}\end{pmatrix}\right)d\zeta\\ &-\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\Delta_{21}}{\Delta_{11}}\begin{pmatrix}\psi_1^{(1)}\tilde{\psi}_2^{(1)}\\ \psi_1^{(1)}\tilde{\psi}_3^{(2)}\end{pmatrix}+\frac{\Delta_{31}}{\Delta_{11}}\begin{pmatrix}\psi_1^{(1)}\tilde{\psi}_2^{(3)}\\ \psi_1^{(1)}\tilde{\psi}_3^{(3)}\end{pmatrix}\right)d\zeta\\ &-2i\sum_{k=1}^{N}\begin{pmatrix}\phi_1^{(1)}(\zeta_k)\tilde{\psi}_2^{(1)}(\zeta_k)\\ \phi_1^{(1)}(\zeta_k)\tilde{\psi}_3^{(1)}(\zeta_k)\end{pmatrix}\\ &-2i\sum_{k=1}^{\bar{N}}\begin{pmatrix}\psi_1^{(1)}(\bar{\zeta}_k)\tilde{\phi}_2^{(1)}(\bar{\zeta}_k)\\ \psi_1^{(1)}(\bar{\zeta}_k)\tilde{\phi}_3^{(1)}(\bar{\zeta}_k)\end{pmatrix}.\end{aligned}$$

The eigenparameter ζ appears in the scattering equations (see the Appendix), while ζ_k and $\bar{\zeta}_k$ are the zeroes of S_{11} and Δ_{11}, respectively. We identify the radiation field as the integrals of the above equation. The discrete sum gives the contribution to $\boldsymbol{q}(x,t)$ from a general N-soliton state. From symmetries associated with VNSE, it follows that $\bar{N}=N$, $\bar{\zeta}=\zeta^*$. Since here $N=1$ the latter contribution is simply $\boldsymbol{q}_s$, while $\boldsymbol{\delta q}(x,t)=(\delta q_1,\delta q_2)^T$ is obtained from

$$\begin{aligned}\begin{pmatrix}\delta q_1\\ \delta q_2\end{pmatrix}=&\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{S_{21}}{S_{11}}\begin{pmatrix}\psi_1^{(2)}\tilde{\psi}_2^{(1)}\\ \psi_1^{(2)}\tilde{\psi}_3^{(1)}\end{pmatrix}+\frac{S_{31}}{S_{11}}\begin{pmatrix}\psi_1^{(3)}\tilde{\psi}_2^{(1)}\\ \psi_1^{(3)}\tilde{\psi}_3^{(1)}\end{pmatrix}\right)d\zeta\\ &-\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{\Delta_{21}}{\Delta_{11}}\begin{pmatrix}\psi_1^{(1)}\tilde{\psi}_2^{(2)}\\ \psi_1^{(1)}\tilde{\psi}_3^{(2)}\end{pmatrix}+\frac{\Delta_{31}}{\Delta_{11}}\begin{pmatrix}\psi_1^{(1)}\tilde{\psi}_2^{(3)}\\ \psi_1^{(1)}\tilde{\psi}_3^{(3)}\end{pmatrix}\right)d\zeta.\end{aligned} \tag{11}$$

Again $\psi_j^{(i)}$, $\tilde{\psi}_j^{(i)}$ are components of Jost functions, while S_{ij} and Δ_{ij}, which depend on ζ and x, are elements of the scattering data. For first order in perturbation theory, $\psi_j^{(i)}$ etc. will be approximated by their solitonic expressions (all known, see the appendix), while S_{ij} and $\Delta_{ij}=S_{ij}^*$ evolve from an initial value $S_{ij}(x=0,\zeta)=0$ in accordance with a soliton input to the fiber.

Note that Eqs. (8) and (9) are the direct extension of the application of the Fourier transform to linear systems, as appropriate to the integrable VNSE equation. Indeed, in the limit where the pulse $\boldsymbol{q}(x,t)$ has no soliton component and simply represents a weak radiation field, $\boldsymbol{\delta q}(x,t)$ say, Eqs. (8) and (9) reduce to

$$\begin{pmatrix}S_{21}\\ S_{31}\end{pmatrix}=-\int_{-\infty}^{+\infty}\begin{pmatrix}\delta q_1^*\\ \delta q_2^*\end{pmatrix}\exp(-i\omega t)dt,$$

$$\begin{pmatrix}\delta q_1^*\\ \delta q_2^*\end{pmatrix}=-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\begin{pmatrix}S_{21}\\ S_{31}\end{pmatrix}\exp(i\omega t)dt,$$

where $*$ denotes complex conjugation, and $\omega=2\xi=2\mathrm{Re}\{\zeta\}$. Each component δq_1^*, δq_2^* of $\boldsymbol{\delta q}^*$ is here linked to the one piece of scattering data, S_{21} and S_{31}, respectively. This simplifying feature is lost for the full (nonlinear) system.

The intention here is the following: for a choice of perturbations $\boldsymbol{F}$, the integral in Eq. (10) is evaluated first, assuming solitonic forms for $\phi^{(j)}$ and $\tilde{\psi}^{(i)}$, after which the differential equation is solved to yield $S_{ij}(x,\zeta)$. This is then substituted into Eq. (11), yielding the required forms for the radiation fields δq_1 and δq_2, namely, Eqs. (20).

C. The conserved quantities

Any member of the vector NSE family has infinitely many conserved quantities. The conserved quantities are obtained by examining the asymptotics of $\phi_1^{(1)}$ as $t\to+\infty$ [specifically, we examine $\ln(\phi_1^{(1)}\exp(i\xi t))\equiv\ln S_{11}$, cf. Eq. (15), as a formal asymptotic expansion in inverse powers of ξ, taken in limit $\xi\to+\infty$], together with the result that S_{11} does not vary with x. Denoting the conserved quantities by C_n, $n=0,1,2,\ldots,$ it is thus found that

$$C_n=\int_{-\infty}^{+\infty}(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\rho}_n)dt, \tag{12}$$

where

$$\boldsymbol{\rho}_0=\boldsymbol{q}^*,$$

$$\boldsymbol{\rho}_1=\boldsymbol{q}_t^*,$$

and

$$\boldsymbol{\rho}_{n+1}=\boldsymbol{\rho}_{n,t}+\sum_{i=0}^{n-1}\boldsymbol{\rho}_i\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\rho}_{n-i-1}.$$

Note that $\boldsymbol{\rho}_i$ are two-component vectors. In particular,

$$C_2=\int_{-\infty}^{+\infty}[\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}_{tt}^*+(\boldsymbol{q}^\dagger\boldsymbol{q})^2]dt$$

is the Hamiltonian functional for Eq. (2), when the term F is set to zero.

As for the scalar problem, it is possible to introduce a set of trace formulas for the VNSE. Introduce the integral

$$I(\zeta)=\int_{-\infty}^{+\infty}\frac{S_{11}'}{S_{11}}\frac{d\xi}{\xi-\zeta}+\int_{-\infty}^{+\infty}\frac{\Delta_{11}'}{\Delta_{11}}\frac{d\xi}{\xi-\zeta},\tag{13}$$

where the prime denotes a derivative with respect to ξ, and ζ has a positive imaginary part, i.e., $\mathrm{Im}\{\zeta\}>0$. Then, since S_{11} and Δ_{11} are known to behave like $1+O(1/\zeta^2)$ as $|\zeta|\to\infty$, Eq. (13) can be evaluated by the usual techniques of contour integration, by considering semicircular paths in the upper half plane for the first term, and the lower half plane for the second, giving

$$I(\zeta)=2\pi i\sum_{n=1}^{N}\left(\frac{1}{\zeta-\overline{\zeta_n}}-\frac{1}{\zeta-\zeta_n}\right)+2\pi i\frac{d}{d\zeta}\ln S_{11}(\zeta).\tag{14}$$

Again, ζ_n and $\overline{\zeta_n}$ are the zeroes of S_{11} and Δ_{11}, respectively. Now consider Eq. (14) in the limit $\mathrm{Im}\{\zeta\}\to 0$, $\mathrm{Re}\{\zeta\}\to+\infty$. Identifying C_n as the coefficients of a formal expansion of $\ln S_{11}$, i.e.,

$$\ln S_{11}=\sum_{n=0}^{\infty}\frac{C_n}{(2i\zeta)^{n+1}},\tag{15}$$

we find

$$C_n=\sum_{m=1}^{N}[(2i\overline{\zeta_m})^{n+1}-(2i\zeta_m)^{n+1}]-\frac{1}{\pi}\int_{-\infty}^{+\infty}(2i\xi)^n\ln(1-|S_{21}|^2-|S_{31}|^2)d\xi.\tag{16}$$

The discrete sum gives the contribution to C_n from an arbitrary N-soliton state, whereas the integral denotes the contribution from the continuum radiation modes. The equivalent form for C_n, Eqs. (12) and (16), constitute the trace formulas for the VNSE.

For the unperturbed system, each C_n satisfies the evolution equation $dC_n/dx=0$. With the introduction of the perturbing term $i\boldsymbol{F}$, these become

$$\frac{dC_n}{dx}=2(-i)^n\mathrm{Re}\left\{\int_{-\infty}^{+\infty}\boldsymbol{F}^\dagger(-i\mathcal{D})^n\boldsymbol{q}dt\right\}.$$

Note that the evolution of each C_n is determined by the projection of the perturbing term $\boldsymbol{F}^\dagger$ onto the nth flow of the VNSE family, i.e., onto $(-i\mathcal{D})^n\boldsymbol{q}$.

With $\boldsymbol{F}$ corresponding to strong birefringence, i.e., $\boldsymbol{F}=-\mu\sigma_3\boldsymbol{q}_t$ (or strong birefringence plus third-order dispersion where $\boldsymbol{F}=-\mu\sigma_3\boldsymbol{q}_t+\delta\boldsymbol{q}_{ttt}$), it is easy to show that both C_0 and C_1 are conserved implying in turn that the soliton parameters ξ_1 and η_1 are similarly constants of the motion. That is, throughout this article, any perturbations $\boldsymbol{F}$ will be considered small, $O(\epsilon)$ say. Then from Eq. (10), the inhomogeneous term generating $S_{ij}(x)$ is $O(\epsilon)$ so, if $S_{ij}(x=0)$ is zero—as appropriate for soliton input to the fiber—$S_{ij}(x)$ is $O(\epsilon)$. The integral term in Eq. (16) is therefore $O(\epsilon^2)$, as are changes in η_1 and ξ_1 whenever C_0 and C_1 are conserved quantities.

Using the results of a related perturbation theory developed elsewhere [13], we can similarly show that the above choices for $\boldsymbol{F}$ result in $d\theta/dx=0$, where θ is the polarization angle. In other words, the polarization state of the pulse is not altered. With no loss of generality we assume that pulses are linearly polarized with polarization angle θ within the fiber.

IV. APPLICATIONS

A. The associate field formalism

We are interested in the equation

$$i\boldsymbol{q}_x-\boldsymbol{q}_{tt}-2\boldsymbol{q}^\dagger\boldsymbol{q}\boldsymbol{q}=i\boldsymbol{F},\tag{17}$$

where $\boldsymbol{F}=-\mu\sigma_3\boldsymbol{q}_t$ represents the effect of strong birefringence within the fiber. The intention is to analyze Eq. (17) using the perturbation theory developed in the preceding section.

Evaluating the integrals in Eq. (10) produces

$$\begin{pmatrix}S_{21}\\S_{31}\end{pmatrix}_x=-4i\zeta^2\begin{pmatrix}S_{21}\\S_{31}\end{pmatrix}-2i\zeta\mu\begin{pmatrix}S_{21}\\-S_{31}\end{pmatrix}+\frac{i\mu}{2}\sin(2\theta)(2\zeta-i)\times\begin{pmatrix}\sin\theta\\-\cos\theta\end{pmatrix}\hat{q}_s^*.\tag{18}$$

The additional contribution

$$-2i\zeta\mu\begin{pmatrix}S_{21}\\-S_{31}\end{pmatrix}$$

has been introduced as derived in Appendix B. This is precisely the additional term required to ensure that S_{21} and S_{31} follow their respective characteristics. We now set $\zeta=\xi\in\mathbb{R}$ to generate the continuum field. More details about the origin of this term can be found in the appendix.

The evolution of the spectral data is now governed by Eq. (18), subject to the initial condition that $S_{21}(0,\xi)=S_{31}(0,\xi)=0$. As for the scalar problem [15], it is useful to introduce two quantities related to S_{21} and S_{31}, namely, the *associate fields* $f_1(x,t)$ and $f_2(x,t)$. Define

$$\hat{f}_1(x,\xi)=\frac{S_{21}^*(x,\xi)}{4\xi^2+1},$$

$$\hat{f}_2(x,\xi)=\frac{S_{31}^*(x,\xi)}{4\xi^2+1},$$

where $\hat{f}(x,\xi)\equiv\mathcal{F}\{f(x,t)\}=\int_{-\infty}^{+\infty}\exp(-2i\xi t)f(x,t)dt$ is the Fourier transforms of $f(x,t)$. Then, Eq. (18) becomes

$$-i\begin{pmatrix}\hat{f}_1\\ \hat{f}_2\end{pmatrix}_x=4\xi^2\begin{pmatrix}\hat{f}_1\\ \hat{f}_2\end{pmatrix}+2\xi\mu\begin{pmatrix}\hat{f}_1\\ -\hat{f}_2\end{pmatrix}+\frac{2\mu\sin(2\theta)}{2\xi+i}\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}\hat{q}_s^*,$$

or in t space

$$-i\begin{pmatrix}f_1\\ f_2\end{pmatrix}_x=\begin{pmatrix}f_1\\ f_2\end{pmatrix}_{tt}+i\mu\begin{pmatrix}f_1\\ -f_2\end{pmatrix}_t-\frac{i\mu}{2}\sin(2\theta)\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix}q_s\otimes h, \qquad (19)$$

where q_s is the (scalar) soliton expression Eq. (4),

$$h(t)=\begin{cases}\exp(t), & t<0\\ 0, & t>0\end{cases}$$

and $\otimes$ denotes convolution product. From Eq. (11), using solitonic expressions for $\psi_j^{(i)}$, $S_{11}(\xi)=(2\xi-i)/(2\xi+i)=\Delta_{11}^*(\xi)$, and then evaluating the various integrals, the following expressions are obtained for δq_1 and δq_2:

$$-\delta q_1=(M-N\sin^2\theta)f_1+\tfrac{1}{2}\sin(2\theta)Nf_2-q_s^2\cos\theta(f_1^*\cos\theta+f_2^*\sin\theta) \qquad (20a)$$

$$-\delta q_2=(M-N\cos^2\theta)f_2+\tfrac{1}{2}\sin(2\theta)Nf_1-q_s^2\sin\theta(f_2^*\sin\theta+f_1^*\cos\theta). \qquad (20b)$$

To make these awkward expressions more manageable, we have introduced the operators

$$M=\frac{\partial^2}{\partial t^2}-2\tanh t\frac{\partial}{\partial t}+\tanh^2 t,$$

$$N=(1-\tanh t)\frac{\partial}{\partial t}+\tanh^2 t-\tanh t.$$

The algorithm for finding δq_1 and δq_2 is first to solve Eqs. (19) for $f_1(x,t)$ and $f_2(x,t)$ subject to the initial condition that $f_1(0,t)$ and $f_2(0,t)$ are both zero (so that the limitations of the method derived in Ref. [17] do not apply here). This is now straightforward since both f_1 and f_2 satisfy *linear* differential equations, and can be easily obtained using standard (Fourier) transform methods. We find δq_1 and δq_2 simply by using Eqs. (20). This is, again, relatively straightforward requiring only differentiation of the known functions f_1 and f_2.

Using the fact that the Manakov system in invariant under rotation we project Eqs. (20) onto the soliton polarization states. By introducing the quantities $f_\perp=f_1\sin\theta-f_2\cos\theta$ and $f_\parallel=f_1\cos\theta+f_2\sin\theta$, the projections of the vector $(f_1,f_2)^{\mathrm{T}}$ onto the polarization modes orthogonal ($\boldsymbol{e}_\perp$) and parallel ($\boldsymbol{e}_\parallel$) to the soliton pulse, respectively, we obtain

$$f_{\perp,tt}-(\tanh t+1)f_{\perp,t}+\tanh t f_\perp=-(\delta q_1\sin\theta-\delta q_2\cos\theta)=-\delta q_\perp,$$

$$f_{\parallel,tt}-2\tanh t f_{\parallel,t}+\tanh^2 t f_\parallel-\mathrm{sech}^2 t f_\parallel^*\exp(-2ix)=-(\delta q_1\cos\theta+\delta q_2\sin\theta)=-\delta q_\parallel.$$

Both modes contribute to the radiation field, unlike observations made elsewhere [7,8] (also see Sec. IV A 2).

When θ is 0 or $\pi/2$, the source term in Eq. (19) vanishes and hence both f_1 and f_2 remain at their initial value of zero; in consequence δq_1 and δq_2 are both zero. Simply setting $\theta=0$, then $\pi/2$, in Eqs. (20) produces the relations $\delta q_2=Mf_1-q_s^2f_1^*$ and $\delta q_2=Mf_2-q_s^2f_2^*$, from the first and second equations, respectively; these are just the expressions obtained for the scalar problem [15].

Projecting the evolution Eqs. (19) onto the polarization vectors $(\cos\theta,\sin\theta)^{\mathrm{T}}$ and $(-\sin\theta,\cos\theta)^{\mathrm{T}}$ results in evolution equations where the source term is first zero, then $(i\mu\sin(2\theta)/2)q_s\otimes h$, respectively. In other words, the source generates radiation orthogonally polarized to the soliton pulse [to $O(\epsilon)$]. There is an interesting asymmetry in the source term. The term $q_s\otimes h$ peaks at a slightly larger value of t than does q_s. At first sight this appears odd; there is nothing in the formulation of this problem nor in our choice of perturbation (strong birefringence) to have allowed one to anticipate this loss of t symmetry in the perturbing term. However, we believe it may be related to the generation of the shadow, which in turn is related to the fast polarization mode instability reported elsewhere [18].

The qualitative features of Eqs. (19) and (20) are straightforward: dispersive radiation is generated, which then propagates along the characteristics $x\pm\mu t$. Both these contribute to the generation of both δq_1 and δq_2, in accordance with Eqs. (20). Near the soliton, δq_1 and δq_2 have a complicated structure with no readily discernable features. Away from the soliton—that is, at large values of $|t|$—we expect the radiation field to evolve in accordance with the linear theory: that is, a predominance of δq_1 should appear in the slow polarization mode, δq_2 in the fast, with each field propagating away from the (source) soliton pulse at a group velocity determined by the frequency shifts $\delta\omega=\pm\mu/2$. At large values of $|t|$, the cross terms proportional to q_s^2 can be ignored in Eqs. (20), and we may approximate $\tanh t\simeq\pm1$ as appropriate. Then,

$$M \simeq \left(\frac{\partial}{\partial t} \mp 1\right)^2$$

as $t \to \pm\infty$, while

$$N \simeq \begin{cases} 0 & \text{as } t \to 0, \\ 2\left(\frac{\partial}{\partial t}+1\right) & \text{as } t \to -\infty. \end{cases}$$

Hence, as $t \to +\infty$

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \end{pmatrix} \simeq \left(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t} + 1\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{21}$$

and as $t \to -\infty$

$$\begin{pmatrix} \delta q_1 \\ \delta q_2 \end{pmatrix} \simeq \left(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t} + 1\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} 2\left(\frac{\partial}{\partial t}+1\right) f_\perp . \tag{22}$$

We have an interesting asymmetry with no ready explanation. For large values of the parameter μ (let us assume, for the moment, that the perturbation theory continues to hold), one expects f_1 to dominate f_2 as $t \to +\infty$, since the characteristic for f_1 is $t-\mu x$, and hence we expect δq_1 to dominate δq_2; this would be in accord with simple intuition. The same intuition—with f_2 now dominating f_1—fails at $t \to -\infty$ because of the presence of $f_\perp$ in Eq. (22); here, now, (large) f_2 will also contribute to δq_1. If the latter terms were missing, the other difference between Eqs. (21) and (22) can be explained in terms of the phase shift induced by the presence of the soliton pulse: i.e., $(\partial_t - 1)^2/(\partial t + 1)^2 \to (\omega + i)^2/(\omega - i)^2$ in frequency space, which is the phase shift experienced by a linear plane wave $\exp(i\omega t)$ on passing from $t \to +\infty$ to $t \to -\infty$ through a soliton pulse [15].

1. Third-order dispersion

We shall now add an additional perturbing term in the birefringent VNSE system, which represents the effect of third-order dispersion. The total effect will be mathematically modeled by $\boldsymbol{F} = -\mu \boldsymbol{q}_t + \delta \boldsymbol{q}_{ttt}$. The second term represents third-order dispersion and the parameter δ, of $O(\epsilon)$, suggests that this term is of the same order as the birefringence μ term.

Evaluating the integrals in Eq. (10) gives

$$\begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix}_x = -4i\zeta^2 \begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix} - 2i\zeta\mu \begin{pmatrix} S_{21} \\ -S_{31} \end{pmatrix} + i\mu \sin(2\theta)(\zeta - i\eta_1) \times \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \hat{q}_s^* + 4i\delta\zeta(\zeta^2+\eta_1^2)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\hat{q}_s^* - 8i\delta\zeta^3 \begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix},$$

where $\hat{q}_s^* = \pi \exp(ix)\operatorname{sech}(\pi\zeta)$ and again we set $2\eta_1 = 1$. Note the "extra" term

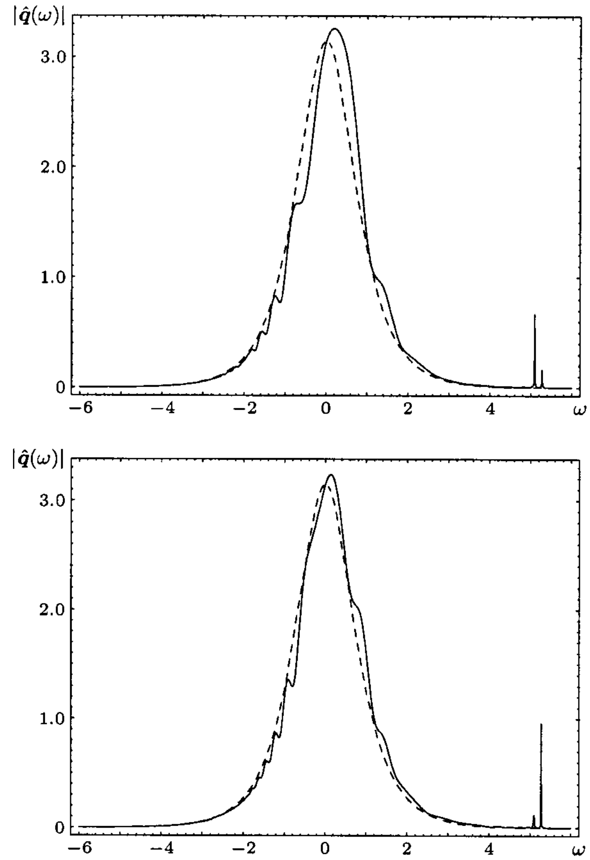

FIG. 1. Pulse spectrum $|\hat{\boldsymbol{q}}(\omega)|$ vs ω at "distances" $x=5$ (top) and $x=7.5$ (bottom) down the fiber with $\mu=0.1$, $\delta=0.2$, and $\theta=\pi/4$. All variables are dimensionless. The dashed line is the soliton.

$$-8i\delta\zeta^3 \begin{pmatrix} S_{21} \\ S_{31} \end{pmatrix}.$$

This is obtained in a similar manner as the one described in the appendix for the strong birefringence term. Also recall that to $O(\epsilon)$ the soliton parameters remain constants of the motion.

Using the associate field formalism we obtain

$$\hat{f}_{1,x} = i(\omega^2 - \mu\omega - \delta\omega^3)\hat{f}_1 + \left(\frac{2i\mu\sin(2\theta)}{\omega+i}\sin\theta + 2i\delta\omega\cos\theta\right)\hat{q}_s, \tag{23a}$$

$$\hat{f}_{2,x} = i(\omega^2 + \mu\omega - \delta\omega^3)\hat{f}_2 + \left(-\frac{2i\mu\sin(2\theta)}{\omega+i}\cos\theta + 2i\delta\omega\sin\theta\right)\hat{q}_s, \tag{23b}$$

where $\omega = -2\xi$ is the frequency (recall that we set $\zeta = \xi \in \mathbb{R}$ to generate the continuum). These simple equations will be shown to account for all features noted in the numerically produced spectra of Fig. 1.

The explicit x dependance represented by the factor $\exp(-ix)$ contained in $\hat{q}_s$ is easily removed by

$$\hat{\boldsymbol{f}}=\begin{pmatrix}\hat{f}_1\\ \hat{f}_2\end{pmatrix}\rightarrow\exp(ix)\begin{pmatrix}\hat{f}_1\\ \hat{f}_2\end{pmatrix},$$

then Eqs. (23) become

$$\hat{f}_{1,x}=iD_{-}(\omega)\hat{f}_1+\mathrm{sech}(\pi\omega)\left(\frac{2i\mu\sin(2\theta)}{\omega+i}\sin\theta+2i\delta\omega\cos\theta\right),\tag{24a}$$

$$\hat{f}_{2,x}=iD_{+}(\omega)\hat{f}_2-\mathrm{sech}(\pi\omega)\left(\frac{2i\mu\sin(2\theta)}{\omega+i}\cos\theta-2i\delta\omega\sin\theta\right),\tag{24b}$$

where

$$D_{\mp}(\omega)=\omega^2\mp\mu\omega-\delta\omega^3+1.$$

The most prominent feature are the resonance peaks in the pulse spectrum, which are observed to occur at $D_{\mp}=0$; this is the origin of these resonance peaks. For any value of propagation distance x, $\hat{\boldsymbol{f}}$ will be returned to zero at those frequency components satisfying $xD_{\mp}(\omega)=2n\pi$, where $n\in\mathbb{Z}$.

The addition of further dispersive terms will not alter things in any significant way. One then must find the new dispersion function and the zeros (perhaps more than two) will be on the points where secular growth occurs.

2. Polarization mode dispersion

Polarization mode dispersion (PMD) is a factor that must be taken into account when transmitting over long fiber distances. In fibres, PMD is caused by the refractive index not exhibiting perfect rotational symmetry around the fiber axis. As a result, the two possible polarization states of the fiber propagate light with slightly different speeds. This difference in propagation speed between the slow and fast fiber axis leads to a broadening of the transmitted bits. PMD is currently a research topic attracting much attention. Its analytical treatment is quite complex in general because of its statistical nature [19].

Birefringence creates differing optical axes that generally correspond to the fast and slow propagation mode axes. It causes one polarization mode to travel faster than the other, resulting in a difference in the propagation time. In a linear system, pulse broadening can be estimated from the time delay Δt between the two polarization components during propagation of the pulse. We discuss here the complementary effect for soliton systems.

We define the relative time displacement between the two polarization modes as

$$\Delta t(x)=\mathrm{Re}\left\{\int_{-\infty}^{+\infty}t\,\delta q_2 q_{2s}^{*}dt\right\}-\mathrm{Re}\left\{\int_{-\infty}^{+\infty}t\,\delta q_1 q_{1s}^{*}dt\right\}=\Delta t_2(x)-\Delta t_1(x).$$

Recall that the soliton is in one polarization state $\boldsymbol{e}_{\parallel}$, whereas the radiation field is in two polarization modes $(\boldsymbol{e}_{\parallel},\boldsymbol{e}_{\perp})$ orthogonal to each other. For that we introduce ΔT_1 and ΔT_2 in order to have the two orthogonal modes in the integrals, so that

$$\Delta T_1=\Delta t_1\tan\theta-\Delta t_2\cot\theta,$$

$$\Delta T_2=\Delta t_1+\Delta t_2,$$

then

$$\Delta t(x)=-\sin(2\theta)\,\Delta T_1-\cos(2\theta)\,\Delta T_2.$$

Using Eqs. (11), it can be shown that, $\Delta T_2\equiv 0$ and finally

$$\Delta t(x)=\sin(2\theta)\,\Delta T_1=\sin(2\theta)\times\mathrm{Re}\left\{\exp(ix)\int_{-\infty}^{+\infty}\exp(-t)\,\mathrm{sech}^2 t F(x,t)dt\right\}.\tag{25}$$

After using the identity

$$\int_{-\infty}^{+\infty}f(t)g^{*}(t)dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega)\hat{g}^{*}(\omega)d\omega,$$

the relative time displacement, Eq. (25), can be expressed as

$$\Delta t(x)=\sin(2\theta)\mathrm{Re}\left\{\frac{i\exp(ix)}{2}\int_{-\infty}^{+\infty}(\omega-i)\times\mathrm{sech}(\pi\omega/2)\hat{f}_{\perp}(x,\omega)d\omega\right\},\tag{26}$$

where

$$\hat{f}_{\perp}(x,\omega)=\frac{2\mu\pi\sin(2\theta)\mathrm{sech}(\pi\omega/2)}{\omega-i}\times\left(\frac{\sin^2\theta}{1+\omega_{+}^2}[\exp(-ix)-\exp(i\omega_{+}^2x)]+\frac{\cos^2\theta}{1+\omega_{-}^2}[\exp(-ix)-\exp(i\omega_{-}^2x)]\right),\tag{27}$$

and $\omega_{\pm}^2=\omega^2\pm\mu\omega$.

Figure 2 shows the relative time displacement obtained using Eqs. (26) and (27) versus normalized distance with parameters $\theta=\pi/3$ and $\mu=0.01$. The graph shows a rapid decay at first and then Δt approaches a $1/\sqrt{x}$ behavior, as

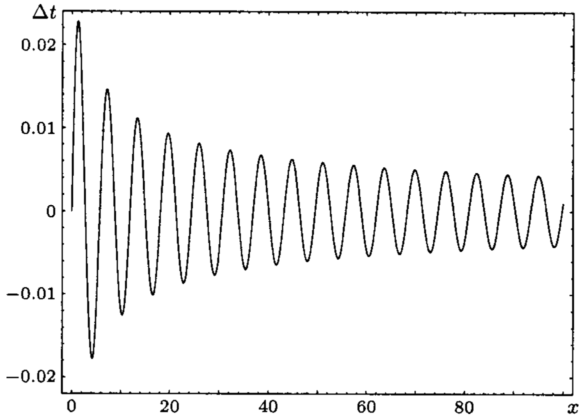

FIG. 2. Time displacement Δt corresponding to Eq. (26), with birefringence parameter $\mu = 0.01$ and polarization angle $\theta = \pi/3$. All quantities are dimensionless.

observed in Ref. [7]. This can now be verified by approximating Eq. (26) for large values of x using the method of steepest descents.

Comment. These results do not imply that the continuum is zero at the orthogonal polarization state $\boldsymbol{e}_{\parallel}$, as argued in Ref. [8]. Recall that

$$\delta q_{\parallel} = -M f_{\parallel} + q_s^2 f_{\parallel}^*,$$

since f_1 and f_2 are nonzero [cf., Eq. (19)], so is $\delta q_{\parallel}$. The interesting feature here is that radiation generated parallel to the vector soliton is nonoscillatory (in the sense discussed above), evolving very much in the manner of the radiation field associated with the scalar problem [15]. Conversely, the perpendicular component $\delta q_{\perp}$ exhibits strong oscillatory motion.

In a "real" optical fiber, the birefringence axes vary along the fiber length in a random manner, giving rise to a similar random variation in the pulse velocity. This is the origin of PMD jitter, an important consideration in the design of a soliton based communication system. The problem is analyzed in detail in Refs. [13,14], and references therein.

B. The soliton shadow

Equations (11) and (18) describe in full the properties of the radiation field generated as a result of birefringence in an optical fiber. This is clearly a complicated system to analyze, so we begin with some preliminary comments. Consider Eq. (18) first: we see here that both S_{21} and S_{31} contribute to the generation of both δq_1 and δq_2, contrary to what one might have anticipated. A simple rearrangement of Eq. (18) permits it to be written in the form

$$\boldsymbol{\delta q} = \boldsymbol{\delta q}_{\parallel} + \boldsymbol{\delta q}_{\perp}, \tag{28}$$

with

$$\boldsymbol{\delta q}_{\parallel} = \delta q_{\parallel} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \boldsymbol{\delta q}_{\perp} = \delta q_{\perp} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

where

$$\delta q_{\parallel} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\frac{b_{\parallel}}{a}\psi_2^2 + \frac{b_{\parallel}^*}{a^*}\psi_1^2\right) d\xi, \tag{29a}$$

$$\delta q_{\perp} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{b_{\perp}^*}{a^*}\exp(-i\xi t)\psi_1 d\xi. \tag{29b}$$

Here, we have introduced

$$b_{\parallel} = S_{21}\cos\theta + S_{31}\sin\theta, \tag{30a}$$

$$b_{\perp} = -S_{21}\sin\theta + S_{31}\cos\theta, \tag{30b}$$

$$a = S_{11}, \tag{30c}$$

while

$$\psi_1 = \frac{\exp(-i\xi t)}{\xi - i\eta_1}[\xi - i\eta_1 \tanh(2\eta_1 t)], \tag{31a}$$

$$\psi_2 = -\frac{i\eta_1}{\xi + i\eta_1}\exp(i\xi t - 4i\eta_1^2 x)\operatorname{sech}(2\eta_1 t). \tag{31b}$$

Note that Eq. (28) resolves $\boldsymbol{\delta q}$ into components polarized parallel and orthogonal to the soliton pulse. Moreover, the reconstruction formula, Eq. (29a), for scalar $\delta q_{\parallel}$ is precisely that obtained for the scalar problem, with ψ_1 and ψ_2 the appropriate scalar Jost function components quoted in Eqs. (31a) and (31b). Of course, the evolution equations for $b_{\parallel}$ and $b_{\perp}$ have no counterpart in the scalar problem: these now read [cf. Eqs. (18), (30a), and (30b) above]

$$b_{\parallel,x} = -4i\xi^2 b_{\parallel} - 2i\xi\mu[b_{\parallel}\cos(2\theta) - b_{\perp}\sin(2\theta)], \tag{32a}$$

$$b_{\perp,x} = -4i\xi^2 b_{\perp} + 2i\xi\mu[b_{\parallel}\sin(2\theta) + b_{\perp}\cos(2\theta)] - i\mu\sin(2\theta)(\xi - i\eta_1)\hat{q}_s. \tag{32b}$$

Here, $\hat{q}_s = \exp(-4i\eta_1^2 x)\operatorname{sech}(\pi\xi/2\eta_1)$ is the Fourier transform of (scalar) $q_s = 2\eta_1\exp(-4i\eta_1^2 x)\operatorname{sech}(2\eta_1 t)$. Note that the inhomogeneous term is in the equation for $b_{\perp}$, which in turn generates $\delta q_{\perp}$, the radiation field orthogonally polarized to the soliton pulse. The above equations uncouple for values of $\theta = 0, \pi/2$, where the VNSE similarly reduces to the scalar form for one or other of the components q_1 and q_2.

When writing Eqs. (9), an important assumption was made which was not discussed then, but which is appropriate to mention now. Namely, it was assumed that the potentials q_1 and q_2 are on compact support: that is, they vanish faster than $\exp(-\lambda|t|)$ as $|t|\to\infty$ for any positive value of λ. When this assumption is not appropriate—such as when $\boldsymbol{q}(0,t)$ has the sech profile of the soliton pulse, Eq. (9) must be modified by moving the contours C and $\overline{C}$ onto the real axis, collecting discrete contributions from the poles (assumed simple) of S_{11}^{-1} and Δ_{11}^{-1} in the upper and lower half planes in the process. These discrete terms are of course the soliton contribution to $\boldsymbol{q}$ (here assumed to be a single soliton), while the remaining integral along the real axis is the continuum contribution (i.e., the radiation field); see Ref. [16] for details and further comment.

We now return to Eq. (9) and, before taking the contours C and $\overline{C}$ down to the real axis as discussed above, we first project the right hand side of the equation onto the orthogonal polarization modes $(\cos\theta,\sin\theta)$ and $(-\sin\theta,\cos\theta)$ (i.e., parallel to and orthogonal to the polarization state of the soliton pulse). When S_{11} has a single zero at $\zeta=i/2$ in the upper half plane (recall we have set $2\eta_1=1$), then the parallel contribution from the discrete terms is just

$$\boldsymbol{q}_{\parallel}=\boldsymbol{q}_s\,,$$

as one might expect. The *orthogonal* contribution is not equal to zero, but rather is given by

$$\boldsymbol{\delta q}_{\perp}=-b_{\perp}^{*}|_{\zeta=-i/2}\mathrm{sech}\, t\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}.$$

Here, we have used $\boldsymbol{\delta q}_{\perp}$ rather than $\boldsymbol{q}_{\perp}$ to emphasize that this is a small quantity. For this to make sense, of course, $b_{\perp}|_{\zeta=-i/2}$ must have meaning, which it will not have unless we reintroduce the assumption of compact support. If we do this, then the evolution equation for $b_{\perp}^{*}|_{\zeta=-i/2}$ is readily obtained from Eqs. (18) and (32b),

$$b_{\perp,x}^{*}|_{\zeta=-i/2}=-ib_{\perp}|_{\zeta=-i/2}-\frac{\mu}{\pi}\sin(2\theta)\exp(-ix).$$

To obtain this, we note that

$$\lim_{\zeta\to-i/2}(\zeta+i/2)\hat{q}_s(\zeta)=\frac{i\,\exp(-ix)}{\pi}.$$

This has the solution

$$b_{\perp}(x)=-\frac{\mu}{\pi}x\sin(2\theta)\exp(-ix),$$

and so

$$\boldsymbol{\delta q}_{\perp}=x\frac{\mu}{\pi}\sin(2\theta)q_s\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}, \tag{33}$$

where q_s is the scalar soliton, Eq. (4), with $\xi_1=0$, $2\eta_1=1$.

The interesting feature here is the linear growth with x, which indicates a *secular* interaction of the soliton pulse with the birefringence medium. Expression (33) is valid only for propagation distances $\mu x\ll 1$. We propose that the contribution (33) is a true description of the early evolution of the soliton shadow. It is in a polarization direction perpendicular to the soliton pulse $\boldsymbol{q}_s$, is a "true" radiation mode in the sense that it does not arise from a simple "tilt" of the polarization angle θ (actually $d\theta/dx=0$ for the choice of the perturbation $\boldsymbol{F}$ taken here), and is a resonant—and localized—contribution to the radiation field in the sense that growth is secular. Moreover, the composite pulse $\boldsymbol{q}_s+\boldsymbol{\delta q}_{\perp}$ on shedding radiation in the manner discussed in preceding sections evolves towards its asymptotic soliton state, Eq. (6). Note that, the sense of the early tilt (toward smaller values of θ) is consistent with the findings of studies on the polarization fast mode instability [18]. Some final comments on the nature of the soliton shadow are in order. First, there is no concise definition of the soliton shadow in the literature. It has been identified as the small orthogonal complement of a pulse close to a polarization mode [20], a second eigenparameter in the scattering data which appears when the parameter μ exceeds some critical value [10,11], or associated with soliton collisions [20]. In his article, Malomed discusses the VNSE for the case when $\boldsymbol{q}=(q_1,\delta q_2)$ obtains a linearized evolution equation for δq_2, and finds an eigenmode solution to this which is effectively our Eq. (33), but without the appearance of "x," that is, without the secular growth. We would like to propose that the term "soliton shadow" be reserved for the localized mode orthogonal to, and resonant with, the vector soliton discussed above.

V. CONCLUSIONS

We have developed a perturbation theory to analyze perturbed forms of the VNSE, as appropriate to studies on pulse propagation down an anomalously dispersive, birefringent optical fiber. We described the radiation shed by the soliton as it propagates down the fiber as a set of linear differential equations. These equations uncouple when projected on the soliton polarization states. Moreover, unlike other approaches, we have shown that both modes contribute to the generation of this field. The theory was finally applied to different examples, namely, the study of third-order dispersion and polarization mode dispersion. We also proposed an analytical treatment for the study of the effect of the soliton shadow.

APPENDIX A: INVERSE SCATTERING FOR THE MANAKOV SYSTEM

The linear eigenvalue problem associated with the unperturbed form ($\boldsymbol{F}=\boldsymbol{0}$) of Eq. (2) is

$$u_{1t}+i\zeta u_1=q_1u_2+q_2u_3, \tag{A1a}$$

$$u_{2t}-i\zeta u_2=-q_1^{*}u_1, \tag{A1b}$$

$$u_{3t}-i\zeta u_3=-q_2^{*}u_1. \tag{A1c}$$

We define the fundamental (or Jost) solutions $\boldsymbol{\phi}^{(i)}$ and $\boldsymbol{\psi}^{(i)}$, $i=1,2,3$, for real $\zeta=\xi$ by the requirements that

$$\boldsymbol{\phi}^{(1)}\sim\boldsymbol{e}_1\,\exp(-i\xi t),$$

$$\boldsymbol{\phi}^{(2)}\sim\boldsymbol{e}_2\,\exp(i\xi t),$$

$$\boldsymbol{\phi}^{(3)}\sim\boldsymbol{e}_3\,\exp(i\xi t),$$

as $t\to-\infty$, and

$$\boldsymbol{\psi}^{(1)}\sim\boldsymbol{e}_1\,\exp(-i\xi t),$$

$$\boldsymbol{\psi}^{(2)}\sim\boldsymbol{e}_2\,\exp(i\xi t),$$

$$\boldsymbol{\psi}^{(3)} \sim \boldsymbol{e}_3 \exp(i\xi t),$$

as $t\to+\infty$, where $\boldsymbol{e}_1=(1,0,0)^{\mathrm{T}}$ etc. Since $\boldsymbol{\phi}^{(i)}$ and $\boldsymbol{\psi}^{(i)}$ are independent sets of solutions, we can write

$$\boldsymbol{\phi}^{(i)}=\sum_{j=1}^{3} S_{ji}(\zeta)\boldsymbol{\psi}^{(j)}, \tag{A2}$$

which defines the scattering data $S_{ji}(\zeta)$. For $\zeta=\xi$ real, S is a 3×3 unitary unimodular matrix.

We also require an adjoint scattering problem which is taken to be

$$\tilde{v}_{1t}-i\zeta\tilde{v}_1=q_1^*\tilde{v}_2+q_2^*\tilde{v}_3, \tag{A3a}$$

$$\tilde{v}_{2t}+i\zeta\tilde{v}_2=-q_1\tilde{v}_1, \tag{A3b}$$

$$\tilde{v}_{3t}+i\zeta\tilde{v}_3=-q_2\tilde{v}_2, \tag{A3c}$$

where the symbol $\sim$ is used to denote solutions of the adjoint problem. As with the direct problem, we define the fundamental solutions $\tilde{\boldsymbol{\phi}}^{(i)}$ and $\tilde{\boldsymbol{\psi}}^{(i)}$ of the adjoint problem by the requirement that

$$\tilde{\boldsymbol{\phi}}^{(1)}\sim\boldsymbol{e}_1 \exp(i\xi t),$$

$$\tilde{\boldsymbol{\phi}}^{(2)}\sim\boldsymbol{e}_2 \exp(-i\xi t),$$

$$\tilde{\boldsymbol{\phi}}^{(3)}\sim\boldsymbol{e}_3 \exp(-i\xi t),$$

as $t\to-\infty$, and

$$\tilde{\boldsymbol{\psi}}^{(1)}\sim\boldsymbol{e}_1 \exp(i\xi t),$$

$$\tilde{\boldsymbol{\psi}}^{(2)}\sim\boldsymbol{e}_2 \exp(-i\xi t),$$

$$\tilde{\boldsymbol{\psi}}^{(3)}\sim\boldsymbol{e}_3 \exp(-i\xi t),$$

as $t\to+\infty$. Since, by construction $\tilde{\boldsymbol{\psi}}^{(i)\mathrm{T}}\boldsymbol{\psi}^{(j)}=\Delta_{ij}$, it follows that $S_{ij}=\tilde{\boldsymbol{\psi}}^{(i)\mathrm{T}}\boldsymbol{\phi}^{(j)}$. The scattering data Δ_{ij} for the adjoint scattering problem are introduced in an analogous manner to Eq. (A2) by

$$\tilde{\boldsymbol{\phi}}^{(i)}=\sum_{j=1}^{3}\Delta_{ji}(\zeta)\tilde{\boldsymbol{\psi}}^{(j)}.$$

By virtue of the unitary nature of S, it is easily demonstrated that $\Delta_{ji}(\zeta)$ is the cofactor of the element $S_{ij}(\zeta)$, and that

$$\Delta_{ij}(\zeta)=S_{ij}^*(\zeta),$$

where * denotes complex conjugate.

APPENDIX B

It is required to evaluate the integral

$$I=-\mu\int_{-\infty}^{+\infty}(\boldsymbol{\phi}^{(j)}\wedge\tilde{\boldsymbol{\psi}}^{(i)})^{\mathrm{T}}\sigma_3\begin{pmatrix}\boldsymbol{q}_t\\ -\boldsymbol{q}_t^*\end{pmatrix}dt.$$

Introduce the quantities α and $\boldsymbol{\beta}$ so that the scattering equations (A1a)–(A1c) and their adjoints (A3a)–(A3c) become, respectively,

$$\alpha_t+i\zeta\alpha=\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\beta},$$

$$\boldsymbol{\beta}_t-i\zeta\boldsymbol{\beta}=-\alpha\boldsymbol{q}^*,$$

and

$$\tilde{\alpha}_t-i\zeta\tilde{\alpha}=\boldsymbol{q}^{\dagger}\tilde{\boldsymbol{\beta}},$$

$$\tilde{\boldsymbol{\beta}}_t-i\zeta\tilde{\boldsymbol{\beta}}=-\tilde{\alpha}\boldsymbol{q}.$$

We then write the integral I as

$$I=-\mu\int_{-\infty}^{+\infty}(\boldsymbol{l}^{\mathrm{T}},\boldsymbol{m}^{\mathrm{T}})\sigma_3\begin{pmatrix}\boldsymbol{q}\\ -\boldsymbol{q}^*\end{pmatrix}dt,$$

where $\boldsymbol{l}=\tilde{\alpha}\boldsymbol{\beta}$ and $\boldsymbol{m}=\alpha\tilde{\boldsymbol{\beta}}$. Integrating by parts and evaluating the derivatives $\boldsymbol{l}_t$ and $\boldsymbol{m}_t$ gives

$$I=-2i\zeta\mu\int_{-\infty}^{+\infty}(\boldsymbol{l}^{\mathrm{T}}\sigma_3\boldsymbol{q}+\boldsymbol{m}^{\mathrm{T}}\sigma_3\boldsymbol{q}^*)dt.$$

Finally introduce the quantity $\boldsymbol{k}=\boldsymbol{\beta}^{\mathrm{T}}\sigma_3\tilde{\boldsymbol{\beta}}$. Evaluate its derivative

$$\boldsymbol{k}_t=-\boldsymbol{l}^{\mathrm{T}}\sigma_3\boldsymbol{q}-\boldsymbol{m}^{\mathrm{T}}\sigma_3\boldsymbol{q}^*,$$

so that

$$I=2i\zeta\mu\boldsymbol{k}\Big|_{-\infty}^{+\infty}+\text{other terms}=-2i\zeta\mu\begin{pmatrix}S_{21}\\ -S_{31}\end{pmatrix}+\text{other terms}.$$

The "other terms" correspond to the rest of the term in Eq. (18) which are obtained directly from Eq. (10) as discussed in the text.

APPENDIX C: THE JOST FUNCTIONS

We will list here the components for the Jost functions $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$. These are obtained by direct solution of the scattering problem with appropriate boundary conditions as $t\to\pm\infty$, with solitonic expressions for $\boldsymbol{q}_s$. The adjoint Jost functions are obtained from the relationships $\tilde{\phi}_j^{(i)}(\zeta,t)=\phi_j^{(i)}(\zeta,t)^*$ and $\tilde{\psi}_j^{(i)}(\zeta,t)=\psi_j^{(i)}(\zeta,t)^*$, where * denotes complex conjugate,

$$\phi_1^{(1)}=\frac{\exp(-i\zeta t)}{\zeta+i\eta_1}[\zeta-i\eta_1\tanh(2\eta_1 t)],$$

$$\phi_2^{(1)}=-\frac{i\eta_1}{\zeta+i\eta_1}\exp(-i\zeta t+4i\eta_1^2 x)\mathrm{sech}(2\eta_1 t)\cos\theta,$$

$$\phi_3^{(1)}=-\frac{i\eta_1}{\zeta+i\eta_1}\exp(-i\zeta t+4i\eta_1^2 x)\mathrm{sech}(2\eta_1 t)\sin\theta,$$

$$\phi_1^{(2)}=-\frac{i\eta_1}{\zeta-i\eta_1}\exp(i\zeta t-4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\cos\theta,$$

$$\phi_2^{(2)}=\frac{\exp(i\zeta t)}{\zeta-i\eta_1}(\zeta+i\eta_1[\cos^2\theta\tanh(2\eta_1 t)-\sin^2\theta]),$$

$$\phi_3^{(2)}=\frac{i\eta_1}{\zeta-i\eta_1}\exp(i\zeta t)[1+\tanh(2\eta_1 t)]\sin\theta\cos\theta,$$

$$\phi_1^{(3)}=-\frac{i\eta_1}{\zeta-i\eta_1}\exp(i\zeta t-4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\sin\theta,$$

$$\phi_2^{(3)}=\frac{i\eta_1}{\zeta-i\eta_1}\exp(i\zeta t)[1+\tanh(2\eta_1 t)]\sin\theta\cos\theta,$$

$$\phi_3^{(3)}=\frac{\exp(i\zeta t)}{\zeta-i\eta_1}(\zeta+i\eta_1[\sin^2\theta\tanh(2\eta_1 t)-\cos^2\theta]),$$

$$\psi_1^{(1)}=\frac{\exp(-i\zeta t)}{\zeta-i\eta_1}[\zeta-i\eta_1\tanh(2\eta_1 t)],$$

$$\psi_2^{(1)}=-\frac{i\eta_1}{\zeta-i\eta_1}\exp(-i\zeta t+4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\cos\theta,$$

$$\psi_3^{(1)}=-\frac{i\eta_1}{\zeta-i\eta_1}\exp(-i\zeta t+4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\sin\theta,$$

$$\psi_1^{(2)}=-\frac{i\eta_1}{\zeta+i\eta_1}\exp(i\zeta t-4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\cos\theta,$$

$$\psi_2^{(2)}=\frac{\exp(i\zeta t)}{\zeta+i\eta_1}(\zeta+i\eta_1[\cos^2\theta\tanh(2\eta_1 t)+\sin^2\theta]),$$

$$\psi_3^{(2)}=\frac{i\eta_1}{\zeta+i\eta_1}\exp(i\zeta t)[-1+\tanh(2\eta_1 t)]\sin\theta\cos\theta,$$

$$\psi_1^{(3)}=-\frac{i\eta_1}{\zeta+i\eta_1}\exp(i\zeta t-4i\eta_1^2x)\operatorname{sech}(2\eta_1 t)\sin\theta,$$

$$\psi_2^{(3)}=\frac{i\eta_1}{\zeta+i\eta_1}\exp(i\zeta t)[-1+\tanh(2\eta_1 t)]\sin\theta\cos\theta,$$

$$\psi_3^{(3)}=\frac{\exp(i\zeta t)}{\zeta+i\eta_1}(\zeta+i\eta_1[\sin^2\theta\tanh(2\eta_1 t)+\cos^2\theta]).$$

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