

## INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH NONLOCAL FRACTIONAL BOUNDARY CONDITIONS ASSOCIATED WITH FINANCIAL ASSET MODEL

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**ABSTRACT.** In this article, we discuss the existence of solutions for a boundary-value problem of integro-differential equations of fractional order with nonlocal fractional boundary conditions by means of some standard tools of fixed point theory. Our problem describes a more general form of fractional stochastic dynamic model for financial asset. An illustrative example is also presented.

### 1. FORMULATION AND BASIC RESULT

Fractional calculus, regarded as a branch of mathematical analysis dealing with derivatives and integrals of arbitrary order, has been extensively developed and applied to a variety of problems appearing in sciences and engineering. It is worthwhile to mention that this branch of mathematics has played a crucial role in exploring various characteristics of engineering materials such as viscoelastic polymers, foams, gels, and animal tissues, and their engineering and scientific applications. For a recent detailed survey of the activities involving fractional calculus, we refer a recent paper by Machado, Kiryakova and Mainardi [16]. Some recent work on the topic can be found in [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 17] and references therein.

The underlying dynamics of equity prices following a jump process or a Levy process provide a basis for modeling of financial assets. The CGMY, KoBoL and FMLS are examples of some interesting financial models involving the dynamics of stock prices. In [8], it is shown that the prices of financial derivatives are expressible in terms of fractional derivative.

In [15], the author described the dynamics of a financial asset by the fractional stochastic differential equation of order  $\mu$  (representing the dynamical memory effects in the market stochastic evolution) with fractional boundary conditions. In the present paper, we study a more general model associated with financial asset.

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Precisely, we consider the following problem:

$$\begin{aligned} -D^\alpha x(t) &= Af(t, x(t)) + BI^\beta g(t, x(t)), \quad (n-1) < \alpha \leq n, \quad t \in [0, 1], \\ D^\delta x(0) &= 0, \quad D^{\delta+1}x(0) = 0, \dots, \quad D^{\delta+(n-2)}x(0) = 0, \quad D^\delta x(1) = \int_0^\eta D^\delta x(s) ds, \end{aligned} \quad (1.1)$$

where  $0 < \delta \leq 1$ ,  $\alpha - \delta > n$ ,  $0 < \beta < 1$ ,  $0 < \eta < 1$ ,  $D^{(\cdot)}$  denotes the Riemann-Liouville fractional derivative of order  $(\cdot)$ ,  $f, g$  are given continuous function, and  $A, B$  are real constants.

We remark that the problem (1.1) also arises in real estate asset securitization modeling [18].

By the substitution  $x(t) = I^\delta y(t) = D^{-\alpha}y(t)$ , the problem (1.1) takes the form

$$\begin{aligned} -D^{\alpha-\delta}y(t) &= Af(t, I^\delta y(t)) + BI^\beta g(t, I^\delta y(t)), \quad t \in [0, 1], \\ y(0) &= 0, \quad y'(0) = 0, \dots, \quad y^{(n-2)}(0) = 0, \quad y(1) = \int_0^\eta y(s) ds. \end{aligned} \quad (1.2)$$

**Lemma 1.1.** *For any  $h \in C(0, 1) \cap L(0, 1)$ , the unique solution of the linear fractional boundary-value problem*

$$\begin{aligned} -D^{\alpha-\delta}y(t) &= h(t), \quad t \in [0, 1], \\ y(0) &= 0, \quad y'(0) = 0, \dots, \quad y^{(n-2)}(0) = 0, \quad y(1) = \int_0^\eta y(s) ds, \end{aligned} \quad (1.3)$$

is

$$y(t) = -I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}} \left( I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta) \right),$$

where  $I^{(\cdot)}(\cdot)$  denotes Riemann-Liouville integral.

*Proof.* It is well known that the solutions of fractional differential equation in (1.1) can be written as

$$y(t) = -I^{\alpha-\delta}h(t) + c_1 t^{\alpha-\delta-1} + c_2 t^{\alpha-\delta-2} + c_3 t^{\alpha-\delta-3} + \dots + c_n t^{\alpha-\delta-n}, \quad (1.4)$$

where  $c_1, c_2, \dots, c_n \in \mathbb{R}$  are arbitrary constants [12]. Using the given boundary conditions, we find that  $c_2 = 0, c_3 = 0, \dots, c_n = 0$  and

$$c_1 = \frac{\alpha-\delta}{\alpha-\delta-\eta^{\alpha-\delta}} \left( I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta) \right).$$

Substituting these values in (1.1) yields

$$y(t) = -I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}} \left( I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta) \right).$$

This completes the proof.  $\square$

Thus, the solution of the linear variant of the problem (1.1) can be written as

$$\begin{aligned} x(t) &= I^\delta y(t) \\ &= I^\delta \left[ -I^{\alpha-\delta}h(t) + \frac{(\alpha-\delta)t^{\alpha-\delta-1}}{\alpha-\delta-\eta^{\alpha-\delta}} \left( I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta) \right) \right] \\ &= -I^\alpha h(t) + \frac{(\alpha-\delta)}{\alpha-\delta-\eta^{\alpha-\delta}} \left( I^{\alpha-\delta}h(1) - I^{\alpha-\delta+1}h(\eta) \right) \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} s^{\alpha-\delta-1} ds \end{aligned}$$

$$= -I^\alpha h(t) + \frac{(\alpha - \delta)}{\alpha - \delta - \eta^{\alpha - \delta}} \left( I^{\alpha - \delta} h(1) - I^{\alpha - \delta + 1} h(\eta) \right) \times \\ \times \left\{ \frac{t^{\alpha - 1}}{\Gamma(\delta)} \int_0^1 (1 - \nu)^{\delta - 1} \nu^{\alpha - \delta - 1} d\nu \right\},$$

where we have used the substitution  $s = \nu t$  in the integral of the last term. Using the relation for Beta function  $B(\cdot, \cdot)$ :

$$B(\beta + 1, \alpha) = \int_0^1 (1 - u)^{\alpha - 1} u^\beta du = \frac{\Gamma(\alpha)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)},$$

we obtain

$$x(t) = -I^\alpha h(t) + \frac{\Gamma(\alpha - \delta + 1)t^{\alpha - 1}}{(\alpha - \delta - \eta^{\alpha - \delta})\Gamma(\alpha)} \left( I^{\alpha - \delta} h(1) - I^{\alpha - \delta + 1} h(\eta) \right). \quad (1.5)$$

The solution of the original nonlinear problem (1.1) can be obtained by replacing  $h$  with the right hand side of the fractional equation of (1.1) in (1.5).

Let  $\mathcal{C} = C([0, 1], \mathbb{R})$  denote the Banach space of all continuous functions from  $[0, 1] \rightarrow \mathbb{R}$  endowed with the norm defined by  $\|x\| = \sup\{|x(t)|, t \in [0, 1]\}$ .

In relation to problem (1.1), we define an operator  $\mathcal{U} : \mathcal{C} \rightarrow \mathcal{C}$  as

$$(\mathcal{U}x)(t) \\ = -A \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(s)) ds - B \int_0^t \frac{(t - s)^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)} g(s, x(s)) ds \\ + Qt^{\alpha - 1} \left[ A \int_0^1 \frac{(1 - s)^{\alpha - \delta - 1}}{\Gamma(\alpha - \delta)} f(s, x(s)) ds + B \int_0^1 \frac{(1 - s)^{\alpha - \delta + \beta - 1}}{\Gamma(\alpha - \delta + \beta)} g(s, x(s)) ds \right. \\ \left. - A \int_0^\eta \frac{(\eta - s)^{\alpha - \delta}}{\Gamma(\alpha - \delta + 1)} f(s, x(s)) ds - B \int_0^\eta \frac{(\eta - s)^{\alpha - \delta + \beta}}{\Gamma(\alpha - \delta + \beta + 1)} g(s, x(s)) ds \right],$$

where

$$Q = \frac{\Gamma(\alpha - \delta + 1)}{(\alpha - \delta - \eta^{\alpha - \delta})\Gamma(\alpha)}, \quad \alpha \neq \delta + \eta^{\alpha - \delta}.$$

For the sake of convenience, we set

$$\Omega = \sup_{t \in [0, 1]} \left\{ |A| \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} + |Q| t^{\alpha - 1} \left( \frac{1}{\Gamma(\alpha - \delta + 1)} + \frac{\eta^{\alpha - \delta + 1}}{\Gamma(\alpha - \delta + 2)} \right) \right] \right. \\ \left. + |B| \left[ \frac{t^{\alpha + \beta}}{\Gamma(\alpha + \beta + 1)} + |Q| t^{\alpha - 1} \left( \frac{1}{\Gamma(\alpha - \delta + \beta + 1)} + \frac{\eta^{\alpha - \delta + \beta + 1}}{\Gamma(\alpha - \delta + \beta + 2)} \right) \right] \right\}. \quad (1.6)$$

### 1.1. Existence results via Banach's fixed point theorem.

**Theorem 1.2.** *Assume that  $f, g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions satisfying the condition:*

$$(A1) \quad |f(t, x) - f(t, y)| \leq L_1 |x - y|, \quad |g(t, x) - g(t, y)| \leq L_2 |x - y|, \quad \text{for all } t \in [0, 1], \\ L_1, L_2 > 0, \quad x, y \in \mathbb{R}.$$

*Then the boundary-value problem (1.1) has a unique solution if  $L < 1/\Omega$ , where  $L = \max\{L_1, L_2\}$  and  $\Omega$  is given by (1.6).*

*Proof.* Let us define  $M = \max\{M_1, M_2\}$ , where  $M_1, M_2$  are finite numbers given by  $\sup_{t \in [0,1]} |f(t, 0)| = M_1$ ,  $\sup_{t \in [0,1]} |g(t, 0)| = M_2$ . Selecting  $r \geq \frac{\Omega M}{1-L\Omega}$ , we show that  $\mathcal{U}B_r \subset B_r$ , where  $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ . Using that  $|f(s, x(s))| \leq |f(s, x(s)) - f(s, 0)| + |f(s, 0)| \leq L_1 r + M_1$ ,  $|g(s, x(s))| \leq |g(s, x(s)) - g(s, 0)| + |g(s, 0)| \leq L_2 r + M_2$  for  $x \in B_r$  and (1.6), it can easily be shown that

$$\begin{aligned} & \|(\mathcal{U}x)\| \\ & \leq (Lr + M) \sup_{t \in [0,1]} \left\{ |A| \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + 1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha - \delta + 2)} \right) \right] \right. \\ & \quad \left. + |B| \left[ \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + \beta + 1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha - \delta + \beta + 2)} \right) \right] \right\} \\ & = (Lr + M)\Omega \leq r, \end{aligned}$$

which implies that  $\mathcal{U}B_r \subset B_r$ . Now, for  $x, y \in \mathcal{C}$  we obtain

$$\begin{aligned} & \| \mathcal{U}x - \mathcal{U}y \| \\ & \leq \sup_{t \in [0,1]} \left\{ |A| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| ds \right. \\ & \quad + |B| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} |g(s, x(s)) - g(s, y(s))| ds \\ & \quad + |Q| t^{\alpha-1} \left[ |A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha - \delta)} |f(s, x(s)) - f(s, y(s))| ds \right. \\ & \quad + |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha - \delta + \beta)} |g(s, x(s)) - g(s, y(s))| ds \\ & \quad + |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha - \delta + 1)} |f(s, x(s)) - f(s, y(s))| ds \\ & \quad \left. \left. + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha - \delta + \beta + 1)} |g(s, x(s)) - g(s, y(s))| ds \right] \right\} \\ & \leq L \sup_{t \in [0,1]} \left\{ |A| \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + 1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha - \delta + 2)} \right) \right] \right. \\ & \quad \left. + |B| \left[ \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + \beta + 1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha - \delta + \beta + 2)} \right) \right] \right\} \\ & \quad \times \|x - y\| \\ & = L\Omega \|x - y\|. \end{aligned}$$

By the given assumption,  $L < 1/\Omega$ . Therefore  $\mathcal{U}$  is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem).  $\square$

Now we present another variant of existence-uniqueness result. This result is based on the Hölder's inequality.

**Theorem 1.3.** *Suppose that the continuous functions  $f$  and  $g$  satisfy the following assumptions:*

$$(H1) \quad |f(t, x) - f(t, y)| \leq m(t)|x - y|, \quad |g(t, x) - g(t, y)| \leq n(t)|x - y|, \quad \text{for } t \in [0, 1], \\ x, y \in \mathbb{R}, \text{ and } m, n \in L^{\frac{1}{\gamma}}([0, 1], \mathbb{R}^+), \quad \gamma \in (0, \alpha - \delta - n).$$

(H2)  $|A|\|m\|Z_1 + |B|\|n\|Z_2 < 1$ , where

$$Z_1 = \frac{1}{\Gamma(\alpha)} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta)} \left( \frac{1-\gamma}{\alpha-\delta-\gamma} \right)^{1-\gamma} \\ + \frac{|Q|}{\Gamma(\alpha-\delta+1)} \left( \frac{1-\gamma}{\alpha-\delta+1-\gamma} \right)^{1-\gamma} \eta^{\alpha-\delta+1-\gamma},$$

$$Z_2 = \frac{1}{\Gamma(\alpha+\beta)} \left( \frac{1-\gamma}{\alpha+\beta-\gamma} \right)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta)} \left( \frac{1-\gamma}{\alpha-\delta+\beta-\gamma} \right)^{1-\gamma} \\ + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} \left( \frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma} \right)^{1-\gamma} \eta^{\alpha-\delta+\beta+1-\gamma},$$

and  $\|\mu\| = \left( \int_0^1 |\mu(s)|^{\frac{1}{\gamma}} ds \right)^\gamma$ ,  $\mu = m, n$ . Then the boundary value problem (1.1) has a unique solution.

*Proof.* For  $x, y \in \mathbb{R}$  and for each  $t \in [0, 1]$ , by Hölder inequality, we have

$$\begin{aligned} & \|\mathcal{U}x - \mathcal{U}y\| \\ & \leq \sup_{t \in [0,1]} \left\{ |A| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s) |x(s) - y(s)| ds \right. \\ & \quad + |B| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} n(s) |x(s) - y(s)| ds \\ & \quad + |Q| \left[ |A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} m(s) |x(s) - y(s)| ds \right. \\ & \quad + |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} n(s) |x(s) - y(s)| ds \\ & \quad + |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} m(s) |x(s) - y(s)| ds \\ & \quad \left. + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} n(s) |x(s) - y(s)| ds \right] \Big\} \\ & \leq \sup_{t \in [0,1]} \left\{ \frac{|A|\|m\|}{\Gamma(\alpha)} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} t^{\alpha-\gamma} + \frac{|B|\|n\|}{\Gamma(\alpha+\beta)} \left( \frac{1-\gamma}{\alpha+\beta-\gamma} \right)^{1-\gamma} t^{\alpha+\beta-\gamma} \right. \\ & \quad + |Q| \left[ \frac{|A|\|m\|}{\Gamma(\alpha-\delta)} \left( \frac{1-\gamma}{\alpha-\delta-\gamma} \right)^{1-\gamma} + \frac{|B|\|n\|}{\Gamma(\alpha-\delta+\beta)} \left( \frac{1-\gamma}{\alpha-\delta+\beta-\gamma} \right)^{1-\gamma} \right. \\ & \quad + \frac{|A|\|m\|}{\Gamma(\alpha-\delta+1)} \left( \frac{1-\gamma}{\alpha-\delta+1-\gamma} \right)^{1-\gamma} \eta^{\alpha-\delta+1-\gamma} \\ & \quad \left. + \frac{|B|\|n\|}{\Gamma(\alpha-\delta+\beta+1)} \left( \frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma} \right)^{1-\gamma} \eta^{\alpha-\delta+\beta+1-\gamma} \right] \Big\} \|x - y\| \\ & \leq |A|\|m\| \left[ \frac{1}{\Gamma(\alpha)} \left( \frac{1-\gamma}{\alpha-\gamma} \right)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta)} \left( \frac{1-\gamma}{\alpha-\delta-\gamma} \right)^{1-\gamma} \right. \\ & \quad \left. + \frac{|Q|}{\Gamma(\alpha-\delta+1)} \left( \frac{1-\gamma}{\alpha-\delta+1-\gamma} \right)^{1-\gamma} \right] \|x - y\| \\ & \quad + |B|\|n\| \left[ \frac{1}{\Gamma(\alpha+\beta)} \left( \frac{1-\gamma}{\alpha+\beta-\gamma} \right)^{1-\gamma} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta)} \left( \frac{1-\gamma}{\alpha-\delta+\beta-\gamma} \right)^{1-\gamma} \right. \\ & \quad \left. + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} \left( \frac{1-\gamma}{\alpha-\delta+\beta+1-\gamma} \right)^{1-\gamma} \right] \|x - y\| \end{aligned}$$

$$\begin{aligned}
& + \frac{|Q|}{\Gamma(\alpha - \delta + \beta + 1)} \left( \frac{1 - \gamma}{\alpha - \delta + \beta + 1 - \gamma} \right)^{1-\gamma} \eta^{\alpha - \delta + \beta + 1 - \gamma} \|x - y\| \\
& = [|A| \|m\| Z_1 + |B| \|n\| Z_2] \|x - y\|.
\end{aligned}$$

In view of condition (H2), it follows that  $\mathcal{U}$  is a contraction mapping. Hence, Banach's fixed point theorem applies and  $\mathcal{U}$  has a unique fixed point which is the unique solution of problem (1.1). This completes the proof.  $\square$

## 1.2. Existence result via Leray-Schauder Alternative.

**Lemma 1.4** (Nonlinear alternative for single valued maps [11]). *Let  $E$  be a Banach space,  $C$  a closed, convex subset of  $E$ ,  $U$  an open subset of  $C$  and  $0 \in U$ . Suppose that  $F : \bar{U} \rightarrow C$  is a continuous, compact (that is,  $F(\bar{U})$  is a relatively compact subset of  $C$ ) map. Then either*

- (i)  $F$  has a fixed point in  $\bar{U}$ , or
- (ii) there is a  $u \in \partial U$  (the boundary of  $U$  in  $C$ ) and  $\lambda \in (0, 1)$  with  $u = \lambda F(u)$ .

**Theorem 1.5.** *Assume that  $f, g : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions. Assume that:*

- (A3) *There exist functions  $p_1, p_2 \in L^1([0, 1], \mathbb{R}^+)$ , and nondecreasing functions  $\psi_1, \psi_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that*

$$|f(t, x)| \leq p_1(t)\psi_1(\|x\|), \quad |g(t, x)| \leq p_2(t)\psi_2(\|x\|),$$

for all  $(t, x) \in [0, 1] \times \mathbb{R}$ .

- (A4) *There exists a constant  $M > 0$  such that*

$$\frac{M}{|A|\Lambda_1\psi_1(M)\|p_1\|_{L^1} + |B|\Lambda_1\psi_2(M)\|p_2\|_{L^1}} > 1,$$

where

$$\begin{aligned}
\Lambda_1 &= \frac{1}{\Gamma(\alpha + 1)} + \frac{|Q|}{\Gamma(\alpha - \delta + 1)} + \frac{|Q|}{\Gamma(\alpha - \delta + 2)}, \\
\Lambda_2 &= \frac{1}{\Gamma(\alpha + \beta + 1)} + \frac{|Q|}{\Gamma(\alpha - \delta + \beta + 1)} + \frac{|Q|}{\Gamma(\alpha - \delta + \beta + 2)}.
\end{aligned}$$

Then the boundary-value problem (1.1) has at least one solution on  $[0, 1]$ .

*Proof.* Consider the operator  $\mathcal{U} : \mathcal{C} \rightarrow \mathcal{C}$  with  $x = \mathcal{U}x$ , where

$$\begin{aligned}
& (\mathcal{U}x)(t) \\
& = -A \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) ds - B \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(s, x(s)) ds \\
& \quad + Qt^{\alpha-1} \left[ A \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f(s, x(s)) ds + B \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} g(s, x(s)) ds \right. \\
& \quad \left. - A \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} f(s, x(s)) ds - B \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} g(s, x(s)) ds \right].
\end{aligned}$$

We show that  $F$  maps bounded sets into bounded sets in  $C([0, 1], \mathbb{R})$ . For a positive number  $r$ , let  $B_r = \{x \in C([0, 1], \mathbb{R}) : \|x\| \leq r\}$  be a bounded set in  $C([0, 1], \mathbb{R})$ . Then

$$|(\mathcal{U}x)(t)|$$

$$\begin{aligned}
&\leq |A| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p_1(s) \psi_1(\|x\|) ds + |B| \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} p_2(s) \psi_2(\|x\|) ds \\
&\quad + |Q| t^{\alpha-1} \left[ |A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} p_1(s) \psi_1(\|x\|) ds \right. \\
&\quad + |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} p_2(s) \psi_2(\|x\|) ds \\
&\quad + |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} p_1(s) \psi_1(\|x\|) ds \\
&\quad \left. + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} p_2(s) \psi_2(\|x\|) ds \right] \\
&\leq |A| \|\psi_1(r)\| p_1 \|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+2)} \right\} \\
&\quad + |B| \|\psi_2(r)\| p_2 \|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)} \right\}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\|\mathcal{U}x\| \\
&\leq |A| \|\psi_1(r)\| p_1 \|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+2)} \right\} \\
&\quad + |B| \|\psi_2(r)\| p_2 \|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)} \right\}.
\end{aligned}$$

Next we show that  $F$  maps bounded sets into equicontinuous sets of  $C([0, 1], \mathbb{R})$ . Let  $t_1, t_2 \in [0, 1]$  with  $t_1 < t_2$  and  $x \in B_r$ , where  $B_r$  is a bounded set of  $C([0, 1], \mathbb{R})$ . Then we obtain

$$\begin{aligned}
&\|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)\| \\
&\leq \left\| \frac{|A|}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] f(s, x(s)) ds \right. \\
&\quad + \frac{|A|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds \\
&\quad + \frac{|B|}{\Gamma(\alpha+\beta)} \int_0^{t_1} [(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}] g(s, x(s)) ds \\
&\quad + \frac{|B|}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} g(s, x(s)) ds \\
&\quad + |Q| [(t_2)^{\alpha-1} - (t_1)^{\alpha-1}] \left[ |A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} |f(s, x(s))| ds \right. \\
&\quad + |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} |g(s, x(s))| ds \\
&\quad \left. + |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} |f(s, x(s))| ds + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} |g(s, x(s))| ds \right] \Big\| \\
&\leq \left\| \frac{|A|}{\Gamma(\alpha)} \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] p_1(s) \psi_1(r) ds \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{|A|}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} p_1(s) \psi_1(r) ds \\
& + \frac{|B|}{\Gamma(\alpha + \beta)} \int_0^{t_1} [(t_2 - s)^{\alpha+\beta-1} - (t_1 - s)^{\alpha+\beta-1}] p_2(s) \psi_2(r) ds \\
& + \frac{|B|}{\Gamma(\alpha + \beta)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha+\beta-1} p_2(s) \psi_2(r) ds \\
& + |Q| [(t_2)^{\alpha-1} - (t_1)^{\alpha-1}] \left[ |A| \int_0^1 \frac{(1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} p_1(s) \psi_1(r) ds \right. \\
& + |B| \int_0^1 \frac{(1-s)^{\alpha-\delta+\beta-1}}{\Gamma(\alpha-\delta+\beta)} p_2(s) \psi_2(r) ds \\
& \left. + |A| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} p_1(s) \psi_1(r) ds + |B| \int_0^\eta \frac{(\eta-s)^{\alpha-\delta+\beta}}{\Gamma(\alpha-\delta+\beta+1)} p_2(s) \psi_2(r) ds \right] \Big\| .
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of  $x \in B_r$  as  $t_2 - t_1 \rightarrow 0$ . As  $\mathcal{U}$  satisfies the above assumptions, therefore it follows by the Arzelà-Ascoli theorem that  $\mathcal{U} : C([0, 1], \mathbb{R}) \rightarrow C([0, 1], \mathbb{R})$  is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 1.4) once we have proved the boundness of the set of all solutions to equations  $x = \lambda \mathcal{U}x$  for  $\lambda \in [0, 1]$ .

Let  $x$  be a solution. Then, for  $t \in [0, 1]$ , and using the computations in proving that  $\mathcal{U}$  is bounded, we have

$$\begin{aligned}
& |x(t)| \\
& = |\lambda(\mathcal{U}x)(t)| \leq |A| \psi_1(\|x\|) \|p_1\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+2)} \right\} \\
& \quad + |B| \psi_2(\|x\|) \|p_2\|_{L^1} \left\{ \frac{1}{\Gamma(\alpha+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+1)} + \frac{|Q|}{\Gamma(\alpha-\delta+\beta+2)} \right\}.
\end{aligned}$$

Consequently,

$$\frac{\|x\|}{|A| \Lambda_1 \psi_1(\|x\|) \|p_1\|_{L^1} + |B| \Lambda_1 \psi_2(\|x\|) \|p_2\|_{L^1}} \leq 1.$$

In view of (A4), there exists  $M$  such that  $\|x\| \neq M$ . Let us set

$$U = \{x \in C([0, 1], X) : \|x\| < M\}.$$

Note that the operator  $\mathcal{U} : \bar{U} \rightarrow C([0, 1], \mathbb{R})$  is continuous and completely continuous. From the choice of  $U$ , there is no  $x \in \partial U$  such that  $x = \lambda \mathcal{U}(x)$  for some  $\lambda \in (0, 1)$ . Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 1.4), we deduce that  $\mathcal{U}$  has a fixed point  $x \in \bar{U}$  which is a solution of the problem (1.1). This completes the proof.  $\square$

**Example.** Consider a boundary-value problem of integro-differential equations of fractional order with nonlocal fractional boundary conditions given by

$$\begin{aligned}
& -D^{5/2}x(t) = Af(t, x(t)) + BI^\beta g(t, x(t)), \quad t \in [0, 1], \\
& D^{1/4}x(0) = 0, \quad D^{5/4}x(0) = 0, \quad D^{1/4}x(1) = \int_0^\eta D^{1/4}x(s) ds,
\end{aligned} \tag{1.7}$$



where  $n = 3$ ,  $A = B = 1$ ,  $\beta = 3/4$ ,  $\eta = 2/3$ ,  $f(t, x) = \frac{3|x|(2+|x|)}{8(1+|x|)} + 4t$ ,  $g(t, x) = \frac{1}{2} \tan^{-1} x + \sin^2 t$ . With the given data, we find that

$$Q = \frac{\Gamma(\alpha - \delta + 1)}{(\alpha - \delta - \eta^{\alpha-\delta})\Gamma(\alpha)} = 1.037485,$$

and

$$\begin{aligned} \Omega &= \sup_{t \in [0,1]} \left\{ |A| \left[ \frac{t^\alpha}{\Gamma(\alpha + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + 1)} + \frac{\eta^{\alpha-\delta+1}}{\Gamma(\alpha - \delta + 2)} \right) \right] \right. \\ &\quad \left. + |B| \left[ \frac{t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)} + |Q| t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha - \delta + \beta + 1)} + \frac{\eta^{\alpha-\delta+\beta+1}}{\Gamma(\alpha - \delta + \beta + 2)} \right) \right] \right\} \\ &= 1.043555, \end{aligned}$$

and  $L_1 = 3/4$ ,  $L_2 = 1/2$  as  $|f(t, x) - f(t, y)| \leq \frac{3}{4}|x - y|$ ,  $|g(t, x) - g(t, y)| \leq \frac{1}{2}|x - y|$ . Clearly  $L = \max\{L_1, L_2\} = 3/4$  and  $L < 1/\Omega$ . Thus all the assumptions of Theorem 1.2 are satisfied. Hence, by the conclusion of Theorem 1.2, the problem (1.7) has a unique solution.

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