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EXISTENCE RESULTS FOR SEMI-LINEAR INTEGRODIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper, we shall establish sufficient conditions for the existence of solutions for semi-linear integrodifferential inclusions in Banach spaces with nonlocal conditions. By using suitable fixed point theorems we study the case when the multi-valued map has convex as well as nonconvex values.

1. Introduction. In this paper, we shall prove existence results, for the following semi-linear integrodifferential inclusions, with nonlocal conditions, of the form

$$y'(t)-Ay(t) \in F\left(t, y(t), \int_0^t k(t,s,y(s))\,ds\right), \quad \text{a.e. } t \in J := [0,b] \tag{1}$$

$$y(0) + f(y) = y_0, \tag{2}$$

where $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, $F : J \times E \times E \to \mathcal{P}(E)$ a multi-valued map, $k : J \times J \times E \to E$ a continuous function, $f : C(J,E) \to E$, $y_0 \in E$, $\mathcal{P}(E)$ is the family of all subsets of E and E is a real separable Banach space with norm $\|\cdot\|$.

The work on nonlocal evolution initial value problems (IVP for short) was initiated by Byszewski. In [**9, 10**] Byszewski using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order IVP. For the importance of nonlocal conditions in different fields, the interested reader is referred to [**9, 10**] and the references cited therein.

In [**5**] Benchohra and Ntouyas studied existence results for integrodifferential inclusions on infinite intervals, in the case where the multi-valued map has bounded, closed and convex values, by using the fixed point theorem of Ma [**18**].

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Here we study existence results on compact intervals, when the multi-valued F has convex or nonconvex values. In the first case a fixed point theorem due to Martelli [**19**] is used and in the latter a fixed point theorem for contraction multi-valued maps, due to Covitz and Nadler [**12**].

2. Preliminaries. In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper.

$C(J,E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of the Bochner integral, see Yosida [**21**].)

$L^1(J,E)$ denotes the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b \|y(t)\|\,dt \quad \text{for all } y \in L^1(J,E).$$

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$. G is called *upper semi-continuous* (*u.s.c.*) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty, closed subset of X, and if, for each open set V of X containing $G(x_0)$, there exists an open neighborhood U of x_0 such that $G(U) \subseteq V$. G is said to be *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_0$, $y_n \to y_0$, $y_n \in G(x_n)$ imply $y_0 \in G(x_0)$. G has a *fixed point* if there is an $x \in X$ such that $x \in G(x)$.

$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \varnothing\}$, $P_{cl}(X) = \{Y \in P(X) : Y$ closed$\}$, $P_b(X) = \{Y \in P(X) : Y$ bounded$\}$, $P_{cp} = \{Y \in P(X) : Y$

compact} and $P_c(X) = \{Y \in P(X) : Y \text{ convex}\}$. A multi-valued map $G : J \to P_{cl}(X)$ is said to be *measurable* if, for each $x \in X$, the function $Y : J \to \mathbf{R}_+$, defined by

$$Y(t) = d(x, G(t)) = \inf\{\|x - z\| : z \in G(t)\},$$

is measurable. For more details on multi-valued maps we refer to the books of Deimling [**13**], Gorniewicz [**15**] and Hu and Papageorgiou [**16**].

An upper semi-continuous map $G : X \longrightarrow \mathcal{P}(X)$ is said to be *condensing* if, for any bounded subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [**4**]. We remark that a completely continuous multi-valued map is the easiest example of a condensing map.

For properties of semi-groups theory, we refer the interested reader to the books of Goldstein [**14**] and Pazy [**20**].

3. Existence result: The convex case. Assume in this section that $F : J \times E \times E \to \mathcal{P}(E)$ is a bounded, closed and convex valued multi-valued map.

Let us list the following hypotheses:

(H1) A is the infinitesimal generator of a semi-group of bounded linear operators $T(t)$ in E such that $\|T(t)\|_{B(E)} \leq M$, for some $M > 0$;

(H2) $F : J \times E \times E \to P_{b,cl,c}(E)$; $(t, w, v) \mapsto F(t, w, v)$ is measurable with respect to t for each $w, v \in E$, and u.s.c. with respect to w, v for each $t \in J$ and the set

$$S_{F,y} := \Big\{ g \in L^1(J, E) : g(t) \in F\Big(t, y(t), \int_0^t k(t, s, y(s))ds\Big) \text{ for a.e. } t \in J \Big\}$$

is nonempty;

(H3) $f : C(J, E) \longrightarrow E$ is continuous and there exists a constant $L > 0$ such that $\|f(y)\| \leq L$ for each $y \in C(J, E)$;

(H4) There exists a function $\alpha \in C(J, \mathbf{R}_+) \cap L^2(J, \mathbf{R}_+)$, such that

$$\Big\| \int_0^t k(t, s, y)\, ds \Big\| \leq \alpha(t)\|y\| \quad \text{for each } t \in J \quad \text{and } y \in E;$$

(H5) There exist $\beta \in L^2(J, \mathbf{R}_+)$ such that

$$\begin{aligned} \|F(t,y,z)\| &= \sup\{\|v\| : v \in F(t,y,z)\} \\ &\leq \beta(t)\psi(\|y\| + \|z\|) \quad \text{for a.e. } t \in J \text{ and } y, z \in E, \end{aligned}$$

where $\psi : \mathbf{R}_+ \to (0, \infty)$ is a continuous and increasing function with

$$\psi(\alpha(t)\|y\|) \leq \alpha(t)\psi(\|y\|) \quad \text{for each } t \in J \quad \text{and } y \in E,$$

and

$$M \int_0^b \beta(t)(1 + \alpha(t))\, ds < \int_c^\infty \frac{du}{\psi(u)}, \quad c = M(\|y_0\| + L).$$

(H6) For each bounded $B \subset C(J, E)$ and $t \in J$ the set

$$\left\{ T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)g(s)\, ds : g \in S_{F,B} \right\}$$

is relatively compact in E, where $S_{F,B} = \cup\{S_{F,y} : y \in B\}$.

Definition 3.1. A function $y \in C(J, E)$ is called a mild solution of (1)–(2) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t), \int_0^t k(t, s, y(s))\, ds)$ almost everywhere on J and

$$y(t) = T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)v(s)\, ds.$$

The following lemmas are crucial in the proof of our main theorem.

Lemma 3.1 [**17**]. *Let I be a compact real interval and X a Banach space. Let F be a multi-valued map satisfying* (H2)*, and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\begin{gathered} \Gamma \circ S_F : C(I, X) \longrightarrow P_{b,cl,c}(C(I, X)), \\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y}) \end{gathered}$$

is a closed graph operator in $C(I,X)\times C(I,X)$.

Lemma 3.2 [**19**]. *Let* X *be a Banach space and* $N : X \to P_{b,cl,c}(X)$ *an upper semi-continuous and condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded, then N *has a fixed point.*

Now we are able to state and prove our main theorem in this section.

Theorem 3.1. *Assume that hypotheses* (H1)–(H6) *are satisfied. Then the problem* (1)–(2) *has at least one mild solution on* J.

Proof. We transform the problem (1)–(2) into a fixed point problem. Consider the multi-valued map, $N : C(J,E) \to \mathcal{P}(C(J,E))$ defined by

$$N(y) := \Big\{ h \in C(J,E) : h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)g(s)\,ds : g \in S_{F,y} \Big\}$$

where

$$S_{F,y} = \Big\{ g \in L^1(J,E) : g(t) \in F\Big(t, y(t), \int_0^t k(t,s,y(s))\,ds\Big) \text{ for a.e. } t \in J \Big\}.$$

Remark 3.1. It is clear that the fixed points of N are mild solutions to (1)–(2).

We shall show that N is completely continuous with bounded, closed, convex values and it is upper semi-continuous. The proof will be given in several steps.

Step 1. $N(y)$ is convex for each $y \in C(J,E)$.

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$h_i(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)g_i(s)\,ds, \quad i = 1, 2.$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$\begin{aligned}(kh_1 + (1-k)h_2)(t) &= T(t)[y_0 - f(y)] \\ &\qquad + \int_0^t T(t-s)[kg_1(s) + (1-k)g_2(s)]\,ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)h_2 \in N(y).$$

Step 2. N is bounded on bounded sets of $C(J,E)$.

Indeed, it is enough to show that there exists a positive constant l such that, for each $h \in N(y)$, $y \in B_r = \{y \in C(J,E) : \|y\|_\infty \le r\}$, one has $\|h\|_\infty \le l$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)g(s)\,ds.$$

By (H1) and (H4)–(H5) we have, for each $t \in J$, that

$$\begin{aligned}\|h(t)\| &\le \|T(t)\|_{B(E)}\|y_0\| + \|T(t)\|_{B(E)}\|f(y)\| + \int_0^t \|T(t-s)g(s)\|\,ds \\ &\le M\|y_0\| + ML + M\int_0^t \beta(s)\psi(\|y(s)\| + \alpha(s)\|y(s)\|)\,ds \\ &\le M\|y_0\| + ML + M\sup_{t\in J}\psi(\|y(t)\|)\int_0^t \beta(s)(1+\alpha(s))\,ds.\end{aligned}$$

Then, for each $h \in N(y)$, we have

$$\|h\|_\infty \le M\|y_0\| + ML + M\sup_{t\in J}\psi(r)\int_0^b \beta(s)(1+\alpha(s))\,ds := l.$$

Step 3. N sends bounded sets into equicontinuous sets of $C(J,E)$.

Let t_1, $t_2 \in J$, $t_1 < t_2$ and B_r be a bounded set in $C(J,E)$. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)g(s)\,ds, \quad t \in J.$$

Thus

$$\begin{aligned}
&\|h(t_2) - h(t_1)\| \\
&\qquad \le \|(T(t_2) - T(t_1))y_0\| + \|[T(t_2) - T(t_1)]f(y)\| \\
&\qquad\quad + \left\| \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]g(s)\,ds \right\| + \left\| \int_{t_1}^{t_2} T(t_2 - s)g(s)\,ds \right\| \\
&\qquad \le \|(T(t_2) - T(t_1))y_0\| + \|[T(t_2) - T(t_1)]f(y)\| \\
&\qquad\quad + \left\| \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]g(s)\,ds \right\| + M\int_{t_1}^{t_2} \|g(s)\|\,ds.
\end{aligned}$$

As $t_2 \to t_1$ the righthand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H6) together with the Arzela-Ascoli theorem, we can conclude that N is completely continuous, and therefore, a condensing map.

Step 4. N has a closed graph.

Let $y_n \to y^*$, $h_n \in N(y_n)$, and $h_n \to h^*$. We shall prove that $h^* \in N(y^*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)y_0 - T(t)f(y_n) + \int_0^t T(t-s)g_n(s)\,ds.$$

We have to prove that there exists $g^* \in S_{F,y^*}$ such that

$$h^*(t) = T(t)y_0 - T(t)f(y^*) + \int_0^t T(t-s)g^*(s)\,ds, \quad t \in J.$$

Consider the linear continuous operator

$$\begin{gathered}
\Gamma : L^1(J,E) \longrightarrow C(J,E) \\
g \longmapsto \Gamma(g)(t) = \int_0^t T(t-s)g(s)\,ds.
\end{gathered}$$

Clearly we have that

$$\|(h_n - T(t)y_0 + T(t)f(y_n)) - (h^* - T(t)[y_0 - f(y^*)])\|_\infty \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

From Lemma 3.1, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - T(t)y_0 + T(t)f(y_n) \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y^*$, it follows from Lemma 3.1 that

$$h^*(t) - T(t)y_0 + T(t)f(y^*) = \int_0^t T(t-s)g^*(s)\,ds$$

for some $g^* \in S_{F,y^*}$.

Step 5. The set

$$\Omega := \{y \in C(J,E) : \lambda y \in N(y), \text{ for some } \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus, there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)y_0 - \lambda^{-1}T(t)f(y) + \lambda^{-1}\int_0^t T(t-s)g(s)\,ds, \quad t \in J.$$

Consequently, by (H1) and (H3)–(H5), we have for each $t \in J$ that

$$\begin{aligned}\|y(t)\| &\le M\|y_0\| + ML + M\int_0^t \beta(s)\psi(\|y(s)\| + \alpha(s)\|y(s)\|)ds \\ &\le M\|y_0\| + ML + M\int_0^t \beta(s)(1+\alpha(s))\psi(\|y(s)\|)\,ds.\end{aligned}$$

Let us take the righthand side of the above inequality as $v(t)$. Then we obtain

$$v(0) = M(\|y_0\| + L), \quad \|y(t)\| \le v(t), \quad t \in J,$$

and

$$v'(t) = M\beta(t)(1+\alpha(t))\psi(\|y(t)\|), \quad t \in J.$$

Applying the nondecreasing character of ψ we get

$$v'(t) \leq M\beta(t)(1+\alpha(t))\psi(v(t)), \quad t \in J.$$

The above inequality implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq M \int_0^b \beta(t)(1+\alpha(t))\, ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)}.$$

Therefore, there exists a constant d such that $v(t) \leq d$, $t \in J$, and hence $\|y\|_\infty \leq d$, where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J, E)$. As a consequence of Lemma 3.2, we deduce that N has a fixed point, which is a mild solution of (1)–(2).

4. Existence result: The nonconvex case. In this section we consider the problems (1)–(2), with a nonconvex valued righthand side.

Let (X, d) be the metric space induced from the normed space $(X, \|\cdot\|)$.

Consider $H_d : P(X) \times P(X) \to \mathbf{R}_+ \cup \{\infty\}$, given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$.

Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition 4.1. A multi-valued operator $N : X \to P_{cl}(X)$ is called

a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x,\ y \in X,$$

b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

c) N has *a fixed point* if there is an $x \in X$ such that $x \in N(x)$. The fixed point set of the multi-valued operator N will be denoted by $\operatorname{Fix} N$.

Our considerations are based on the following fixed point theorem for contraction multi-valued operators given by Covitz and Nadler in 1970 [**12**], see also Deimling [**13**, Theorem 11.1].

Lemma 4.1. *Let* (X, d) *be a complete metric space. If* $N : X \to P_{cl}(X)$ *is a contraction, then* $\operatorname{Fix} N \neq \varnothing$.

Theorem 4.1. *Assume that*:

(A1) $F : J \times E \times E \longrightarrow P_{cp}(E)$ *has the property that* $F(\cdot, u, v) : J \to P_{cp}(E)$ *is measurable for each* $u, v \in E$;

(A2) *There exists* $l \in L^1(J, \mathbf{R})$ *such that*

$$H_d(F(t,u,v), F(t,\bar{u},\bar{v})) \leq l(t)[\|u-\bar{u}\| + \|v-\bar{v}\|],$$

for each $t \in J$ *and* $u, \bar{u}, v, \bar{v} \in E$, *and*

$$d(0, F(t,0,0)) \leq l(t), \quad \text{for almost each} \quad t \in J.$$

(A3) $\|f(y) - f(\bar{y})\| \leq c\|y - \bar{y}\|_\infty$, *for each* $y, \bar{y} \in C(J, E)$, *where* c *is a nonnegative constant.*

(A4) $\|k(t,s,z) - k(t,s,\bar{z})\| \leq K_1\|z - \bar{z}\|$, *for each* $t, s \in J$, $z, \bar{z} \in E$, *where* K_1 *is a nonnegative constant.*

Then the IVP (1)–(2) *has at least one solution on* J.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the multi-valued operator, $N : C([0,b], E) \to \mathcal{P}(C([0,b], E))$ defined by:

$$N(y) := \left\{ h \in C([0,b], E) : h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)g(s)\,ds \right\},$$

where $g \in S_{F,y}$.

Remark 4.1. (i) It is clear that the fixed points of N are solutions to (1)–(2).

(ii) For each $y \in C([0,b], E)$ the set $S_{F,y}$ is nonempty since, by (A1), F has a measurable selection, see [**11**, Theorem III.6].

Consider the following Bielecki-type norm, see [**8**], on $C(J,E)$ defined by
$$\|y\|_B = \max_{t\in J}\{\|y(t)\|e^{-\tau L(t)}\},$$
where $L(t)=\int_0^t l(s)\,ds$. Since
$$e^{-\tau L(b)}\|y\|_\infty \le \|y\|_B \le \|y\|_\infty,$$
the norms $\|y\|_B$ and $\|y\|_\infty$ are equivalent.

We shall show that N satisfies the assumptions of Lemma 4.1. The proof will be given in two steps.

Step 1. $N(y)\in P_{cl}(C[0,b],E)$ for each $y\in C([0,b],E)$.

Indeed, let $(y_n)_{n\ge 0}\in N(y)$ be such that $y_n \longrightarrow \tilde{y}$ in $C[0,b],E)$. Then $\tilde{y}\in C([0,b],E)$ and
$$\begin{aligned} y_n(t) \in{}& T(t)[y_0 - f(y)] \\ &+ \int_0^t T(t-s)F\left(s, y(s), \int_0^s k(s,w,y(w)\,dw\right)ds \quad \text{for each } t\in[0,b]. \end{aligned}$$

Using the closedness property of the values of F and the second part of (A2) we can prove that $\int_0^t T(t-s)F(s,y(s),\int_0^s k(s,w,y(w))\,dw)\,ds$ is closed for each $t\in[0,b]$. Then
$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) \in{}& T(t)[y_0 - f(y)] \\ &+ \int_0^t T(t-s)F\left(s, y(s), \int_0^s k(s,w,y(w))\,dw\right)ds, \qquad \text{for } t\in[0,b]. \end{aligned}$$

So $\tilde{y}\in N(y)$.

Step 2. $H_d(N(y_1),N(y_2)) \le \gamma\|y_1-y_2\|_B$ for each $y_1,y_2\in C([0,b],E)$, where $\gamma<1$.

Let $y_1,y_2\in C([0,b],E)$ and $h_1\in N(y_1)$. Then there exists
$$g_1(t)\in F\left(t, y_1(t), \int_0^t k(t,s,y_1(s))\,ds\right)$$

such that

$$h_1(t) = T(t)[y_0 - f(y_1)] + \int_0^t T(t-s)g_1(s)\,ds, \quad t \in [0,b].$$

From (A2) it follows that

$$H_d\bigg(F\Big(t, y_1(t), \int_0^t k(t,s,y_1(s))\,ds\Big), F\Big(t, y_2(t), \int_0^t k(t,s,y_2(s))\,ds\Big)\bigg) \\ \leq l(t)[\|y_1(t) - y_2(t)\| + bK_1\|y_1(t) - y_2(t)\|].$$

Hence there is $w \in F(t, y_2(t), \int_0^t k(t,s,y_2(s))\,ds)$ such that

$$\|g_1(t) - w\| \leq l(t)[\|y_1(t) - y_2(t)\| + bK_1\|y_1(t) - y_2(t)\|], \quad t \in [0,b].$$

Consider $U : [0,b] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \\ \leq l(t)[\|y_1(t) - y_2(t)\| + bK_1\|y_1(t) - y_2(t)\|]\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, y_2(t), \int_0^t k(t,s,y_2(s))\,ds)$ is measurable, see Proposition III.4 in [**11**], there exists $g_2(t)$ a measurable selection for V. So,

$$g_2(t) \in F\bigg(t, y_2(t), \int_0^t k(t,s,y_2(s))\,ds\bigg)$$

and

$$\|g_1(t) - g_2(t)\| \leq l(t)[\|y_1(t) - y_2(t)\| + bK_1\|y_1(t) - y_2(t)\|], \\ \text{for each } t \in J.$$

Let us define for each $t \in J$

$$h_2(t) = T(t)[y_0 - f(y_2)] + \int_0^t T(t-s)g_2(s)\,ds.$$

Then we have

$$
\begin{aligned}
&\|h_1-h_2\|_B \\
&= \max_{t\in J} e^{-\tau L(t)} \left\| T(t)[f(y_1)-f(y_2)] + \int_0^t T(t-s)[g_1(s)-g_2(s)]\,ds \right\| \\
&\le \max_{t\in J} e^{-\tau L(t)} Mc\|y_1-y_2\|_\infty + M \max_{t\in J} e^{-\tau L(t)} \int_0^t l(s)[\|y_1(s)-y_2(s)\| \\
&\qquad\qquad + bK_1\|y_1(s)-y_2(s)\|]\,ds \\
&\le Mce^{\tau L(b)}\|y_1-y_2\|_B \\
&\quad + \max_{t\in J} e^{-\tau L(t)} M(1+bK_1)\|y_1-y_2\|_B \int_0^t l(s)e^{\tau L(s)}\,ds \\
&\le Mce^{\tau L(b)}\|y_1-y_2\|_B + M(1+bK_1)\|y_1-y_2\|_B \frac{1}{\tau} \max_{t\in J} e^{-\tau L(t)}(e^{\tau L(t)}-1) \\
&\le Mce^{\tau L(b)}\|y_1-y_2\|_B + M(1+bK_1)\|y_1-y_2\|_B \frac{1}{\tau}(1-e^{-\tau L(b)}) \\
&\le Mce^{\tau L(b)}\|y_1-y_2\|_B + M(1+bK_1)\|y_1-y_2\|_B \frac{1}{\tau}.
\end{aligned}
$$

Then

$$\|h_1-h_2\|_B \le \left[Mce^{\tau L(b)} + \frac{M(1+bK_1)}{\tau}\right]\|y_1-y_2\|_B.$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1),N(y_2)) \le \left[Mce^{\tau L(b)} + \frac{M(1+bK_1)}{\tau}\right]\|y_1-y_2\|_B.$$

Let τ and c be such that $\gamma = Mce^{\tau L(b)} + (M(1+bK_1))/\tau < 1$. Then N is a contraction and thus, by Lemma 4.1, it has a fixed point y, which is the solution to (1)–(2).

5. Applications. As applications of our results, we shall give controllability results for first order semi-linear integrodifferential inclusions of the form

$$y'(t) \in Ay(t) + F\left(t, y(t), \int_0^t k(t,s,y(s))\,ds\right) + (Bu)(t), \quad t\in J=[0,b], \tag{3}$$

$$y(0) + f(y) = y_0, \tag{4}$$

where the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U as a Banach space and B a bounded linear operator from U to E. For recent controllability results of nonlinear ordinary, functional and neutral functional differential and integrodifferential systems in Banach spaces, by using different tools of fixed point arguments, we refer to the papers by Benchohra and Ntouyas [**6, 7**] and Balachandran et al. [**1, 2**] and [**3**].

Definition 5.1. A function $y \in C(J,E)$ is called a mild solution of (3)–(4) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t, y(t), \int_0^t k(t,s,y(s))\, ds)$ almost everywhere on J, and

$$y(t) = T(t)y_0 - T(t)f(y) + \int_0^t T(t-s)(Bu)(s)\, ds + \int_0^t T(t-s)v(s)\, ds.$$

Definition 5.2. The system (3)–(4) is said to be nonlocally controllable on the interval J, if for every y_0, $x_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the mild solution $y(t)$ of (3)–(4) satisfies $y(b) + f(y) = x_1$.

We will need the following additional assumption:

(H7) The linear operator $W : L^2(J,U) \to E$, defined by

$$Wu = \int_0^b T(b-s)Bu(s)\, ds,$$

has an invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)/\ker W$ and there exist positive constants M_1 and M_2 such that $\|B\| \le M_1$ and $\|\widetilde{W}^{-1}\| \le M_2$.

Theorem 5.1. *Let $F : J \times E \times E \to \mathcal{P}(E)$ be a bounded, closed and convex valued multi-valued map. Assume that hypotheses* (H1)–(H7) *are satisfied. Then the problem* (3)–(4) *is nonlocally controllable on J.*

Theorem 5.2. *Let $F : J \times E \times E \to \mathcal{P}(E)$ be a nonconvex valued multi-valued map. Assume that hypotheses* (A1)–(A4) *and* (H7) *are satisfied. Then the problem* (3)–(4) *is nonlocally controllable on J.*

Using hypothesis (H7), for an arbitrary function $y(\cdot)$, define the control

$$u_y(t) = \widetilde{W}^{-1}\left[x_1 - f(y) - T(b)y_0 + T(b)f(y) - \int_0^b T(b-s)g(s)\,ds\right](t)$$

where $g \in S_{F,y}$.

We shall now show that, when using this control, the operator N defined by

$$N(y) := \Big\{h \in C(J,E) : h(t) = T(t)(y_0 - f(y)) + \int_0^t T(t-s)(Bu_y)(s)\,ds + \int_0^t T(t-s)g(s)\,ds : g \in S_{F,y}\Big\}$$

has a fixed point. This fixed point is then a solution of the system (3)–(4).

Clearly $x_1 - f(y) \in N(y)(b)$.

We shall show that N for Theorem 5.1 is completely continuous with bounded closed convex values and it is upper semi-continuous and for Theorem 5.2 that N has closed values and it is a contraction multi-valued map. The steps for the proofs are parallel to that of Theorems 3.1 and 4.1. So we omit the details.

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