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EXISTENCE RESULTS FOR IMPULSIVE PARTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove existence results for first order semilinear impulsive neutral functional differential inclusions under the mixed Lipschitz and Carathéodory conditions.

1. Introduction

The theory of impulsive differential equations is emerging as an important area of investigation since it is much richer that the corresponding theory of differential equations; see the monograph of Lakshmikantham *et al* [3]. In this paper, we study the existence of solutions for initial value problems for first order impulsive semilinear neutral functional differential inclusions. More precisely in Section 3 we consider first-order impulsive semilinear neutral functional differential inclusions of the form

$$\frac{d}{dt}[x(t)-f(t,x_t)] \in Ax(t)+G(t,x_t) \quad \text{a.e. } t\in J:=[0,T], \quad t\neq t_k \quad k=1,\dots,m, \tag{1.1}$$

$$x(t_k^+)-x(t_k^-)=I_k(x(t_k^-)), \quad k=1,\dots,m, \tag{1.2}$$

$$x(t)=\phi(t), \quad t\in[-r,0], \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $S(t), t\geq 0$ on a Banach space X, $f: J\times\mathcal{D}\to X$ and $G: J\times\mathcal{D}\to\mathcal{P}(X)$; $\mathcal{D}$ consists of functions $\psi:[-r,0]\to X$ such that ψ is continuous everywhere except for a finite number of points s at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-)=\psi(s)$; $\phi\in\mathcal{D}$, $(0<r<\infty)$, $0=t_0<t_1<\dots<t_m<t_{m+1}=T$, $I_k: X\to X$ $(k=1,2,\dots,m)$, $x(t_k^+)$ and $x(t_k^-)$ are respectively the right and the left limit of x at $t=t_k$, and $\mathcal{P}(X)$ denotes the class of all nonempty subsets of X.

For any continuous function x defined on the interval $[-r,T]\setminus\{t_1,\dots,t_m\}$ and any $t\in J$, we denote by x_t the element of $\mathcal{D}$ defined by

$$x_t(\theta)=x(t+\theta), \quad \theta\in[-r,0].$$

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For $\psi \in \mathcal{D}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{D}} = \sup\{|\psi(\theta)|, \theta \in [-r, 0]\}.$$

The main tools used in the study is a fixed point theorem proved by Dhage [1]. In the following section, we give some auxiliary results needed in the subsequent part of the paper.

2. Auxiliary results

Throughout this paper, X will be a separable Banach space provided with norm $\|\cdot\|$ and $A : D(A) \to X$ will be the infinitesimal generator of an analytic semigroup, $S(t),\ t \geq 0$, of bounded linear operators on X. For the theory of strongly continuous semigroup, refer to Pazy [5]. If $S(t),\ t \geq 0$, is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fraction power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha$$

defines a norm on $D(-A)^\alpha$. Hereafter we denote by X_α the Banach space $D(-A)^\alpha$ normed with $\|\cdot\|_\alpha$. Then for each $0 < \alpha \leq 1$, X_α is a Banach space, and $X_\alpha \hookrightarrow X_\beta$ for $0 < \beta \leq \alpha \leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. Also for every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha S(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq T. \tag{2.1}$$

Let $\mathcal{P}(X)$ denote the class of all nonempty subsets of X. Let $\mathcal{P}_{bd,cl}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote respectively the classes of all bounded-closed and compact-convex subsets of X. For $x \in X$ and $Y, Z \in \mathcal{P}_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| : y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$.

Define the function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The function H is called a Hausdorff metric on X. Note that $\|Y\| = H(Y, \{0\})$.

A correspondence $G : X \to \mathcal{P}(X)$ is called a multi-valued mapping on X. A point $x_0 \in X$ is called a *fixed point of the multi-valued operator* $G : X \to \mathcal{P}(X)$ *if* $x_0 \in G(x_0)$. The fixed points set of G will be denoted by $\mathrm{Fix}(G)$.

Definition 2.1. Let $G : X \to \mathcal{P}_{bd,cl}(X)$ be a multi-valued operator. Then G is called a multi-valued contraction if there exists a constant $k \in (0, 1)$ such that for each $x, y \in X$ we have

$$H(G(x), G(y)) \leq k\|x - y\|.$$

The constant k is called a contraction constant of G.

A multi-valued mapping $G : X \to \mathcal{P}(X)$ is called *lower semi-continuous* (shortly l.s.c.) (resp. *upper semi-continuous* (shortly u.s.c.)) if B is any open subset of X then $\{x \in X : Gx \cap B \neq \emptyset\}$(resp. $\{x \in X : Gx \subset B\}$) is an open subset of X. The multi-valued operator G is called *compact* if $\overline{G(X)}$ is a compact subset of X. Again G is called *totally bounded* if for any bounded subset S of X, $G(S)$ is a totally bounded subset of X. A multi-valued operator $G : X \to \mathcal{P}(X)$ is called *completely continuous* if it is upper semi-continuous and totally bounded on X, for each bounded $B \in \mathcal{P}(X)$. Every compact multi-valued operator is totally bounded but the converse may not be true. However the two notions are equivalent on a bounded subset of X.

We apply the following form of the fixed point theorem by Dhage [1] in the sequel.

Theorem 2.2. *Let X be a Banach space, $A : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : X \to \mathcal{P}_{cp,cv}(X)$ two multi-valued operators satisfying*

(a) *A is contraction with a contraction constant k, and*
(b) *B is completely continuous.*

Then either

(i) *The operator inclusion $\lambda x \in Ax + Bx$ has a solution for $\lambda = 1$, or*
(ii) *The set $\mathcal{E} = \{u \in X : \lambda u \in Au + Bu,\ \lambda > 1\}$ is unbounded.*

3. Existence results

Let us state what we mean by a solution of problem (1.1)–(1.3). For this purpose, we consider the space $PC([-r,T],X)$ consisting of functions $x : [-r,T] \to X$ such that $x(t)$ is continuous almost everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+)$, $k = 1,\dots,m$ exist and $x(t_k^-) = x(t_k)$.

Obviously, for any $t \in [0,T]$ we have $x_t \in \mathcal{D}$ and $PC([-r,T],X)$ is a Banach space with the norm

$$\|x\| = \sup\{|x(t)| : t \in [-r,T]\}.$$

In the following we set for convenience

$$\Omega = PC([-r,T],X).$$

Also we denote by $AC(J,X)$ the space of all absolutely continuous functions $x : J \to X$.

A function $x \in \Omega \cap AC((t_k,t_{k+1}),X)$, $k = 1,\dots,m$, is said to be a solution of (1.1)–(1.3) if $x(t) - f(t,x_t)$ is absolutely continuous on $J \setminus \{t_1,\dots,t_m\}$ and (1.1)–(1.3) are satisfied.

A multi-valued map $G : J \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function $t \to d(y,G(t)) = \inf\{\|y-x\| : x \in G(t)\}$ is measurable.

A multi-valued map $G : J \times \mathcal{D} \to \mathcal{P}_{cl}(X)$ is said to be L^1-Carathéodory if

(i) $t \mapsto G(t,x)$ is measurable for each $x \in \mathcal{D}$,
(ii) $x \mapsto G(t,x)$ is upper semi-continuous for almost all $t \in J$, and
(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1(J,\mathbb{R}^+)$ such that
$$\|G(t,u)\| := \sup\{\|v\| : v \in G(t,u)\} \le h_\rho(t), \quad a.e. \quad t \in J$$
for all $u \in \mathcal{D}$ with $\|u\|_{\mathcal{D}} \le \rho$.

Then we have the following lemmas due to Lasota and Opial [4].

Lemma 3.1. *If $\dim(X) < \infty$ and $F : J \times X \to \mathcal{P}(X)$ is L^1-Carathéodory, then $S_G^1(x) \neq \emptyset$ for each $x \in X$.*

Lemma 3.2. *Let X be a Banach space, G an L^1-Carathéodory multi-valued map with $S_G^1 \neq \emptyset$ where*

$$S_G^1(x) := \{v \in L^1(I,\mathbb{R}^n) : v(t) \in G(t,x_t) \text{ a.e. } t \in J\},$$

and $\mathcal{K} : L^1(J, X) \to C(J, X)$ be a linear continuous mapping. Then the operator
$$\mathcal{K} \circ S^1_G : C(J, X) \to \mathcal{P}_{cp,cv}(C(J, X))$$
is a closed graph operator in $C(J, X) \times C(J, X)$.

We need also the following result from [2].

Lemma 3.3. *Let $v(\cdot), w(\cdot) : [0, T] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $\theta > 0$, $0 < \alpha < 1$ such that*
$$v(t) \le w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}\, ds, \quad t \in [0, T],$$
then
$$v(t) \le e^{\theta^n \Gamma(\alpha)^n t^{n\alpha}/\Gamma(n\alpha)} \sum_{J=0}^{n-1} \big(\frac{\theta T^\alpha}{\alpha}\big)^j w(t),$$
for every $t \in [0, T]$ and every $n \in \mathbb{N}$ such that $n\alpha > 1$, and $\Gamma(\cdot)$ is the Gamma function.

We consider the following set of assumptions in the sequel.

- (H1) There exist constants $0 < \beta < 1, c_1, c_2, L_f$ such that f is X_β-valued, $(-A)^\beta f$ is continuous, and
  - (i) $\|(-A)^\beta f(t, x)\| \le c_1 \|x\|_{\mathcal{D}} + c_2$, $(t, x) \in J \times \mathcal{D}$
  - (ii) $\|(-A)^\beta f(t, x_1) - (-A)^\beta f(t, x_2)\| \le L_f \|x_1 - x_2\|_{\mathcal{D}}$, $(t, x_i) \in J \times \mathcal{D}$, $i = 1, 2$, with
$$L_f \big\{ \|(-A)^{-\beta}\| + \frac{C_{1-\beta} T^\beta}{\beta} \big\} < 1.$$
- (H2) The multivalued map $G(t, x)$ has compact and convex values for each $(t, x) \in J \times \mathcal{D}$.
- (H3) The semigroup $S(t)$ is compact for $t > 0$, and there exists $M \ge 1$ such that
$$\|S(t)\| \le M, \quad \text{for all } t \ge 0.$$
- (H4) G is L^1-Carathéodory.
- (H5) There exists a function $q \in L^1(I, \mathbb{R})$ with $q(t) > 0$ for a.e. $t \in J$ and a nondecreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ such that
$$\|G(t, x)\| := \sup\{\|v\| : v \in G(t, x)\} \le q(t)\psi(\|x\|_{\mathcal{D}}) \ \text{ a.e. } t \in J,$$
for all $x \in \mathcal{D}$.
- (H6) The impulsive functions I_k are continuous and there exist constants c_k such that $\|I_k(x)\| \le c_k$, $k = 1, \dots, m$ for each $x \in X$.

Theorem 3.4. *Assume that (H1)–(H6) hold. Suppose that*
$$bK_2 \int_0^T q(s)\, ds < \int_{K_0}^\infty \frac{ds}{s + \psi(s)},$$
where
$$K_0 = \frac{F}{1 - c_1 \|(-A)^{-\beta}\|}, \quad K_2 = \frac{M}{1 - c_1 \|(-A)^{-\beta}\|},$$
$$b = e^{K_1^n (\Gamma(\beta))^n T^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \big(\frac{K_1 T^\beta}{\beta}\big)^j,$$

and

$$F = M\|\phi\|_{\mathcal{D}}\{1 + c_1\|(-A)^{-\beta}\|\} + c_2\|(-A)^{-\beta}\|\{M+1\} + M\sum_{k=1}^{m} c_k + \frac{C_{1-\beta}c_2T^{\beta}}{\beta}.$$

Then the initial-value problem (1.1)–(1.3) has at least one solution on $[-r,T]$.

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$\begin{aligned} Nx(t) = \Big\{ & h \in \Omega : h(t) = \phi(t) \text{ for } t \in [-r,0], \text{ and } h(t) = S(t)[\phi(0) - f(0,\phi(0))] \\ & + f(t,x_t) + \int_0^t AS(t-s)f(s,x_s)ds + \int_0^t S(t-s)v(s)ds \\ & + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) \text{ for } t \in J \Big\}, \end{aligned}$$

where $v \in S^1_G(x)$.

Now, we define two operators as follows. $A : \Omega \to \Omega$ by

$$Ax(t) = \begin{cases} 0, & \text{if } t \in [-r,0], \\ \big\{-S(t)f(0,\phi) + f(t,x_t) + \int_0^t AS(t-s)f(s,x_s)ds\big\}, & \text{if } t \in J, \end{cases} \tag{3.1}$$

and the multi-valued operator $B : \Omega \to \mathcal{P}(\Omega)$ by

$$\begin{aligned} Bx(t) = \Big\{ & h \in \Omega : h(t) = \phi(t) \text{ for } t \in [-r,0], \text{ and } h(t) = S(t)\phi(0) \\ & + \int_0^t S(t-s)v(s)\,ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) \text{ for } t \in J \Big\}. \end{aligned} \tag{3.2}$$

Then $N = A + B$. We shall show that the operators A and B satisfy all the conditions of Theorem 2.2 on $[-r,T]$. For better readability, we break the proof into a sequence of steps.

Step I. First we remark that A for each $x \in \Omega$, has closed, convex values on Ω. Next we show that A has bounded values for bounded sets in X. To show this, let S be a bounded subset of Ω, with bound ρ. Then, for any $x \in S$ one has

$$\begin{aligned} \|Ax(t)\| \le & M\|f(0,\phi)\| + \|(-A)^{-\beta}\|[c_1\|x_t\|_{\mathcal{D}} + c_2] \\ & + \int_0^t \|(-A)^{1-\beta}S(t-s)\|\|(-A)^{\beta}f(s,x_s)\|ds \\ \le & M\|f(0,\phi)\| + \|(-A)^{-\beta}\|[c_1\|x_t\|_{\mathcal{D}} + c_2] \\ & + \int_0^t \frac{C_{1-\beta}c_1}{(t-s)^{1-\beta}}\|x_s\|_{\mathcal{D}}ds + \frac{C_{1-\beta}c_2T^{\beta}}{\beta} \\ \le & M\|f(0,\phi)\| + \|(-A)^{-\beta}\|[c_1\rho + c_2] + \frac{C_{1-\beta}T^{\beta}}{\beta}[\rho c_1 + c_2], \end{aligned}$$

and consequently

$$\|Ax\| \le M\|f(0,\phi)\| + \|(-A)^{-\beta}\|[c_1\rho + c_2] + \frac{C_{1-\beta}T^{\beta}}{\beta}[\rho c_1 + c_2].$$

Hence A is bounded on bounded subsets of Ω.

Step II. Next we prove that Bx is a convex subset of Ω for each $x \in \Omega$. Let $u_1, u_2 \in Bx$. Then there exists v_1 and v_2 in $S^1_G(x)$ such that

$$u_j(t) = S(t)\phi(0) + \sum_{0<t_k<t} S(t-s_k)I_k(x(t_k^-)) + \int_0^t S(t-s)v_j(s)\,ds, \quad j=1,2.$$

Since $G(t,x)$ has convex values, one has for $0 \le \mu \le 1$,

$$[\mu v_1 + (1-\mu)v_2](t) \in S^1_G(x)(t), \quad \forall t \in J.$$

As a result we have

$$\begin{aligned}
&[\mu u_1 + (1-\mu)u_2](t)\\
&= S(t)\phi(0) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) + \int_0^t S(t-s)[\mu v_1(s) + (1-\mu)v_2(s)]\,ds.
\end{aligned}$$

Therefore, $[\mu u_1 + (1-\mu)u_2] \in Bx$ and consequently Bx has convex values in Ω. Thus we have $B : \Omega \to \mathcal{P}_{cv}(\Omega)$.

Step III. We show that A is a contraction on Ω. Let $x, y \in X$. By hypothesis (H1)

$$\begin{aligned}
\|Ax(t) - Ay(t)\| &\le \|f(t,x_t) - f(t,y_t)\| + \Big\| \int_0^t AS(t-s)[f(s,x_s) - f(s,y_s)]\,ds \Big\|\\
&\le \|(-A)^{-\beta}\| L_f \|x_t - y_t\|_{\mathcal{D}} + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}\,ds\, L_f \|x_t - y_t\|_{\mathcal{D}}\\
&\le L_f \Big\{ \|(-A)^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta} \Big\} \|x_t - y_t\|_{\mathcal{D}}.
\end{aligned}$$

Taking supremum over t,

$$\|Ax - Ay\| \le L_0 \|x-y\|_{\mathcal{D}}, \quad L_0 := L_f \Big\{ \|(-A)^{-\beta}\| + \frac{C_{1-\beta}T^\beta}{\beta} \Big\}.$$

This shows that A is a multi-valued contraction, since $L_0 < 1$.

Step IV. Now we show that the multi-valued operator B is completely continuous on Ω. First we show that B maps bounded sets into bounded sets in Ω. To see this, let Q be a bounded set in Ω. Then there exists a real number $\rho > 0$ such that $\|x\| \le \rho, \forall x \in Q$.

Now for each $u \in Bx$, there exists a $v \in S^1_G(x)$ such that

$$u(t) = S(t)\phi(0) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) + \int_0^t S(t-s)v(s)\,ds, \quad t \in J.$$

Then for each $t \in J$,

$$\begin{aligned}
\|u(t)\| &\le M\|\phi(0)\| + M\sum_{k=1}^m c_k + M\int_0^t |v(s)|\,ds\\
&\le M\|\phi\|_{\mathcal{D}} + M\sum_{k=1}^m c_k + M\int_0^t h_\rho(s)\,ds\\
&\le M\|\phi\|_{\mathcal{D}} + M\sum_{k=1}^m c_k + M\|h_\rho\|_{L^1}.
\end{aligned}$$

This implies

$$\|u\| \leq M\|\phi\|_{\mathcal{D}} + M\sum_{k=1}^{m} c_k + M\|h_\rho\|_{L^1}$$

for all $u \in Bx \subset B(Q) = \bigcup_{x\in Q} B(x)$. Hence $B(Q)$ is bounded.

Next we show that B maps bounded sets into equi-continuous sets. Let Q be, as above, a bounded set and $h \in Bx$ for some $x \in Q$. Then there exists $v \in S^1_G(x)$ such that

$$h(t) = S(t)\phi(0) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) + \int_0^t S(t-s)v(s)\,ds, \quad t \in J.$$

Let $\tau_1, \tau_2 \in J\backslash\{t_1,\dots,t_m\},\ \tau_1 < \tau_2$. Then we have

$$\begin{aligned}
&\|h(\tau_2) - h(\tau_1)\| \\
&\leq \|[S(\tau_2) - S(\tau_1)]\phi(0)\| + \int_0^{\tau_1-\epsilon} \|S(\tau_2 - s) - S(\tau_1 - s)\|\varphi_q(s)ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|\varphi_q(s)ds + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|\varphi_q(s)ds \\
&\quad + \sum_{0<t_k<\tau_2-\tau_1} Mc_k + \sum_{0<t_k<\tau_2} \|S(\tau_2 - t_k) - S(\tau_1 - t_k)\|c_k.
\end{aligned}$$

As $\tau_2 \to \tau_1$ and ϵ becomes sufficiently small the right-hand side of the above inequality tends to zero, since $S(t)$ is a strongly continuous operator and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology.

This proves the equicontinuity for the case where $t \neq t_i,\ i = 1,\dots,m+1$. It remains to examine the equicontinuity at $t = t_i$. Set

$$h_1(t) = S(t)\phi(0) + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-))$$

and

$$h_2(t) = \int_0^t S(t-s)v(s)ds.$$

First we prove equicontinuity at $t = t_i^-$. Fix $\delta_1 > 0$ such that $\{t_k :\ k \neq i\} \cap [t_i - \delta_1, t_i + \delta_1] = \emptyset$,

$$\begin{aligned}
h_1(t_i) &= S(t_i)\phi(0) + \sum_{0<t_k<t_i} S(t-t_k)I_k(y(t_k^-)) \\
&= S(t_i)\phi(0) + \sum_{k=1}^{i-1} T(t_i - t_k)I_k(y(t_k^-)).
\end{aligned}$$

For $0 < h < \delta_1$ we have

$$\begin{aligned}
&\|h_1(t_i - h) - h_1(t_i)\| \\
&\leq \|(S(t_i - h) - S(t_i))\phi(0) + \sum_{k=1}^{i-1} |[S(t_i - h - t_k) - S(t_i - t_k)]I(y(t_k^-))\|.
\end{aligned}$$

The right-hand side tends to zero as $h \to 0$. Moreover

$$\|h_2(t_i - h) - h_2(t_i)\| \leq \int_0^{t_i-h} \|[S(t_i - h - s) - S(t_i - s)]v(s)\|ds + \int_{t_i-h}^{t_i} M\phi_q(s)ds,$$

which tends to zero as $h \to 0$. Define

$$\hat{h}_0(t) = h(t), \quad t \in [0, t_1]$$

and

$$\hat{h}_i(t) = \begin{cases} h(t), & \text{if } t \in (t_i, t_{i+1}], \\ h(t_i^+), & \text{if } t = t_i \end{cases}$$

Next we prove equicontinuity at $t = t_i^+$. Fix $\delta_2 > 0$ such that $\{t_k : k \neq i\} \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then

$$\hat{h}(t_i) = S(t_i)\phi(0) + \int_0^{t_i} S(t_i - s)v(s) + \sum_{k=1}^{i} S(t_i - t_k)I_k(y(t_k)).$$

For $0 < h < \delta_2$ we have

$$\begin{aligned} &\|\hat{h}(t_i + h) - \hat{h}(t_i)\| \\ &\leq \|(S(t_i + h) - S(t_i))\phi(0)\| + \int_0^{t_i} \|[S(t_i + h - s) - S(t_i - s)]v(s)\| ds \\ &\quad + \int_{t_i}^{t_i+h} M\varphi_q(s)ds + \sum_{k=1}^{i} \|[S(t_i + h - t_k) - S(t_i - t_k)]I(y(t_k^-))\|. \end{aligned}$$

The right-hand side tends to zero as $h \to 0$.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ follows from the uniform continuity of ϕ on the interval $[-r, 0]$. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem it suffices to show that B maps Q into a precompact set in X.

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $x \in Q$ we define

$$\begin{aligned} &h_\epsilon(t) \\ &= S(t)\phi(0) + S(\epsilon)\int_0^{t-\epsilon} S(t - s - \epsilon)v_1(s)ds + S(\epsilon)\sum_{0<t_k<t-\epsilon} S(t - t_k - \epsilon)I_k(y(t_k^-)), \end{aligned}$$

where $v_1 \in S_F^1$. Since $S(t)$ is a compact operator, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in N(y)\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $h \in N(y)$ we have

$$|h(t) - h_\epsilon(t)| \leq \int_{t-\epsilon}^{t} \|S(t - s)\|\varphi_q(s)ds + \sum_{t-\epsilon<t_k<t} \|S(t - t_k)\|c_k.$$

Therefore, there are precompact sets arbitrarily close to the set $H(t) = \{h_\epsilon(t) : h \in N(y)\}$. Hence the set $H(t) = \{h(t) : h \in B(Q)\}$ is precompact in X. Hence, the operator $B : \Omega \to \mathcal{P}(\Omega)$ is completely continuous.

Step V. Next we prove that B *has a closed graph.* Let $\{x_n\} \subset \Omega$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S_G^1(x_n)$ such that

$$y_n(t) = \phi(0) + \sum_{0<t_k<t} S(t - t_k)I_k(y_n(t_k^-)) + \int_0^t v_n(s)\, ds.$$

Consider the linear and continuous operator $\mathcal{K} : L^1(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ defined by

$$\mathcal{K}v(t) = \int_0^t v_n(s)\,ds.$$

Now

$$\begin{aligned}&\Big\| y_n(t) - \phi(0) - \sum_{0<t_k<t} S(t-t_k)I_k(y_n(t_k^-)) \\ &- \Big(y_*(t) - \phi(0) - \sum_{0<t_k<t} S(t-t_k)I_k(y_*(t_k^-))\Big)\Big\| \to 0,\end{aligned}$$

as $n \to \infty$. From Lemma 3.2 it follows that $(\mathcal{K} \circ S^1_G)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) - \sum_{0<t_k<t} S(t-t_k)I_k(y_n(t_k^-)) \in (\mathcal{K} \circ S^1_F(y_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S^1_G(x_*)$ such that

$$y_*(t) = \phi(0) + \sum_{0<t_k<t} S(t-t_k)I_k(y_*(t_k^-)) + \int_0^t v_*(s)\,ds.$$

Hence the multi-valued operator B is an upper semi-continuous operator on Ω.

Step VI. Finally we show that the set

$$\mathcal{E} = \{u \in \Omega : \lambda u \in Au + Bu \text{ for some } \lambda > 1\}$$

is bounded. Let $u \in \mathcal{E}$ be any element. Then there exists $v \in S^1_G(u)$ such that

$$\begin{aligned}u(t) =&\lambda^{-1}S(t)[\phi(0) - f(0,\phi(0))] + \lambda^{-1}f(t,x_t) \\ &+ \lambda^{-1}\int_0^t AS(t-s)f(s,x_s)ds + \lambda^{-1}\int_0^t S(t-s)v(s)ds \\ &+ \lambda^{-1}\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).\end{aligned}$$

Then

$$\begin{aligned}
\|u(t)\| &\le M\|\phi\|_{\mathcal{D}} + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}}+c_2] + \|(-A)^{-\beta}\|[c_1\|u_t\|_{\mathcal{D}}+c_2] \\
&\quad + \int_0^t \|(-A)^{1-\beta}S(t-s)\|\|(-A)^{\beta}f(s,x_s)\|\,ds \\
&\quad + M\int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})ds + M\sum_{k=1}^m c_k \\
&\le M\|\phi\|_{\mathcal{D}} + M\|(-A)^{-\beta}\|[c_1\|\phi\|_{\mathcal{D}}+c_2] + \|(-A)^{-\beta}\|[c_1\|u_t\|_{\mathcal{D}}+c_2] \\
&\quad + \int_0^t \frac{C_{1-\beta}c_1}{(t-s)^{1-\beta}}\,\|u_s\|_{\mathcal{D}}\,ds + \frac{C_{1-\beta}c_2T^{\beta}}{\beta} \\
&\quad + M\int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})ds + M\sum_{k=1}^m c_k \\
&\le F + c_1\|(-A)^{-\beta}\|\|u_t\|_{\mathcal{D}} \\
&\quad + \int_0^t \frac{C_{1-\beta}c_1}{(t-s)^{1-\beta}}\,\|u_s\|_{\mathcal{D}}\,ds + M\int_0^t q(s)\psi(\|u_s\|_{\mathcal{D}})ds, \quad t\in J,
\end{aligned}$$

where

$$F = M\|\phi\|_{\mathcal{D}}\{1+c_1\|(-A)^{-\beta}\|\} + c_2\|(-A)^{-\beta}\|\{M+1\} + M\sum_{k=1}^m c_k + \frac{C_{1-\beta}c_2T^{\beta}}{\beta}.$$

Put $w(t)=\max\{\|u(s)\| : -r\le s\le t\}$, $t\in J$. Then $\|u_t\|_{\mathcal{D}}\le w(t)$ for all $t\in J$ and there is a point $t^*\in[-r,t]$ such that $w(t)=\|u(t^*)\|$. Hence we have

$$\begin{aligned}
w(t) &= \|u(t^*)\| \\
&\le F + c_1\|(-A)^{-\beta}\|\|u_{t^*}\|_{\mathcal{D}} + C_{1-\beta}c_1\int_0^{t^*}\frac{\|u_s\|_{\mathcal{D}}}{(t-s)^{1-\beta}}\,ds \\
&\quad + M\int_0^{t^*} q(s)\psi(\|u_s\|_{\mathcal{D}})ds \\
&\le F + c_1\|(-A)^{-\beta}\|w(t) + C_{1-\beta}c_1\int_0^t\frac{w(s)}{(t-s)^{1-\beta}}\,ds + M\int_0^t q(s)\psi(w(s))\,ds,
\end{aligned}$$

or

$$\begin{aligned}
w(t) &\le \frac{F}{1-c_1\|(-A)^{-\beta}\|} \\
&\quad + \frac{1}{1-c_1\|(-A)^{-\beta}\|}\Big\{C_{1-\beta}c_1\int_0^t\frac{w(s)}{(t-s)^{1-\beta}}\,ds + M\int_0^t q(s)\psi(w(s))\,ds\Big\} \\
&\le K_0 + K_1\int_0^t\frac{w(s)}{(t-s)^{1-\beta}}\,ds + K_2\int_0^t q(s)\psi(w(s))\,ds, \quad t\in I,
\end{aligned}$$

where

$$K_0 = \frac{F}{1-c_1\|(-A)^{-\beta}\|}, \quad K_1 = \frac{C_{1-\beta}c_1}{1-c_1\|(-A)^{-\beta}\|} \quad \text{and} \quad K_2 = \frac{M}{1-c_1\|(-A)^{-\beta}\|}.$$

From Lemma 3.3 we have

$$w(t) \le b\Big(K_0 + K_2 \int_0^t q(s)\psi(w(s))\,ds\Big),$$

where

$$b = e^{K_1^n(\Gamma(\beta))^n T^{n\beta}/\Gamma(n\beta)} \sum_{j=0}^{n-1} \Big(\frac{K_1 T^\beta}{\beta}\Big)^j.$$

Let

$$m(t) = b\Big(K_0 + K_2 \int_0^t q(s)\psi(w(s))\,ds\Big), \quad t \in J.$$

Then we have $w(t) \le m(t)$ for all $t \in J$. Differentiating with respect to t, we obtain

$$m'(t) = bK_2 q(t)\psi(w(t)), \quad \text{a.e. } t \in J,\ m(0) = K_0.$$

This implies $m'(t) \le bK_2 q(t)\psi(m(t))$ a.e. $t \in J$; that is,

$$\frac{m'(t)}{\psi(m(t))} \le bK_2 q(t), \quad \text{a.e. } t \in J.$$

Integrating from 0 to t, we obtain

$$\int_0^t \frac{m'(s)}{\psi(m(s))}\,ds \le bK_2 \int_0^t q(s)\,ds.$$

By the change of variable,

$$\int_{K_0}^{m(t)} \frac{ds}{\psi(s)} \le bK_2 \int_0^T q(t)\,ds < \int_{K_0}^{\infty} \frac{ds}{\psi(s)}.$$

Hence there exists a constant M such that $m(t) \le M$ for all $t \in J$, and therefore

$$w(t) \le m(t) \le M \quad \text{for all } t \in J.$$

Now from the definition of w it follows that

$$\|u\| = \sup_{t\in[-r,T]} \|u(t)\| = w(T) \le m(T) \le M,$$

for all $u \in \mathcal{E}$. This shows that the set $\mathcal{E}$ is bounded in Ω. As a result the conclusion (ii) of Theorem 2.2 does not hold. Hence the conclusion (i) holds and consequently the initial value problem (1.1)–(1.3) has a solution x on $[-r, T]$. This completes the proof. $\square$

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