Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 41, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Existence results for functional differential inclusions *

M. Benchohra & S. K. Ntouyas

Abstract

In this note we investigate the existence of solutions to functional differential inclusions on compact intervals. We use the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps.

1 Introduction

This note is concerned with the existence of solutions defined on a compact real interval for the following initial and boundary-value problems of the functional differential inclusions

$$y' \in F(t, y_t), \quad \text{a. e. } t \in [0,T] \tag{1.1}$$
$$y(t) = \phi(t), \quad t \in [-r, 0] \tag{1.2}$$

and

$$y'' \in F(t, y_t), \quad \text{a. e. } t \in [0,T] \tag{1.3}$$
$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y'(0) = \eta, \tag{1.4}$$

where $F : J \times C([-r, 0], E) \to \mathcal{P}(E)$ is a multi-valued map, $\phi \in C([-r, 0], E)$, $\eta \in E$, $\mathcal{P}(E)$ is the family of all subsets of a real separable Banach space E with norm $\|\cdot\|$.

For any continuous function y defined on the interval $[-r, T]$ and any $t \in [0, T]$, we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ to the time t.

The approach used here is to reduce the existence of solutions to problems (1.1)-(1.2) and (1.3)-(1.4) to the search for fixed points of a suitable multi-valued map on the Banach space $C([-r, T], E)$. To prove the existence of fixed points,

* *Mathematics Subject Classifications:* 34A60, 34K10.
Key words: Functional differential inclusions, measurable selection, contraction multi-valued map, existence, fixed point, Banach space.

Submitted March 30, 2001. Published June 4, 2001.

we use a fixed point theorem for Contraction multi-valued maps, introduced by Covitz and Nadler [2] (see also Deimling [3]).

For a review of recent results on boundary-value problems for functional differential equations we refer the reader to the books by Erbe, Qingai and Zhang [5] and by Henderson [7], to the papers by Ntouyas [12], by Nieto, Jiang and Jurang [11], by Liz and Nieto [10], and the references cited therein. The methods used in these problems are usually the topological transversality by Granas [4] and the monotone iterative method combined with upper and lower solutions [9].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this note.

Let $C([-r,0],E)$ be the Banach space consisting of all continuous functions from $[-r,0]$ to E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Similarly $C([0,T],E)$ denotes the Banach space of continuous functions on $[0,T]$ with norm $\|\cdot\|_{[0,T]}$.

Let $L^1([0,T],E)$ denote the Banach space of measurable functions $y : [0,T] \to E$ which are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\|dt.$$

For a metric space (X,d), we define

$$P(X) = \{Y \in \mathcal{P}(X) : Y \neq \emptyset\},$$
$$P_{cl}(X) = \{Y \in P(X) : Y \text{ is closed }\},$$
$$P_b(X) = \{Y \in P(X) : Y \text{ is bounded }\}.$$

Let $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ be the operator

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space.

Definition Let $N : X \to P_{cl}(X)$ be a multi-valued operator. Then

- N is γ-Lipschitz if there exists $\gamma > 0$ such that for each x and y in X, $H(N(x), N(y)) \le \gamma d(x,y)$.
- N is a contraction if N is γ-Lipschitz with $\gamma < 1$.

- N is completely continuous if $N(B)$ is relatively compact for every $B \in P_b(X)$.
- N has a fixed point if there is $x \in X$ such that $x \in N(x)$. The fixed point set of the multi-valued operator N will be denoted by $\operatorname{Fix} N$.

For more details on multi-valued maps and the proof of the results cited in this section, we refer the reader to the books by Deimling [3], by Gorniewicz [6], and by Hu and Papageorgiou [8].

Our results are based on the following fixed point theorem for contraction multi-valued operators intorduced by Covitz and Nadler in 1970 [2] (see also Deimling, [3, Theorem 11.1]).

Lemma 2.1 *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

3 Main Results

Now, we are able to state and prove our main theorems. The first result of this note concerns the initial value problem (1.1)–(1.2). Before stating and proving this result, we give the definition of solution.

Definition A function $y : [-r, 0] \to E$ is called solution of (1.1)-(1.2) if $y \in C([-r, T], E) \cap AC([0, T], E)$ and satisfies the differential inclusion (1.1) a.e. on $[0, T]$ and the past conditions (1.2).

Theorem 3.1 *Assume that:*

(H1) $F : [0, T] \times C([-r, 0], E) \to P_{cl}(E)$ *has the property that* $F(\cdot, u) : [0, T] \to P_{cl}(E)$ *is measurable for each* $u \in C([-r, 0], E)$*;*

(H2) $H(F(t, u), F(t, \overline{u})) \leq l(t)\|u - \overline{u}\|$, *for each* $t \in [0, T]$ *and* $u, \overline{u} \in C([-r, 0], E)$, *where* $l \in L^1([0, T], \mathbb{R})$.

Then (1.1)-(1.2) has at least one solution on $[-r, T]$.

Proof Transform the problem into a fixed point problem. Consider the multi-valued operator, $N : C([-r, T], E) \to \mathcal{P}(C([-r, T], E))$ defined by:

$$N(y) := \Big\{ h \in C([-r, T], E) : h(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0] \\ \phi(0) + \int_0^t g(s)\, ds & \text{if } t \in [0, T], \end{cases} \Big\}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0, T], E) : g(t) \in F(t, y_t) \quad \text{for a.e. } t \in [0, T] \Big\}.$$

Remarks:

(i) It is clear that the fixed points of N are solutions to (1.1)-(1.2).

(ii) For each $y \in C([-r,T],E)$ the set $S_{F,y}$ is nonempty since by (H1) F has a measurable selection [1, Theorem III.6].

We shall show that N satisfies the assumptions of Lemma 2.1. The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}(C(-r,T],E)$ for each $y \in C([-r,T],E)$. Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $C[-r,T],E)$. Then $\tilde{y} \in C[-r,T],E)$ and

$$y_n(t) \in \phi(0) + \int_0^t F(s,y_s)\,ds \quad \text{for each } t \in [0,T].$$

Because $\int_0^t F(s,y_s)\,ds$ is closed for each $t \in [0,T]$, then

$$y_n(t) \to \tilde{y}(t) \in \phi(0) + \int_0^t F(s,y_s)\,ds, \quad \text{for } t \in [0,T].$$

So $\tilde{y} \in N(y)$.

Step 2: $H(N(y_1),N(y_2)) \leq \gamma\|y_1 - y_2\|$ for each $y_1, y_2 \in C[-r,T],E)$ with $\gamma < 1$. Let $y_1, y_2 \in C[-r,T],E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t,y_t)$ such that

$$h_1(t) = \phi(0) + \int_0^t g_1(s)\,ds, \quad t \in [0,T].$$

From (H2) it follows that

$$H(F(t,y_{1t}),F(t,y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|.$$

Hence there is $w \in F(t,y_{2t})$ such that

$$\|g_1(t) - w\| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in [0,T].$$

Consider $U : [0,T] \to \mathcal{P}(E)$, given by

$$U(t) = \{w \in E : \|g_1(t) - w\| \leq l(t)\|y_{1t} - y_{2t}\|\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t,y_{2t})$ is measurable [1, Prop. III.4] there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t,y_{2t})$ and

$$\|g_1(t) - g_2(t)\| \leq l(t)\|y_1 - y_2\|, \quad \text{for each } t \in J.$$

For $t \in J$, let $h_2(t) = \phi(0) + \int_0^t g_2(s)\,ds$. Then

$$\begin{aligned}
\|h_1(t) - h_2(t)\| &\leq \int_0^t \|g_1(s) - g_2(s)\|\,ds \\
&\leq \int_0^t l(s)\|y_1(s) - y_2(s)\|ds \\
&= \int_0^t l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|\,ds \\
&\leq \|y_1 - y_2\|_B \int_0^t l(s)e^{\tau L(s)}ds \\
&= \|y_1 - y_2\|_B \frac{1}{\tau}\int_0^t (e^{\tau L(s)})' ds \\
&\leq \frac{\|y_1 - y_2\|_B}{\tau} e^{\tau L(t)} ds,
\end{aligned}$$

where $L(t) = \int_0^t l(s)\,ds$, $\tau > 1$, and $\|\cdot\|_B$ is the Bielecki-type norm on $C([0,T], E)$,

$$\|y\|_B = \max_{t\in[0,T]}\{\|y(t)\|e^{-\tau L(t)}\}.$$

Then $\|h_1 - h_2\|_B \leq \frac{1}{\tau}\|y_1 - y_2\|_B$. By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H(N(y_1), N(y_2)) \leq \frac{1}{\tau}\|y_1 - y_2\|_B.$$

Therefore, N is a contraction and thus, by Lemma 2.1, it has a fixed point y, which is a solution to (1.1)-(1.2). $\diamondsuit$

The next theorem gives an existence result for the boundary-value problem (1.3)–(1.4).

Definition A function $y : [-r, T] \to E$ is called solution of (1.3)-(1.4) if $y \in C([-r, 0], E) \cap AC^1([0, T], E)$ and satisfies the differential inclusion (1.3) a.e. on $[0, T]$ and the condition (1.4).

Theorem 3.2 *Let F satisfy (H1) and (H2). Then (1.3)-(1.4) has at least one solution on $[-r, T]$.*

Proof As in Theorem 3.1 we transform the problem into a fixed point problem. Consider the multi-valued operator, $N_1 : C([-r, T], E) \to \mathcal{P}(C([-r, T], E))$ defined by

$$N_1(y) := \Big\{ h \in C([-r,T],E) \text{ such that } h(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0] \\ \phi(0) + t\eta + \int_0^t (t-s)g(s)\,ds & \text{if } t \in [0, T], \end{cases} \Big\}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,T],E) : g(t) \in F(t,y_t) \quad \text{for a.e. } t \in [0,T] \Big\}.$$

Remark It is clear that the fixed points of N_1 are solutions to (1.3)-(1.4).

We shall show that N_1 satisfies the assumptions of Lemma 2.1. Using the same reasoning as in Step 1 of Therem 3.1 we can show that $N_1(y) \in P_{cl}(C(-r,T],E)$, for each $y \in C([-r,T],E)$.

N_1 is a contraction multi-valued map. Indeed, let $y_1, y_2 \in C[-r,T],E)$ and $h_1 \in N_1(y_1)$. Then there exists $g_1(t) \in F(t,y_t)$ such that

$$h_1(t) = \phi(0) + t\eta + \int_0^t (t-s)g_1(s)\,ds, \quad t \in [0,T].$$

From (H2) it follows that

$$H(F(t,y_{1t}), F(t,y_{2t})) \le l(t)\|y_{1t} - y_{2t}\|.$$

Hence there is $w \in F(t,y_{2t})$ such that

$$\|g_1(t) - w\| \le l(t)\|y_{1t} - y_{2t}\|, \quad t \in [0,T].$$

Consider $U : [0,T] \to \mathcal{P}(E)$, given by

$$U(t) = \{ w \in E : \|g_1(t) - w\| \le l(t)\|y_{1t} - y_{2t}\| \}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t,y_{2t})$ is measurable [1, Prop. III.4] there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t,y_{2t})$ and

$$\|g_1(t) - g_2(t)\| \le l(t)\|y_1 - y_2\|, \quad \text{for each } t \in J.$$

For t in J, let us define $h_2(t) = \phi(0) + t\eta + \int_0^t (t-s)g_2(s)\,ds$. Then we have

$$\begin{aligned}
\|h_1(t) - h_2(t)\| &\le \int_0^t (t-s)\|g_1(s) - g_2(s)\|\,ds \\
&\le \int_0^t (t-s)l(s)\|y_1(s) - y_2(s)\|ds \\
&= \int_0^t (t-s)l(s)e^{-\tau L(s)}e^{\tau L(s)}\|y_1(s) - y_2(s)\|\,ds \\
&\le \|y_1 - y_2\|_B \int_0^t (t-s)l(s)e^{\tau L(s)}ds \\
&\le \|y_1 - y_2\|_B \frac{T}{\tau}\int_0^t (e^{\tau L(s)})'ds \\
&\le \frac{T\|y_1 - y_2\|_B}{\tau} e^{\tau L(t)}ds.
\end{aligned}$$

Then $\|h_1 - h_2\|_B \leq \frac{T}{\tau}\|y_1 - y_2\|_B$. By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H(N_1(y_1), N_1(y_2)) \leq \frac{T}{\tau}\|y_1 - y_2\|_B.$$

Therefore, when $\tau > T$, N_1 is a contraction, and thus, by Lemma 2.1, it has a fixed point y, which is solution to (1.3)-(1.4). $\diamondsuit$

Remark It seems that the reasoning used above can be applied for other boundary value problems for functional differential inclusions such as

$$y'' \in F(t, y_t), \quad \text{a.e. } t \in [0, T] \tag{3.1}$$

$$y(t) = \phi(t) \quad t \in [-r, 0], \; y(T) = \eta. \tag{3.2}$$

References

[1] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[2] H. Covitz and S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.

[3] K. Deimling, Multivalued Differential Equations, Walter de Gruyter, Berlin-New York, 1992.

[4] J. Dugundji and A. Granas, Fixed Point Theory, Monografie Mat. PWN, Warsaw, 1982.

[5] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, Pure and Applied Mathematics, Marcel Dekker, 1994.

[6] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[7] J. Henderson, Boundary Value Problems for Functional Differenial Equations, World Scientific, Singapore, 1995.

[8] Sh. Hu and N. Papageorgiou, Handbook of Multivalued Analysis, Volume I: Theory, Kluwer Academic Publishers, Dordrecht, 1997.

[9] G. S. Ladde, V. Lakshmikantham and A.S. Vatsala, Monotone Iterative Techniques for Nonlinear Differential Equations, Pitman, Boston M. A. 1985.

[10] E. Liz and J. J. Nieto, Periodic BVPs for a class of functional differential equations, *J. Math. Anal. Appl.* **200** (1996), 680-686.

[11] J. J. Nieto, Y. Jiang and Y. Jurang, Monotone iterative method for functional differential equations, *Nonlinear Anal.* **32** (6) (1998), 741-749.

[12] S. K. Ntouyas, Initial and boundary value problems for functional differential equations via the topological transversality method: A survey, *Bull. Greek Math. Soc.* **40** (1998), 3-41.

M. Benchohra
Department of Mathematics, University of Sidi Bel Abbes
BP 89, 22000 Sidi Bel Abbes, Algeria
e-mail: benchohra@yahoo.com

S. K. Ntouyas
Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
e-mail: sntouyas@cc.uoi.gr
http://www.uoi.gr/schools/scmath/math/staff/snt/snt.htm