

**EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS
FOR FUNCTIONAL DIFFERENTIAL EQUATIONS**

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ABSTRACT. In this paper, using a simple and classical application of the Leray-Schauder degree theory, we study the existence of solutions of the following boundary value problem for functional differential equations

$$x''(t) + f(t, x_t, x'(t)) = 0, \quad t \in [0, T]$$

$$x_0 + \alpha x'(0) = h$$

$$x(T) + \beta x'(T) = \eta$$

where $f \in C([0, T] \times C_r \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C_r$, $\eta \in \mathbb{R}^n$ and α, β are real constants.

KEY WORDS AND PHRASES. Boundary value problem, functional differential equations.

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1. INTRODUCTION

Let \mathbb{R}^n be the real euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. Let also, C_r be the space of all continuous functions $x : [-r, 0] \rightarrow \mathbb{R}^n$, $r > 0$, endowed with the sup-norm

$$\|x\| = \sup\{|x(t)| : t \in [-r, 0]\}.$$

For every continuous function $x : [-r, T] \rightarrow \mathbb{R}^n$, $T > 0$ and every $t \in [0, T]$, we denote by x_t the element of C_r defined by

$$x_t(\vartheta) = x(t + \vartheta), \quad \vartheta \in [-r, 0].$$

The main purpose of this paper is to discuss when the functional differential equation

$$x''(t) + f(t, x_t, x'(t)) = 0, \quad t \in [0, T], \quad (1.1)$$

admits a solution x on $[0, T]$ such that the boundary value conditions

$$x_0 + \alpha x'(0) = h \quad (1.2a)$$

$$x(T) + \beta x'(T) = \eta \quad (1.2b)$$

to be satisfied. Here, $f: [0, T] \times C_r \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function, $h \in C_r$, $\eta \in \mathbb{R}^n$ and α, β are real constants such that

$$\alpha \leq 0 \leq \beta \quad (1.2c)$$

By $x'(0)$ and $x'(T)$ we mean $x'(0^+)$ and $x'(T^-)$, respectively. In the next, the boundary value problem (B.V.P.) which constitutes from the equation (1.1) and the boundary conditions (1.2a), (1.2b), (1.2c), will be mentioned briefly as B.V.P. (1.1)-(1.2).

Analogous boundary value problems for ordinary differential equations has been studied by many authors, who used the Leray-Schauder continuation theorem (see Lasota and Yorke [1], Szmanda [2], Traple [3] and others). Usually, in these problems the authors derive a priori estimates of solutions by using inequalities of Wirtinger and Opial type.

Our work is motivated by the recent papers of Fabry and Habets [4], Fabry [5] and Ntouyas [6]. In [6] the author generalizes the results of Fabry and Habets [4] to the functional equation (1.1) with boundary conditions

$$\begin{aligned} x_0 &= h, \quad h(0) = 0, \\ x(T) &= 0. \end{aligned}$$

Here, following Fabry [5] we extend the results of Ntouyas [6].

2. MAIN RESULTS

Before stating our main results we refer some lemmas which simplify the proof of the theorem bellow.

LEMMA 2.1. [4, pp 187]. Let X be a Banach space, $A: X \rightarrow X$ be a completely continuous mapping such that $I-A$ is one to one, and let Ω be a bounded set such that $0 \in (I-A)(\Omega)$. Then the completely continuous mapping $S: \Omega \rightarrow X$ has a fixed point in Ω if for any $\lambda \in (0, 1)$, the equation

$$x = \lambda Sx + (1-\lambda)Ax \quad (2.1)$$

has no solution on the boundary $\partial\Omega$ of Ω .

LEMMA 2.2. [5, pp 133]. Let $X: [0, T] \rightarrow \mathbb{R}^n$ be a twice differentiable function and let $R > 0$ be such that

$$\|x\| \leq R. \quad (2.2)$$

Assume that positive constants c, d exist, with $c < 1$, such that

$$-\langle x(t), x''(t) \rangle \leq c |x'(t)|^2 + d, \quad t \in [0, T]. \quad (2.3)$$

Moreover, assume that positive constants c', d' exist with $c' < (1-c)^2/8R$ such that

$$|\langle x'(t), x''(t) \rangle| \leq (c' |x'(t)|^2 + d') |x'(t)|, \quad t \in [0, T]. \quad (2.4)$$

Then there exists a number K nondepending on x , such that

$$\|x'(t)\| \leq K.$$

LEMMA 3.2. If $\alpha \leq 0 \leq \beta$ the B.V.P

$$x''(t) = kx(t), \quad k > 0$$

$$x(0) + \alpha x'(0) = 0, \quad x(T) + \beta x'(T) = 0$$

has the unique solution $x = 0$.

PROOF. The general solution of the above equation has the form

$$x(t) = c_1 e^{\sqrt{k}t} + c_2 e^{-\sqrt{k}t}.$$

On account of the above boundary conditions we obtain

$$\frac{(1+\alpha\sqrt{k})(1-\beta\sqrt{k})}{(1-\alpha\sqrt{k})(1+\beta\sqrt{k})} \neq e^{2\sqrt{k}T}.$$

Since $e^{2\sqrt{k}T} > 1$, $k > 0$, the last expression is true for every $k > 0$, provided the left hand side is less than or equal to one. But this is clear since $\alpha \leq 0 \leq \beta$.

The next Theorem guarantees existence of solutions for the B.V.P. (1.1)-(1.2) which are bounded by an a priori given function φ . Moreover, the first derivative of a such solution is also bounded by a constant ρ not depending on this solution.

THEOREM. Let $f : [0, T] \times C_r \times \mathbb{R}^n$ be a continuous function which maps bounded sets of $[0, T] \times C_r \times \mathbb{R}^n$ into bounded sets of \mathbb{R}^n . Assume that $\varphi : [0, T] \rightarrow (0, \infty)$ is a twice continuously differentiable function such that

$$-\varphi(0) - |\alpha| \varphi'(0) > |h(0)|, \quad \text{if } \alpha \neq 0$$

$$\varphi(0) > |h(0)|, \quad \text{if } \alpha = 0 \tag{2.5a}$$

and

$$-\varphi(T) + |\beta| \varphi'(T) > |\eta|, \quad \text{if } \beta \neq 0$$

$$\varphi(T) > |\eta|, \quad \text{if } \beta = 0. \tag{2.5b}$$

Also, we suppose that

$$\varphi(t)\varphi''(t) + \langle u(0), f(t, u, v) \rangle \leq 0 \tag{2.6}$$

for any $(t, u, v) \in [0, T] \times C_r \times \mathbb{R}^n$ with $\varphi(t) = |u(0)|$ and $\langle u(0), v \rangle = |u(0)|\varphi'(t)$.

Moreover, assume that there exist positive numbers k_1, k_2 with $k_1 < 1$ and positive numbers k'_1, k'_2 with

$$k'_1 < \frac{1}{8m} (1-k_1)^2, \quad m = \max_{t \in [0, T]} |\varphi(t)|$$

such that

$$\langle u(0), f(t, u, v) \rangle \leq k_2 |v|^2 + k_2, \tag{2.7}$$

$$|\langle v, f(t, u, v) \rangle| \leq (k'_1 |v|^2 + k'_2) |v| \tag{2.8}$$

for any $(t, u, v) \in [0, T] \times C_r \times \mathbb{R}^n$ with $|u(0)| \leq \varphi(t)$.

Then the problem (1.1)-(1.2) has at least one solution x such that $|x(t)| \leq \varphi(t)$, $t \in [0, T]$ and $|x'(t)| \leq \rho$, $t \in [0, T]$.

PROOF. Let $k > 0$ be a constant, such that $k > \max \left\{ \frac{\varphi''(t)}{\varphi(t)}, t \in [0, T] \right\}$ and x a solution of the equation

$$x''(t) + \lambda f(t, x_t, x'(t)) = (1-\lambda)kx(t), \quad \lambda \in (0, 1) \tag{2.9}$$

with $t \in [0, T]$ and $|x(t)| \leq \varphi(t)$.

Multiplying both sides of (2.9) by $x(t)$ and using (2.7) we deduce that

$$-\langle x(t), x''(t) \rangle = \lambda \langle x_t(0), f(t, x_t, x'(t)) \rangle - (1-\lambda)k|x(t)|^2$$

$$\leq \lambda(k_1 |x'(t)|^2 + k_2)$$

$$\leq k_1 |x'(t)|^2 + k_2$$

Similarly, condition (2.8) yields

$$\begin{aligned} | \langle x'(t), x''(t) \rangle | &\leq (k'_1 |x'(t)|^2 + k'_2) |x'(t)| + k |x'(t)|^m \\ &\leq (k'_1 |x'(t)|^2 + \hat{c}) |x'(t)| \end{aligned}$$

where $\hat{c} = k'_2 + km$.

Thus the conditions of Lemma 2.2 are fulfilled and hence there exists a number K not depending on x , such that $|x'(t)| \leq K$.

Let us now consider the Banach space B of all continuous functions $x : [0, T] \rightarrow \mathbb{R}^n$, which are continuously differentiable on $[0, T]$, endowed with the norm

$$\|x\|_1 = \max \left\{ \sup_{t \in [0, T]} |x(t)|, \sup_{t \in [0, T]} |x'(t)| \right\}.$$

Also, for any $x \in B$ we set

$$Sx(t) = \int_0^T G(t, s) f(s, x_s, x'(s)) ds + \frac{1}{\ell} [(T-t)h(0) + \beta h(0) - \alpha t + t\eta], \quad t \in [0, T] \quad (2.10\alpha)$$

where

$$x_s(\vartheta) = \begin{cases} x(s+\vartheta), & \text{if } \vartheta \geq -s \\ h(s+\vartheta) - \alpha x'(0), & \text{if } \vartheta < -s. \end{cases} \quad (2.10\beta)$$

Here, G is the Green function for the B.V.P.

$$y'' = 0$$

$$y(0) + \alpha y'(0) = 0, \quad y(T) + \beta y'(T) = 0$$

and is given by the formula

$$G(t, s) = \frac{1}{\ell} \begin{cases} (t-T-\beta)(s-\alpha), & s \leq t \\ (t-\alpha)(s-T-\beta), & t \leq s, \end{cases}$$

where $\ell = T + \beta - \alpha \neq 0$ because of (1.2c).

Obviously, the operator S is a compact operator defined on B and taking values in B .

Since the B.V.P. (1.1)-(1.2) is equivalent to (2.10\alpha) and (2.10\beta), the purpose of the following proof is to show that the mapping S has a fixed point.

To this end we define an operator $A : B \rightarrow B$, and a subset Ω of B as follows:

$$(Ax)(t) = - \int_0^T G(t, s) k x(t) dt, \quad k \neq 0 \quad (2.11)$$

and

$$\Omega = \{x \in B : \forall t \in [0, T], |x(t)| < \varphi(t), |x'(t)| < K+1\}, \quad (2.12)$$

where k and K are defined as above.

It is clear that Ω is open and bounded in B and A is a completely continuous operator

First we prove that the operator $I-A$ is one to one. Let $(I-A)x = (I-A)y$. If $z(t) = x(t) - y(t)$ then $(I-A)z = 0$ and $z(0) + \alpha z'(0) = 0, z(T) + \beta z'(T) = 0$. Hence, z is a solution of the B.V.P.

$$\begin{aligned} z''(t) &= k z(t) \\ z(0) + \alpha z'(0) &= 0 \\ z(T) + \beta z'(T) &= 0. \end{aligned}$$

By Lemma 2.3 the last problem has the unique solution $z = 0$, and consequently I-A is one to one.

Next, we show that for any $\lambda \in [0,1]$ and $x \in \partial\Omega$ it is the case that

$$x \neq \lambda Sx + (1-\lambda)Ax$$

Indeed, if there exists $\lambda \in [0,1]$ and $x \in \partial\Omega$ satisfying

$$x = \lambda Sx + (1-\lambda)Ax,$$

then the equation

$$x''(t) + \lambda f(t, x_t, x'(t)) = (1-\lambda)kx(t),$$

has a solution $x : [0, T] \rightarrow \mathbb{R}^n$ satisfying

$$\begin{aligned} x_0 + \alpha x'(0) &= h \\ x(T) + \beta x'(T) &= \eta \end{aligned} \tag{2.13a}$$

$$x \in \bar{\Omega}. \tag{2.13\beta}$$

Hence there exist $\xi, r \in [0, T]$ such that either

$$|x(\xi)| = \varphi(\xi) \text{ or } |x'(r)| = K+1. \tag{2.14}$$

Now, we shall prove that, in view of (2.13a), (2.13\beta), the relations in (2.14) cannot hold. Since x is a solution of (2.9) for some $\lambda \in [0,1]$, the computation following (2.9) show that $|x'(t)| \leq K$ and hence $|x'(t)| < K+1, 0 \leq t \leq T$. Hence, the second case in (2.14) cannot hold. Thus it remains to eliminate the first possibility of (2.14). We shall prove that if $x \in \partial\Omega$ is a solution of (2.9), then there exists no $\xi \in [0, T]$ such that $|x(t)|^2 - \varphi^2(t)$ reaches maximum value zero at $t = \xi \in [0, T]$.

Assume the contrary. Then, if $\xi \in (0, T)$, we have the following relations

$$|x(\xi)| = \varphi(\xi) \tag{2.15}$$

$$\langle x(\xi), x'(\xi) \rangle = \varphi(\xi)\varphi'(\xi) \tag{2.16a}$$

$$\langle x_\xi(0), x'(\xi) \rangle = \varphi(\xi)\varphi'(\xi)$$

or

$$\langle x_\xi(0), x'(\xi) \rangle = \varphi(\xi)\varphi'(\xi) \tag{2.16\beta}$$

$$J \equiv \langle x_\xi(0), x''(\xi) \rangle + |x'(\xi)|^2 - \varphi(\xi)\varphi''(\xi) - \varphi'^2(\xi) \leq 0. \tag{2.17}$$

Now assume that x is a solution of (2.9). Then by (2.6), (2.15), (2.16\beta) we obtain

$$\begin{aligned} J &= -\lambda \langle x_\xi(0), f(t, x_\xi, x'(\xi)) \rangle + (1-\lambda)k|x(\xi)|^2 + |x'(\xi)|^2 - \varphi(\xi)\varphi''(\xi) - \varphi'^2(\xi) \\ &\geq (1-\lambda)\{|x'(\xi)|^2 - \varphi'^2(\xi) - \varphi(\xi)\varphi''(\xi) + k|x(\xi)|^2\} \\ &\geq (1-\lambda)\varphi(\xi)\{k\varphi(\xi) - \varphi''(\xi)\}. \end{aligned}$$

Since $k > \frac{\varphi''(t)}{\varphi(t)}, t \in (0, T)$, we get $J > 0, \lambda \in [0,1]$, contradicting (2.17).

Next we show that $\xi \neq T$. If $\xi = T$ and $g(t) = |x(t)|^2 - \varphi^2(t)$ then the following must hold:

$$g'(T) = 2\langle x(T), x'(T) \rangle - 2\varphi(T)\varphi'(T) \geq 0$$

and

$$g(T) = 0.$$

Then $|x(T)| = \varphi(T)$ and $\varphi'(T) \leq |x'(T)|$. But, by the boundary condition (1.2b), we have

$$|\beta| |x'(T)| \leq |\eta| + \varphi(T).$$

Hence

$$|\beta| \varphi'(T) \leq |\eta| + \varphi(T), \text{ if } \beta \neq 0$$

or

$$\varphi(T) \leq |\eta|, \text{ if } \beta = 0$$

which contradicts (2.5β). Therefore $\xi \neq T$ as required.

Finally, we show that $\xi \neq 0$. Assume on the contrary that $\xi = 0$. It is straightforward to see that

$$g(0) = 0 \text{ and } g'(0) \leq 0,$$

imply

$$|x(0)| = \varphi(0) \text{ and } -|x'(0)| \leq \varphi'(0)$$

From the boundary condition (1.2a) we obtain

$$-\varphi(0) \leq |h(0)| + |\alpha| \varphi'(0), \text{ if } \alpha \neq 0$$

or

$$\varphi(0) \leq |h(0)|, \text{ if } \alpha = 0,$$

contradicting (2.5α).

Consequently, no solutions of (2.9) can belong to $\partial\Omega$ for $\lambda \in [0,1)$, completing the proof of the theorem.

3. APPLICATIONS

As an application of the Theorem we consider the equation

$$x''(t) + \ell(t, x_t) x'(t) + p(t, x_t) x(t) + q(t, x_t) = 0, \quad t \in [0, T] \tag{3.1}$$

where ℓ and p are bounded real valued functions defined on $[0, T] \times C_r$ and q is also bounded \mathbb{R}^n -valued function defined on $[0, T] \times C_r$.

We set

$$\bar{\ell} = \sup_{(t,u) \in [0,T] \times C_r} |\ell(t,u)|, \quad \bar{p} = \sup_{(t,u) \in [0,T] \times C_r} |p(t,u)|, \quad \bar{q} = \sup_{(t,u) \in [0,T] \times C_r} |q(t,u)|.$$

Then we have the following

PROPOSITION. If there exists a constant M ,

$$M \geq \max\{\bar{\ell}, \bar{p}, \bar{q}\}$$

such that the inequality

$$\varphi''(t) + M[|\varphi'(t)| + \varphi(t) + 1] \leq 0, \quad t \in [0, T] \tag{3.2}$$

has a strictly positive solution φ , subject to the conditions (2.5α), (2.5β), then the B.V.P. (3.1)-(1.2) has at least one solution satisfying

$$|x(t)| \leq \varphi(t), \quad t \in [0, T].$$

Moreover, there exists ρ not depending on x with

$$|x'(t)| \leq \rho, \quad t \in [0, T].$$

PROOF. It is enough to check the conditions of the theorem for the function

$$f(t, u, v) = \ell(t, u)v + p(t, u)u(0) + q(t, u), \quad (t, u, v) \in [0, T] \times C_r \times \mathbb{R}^n.$$

Indeed, for any $(t, u, v) \in [0, T] \times C_r \times \mathbb{R}^n$, with $|u(0)| = \varphi(t)$ and $\langle u(0), v \rangle = |u(0)|\varphi'(t)$, we obtain

$$\langle u(0), f(t, u, v) \rangle = \ell(t, u)\langle u(0), v \rangle + p(t, u)|u(0)|^2 + \langle u(0), q(t, u) \rangle$$

$$\begin{aligned} &\leq |\ell(t,u)| |u(0)| |\varphi'(t) + p(t,u)| u(0)|^2 + |u(0)| |q(t,u)| \\ &= |\ell(t,u)| \varphi(t) \varphi'(t) + p(t,u) \varphi^2(t) + \varphi(t) |q(t,u)| \\ &\leq \tilde{\ell} \varphi(t) |\varphi'(t)| + \tilde{p} \varphi^2(t) + \tilde{q} \varphi(t) \\ &\leq M \varphi(t) (|\varphi'(t)| + \varphi(t) + 1). \end{aligned}$$

In view of (3.2), the above relation shows that (2.6) holds.

Also, for any $(t,u,v) \in [0,T] \times C_r \times \mathbb{R}^n$ with $|u(0)| \leq \varphi(t)$ we get, obviously,

$$\begin{aligned} \langle u(0), f(t,u,v) \rangle &\leq \tilde{\ell} \varphi(t) |v| + \tilde{p} \varphi^2(t) + \tilde{q} \varphi(t) \\ &\leq c_1 + c_2 |v|, \end{aligned}$$

where $c_1 = \sup_{t \in [0,T]} (\tilde{p} \varphi^2(t) + \tilde{q} \varphi(t))$ and $c_2 = \sup_{t \in [0,T]} (\tilde{\ell} \varphi(t))$.

Moreover,

$$\begin{aligned} \langle v, f(t,u,v) \rangle &\leq \tilde{\ell} |v|^2 + \tilde{p} |v| \varphi(t) + \tilde{q} |v| \\ &\leq c'_1 |v| + \tilde{\ell} |v|^2, \end{aligned}$$

where $c'_1 = \sup_{t \in [0,T]} (\tilde{p} \varphi(t) + \tilde{q})$. Now, if $|v| \geq 1$ then we have $c'_1 |v| + \tilde{\ell} |v|^2 \leq (c'_1 + \tilde{\ell} |v|) |v|$. If $|v| < 1$ then (2.8) follows from the inequality

$$\tilde{\ell} \geq \tilde{\ell} |v| - \ell_1 |v|^2, \text{ for each } \ell_1 \geq 0.$$

Indeed, we have

$$c'_1 + \tilde{\ell} |v| = c'_1 + \ell_1 |v|^2 + \tilde{\ell} |v| - \ell_1 |v|^2 \leq c'_1 + \ell_1 |v|^2 + \tilde{\ell}.$$

Hence (2.8) is satisfied for $k_1^1 = \ell_1$ and $k_2^1 = c'_1 + \tilde{\ell}$.

EXAMPLE. The B.V.P.

$$\begin{aligned} x''(t) + \frac{x(t)}{1 + \|x_t\|} x'(t) &= 0, \quad t \in [0,1] \\ x_0 &= h \\ x(1) + \beta x'(1) &= \eta \end{aligned}$$

has at least one solution x such that

$$|x(t)| \leq 2 - e^{-t}$$

provided that function h and constants β and η are such that

$$|h(0)| < 1 \text{ and } |\beta| + 1 > e(2 + |\eta|).$$

We remark that in this case $\tilde{\ell} = 1$ (and hence $M = 1$) and (3.2) becomes $\varphi''(t) + |\varphi'(t)| \leq 0$, $t \in [0,1]$.

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