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EXISTENCE OF MILD SOLUTIONS OF SECOND ORDER INITIAL VALUE PROBLEMS FOR DELAY INTEGRODIFFERENTIAL INCLUSIONS WITH NONLOCAL CONDITIONS

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Abstract. In this paper we investigate the existence of mild solutions to second order initial value problems for a class of delay integrodifferential inclusions with nonlocal conditions. We rely on a fixed point theorem for condensing maps due to Martelli.

Keywords: initial value problems, convex multivalued map, mild solution, evolution inclusion, nonlocal condition, fixed point

MSC 2000: 34A60, 34G20, 35R10, 47H20

1. Introduction

In this paper we study the existence of mild solutions, defined on a compact real interval J, for second order Initial Value Problems (IVP) for semilinear delay integrodifferential inclusions, with nonlocal conditions, of the form

$$y'' - Ay \in \int_0^t K(t,s)F(s,y(\sigma(s)))\,\mathrm{d}s, \quad t \in J := [0,b], \tag{1.1}$$

$$y(0) + f(y) = y_0, \;\; y'(0) = y_1 \tag{1.2}$$

where $F\colon J \times E \longrightarrow 2^E$ is a nonempty bounded, closed, convex valued multivalued map, $\sigma\colon J \to J$ is a continuous function such that $\sigma(t) \leqslant t, \forall t \in J$, $K\colon D \longrightarrow \mathbb{R}$, $D = \{(t,s) \in J \times J\colon t \geqslant s\}$, $f \in C(J,E) \to E, A$ is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t)\colon t \in \mathbb{R}\}$ in a Banach space $E = (E, |\cdot|), y_0, y_1 \in E$.

A pioneering work on nonlocal evolution problems is due to L. Byszewski. As pointed out by Byszewski [4], [5] the study of IVP with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics. In fact, more authors have paid attention to the research of IVP with nonlocal conditions during the few past years. We refer to Balachandran and Chandrasekaran [1], Byszewski [4], [5], Ntouyas [16] and Ntouyas and Tsamatos [14], [15].

IVP for second order semilinear equations with nonlocal conditions were studied in Ntouyas and Tsamatos [15] and Ntouyas [16].

On the other hand, IVP for first order semilinear delay evolution inclusions with nonlocal conditions were studied by authors in [3].

Here we extend the results of [3] to second order IVP for integrodifferential inclusions, with nonlocal conditions, by using also a fixed point theorem for condensing maps due to Martelli [13].

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts from multivalued analysis which are used throughout the paper.

$C(J,E)$ is the Banach space of continuous functions from J into E normed by

$$\|y\|_\infty = \sup\{|y(t)|\colon\ t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y\colon\ J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Leb esgue integrable. (For properties of the Bochner integral see Yosida [19].) $L^1(J,E)$ denotes the Banach space of measurable functions $y\colon\ J \longrightarrow E$ which are Bochner integrable, normed by

$$\|y\|_{L^1} = \int_J |y(t)|\,\mathrm{d}t \quad \text{for all} \quad y \in L^1(J,E).$$

Let $(X, |\cdot|)$ be a Banach space. A multivalued map $G\colon\ X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x\in B} G(x)$ is bounded in X for any bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y|\colon\ y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set V of X containing $G(x_0)$ there exists an open neighbourhood U of x_0 such that $G(U) \subseteq V$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$. If a multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$,

$y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G\colon J \longrightarrow BCC(X)$ is said to be measurable if, for each $x \in E$, the function $Y\colon J \longrightarrow \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z|\colon z \in G(t)\}$$

is measurable on J. For more details on multivalued maps see Deimling [6] and Hu and Papageorgiou [11].

An upper semi-continuous map $G\colon X \longrightarrow 2^X$ is said to be condensing if, for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure we refer to Banas and Goebel [2].

We remark that a completely continuous multivalued map is the simplest example of a condensing map.

We say that a family $\{C(t)\colon t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,
(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t)\colon t \in \mathbb{R}\}$, associated with a given strongly continuous cosine family $\{C(t)\colon t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y \,\mathrm{d}s, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $A\colon E \longrightarrow E$ of a cosine family $\{C(t)\colon t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{\mathrm{d}^2}{\mathrm{d}t^2} C(t)y|_{t=0}.$$

For more details on strongly continuous cosine and sine families we refer the reader to the boo ks of Goldstein [9], Heikkila and Lakshmikantham [10] and to the papers of Fattorini [7], [8] and Travis and Webb [17], [18].

We will need the following assumptions:

(H1) A is the infinitesimal generator of a strongly continuous and bounded cosine family $\{C(t)\colon t \in J\}$;

(H2) $F\colon J\times E\longrightarrow BCC(E); (t,y)\longmapsto F(t,y)$ is measurable with respect to t for each $y\in E$, u.s.c. with respect to y for each $t\in J$, and for each fixed $y\in C(J,E)$, the set

$$S_{F,y} := \{g\in L^1(J,E)\colon\ g(t)\in F(t,y(\sigma(t))) \text{ for a.e. } \ t\in J\}$$

is nonempty;

(H3) there exists a constant G such that

$$|f(y)|\leqslant G \ \text{ for each } \ y\in C(J,E);$$

(H4) for each $t\in J$, $K(t,s)$ is measurable on $[0,t]$ and

$$K(t) = \operatorname{ess\,sup}\{|K(t,s)|\colon\ 0\leqslant s\leqslant t\}$$

is bounded on J;

(H5) the map $t\longmapsto K_t$ is continuous from J to $L^\infty(J,\mathbb{R})$; here $K_t(s)=K(t,s)$;

(H6) $\|F(t,y)\| := \sup\{|v|\in F(t,y)\}\leqslant p(t)\psi(|y|)$ for almost all $t\in J$ and all $y\in E$, where $p\in L^1(J,\mathbb{R}_+)$ and $\psi\colon\ \mathbb{R}_+\longrightarrow(0,\infty)$ is continuous and increasing with

$$Mb\sup_{t\in J}K(t)\int_0^b p(s)\,\mathrm{d}s < \int_c^\infty \frac{\mathrm{d}u}{\psi(u)}$$

where $c = M|y_0| + MG + Mb|y_1|$ and $M=\sup\{|C(t)|; t\in J\}$;

(H7) for each bounded set $B\subset C(J,E)$ and $t\in J$, the set

$$\left\{C(t)(y_0-f(y)) + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s\colon\ g\in S_{F,B}\right\}$$

is relatively compact in E, where $S_{F,B}=\bigcup\{S_{F,y}\colon\ y\in B\}$.

Remark 2.1. (i) If $\dim E<\infty$, then $S_{F,y}\neq\emptyset$ for each $y\in C(J,E)$ (see Lasota and Opial [12]).

(ii) If $\dim E=\infty$ and $y\in C(J,E)$, then the set $S_{F,y}$ is nonempty if and only if the function $Y\colon\ J\longrightarrow\mathbb{R}$ defined by

$$Y(t) := \inf\{|v|\colon\ v\in F(t,y(\sigma(t)))\}$$

belongs to $L^1(J,\mathbb{R})$ (see Hu and Papageorgiou [11]).

(iii) (H6) is satisfied if F satisfies the standard domination

$$\|F(t,y)\|\leqslant p(t)(\|y\|+1) \ \text{ for almost all } \ t\in J \text{ and all } \ y\in E.$$

(iv) (H7) is satisfied if $\dim E < \infty$, or if $C(t), t > 0$ is completely continuous, or if f is completely continuous and for each $t \in J$ the multivalued map $F(t, .)$ maps bounded sets of $C(J, E)$ into relatively compact sets.

The following lemmas are crucial in the proof of our main theorem.

Lemma 2.1 [12]**.** *Let I be a compact real interval and X a Banach space. Let F be a multivalued map satisfying* (H2) *and let Γ be a linear continuous mapping from $L^1(I, X)$ to $C(I, X)$. Then the operator*

$$\Gamma \circ S_F \colon\ C(I, X) \longrightarrow BCC(C(I, X)),\ y \longmapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Lemma 2.2 [13]**.** *Let X be a Banach space and $N\colon\ X \longrightarrow BCC(X)$ an u.s.c. and condensing map. If the set*

$$\Omega := \{y \in X\colon\ \lambda y \in N(y) \ \ \textit{for some} \ \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3. Main result

A function $y \in C(J, E)$ is said to be a mild solution of (1.1)–(1.2) if there exists a function $v \in L^1(J, E)$ such that $v(t) \in F(t, y(\sigma(t)))$ a.e. on J and

$$y(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s,u)v(u) \,\mathrm{d}u \,\mathrm{d}s.$$

The following theorem is our main result in this article.

Theorem 3.1. *Let $f\colon\ C(J, E) \longrightarrow E$ be a continuous function. Assume that F satisfies* (H1)–(H7)*. Then the IVP* (1.1)–(1.2) *has at least one mild solution.*

P r o o f. We transform the problem (1.1)–(1.2) into a fixed point problem. Consider a multivalued map, $N\colon\ C(J, E) \longrightarrow 2^{C(J,E)}$ defined by

$$N(y) := \Big\{h \in C(J, E)\colon\ h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s,u)g(u) \,\mathrm{d}u \,\mathrm{d}s\colon\ g \in S_{F,y}\Big\}$$

where

$$S_{F,y} = \{g \in L^1(J, E)\colon\ g(t) \in F(t, y(\sigma)(t)) \ \ \text{for a.e.} \ \ t \in J\}.$$

R e m a r k 3.1. It is clear that the fixed points of N are mild solutions to (1.1)–(1.2).

We shall show that N is completely continuous with bounded, closed, convex values and that it is upper semicontinuous. The proof will be given in several steps.

S t e p 1: $N(y)$ is convex for each $y \in C(J,E)$.

Indeed, if $h_1,\ h_2$ belong to Ny, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g_i(u)\,\mathrm{d}u\,\mathrm{d}s, \quad i = 1,2.$$

Let $0 \leqslant \alpha \leqslant 1$. Then for each $t \in J$ we have

$$\begin{aligned}(\alpha h_1 + (1-\alpha)h_2)(t) &= C(t)(y_0 - f(y)) + S(t)y_1 \\ &\quad + \int_0^t S(t-s)\int_0^s K(s,u)[\alpha g_1(u) + (1-\alpha)g_2(u)]\,\mathrm{d}u\,\mathrm{d}s.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values) we conclude

$$\alpha h_1 + (1-\alpha)h_2 \in N(y).$$

S t e p 2: N is bounded on bounded sets of $C(J,E)$.

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in N(y), y \in B_r = \{y \in C(J,E)\colon\ \|y\|_\infty \leqslant r\}$ one has $\|h\|_\infty \leqslant l$. If $h \in N(y)$, then there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s, \quad t \in J.$$

By (H3)–(H6) we have for each $t \in J$ that

$$\begin{aligned}|h(t)| &\leqslant |C(t)||y_0| + |C(t)||f(y)| + |S(t)||y_1| \\ &\quad + \left\| \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s \right\| \\ &\leqslant M|y_0| + MG + Mb|y_1| + M\int_0^t\int_0^s |K(s,u)|p(u)\psi(|y(\sigma(u))|)\,\mathrm{d}u\,\mathrm{d}s \\ &\leqslant M|y_0| + MG + b|y_1| + Mb\sup_{t\in J}K(t)\|p\|_{L^1}\sup_{t\in J}\psi(|y(t)|) = l.\end{aligned}$$

S t e p 3: N maps bounded sets of $C(J,E)$ into equicontinuous sets.

Let $t_1, t_2 \in J,\ t_1 < t_2$ and let B_r be a bounded set of $C(J,E)$. For each $y \in B_r$ and $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$h(t) = C(t)(y_0 - f(y)) + S(t)y_1 + \int_0^t S(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s, \quad t \in J.$$

Thus

$$\begin{aligned}
|h(t_2)-h(t_1)| &\leqslant |(C(t_2)-C(t_1))y_0| + G|C(t_2)-C(t_1)| \\
&\quad + \left\| \int_0^{t_2} [S(t_2-s)-S(t_1-s)] \int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t_1}^{t_2} S(t_1-s) \int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s \right\| \\
&\leqslant |(C(t_2)-C(t_1))y_0| + G|C(t_2)-C(t_1)| \\
&\quad + \sup_{t\in J} K(t) \left\| \int_0^{t_2} [S(t_2-s)-S(t_1-s)] \int_0^s g(u)\,\mathrm{d}u\,\mathrm{d}s \right\| \\
&\quad + M \sup_{t\in J} K(t)(t_2-t_1) \int_0^b \|g(s)\|\,\mathrm{d}s.
\end{aligned}$$

As $t_2 \longrightarrow t_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 2, Step 3 and (H7), by virtue the Arzela-Ascoli theorem, we can conclude that N is completely continuous.

Step 4: N has a closed graph.

Let $y_n \longrightarrow y^*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h^*$. We shall prove that $h^* \in N(y^*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$h_n(t) = C(t)(y_0 - f(y_n)) + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s,u)g_n(u)\,\mathrm{d}u\,\mathrm{d}s, \quad t \in J.$$

We have to prove that there exists $g^* \in S_{F,y^*}$ such that

$$h^*(t) = C(t)(y_0 - f(y^*)) + S(t)y_1 + \int_0^t S(t-s) \int_0^s K(s,u)g^*(u)\,\mathrm{d}u\,\mathrm{d}s, \quad t \in J.$$

Clearly we have

$$\|(h_n - C(t)y_0 + C(t)f(y_n) - S(t)y_1) - (h^* - C(t)y_0 + C(t)f(y^*) - S(t)y_1)\|_\infty \to 0$$

as $n \to \infty$.

We consider the linear continuous operator

$$\begin{aligned}
\Gamma\colon\ & L^1(J,E) \longrightarrow C(J,E), \\
& g \longmapsto \Gamma(g)(t) = \int_0^t S(t-s) \int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

Lemma 2.1 implies that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$h_n(t) - C(t)y_0 + C(t)f(y_n) + S(t)y_1 = \int_0^t S(t-s) \int_0^s K(s,u)g_n(u)\,\mathrm{d}u\,\mathrm{d}s.$$

Since $y_n \longrightarrow y^*$, it follows from Lemma 2.1 that

$$h^*(t) - C(t)y_0 + C(t)f(y^*) + S(t)y_1 = \int_0^t S(t-s)\int_0^s K(s,u)g^*(u)\,\mathrm{d}u\,\mathrm{d}s$$

for some $g^* \in S_{F,y^*}$.

Step 5: The set

$$\Omega := \{y \in C(J,E)\colon\ \lambda y \in N(y), \quad \text{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}C(t)(y_0 - f(y)) + \lambda^{-1}S(t)y_1 + \lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s, \quad t \in J.$$

By (H3)–(H6) this implies that for each $t \in J$ we have

$$\begin{aligned}|y(t)| &\leqslant M|y_0| + MG + Mb|y_1| + M\left\|\int_0^t\int_0^s K(s,u)g(u)\,\mathrm{d}u\,\mathrm{d}s\right\| \\ &\leqslant M|y_0| + MG + Mb|y_1| + M\int_0^t\int_0^s |K(s,u)|p(u)\psi(|y(\sigma(u))|)\,\mathrm{d}u\,\mathrm{d}s \\ &\leqslant M|y_0| + MG + Mb|y_1| + Mb\sup_{t\in J}K(t)\int_0^t p(s)\psi(|y(s)|)\,\mathrm{d}s.\end{aligned}$$

Denoting the right-hand side of the above inequality by $v(t)$ we have

$$v(0) = M|y_0| + MG \quad \text{and} \quad |y(t)| \leqslant v(t), \quad t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leqslant Mb\sup_{t\in J}K(t)p(t)\psi(v(t)), \quad t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)}\frac{\mathrm{d}u}{\psi(u)} \leqslant Mb\sup_{t\in J}K(t)\int_0^b p(s)\,\mathrm{d}s < \int_{v(0)}^{\infty}\frac{\mathrm{d}u}{\psi(u)}.$$

This inequality implies that there exists a constant d such that $v(t) \leqslant d$, $t \in J$, and hence $\|y\|_\infty \leqslant d$ where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J,E)$. As a consequence of Lemma 2.2 we conclude that N has a fixed point which is a mild solution of (1.1)–(1.2). □

4. An example

Consider a partial integrodifferential equation of the form

$$z_{tt}(y,t) - z_{yy}(y,t) = \int_0^t K(t,s)q(s,z(y(s-\tau),s))\,\mathrm{d}s,\ 0 \leqslant y \leqslant \pi,\ t \in J \tag{4.1}$$

$$z(0,t) = z(\pi,t) = 0,$$

$$z(y,0) + z(y,1) = z_0(y), \tag{4.2}$$

$$z_t(y,0) = z_1(y)$$

where $q\colon\ J \times E \to E$, is continuous and $\tau > 0$.

Let $E = L^2[0,\pi]$ and define $A\colon\ E \to E$ by $Aw = w''$ with the domain

$$D(A) = \{w \in E\colon\ w, w' \quad \text{are absolutely continuous}, w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2(w,w_n)w_n,\ w \in D(A)$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\sin ns,\ n = 1,2,\dots$ is an orthogonal set of eigenvectors in A. It is easily shown that A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ in E given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w,w_n)w_n, \quad w \in E$$

and that the associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(w,w_n)w_n, \quad w \in E.$$

Assume that there exists an integrable function $p\colon\ J \to [0,\infty)$ such that

$$|q(t,w(t))| \leqslant p(t)\psi(|w|)$$

where $\psi\colon\ [0,\infty) \to (0,\infty)$ is continuous and nondecreasing with

$$Mb \sup_{t\in J} K(t) \int_0^b p(t)\,\mathrm{d}t < \int_c^{\infty} \frac{\mathrm{d}s}{\psi(s)}$$

where c is a constant.

Then the problem (1.1)– (1.2) is an abstract formulation of (4.1)–(4.2). Since all conditions of Theorem 3.1 are satisfied, the problem (4.1)–(4.2) has at least one mild solution on J.

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