Dispersive perturbations of optical solitons

John N. Elgin* and Theodoros P. Horikis†
Department of Mathematics, Imperial College, London SW7 2BZ, United Kingdom

(Received 2 November 2000; published 24 September 2001)

The precise manner in which radiation disperses away from a soliton in an optical fiber is a topic attracting current attention. The purpose of this paper is to emphasize that there exists a well-developed formalism derived from inverse scattering theory, which has ready application to this problem for the case when the radiation in question forms part of the input pulse to the fiber.

DOI: 10.1103/PhysRevE.64.047602 PACS number(s): 42.65.Tg, 42.81.Dp

In a recent article, Haus, Wong, and Khatri [1] analyzed the continuum generated by input pulses to a fiber corresponding to a perturbed soliton state, and discussed the subsequent evolution of the radiation as the composite pulse propagated down the fiber. This work extended previous results obtained by Gordon [2]. Their main tool was the use of an adjoint function, obtained from the nonlinear Schrödinger equation (NLS), perturbed about the soliton state. However, there is no demonstration that these adjoint functions are complete (actually, they are not) and they have the unfortunate property that the orthogonality condition satisfied by them requires only that the *real part* of an integral over the inner product of the adjoint function with its conjugate should be δ correlated in terms of a spectral parameter Ω. The reason for this undesirable property is that the adjoint states introduced by the authors, though close, are not quite the right set of basis states to use.

It is our objective here to point out that an appropriate set of basis states already exists for such an analysis, and to emphasize that such states are complete, and are properly orthogonal; these are the squared eigenstates of the scattering problem associated with the NLS, as first analyzed by Kaup [3]. We take the NLS in the form

$$iq_x - q_{tt} - 2q|q|^2 = 0 \tag{1}$$

with the single-soliton solution

$$q_s = \exp(-ix)\operatorname{sech} t \tag{2}$$

corresponding to an appropriate choice of (complex) eigenparameter $\zeta_1 = \xi_1 + i\eta_1$ (specifically, $\xi_1 = 0$ and $2\eta_1 = 1$). In Eq. (1), a suffix denotes a partial derivative and q is the complex amplitude; all quantities appear in normalized form. As it is well known, Eq. (1) can be solved using the Zakharov-Shabat scattering problem [4], where the two component Jost functions ϕ, $\overline{\phi}$, ψ, and $\overline{\psi}$ are introduced and defined.

Kaup first introduced the *squared eigenfunctions*

$$\Psi = \begin{bmatrix} \psi_1^2(t,\zeta) \\ \psi_2^2(t,\zeta) \end{bmatrix}, \quad \tilde{\Psi} = \begin{bmatrix} \overline{\psi}_1^2(t,\zeta) \\ \overline{\psi}_2^2(t,\zeta) \end{bmatrix}, \tag{3}$$

$$\Phi = \begin{bmatrix} \phi_2^2(t,\zeta) \\ -\phi_1^2(t,\zeta) \end{bmatrix}, \quad \tilde{\Phi} = \begin{bmatrix} \overline{\phi}_2^2(t,\zeta) \\ -\overline{\phi}_1^2(t,\zeta) \end{bmatrix}, \tag{4}$$

where ψ_1, ψ_2 are the components of the Jost function ψ, etc., and demonstrated that these were orthogonal, in the sense that

$$\int_{-\infty}^{+\infty} \Phi^T(t,\xi')\Psi(t,\xi)dt = -\pi a^2(\xi)\delta(\xi-\xi'), \tag{5}$$

$$\int_{-\infty}^{+\infty} \tilde{\Phi}^T(t,\xi')\Psi(t,\xi)dt = 0, \tag{6}$$

$$\int_{-\infty}^{+\infty} \Phi^T(t,\xi')\tilde{\Psi}(t,\xi)dt = 0, \tag{7}$$

$$\int_{-\infty}^{+\infty} \tilde{\Phi}^T(t,\xi')\tilde{\Psi}(t,\xi)dt = \pi\overline{a}^2(\xi)\delta(\xi-\xi'). \tag{8}$$

Here, $\zeta = \xi \in \mathbb{R}$, and a, $\overline{a}$, b, and $\overline{b}$ are scattering data, defined through the relationships between ϕ, $\overline{\phi}$ and ψ, $\overline{\psi}$ in the usual way:

$$\phi = a\overline{\psi} + b\psi, \tag{9a}$$

$$\overline{\phi} = -\overline{a}\psi + \overline{b\psi}. \tag{9b}$$

Kaup also derived a completeness relation for these eigenstates, permitting, among other things, the potentials to be expressed in terms of integrals over the scattering data, which can be inverted to give a reciprocal relationship as stated below.

Suppose now that the soliton input is perturbed so that a pulse $q = q_s + \delta q$ is inserted into the fiber. Then, the presence of δq will do two things: it will modify the soliton eigenparameter ζ_1, and it will result in a continuum contribution δq_c accompanying the modified soliton into the fiber. Evidently, $\delta q_c = \delta q + q_s - \tilde{q}_s$, where $\tilde{q}_s$ is the soliton state at the appropriately modified soliton parameter. It is straightforward to show that [5–7]

$$\delta q_c = \delta q + i\alpha t q_s - \beta(tq_s)', \tag{10}$$

where $'$ denotes $\partial/\partial t$, and α and β are defined by

*Email address: j.elgin@in.ac.uk
†Email address: theodore.horikis@ic.ac.uk

$$\alpha = \int_{-\infty}^{+\infty} \mathrm{Im}\{\delta q\} q_s' \, dt, \tag{11}$$

$$\beta = \int_{-\infty}^{+\infty} \mathrm{Re}\{\delta q\} q_s \, dt. \tag{12}$$

Here Re$\{\delta q\}$ and Im$\{\delta q\}$ are the real and imaginary parts of δq. The *transform* relation between the potential δq_c and the scattering data, obtained from the completeness properties of the above product states, is

$$\begin{bmatrix} -\delta q_c^* \\ \delta q_c \end{bmatrix} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\frac{b}{a} \tilde{\Phi}(t,\xi) + \frac{\overline{b}}{\overline{a}} \Phi(t,\xi) \right) d\xi \tag{13}$$

which, on using Eqs. (5)–(8), is easily inverted to give

$$\overline{b}\,\overline{a} = \int_{-\infty}^{+\infty} [-\delta q_c^*, \delta q_c] \Psi(t,\xi) \, dt \tag{14}$$

with a similar result for $b\overline{a}$.

Note, that the squared eigenstates now play the role of the exponentials in linear Fourier transforms, and also that both δq_c and δq_c^* appear in Eq. (14), unlike a similar statement in Ref. [1] where only δq_c appears, a consequence of the choice of adjoint functions.

We conclude with two comments. The connection between the scattering data $\overline{b}$ and the associate field f first introduced by Gordon is discussed in Ref. [8], where it is shown that

$$\mathcal{F}\{f\} = \frac{\overline{b(\xi,x)}}{4\xi^2 + 1}, \tag{15}$$

where $\mathcal{F}\{f\} = \int_{-\infty}^{+\infty} \exp(2i\xi t) f(t) dt$ is the Fourier transform of f.

Also, the appropriate forms for the components ψ_1, ψ_2, and $a(\xi)$, which appear in Eq. (14), for the real eigenparameter ξ, are

$$\psi_1 = -\frac{i}{2\xi + i} \exp(i\xi t) \operatorname{sech} t, \tag{16}$$

$$\psi_2 = \frac{1}{2\xi + i} \exp(i\xi t)(2\xi + i \tanh t), \tag{17}$$

$$a = \frac{2\xi - i}{2\xi + i}. \tag{18}$$

Moreover, $\overline{a} = a^*$ and $\overline{b} = b^*$. In conclusion, we have demonstrated a systematic way in which the radiation field accompanying the soliton on input to a fiber can be analyzed utilizing the squared eigenfunctions encountered in inverse scattering theory. These are the natural basis states to use, are complete, are orthogonal, and are the natural extensions of the simple exponential kernel encountered in linear transform theory. A given choice for δq at $x = 0$ results in an appropriate δq_c using Eq. (10), which then determines $\overline{b(\xi,0)}$ through Eq. (14). Evolution of $\overline{b}$ in x is trivial: $\overline{b(\xi,x)} = \overline{b(\xi,0)} \exp(-4i\xi^2 x)$, which on substitution back into Eq. (13), then determines $\delta q(t,x)$ (at least in principle). A similar statement holds true for the vector problem: completeness of a set of product states for the Manakov system has been demonstrated, leading to results for the vector field $\delta \mathbf{q}_c$ similar in spirit to Eqs. (13) and (14) above [9]. These results will be discussed elsewhere.

In summary, the precise manner in which radiation disperses away from an optical soliton in an optical fiber is best examined using Eqs. (13) and (14). These equations define a natural mathematical framework for such studies and are, we believe, superior to a similar set recently reported in Ref. [1]. Moreover, these equations are the natural extensions of a similar set of relations found in the application of Fourier theory to linear systems; the only difference is that the kernels $\exp(i\omega t)$ for the continuum states are now ''dressed'' by the presence of a soliton, resulting in modified kernels such as appear in Eqs. (13) and (14).

One of the authors (T.P.H.) is pleased to acknowledge support in the form of an award from EPSRC.

[1] H. A. Haus, W. S. Wong, and F. I. Khatri, J. Opt. Soc. Am. B **14**, 304 (1997).
[2] J. P. Gordon, J. Opt. Soc. Am. B **9**, 91 (1992).
[3] D. J. Kaup, J. Math. Anal. Appl. **54**, 849 (1976).
[4] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
[5] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. **55**, 284 (1974).
[6] J. N. Elgin, Phys. Lett. **110A**, 441 (1985).
[7] J. P. Gordon and H. A. Haus, Opt. Lett. **11**, 665 (1986).
[8] J. N. Elgin, Phys. Rev. A **47**, 4331 (1993).
[9] R. G. Doksey and J. N. Elgin, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **32**, 54 (2000).