Cohen-Macaulay modules and virtually Gorenstein algebras

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Let Λ be an Artin algebra. We denote by $\mathsf{Mod}\text{-}\Lambda$ the category of all right Λ-modules and by $\mathsf{mod}\text{-}\Lambda$ the full subcategory of finitely generated modules. We let $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$ be the category of *Cohen-Macaulay* modules which is defined as the maximal subcategory of $\mathsf{Mod}\text{-}\Lambda$ which contains the projectives as an Ext-injective cogenerator. Following [3] we let $\mathfrak{P}_\Lambda^{\prec\infty}$ be the subcategory of modules of *virtually finite projective dimension* which is defined as the right Ext-orthogonal subcategory of $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$. The full subcategories $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$ of *CoCohen-Macaulay* modules and $\mathfrak{I}_\Lambda^{\prec\infty}$ of modules of *virtually finite injective dimension* are defined dually. Note that $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$, resp. $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$, is an exact Frobenius definable subcategory of $\mathsf{Mod}\text{-}\Lambda$ and its stable category modulo projectives, resp. injectives, is a monogenic compactly generated triangulated category. Also the subcategories $\mathfrak{P}_\Lambda^{\prec\infty}$ and $\mathfrak{I}_\Lambda^{\prec\infty}$ are resolving and coresolving subcategories of $\mathsf{Mod}\text{-}\Lambda$ and there exist cotorsion pairs $\big(\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda), \mathfrak{P}_\Lambda^{\prec\infty}\big)$ and $\big(\mathfrak{I}_\Lambda^{\prec\infty}, \mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)\big)$ in $\mathsf{Mod}\text{-}\Lambda$. Λ is called *virtually Gorenstein* if $\mathfrak{P}_\Lambda^{\prec\infty} = \mathfrak{I}_\Lambda^{\prec\infty}$, see [3].

In this talk I shall report on some recent results, extracted from [2] and [3], on Cohen-Macaulay modules and virtually Gorenstein algebras. We study the virtual Gorensteinness property by using the above cotorsion pairs in the module category $\mathsf{Mod}\text{-}\Lambda$ and the induced torsion pairs in the stable category of $\mathsf{Mod}\text{-}\Lambda$ modulo projectives or injectives. The class of virtually Gorenstein algebras, which provides a common generalization of Gorenstein algebras and algebras of finite representation or Cohen-Macaulay type, on the one hand is closed under various operations and on the other hand has rich homological structure and satisfies several representation/torsion theoretic finiteness conditions. In this context we characterize the virtually Gorenstein algebras in terms of finitely generated modules by showing, among other equivalent conditions, that Λ is virtually Gorenstein if and only if the class of finitely generated Λ-modules of virtually finite projective dimension (which coincides with the class of finitely generated Λ-modules of virtually finite injective dimension) is contravariantly finite, or equivalently covariantly finite in $\mathsf{mod}\text{-}\Lambda$. Moreover, we show that virtually Gorenstein algebras enjoy the following properties, referring to [2], [3] for more details:

(1) The virtual Gorensteinness property is left-right symmetric.
(2) The class of virtually Gorenstein algebras is closed under derived equivalences and stable equivalences of Morita type.
(3) If the Artin algebra Λ is virtually Gorenstein, then:
    (a) The full subcategories $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$ and $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$ are functorially finite in $\mathsf{Mod}\text{-}\Lambda$, and their full subcategories of finitely generated modules are functorially finite in $\mathsf{mod}\text{-}\Lambda$ with free Grothendieck group of finite rank.
    (b) The full subcategory $\mathfrak{P}_\Lambda^{\prec\infty} = \mathfrak{I}_\Lambda^{\prec\infty}$ is thick, definable and functorially finite in $\mathsf{Mod}\text{-}\Lambda$, and its full subcategory of finitely generated modules

is thick and functorially finite in mod-Λ, hence it has Auslander-Reiten sequences, with free Grothendieck group of finite rank.

(c) The full subcategories $\mathsf{CM}(\mathsf{Mod}\text{-}\Lambda)$, $\mathsf{CoCM}(\mathsf{Mod}\text{-}\Lambda)$, $\mathfrak{P}_{\Lambda}^{\prec\infty}$ and $\mathfrak{I}_{\Lambda}^{\prec\infty}$ are completely determined by their intersection with the finitely generated modules (as their closure under filtered colimits).

(d) The subcategory of compact objects of the compactly generated triangulated category of Cohen-Macaulay modules modulo projectives admits a Serre functor and therefore has Auslander-Reiten triangles.

In addition, virtual Gorensteinness provides a useful tool for the study of the Gorenstein Symmetry Conjecture and modified versions of the Telescope Conjecture for module or stable categories. Recall that the former asserts that Λ is Gorenstein provided it has finite right or left self-injective dimension [1], and a generalized version of the latter asserts that any torsion pair of finite type, in the sense of [3], in a suitable "homotopy" category $\mathcal{C}$ is generated in a certain sense by compact objects induced from $\mathcal{C}$, see [4], [5], [6]. For instance, we show that Λ is virtually Gorenstein if and only if the monogenic compactly generated triangulated category of Cohen-Macaulay modules modulo projectives is smashing if and only if all of its compact objects are induced from finitely generated modules. Moreover, in the context of the above conjectures we show (in particular) the following:

- The Gorenstein Symmetry Conjecture holds for any virtually Gorenstein algebra.
- The Telescope Conjecture holds in the stable category modulo projectives for the torsion pair induced by the Cohen-Macaulay modules over a virtually Gorenstein algebra.

In particular, both conjectures hold for any algebra lying in the derived equivalence class or the stable equivalence class (of Morita type) of an algebra of finite representation or Cohen-Macaulay type.

As it is clear from the above that the class of virtually Gorenstein algebras is rather large, since it contains on the one hand algebras or finite global dimension and self-injective algebras or, more generally, Gorenstein algebras, and on the other hand algebras of finite representation or Cohen-Macaulay type. This gives the motivation for the following:

Problem: Find an Artin algebra which is not virtually Gorenstein.

References

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