Czechoslovak Mathematical Journal, 55 (130) (2005), 581–592

BOUNDARY VALUE PROBLEMS WITH COMPATIBLE BOUNDARY CONDITIONS

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(Received October 11, 2001)

Abstract. If Y is a subset of the space $\mathbb{R}^n \times \mathbb{R}^n$, we call a pair of continuous functions U, V Y-*compatible*, if they map the space $\mathbb{R}^n$ into itself and satisfy $Ux \cdot Vy \geqslant 0$, for all $(x, y) \in Y$ with $x \cdot y \geqslant 0$. (Dot denotes inner product.) In this paper a nonlinear two point boundary value problem for a second order ordinary differential n-dimensional system is investigated, provided the boundary conditions are given via a pair of compatible mappings. By using a truncation of the initial equation and restrictions of its domain, Brouwer's fixed point theorem is applied to the composition of the consequent mapping with some projections and a one-parameter family of fixed points P_δ is obtained. Then passing to the limits as δ tends to zero the so-obtained accumulation points are solutions of the problem.

Keywords: differential equations of second order, two-point boundary value problems

MSC 2000: 34C30, 34C99

1. Introduction

In this paper we combine continuity properties of the evolution of the second order ordinary differential equation

$$\ddot{x} = f(t, x, \dot{x}), \quad t \in [-\sigma, 1 + \sigma], \tag{1.1}$$

with Brouwer's fixed point theorem to establish existence of a solution x satisfying conditions of the form

$$x(0) = U(x(1)), \quad \dot{x}(0) = V(\dot{x}(1)). \tag{1.2}$$

Here σ is a positive real number, the function f satisfies the well known Hartman's condition and U, V are Y-*compatible*, in the sense that they satisfy $Ux \cdot Vy \geqslant$

0 for all $(x,y) \in Y$ with $x \cdot y \geqslant 0$, for a certain Y specified in the text. (Dot denotes inner product.) For instance, if A is an invertible $n \times n$ matrix and B is any positive multiple of the transpose of the inverse A^{-1}, then the pair A, B is Y-compatible, with Y being the whole space $\mathbb{R}^n \times \mathbb{R}^n$. The problem under investigation has been inspired by the periodic problem concerning (1.1), for which the literature is voluminous, as well as by the ones presented in [3], [9]. Notice that in [9] the existence of a Sturm-Liouville boundary value problem is investigated, by transforming the problem into the equivalent form $Lx = Gx$ and then applying Leray-Schauder's continuation theorem. This represents one among the three approaches most widely used in discussing existence of solutions of (1.1) satisfying additional conditions, such as boundary value conditions, periodicity, cost functionals, etc. The second is to examine the existence of a fixed point for an integral operator defined on the family of functions which satisfy some additional conditions. In this case the well-known shooting method is usually applied. And the third is to show that the set of solutions contains an element satisfying the conditions. To follow the last approach several methods have been developed, as, for example (in case of boundary value conditions), methods based on upper and lower solutions, or degree theory arguments (see, e.g., [7], [8] and the references therein), or Ważewski's topological method (see, e.g., [4]). In occasion we would like to refer to [4, p. 338], where by using Ważewski's method it was shown that if in (1.1) the function f satisfies Hartman's condition for all $t \geqslant 0$, x and $y \neq 0$, then there is a $t_0 > 0$ such that $x(t) \cdot x(t)$ is nonincreasing for all $t \geqslant t_0$, where $x(t)$ is the solution of equation (1.1). In this paper we do use Hartman's condition and give more information on the solutions. Also methods based on the application of fixed point theorems applied to the Poincaré-like mapping give good results. For a two-point boundary value problem concerning a more general differential equation in a Hilbert space discussed by the authors in [6] Schauder's fixed point theorem is used. Here we have to mention [3], where the existence of a solution x of a similar problem is discussed with the functions U and V being replaced by nonsingular $n \times n$-square matrices Q_0 and Q_1 such that the former is orthogonal and the pair (Q_0, Q_1) satisfies the inequalities $x \cdot Q_0 Q_1^{-1} y \leqslant 0$ and $x \cdot (Q_0 + Q_1^{-1}) y \leqslant 0$ for all vectors $x, y \in \mathbb{R}^n$ with $x \cdot y \leqslant 0$. The proof of the results are based on a technique of [1], where the degree theory is used.

Our purpose here is to provide sufficient conditions for the existence of solutions of the problem (1.1)–(1.2). Furthermore, our method, which is analytical (Brouwer's fixed point theorem is used), permits us to get information on the location of the solutions. Indeed, under quite natural conditions we are able to obtain C^1 bounds for the solutions.

We denote by I_σ and I the (so-called time-) intervals $[-\sigma, 1+\sigma]$ and $[0,1]$ respectively of the real line $\mathbb{R}$. Also we let E_σ and E be the sets $I_\sigma \times \mathbb{R}^n \times \mathbb{R}^n$ and

$I \times \mathbb{R}^n \times \mathbb{R}^n$, respectively. The Euclidean norm in the space $\mathbb{R}^n$ is denoted by $|\cdot|$ and the open ball centered at 0 and having radius $r > 0$ is denoted by $B(0, r)$. (We write ∂A, $\operatorname{int} A$ and $\operatorname{cl} A$, respectively, for the boundary, interior and closure of a set A.) The graph space of the solutions of equation (1.1) is the space E_σ and Brouwer's fixed point theorem applies on subsets of the boundary of the set

$$N(r, q) := \operatorname{cl} B(0, r) \times \operatorname{cl} B(0, q)$$

for some positive constants r, q. Also for each $\tau \in I$ we let $D_\tau(r, q)$ be the $t = \tau$ cross-section of the set

$$D(r, q) := I \times N(r, q),$$

namely $D_\tau(r, q)$ is the set $\{\tau\} \times N(r, q)$. If $x\colon I_\sigma \to \mathbb{R}^n$ is a function and Z is a subset of the interval I_σ we use the symbol

$$G(x|Z)$$

to denote the set

$$\{(t, x(t), \dot{x}(t)) \colon t \in Z\}.$$

In the sequel we assume that equation (1.1) admits unique solutions and, if x is the solution passing through a point $P \in E_\sigma$ which is defined at least on a maximal (open) interval of the form $(\alpha_P, \beta_P) =: J_P$, we shall denote it by $x(t; P)$, $t \in J_P$.

Let r, q be positive real numbers and consider the set $D(r, q)$. A point $(\tau, \xi, \eta) =: P \in \partial D(r, q)$ is a point of *egress* (with respect to (1.1) and $D(r, q)$), if there is a number $\varepsilon > 0$ such that

$$G(x(\cdot; P)|(\tau - \varepsilon, \tau)) \subseteq \operatorname{int} D(r, q).$$

Furthermore, if for some $\bar{\varepsilon} > 0$ we have

$$G(x(\cdot; P)|(\tau, \tau + \bar{\varepsilon})) \subseteq E_\sigma - D(r, q),$$

then we say that P is a point of *strict egress*, (see, e.g., [3]). By the basic theorem of existence of solutions it follows that any point of the set $D_1(r, q)$ is a point of strict egress. The sets of all egress and all strict egress points of the set $D(r, q)$ are usually denoted by $D(r, q)^e$ and $D(r, q)^{se}$, respectively.

A point $(\tau, \xi, \eta) =: P \in D(r, q)^e$ is a *consequent point* of $P_0 := (\tau_0, \xi_0, \eta_0)$, if $\tau_0 < \tau$, $x(\tau; P_0) = P$ and

$$G(x(\cdot; P_0)|(\tau_0, \tau)) \subseteq \operatorname{int} D(r, q).$$

It is clear that if such a point P exists, then it is unique. The consequent point of P_0 is denoted by $C(P_0)$ and the so defined mapping $C\colon\ P_0 \to C(P_0)$ is the *consequent mapping*.

Now we borrow from [5] the following result (after making the necessary adaptations):

Lemma 1.1. *If P is an interior point of $D_\sigma(r,q)$ and the solution $x(\cdot;P)$ egresses strictly from $D(r,q)$, then the consequent mapping C is continuous at P.*

A point $(\tau,\xi,\eta) =: P$ on the boundary of the set $D(r,q)$ is a point of *ingress* (with respect to (1.1)), if there is a number $\varepsilon > 0$ such that

$$G(x(\cdot;P)|(\tau-\varepsilon,\tau)) \subseteq E_\sigma - \operatorname{cl} D(r,q).$$

Furthermore, if for some $\bar\varepsilon > 0$ we have

$$G(x(\cdot;P)|(\tau,\tau+\bar\varepsilon)) \subseteq \operatorname{int} D(r,q),$$

then we say that P is a point of *strict ingress*. Again, by the theorem of existence of solutions it follows that any point of the set $D_0(r,q)$ is a point of strict ingress.

2. The main results

Consider a continuous function $f\colon\ E_\sigma \to \mathbb{R}^n$ satisfying the following conditions:

(f1) It guarantees uniqueness of the solutions of equation (1.1).

(f2) The function

$$g(r,q) := \sup\{|f(t,x,y)|\colon\ t\in I,\ |x|\leqslant r,\ |y|\leqslant q\},\quad r,q>0$$

is $o(q^2)$ as $q\to+\infty$, for each $r>0$ fixed.

(f3) There is a real number $R>0$ such that

$$4R \leqslant g(R,q),$$

for all $q>0$ and moreover Hartman's condition

$$x\cdot f(t,x,y) + |y|^2 > 0$$

holds for all $(t,x,y)\in E_\sigma$ with $x\cdot y = 0$ and $|x| = R$.

Lemma 2.1. *If the above conditions are satisfied, then the following assertions hold:*

(a) *The infimum i_R of the set $A(R)$ of all real numbers $k > 0$ with the property that, if $q > k$, then $q^2 > 4Rg(R, q)$ is a finite real number.*

We let K_R be any fixed real number greater than i_R.

(b) *If $P := (0, \xi, \eta)$ is a point such that the solution $x(\cdot; P)$ defined on J_P containing the interval I satisfies*

$$|x(t; P)| \leqslant R, \quad t \in J_P,$$

then

$$|\dot{x}(t; P)| < K_R, \quad t \in J_P.$$

Proof. (a) Assume that the set $A(R)$ is empty. Then there is a sequence (q_k) tending to $+\infty$ such that

$$\frac{g(R, q_k)}{q_k^2} \geqslant \frac{1}{4R},$$

contradicting (f2).

(b) We let J be a compact subinterval of J_P containing I and set

$$q := \sup\{|\dot{x}(t; P)| \colon t \in J_P\}.$$

Also we define

$$s := 2t, \quad y(s) := x(t; P), \ t \in J_P.$$

Then q is finite and the new function y satisfies the equation

$$4\ddot{y}(s) = f\Big(\frac{s}{2}, y(s), 2\dot{y}(s)\Big), \quad s \in \{2t \colon t \in J_P\} =: 2J_P.$$

Therefore we have

$$|y(s)| \leqslant R$$

and

$$|\ddot{y}(s)| \leqslant \frac{g(R, q)}{4} =: \gamma$$

for all $s \in 2J_P$, where $R \leqslant \gamma$ because of (f3). Now, since the interval $2J_P$ has length at least equal to 2, we apply Lemma 2 of [2, p. 139] to conclude that

$$|\dot{y}(s)|^2 \leqslant Rg(R, q)$$

for all $s \in 2J_P$. This implies that $q^2 \leqslant 4Rg(R, q)$ and, so, from (a) we get $q \leqslant i_R < K_R$. This implies statement (b).

Next we let

$$H_R := K_R + g(R, K_R).$$

Our main results are given in the following theorem:

Theorem 2.1. *Assume that f is a function satisfying the conditions* (f1)–(f3) *and consider a direct product function $W := U \otimes V$, where*

(a) *U maps the closed ball $\operatorname{cl} B(0,R)$ into itself continuously, it is invertible, it has no fixed point on the boundary of the ball $B(0,R)$ and there is a $\delta_U \in (0,R)$ such that for all $\delta \in (0,\delta_U)$ the inequality $|U(u)| \leqslant R-\delta$ holds for all u with $|u| \leqslant R-\delta$;*

(b) *V maps the closed ball $\operatorname{cl} B(0,K_R)$ into itself continuously, and there is a $\delta_V \in (0,K_R)$ such that for all $\delta \in (0,\delta_V)$ the inequality $|V(v)| \leqslant K_R-\delta$ holds for all v with $|v| \leqslant H_R-\delta$;*

(c) *the pair U,V is Y-compatible, where $Y := \{(x,y)\colon\ x,y \in \mathbb{R}^n,\ |x| = R,\ |y| \leqslant H_R\}$ (see the introduction).*

Then there is a solution $x(\cdot;P_0)$ of the problem (1.1)–(1.2) *for a certain point $P_0 \in D_0(R,K_R)$ defined at least on the interval I. Moreover, this solution satisfies the inequality $|x(t;P_0)| \leqslant R$ for all $t \in I$, thus, by Lemma* 2.1, *$|\dot{x}(t;P_0)| \leqslant K_R$.*

P r o o f. Consider a function $f(t,x,y)$, $(t,x,y) \in E_\sigma$ as above and define a new function

$$F\colon\ E_\sigma \to \mathbb{R}^n$$

by

$$F(t,x,y) := f\Bigl(t, \min\Bigl\{1, \frac{R}{|x|}\Bigr\}x, \min\Bigl\{1, \frac{K_R}{|y|}\Bigr\}y\Bigr)$$

for all $(t,x,y) \in E_\sigma$. On the set $D(R,K_R)$ the function F is identically equal to f and satisfies the inequality

$$|F(t,x,y)| \leqslant g(R,K_R) \tag{2.1}$$

for all $(t,x,y) \in E_\sigma$. We formulate the ordinary differential equation

$$\ddot{x} = F(t,x,\dot{x}), \quad t \in E_\sigma. \tag{2.2}$$

From (2.1) it follows that solutions of (2.2) with initial values in $D_0(R,K_R)$ are defined on the whole interval I_σ and are unique. Moreover, any solution x of the differential equation (2.2) with $G(x|I)$ in $D(R,K_R)$ is also a solution of the original equation (1.1).

Let $\delta > 0$ be a sufficiently small fixed number with $\delta \leqslant \min\{\delta_U,\delta_V\}$ and consider a point $P_0 := (0,\xi,\eta)$ in $D_0(R-\delta,K_R-\delta) \subseteq D_0(R,H_R)$. As we have remarked earlier, such a point is a strict ingress point of the set $D(R,H_R)$. Since $D(R,H_R)$ is a compact subset of E_σ, by the extendability property of the solution $x(\cdot;P_0)$ there is a time $s \geqslant 0$ such that the point $P := (s,x(s;P_0),\dot{x}(s;P_0))$ is a egress point of the set $D(R, H_R)$ and thus it belongs to the boundary. We can assume that s is the smallest time with this property. It is clear that $s > 0$, since in a small neighborhood J of 0 the set $G(x(\cdot; P_0)|J)$ lies in the interior of $I_\sigma \times N(R, H_R)$. Then we must have

$$G(x|(0, s)) \subseteq \operatorname{int} D(R, H_R),$$

and either

$$|x(s; P_0)| \leqslant R, \quad |\dot{x}(s; P_0)| = H_R, \tag{2.3}$$

or

$$|x(s; P_0)| = R, \quad |\dot{x}(s; P_0)| \leqslant H_R. \tag{2.4}$$

We claim that the point P is a strict egress point of the set $D(R, H_R)$, thus the consequent mapping is well defined and (by Lemma 1.1) it is continuous at P_0. This fact is obvious in case $s = 1$, so we discuss the case $s < 1$.

Assume that the relations (2.3) hold. From inequality (2.1) and equation (2.2) it follows that for each $t \in [0, s]$ we have

$$|\ddot{x}(t)| \leqslant g(R, K_R),$$

hence

$$|\dot{x}(s)| \leqslant |\dot{x}(0)| + sg(R, K_R) < K_R + sg(R, K_R) < H_R,$$

a contradiction. Recall that $\dot{x}(0) = \eta \in \operatorname{cl} B(0, K_R - \delta)$. Therefore only the relations (2.4) hold. Define the function

$$m_{P_0}(t) := \frac{1}{2}[|x(t; P_0)|^2 - R^2], \quad t \in I_\sigma,$$

and observe that $m_{P_0}(s) = 0$. It is clear that, if $\dot{m}_{P_0}(s) > 0$, then $P \in D(R, H_R)^{se}$. Assume that $\dot{m}_{P_0}(s) = 0$, hence $x(s; P_0) \cdot \dot{x}(s; P_0) = 0$. Then (f3) implies that

$$\ddot{m}_{P_0}(s) = x(s; P_0) \cdot f(s, x(s; P_0), \dot{x}(s; P_0)) + |\dot{x}(s; P_0)|^2 > 0,$$

which guarantees that $P \in D(R, H_R)^{se}$. Our claim is proved.

Next consider the continuous functions J, Q, M defined by

$$J(u, v) := (0, u, v), \quad Q(t, u, v) := (u, v),$$

and

$$M(u, v) := \Big(\frac{R - \delta}{R} u, \frac{H_R - \delta}{H_R}\Big)$$

for all $u, v \in \mathbb{R}^n$ and $t \in I$. For the direct product function $W := U \otimes V$ observe that

$$\Big|U\Big(\frac{R-\delta}{R}x(s;(0,\xi,\eta))\Big)\Big| \leqslant R-\delta$$

and

$$\Big|V\Big(\frac{H_R-\delta}{H_R}\dot{x}(s;(0,\xi,\eta))\Big)\Big| \leqslant K_R-\delta.$$

Consequently, the composition

$$T := W \circ M \circ Q \circ C \circ J$$

maps the generalized interval $V_\delta := N(R-\delta, K_R-\delta)$ into itself continuously. Now we apply Brouwer's fixed point theorem and get the existence of a point

$$(u_\delta, v_\delta) \in V_\delta$$

such that

$$U\Big(\frac{R-\delta}{R}x(s;(0,u_\delta,v_\delta))\Big) = u_\delta \tag{2.5}$$

and

$$V\Big(\frac{H_R-\delta}{H_R}\dot{x}(s;(0,u_\delta,v_\delta))\Big) = v_\delta \tag{2.6}$$

for some $s_\delta \in I$. Assume that $s_\delta < 1$. Then

$$|x(s_\delta;(0,u_\delta,v_\delta))| = R. \tag{2.7}$$

Suppose that this relation holds for a sequence of δ's converging to zero. Then, since the interval $[0,1]$ and the product $N(R, H_R)$ are compact sets, it follows from the continuity of $x(t,P)$ and $\dot{x}(t,P)$ on the initial point P, uniformly for all t, that there is a time $s' \in [0,1]$ and a pair $(u,v) \in N(R,H_R)$ such that

$$U(x(s';(0,u,v))) = u \tag{2.8}$$

and

$$V(\dot{x}(s');(0,u,v)) = v.$$

If $s' = 0$, then $x(s';(0,u,v)) = u$ and (2.8) gives that u is a fixed point of U on the boundary of the ball $B(0,R)$, contradicting our hypothesis. Thus we must have $s' > 0$. If $|u| < R$, then it follows, from (a) that

$$R = |x(s';(0,u,v))| = |U^{-1}(u)| \leqslant \sup_{|u'|\leqslant|u|} |U^{-1}(u')| \leqslant |u| < R,$$

a contradiction. Let us, finally, assume that $|u| = R$. Since

$$G(x(\cdot;(0,u_\delta,v_\delta))|(0,s_\delta)) \subseteq \operatorname{int} D(R,H_R),$$

from continuity we get

$$G(x(\cdot;(0,u,v)|[0,s']) \subseteq D(R,H_R). \tag{2.9}$$

On the other hand, we have

$$x(s_\delta;(0,u_\delta,v_\delta)) \cdot \dot{x}(s_\delta;(0,u_\delta,v_\delta)) \geqslant 0$$

and so $x(s';(0,u,v)) \cdot \dot{x}(s';(0,u,v)) \geqslant 0$. Hence by the compatibility of the functions U, V we get

$$u \cdot v = U(x(s';(0,u,v))) \cdot V(\dot{x}(s';(0,u,v))) \geqslant 0.$$

This inequality combined with the condition (f3) means that the point (u,v) is a strict egress point, so (2.9) cannot be true.

The previous arguments show that in (2.7) we have $s = 1$ for all small δ's. Now, from (2.5), (2.6), the compactness of the set $D_0(R,K_R)$, the continuity of the function W and the continuity of the solutions with respect to the initial values, it follows that there exists a point $(0,u,v) \in D_0(R,K_R)$ such that (u,v) satisfies the relation

$$(U(x(1;(0,u,v))), V(\dot{x}(1;(0,u,v)))) = (u,v),$$

which is the same as (1.2). Also, for each δ the solution $x(\cdot;(0,u_\delta,v_\delta))$ of equation (2.2) egresses strictly from $D(R,H_R)$ at $s = 1$. Thus we have

$$|x(t;(0,u_\delta,v_\delta))| \leqslant R$$

for all $t \in I$. Therefore

$$|x(t;(0,u,v))| \leqslant R$$

for all $t \in I$ and so, from Lemma 2.1,

$$|\dot{x}(t;(0,u,v))| \leqslant K_R$$

for all $t \in I$. Hence $|v| \leqslant K_R$ and, so, $x(\cdot;(0,u,v))$ is also a solution of equation (1.1). The proof of the theorem is complete. □

Some Applications

(a) Consider Van der Pol's equation

$$\ddot{x} + F(x)\dot{x} + G(x) = 0,$$

where the continuous functions F, G are such that

$$\limsup_{r\to+\infty} r^{-1} \sup_{|x|\leqslant r} |G(x)| > 4$$

and

$$\limsup_{|x|\to+\infty} xG(x) < 0.$$

Also consider two functions U, V defined by $U(x) := -x$ and $V(x) := \varphi(x)$, where φ is a bounded continuous real valued function such that $x\varphi(x) \leqslant 0$ for all large $|x|$. In this case the function $g(r, q)$ is defined by

$$g(r, q) := F_1(r)q + G_1(r),$$

where

$$F_1(r) =: \sup_{|x|\leqslant r} |F(x)|$$

and

$$G_1(r) =: \sup_{|x|\leqslant r} |G(x)|.$$

Choose a certain $R > 0$ with $G_1(R) > 4R$, $RG(R) < 0$ and $\pm R\varphi(\pm R) \leqslant 0$. In this case our theorem above applies where K_R is any number such that

$$K_R > 2RF_1(R) + 2R\sqrt{F_1^2(R) + G_1(R)}$$

and it guarantees the existence of a solution x of Van der Pol's equation such that

$$x(0) = -x(1), \quad \dot{x}(0) = \psi(\dot{x}(1)).$$

(b) Consider the differential equation

$$\ddot{x} = Ax + |\dot{x}|^\gamma (x \cdot \dot{x})b(t), \quad t \in [0, 1] \tag{2.10}$$

where $0 \leqslant \gamma < 1$, the matrix A is strictly positive definite with $|A| =: a \geqslant 4$ and the vector valued continuous function $b(\cdot)$ has sup-norm equal to $|b|$. Assume that we are interested in solutions of equation (2.10) satisfying the boundary conditions (1.2), where the functions U, V are defined by

$$U(u) := (p - |u|)e - u, \quad |u| \leqslant p$$

and

$$U(u) := -u, \quad |u| \geqslant p$$

for some $p > \frac{1}{8}$ and also

$$V(v) := -kv,$$

where e is the vector $(1, 0, \ldots, 0)$ and k is any positive real number such that

$$k < K[K + aR + |b|RK^{1+\gamma}]^{-1}$$

with $R := 2p$ and

$$K := [4(a + |b|)R^2]^{1/(1-\gamma)}.$$

In this case the function g is given by

$$g(r, q) := ar + r|b|q^{1+\gamma}.$$

Thus the conditions (f1)–(f2) are satisfied. Also observe that, whenever $q > K$, then $q^2 > 4Rg(R, q)$, so we can set $K_R := K$.

The conclusion is that there is a (nonzero) solution x defined on $[0, 1]$ such that

$$x_1(0) + x_1(1) \geqslant 0$$

and

$$x_j(0) + x_j(1) = 0$$

for all indices $j = 2, 3, \ldots, n$, as well as

$$\dot{x}(0) + k\dot{x}(1) = 0.$$

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