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BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

P. Ch. Tsamatos, and S. K. Ntouyas, Ioannina

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1. Introduction

In the paper we consider the equations with deviating arguments

$$(\mathrm{E}_1)\qquad x''(t)+f\big(t,x(t),x(\sigma_1(t)),\dots,x(\sigma_k(t))\big)=0$$

and

$$(\mathrm{E}_2)\qquad x''(t)+\hat f\Big(t,x(t),x\big(\sigma_1(t)\big),\dots,x\big(\sigma_k(t)\big),x'(t),x'\big(g_1(t)\big),\dots,x'\big(g_m(t)\big)\Big)=0$$

where $t\in I=[a,b]$ $(a<b)$ and $f\colon I\times(\mathbb{R}^n)^k\to\mathbb{R}^n$, $\hat f\colon I\times(\mathbb{R}^n)^{k+m+2}\to\mathbb{R}^n$ are continuous functions. Also, the arguments σ_i, $i=1,\dots,k$, g_j, $j=1,\dots,m$ are continuous real valued functions defined on I and such that the set $\{t\in I\colon g_j(t)=a \text{ or } g_j(t)=b, j=1,\dots,m\}$ is finite.

We suppose that

$$-\infty<a_0=\min_{1\leqslant i\leqslant k}\min_{t\in I}\sigma_i(t)<a,\ b<\max_{1\leqslant i\leqslant m}\max_{t\in I}\sigma_i(t)=b_0<+\infty$$

and

$$-\infty<\hat a=\min\Big\{\min_{1\leqslant i\leqslant k}\min_{t\in I}\sigma_i(t),\ \min_{1\leqslant j\leqslant m}\min_{t\in I}g_j(t)\Big\}<a,$$
$$b<\max\Big\{\max_{1\leqslant i\leqslant k}\max_{t\in I}\sigma_i(t),\ \max_{1\leqslant j\leqslant m}\max_{t\in I}g_j(t)\Big\}=\hat b<+\infty$$

and we set $E(a)=[a_0,0]$, $E(b)=[b,b_0]$, $\hat E(a)=[\hat a,a]$ and $\hat E(b)=[\hat b,b]$.

Here we seek a solution of (E_1) (resp. (E_2)) which satisfies the following general type boundary conditions:

$$\text{(BC)} \qquad \begin{aligned} &\alpha_0 x(t) + \alpha_1 x'(t) = q_1(t),\ t \in E(a)\ (\text{resp. } t \in \hat{E}(a)),\\ &\beta_0 x(t) + \beta_1 x'(t) = q_2(t),\ t \in E(b)\ (\text{resp. } t \in \hat{E}(b)) \end{aligned}$$

where α_i, β_i, $i = 0, 1$, are real constants satisfying

$$\ell = \alpha_0\beta_0(b-a) + \alpha_0\beta_1 - \alpha_1\beta_0 \neq 0, \tag{1.1}$$

$$\begin{cases} \frac{\alpha_1}{\alpha_0} \leqslant 0 \leqslant \frac{\beta_1}{\beta_0}, & \text{if } \alpha_0\beta_0 \neq 0,\\ \alpha_1 \in \mathbb{R},\ 0 \leqslant \frac{\beta_1}{\beta_0}, & \text{if } \alpha_0 = 0,\\ \frac{\alpha_1}{\alpha_0} \leqslant 0,\ \beta_1 \in \mathbb{R}, & \text{if } \beta_0 = 0. \end{cases} \tag{1.2}$$

Finally we suppose that q_1, q_2 are $\mathbb{R}^n$-valued functions defined and differentiable on $E(a)$, $E(b)$ (resp. $\hat{E}$ (a), $\hat{E}(b)$) respectively.

For the sake of brevity we use the notation B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) for the boundary value problem which consists of the equation (E_1) (resp. (E_2)), the boundary conditions (BC) and the conditions (1.1), (1.2).

By the term solution of the B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) we mean a function $x\colon E(a) \cup I \cup E\ (b) \to \mathbb{R}^n$ (resp. $x\colon \hat{E}(a) \cup I \cup \hat{E}\ (b) \to \mathbb{R}^n$) which is continuous on its domain, differentiable on $E(a)$, $E(b)$ (resp. $\hat{E}(a)$, $\hat{E}(b)$), twice differentiable (resp. twice piecewise differentiable) on I and satisfies the equation (E_1) (resp. (E_2)) and the boundary conditions (BC).

A very interesting method for the proof of existence of solutions for boundary value problems is based on a simple and classical application of the Leray-Schauder degree theory. Recently, Fabry and Habets [3], Fabry [4] and Ntouyas and Tsamatos [5] have used this method to give answers to a series of boundary value problems.

In this paper, we apply this method to our general B.V.Ps (E_1)–(BC) and (E_2)–(BC). In a recent paper [9] we gave some results concerning the existence of solutions of a B.V.P. of the form (E_2)–(BC) by applying the topological transversality method of Granas [2]. More precisely we studied B.V.P.

$$\text{(E}_2)' \qquad \begin{aligned} x''(t) = f\Big(t, x(t), x\big(\sigma_1(t)\big), \ldots, x\big(\sigma_k(t)\big), x'(t), x'\big(g_1(t)\big),\\ \ldots, x'\big(g_m(t)\big)\Big),\ t \in I, \end{aligned}$$

$$\text{(BC)}' \qquad \begin{aligned} -\alpha_0 x(t) + \alpha_1 x'(t) = q_1(t),\ t \in \hat{E}(a),\\ \beta_0 x(t) + \beta_1 x'(t) = q_2(t),\ t \in \hat{E}(b), \end{aligned}$$

where the constants α_0, β_0, β_1 are nonnegative, $\alpha_1 > 0$ and $\ell \neq 0$.

Although the problems (E_2)–(BC), $(E_2)'$–(BC)$'$ seem to be almost the same, the method developed in [9] cannot be applied for the B.V.P. (E_2)–(BC) (see the proof of Lemma 3.1 in [9]). On the other hand the method used here ensures the existence of a solution of the B.V.P. (E_2)–(BC) which is bounded by an a priori given positive function. The remarkable fact is that the assumptions on φ (see conditions (3.1), (3.2) below) do not allow φ to be taken as a constant function. (This can be done only in the case when $\alpha_1 = \beta_1 = 0$.) This does not allow us to conclude that the results of our paper generalize those of [9]. Nevertheless, the results obtained here generalize the results of Fabry and Habets [3] and Fabry [4].

It is noteworthy that the present method can be applied also to the B.V.P. (E_2)–(BC)$'$.

The plan of this paper is as follows: In Section 2 we state some auxiliary lemmas. Main results are given in Section 3, where sufficient conditions are established for the existence of solutions of the B.V.Ps (E_i)–(BC), $i = 1, 2$. In Section 4 some results for smooth solutions of B.V.Ps (E_i)–(BC), $i = 1, 2$ are given. Section 5 includes applications of the result of Section 3.

2. Auxiliary Lemmas

The next Lemma 2.1 is the basic tool of the method which we use in the proof of existence of solutions for the B.V.Ps (E_i)–(BC), $i = 1, 2$.

Lemma 2.1 [3, Theorem 1]. *Let X be a Banach space, $A\colon X \to X$ a compact mapping such that $I - A$ is one to one and Ω an open bounded subset of X such that $0 \in (I - A)(\Omega)$. Then a compact mapping $T\colon \overline{\Omega} \to X$ has a fixed point in Ω if for any $\lambda \in (0, 1)$ the equation*

$$x = \lambda T x + (1 - \lambda) A x$$

has no solution x on the boundary $\partial\Omega$ of Ω.

Also we need the following lemma from [7] whose basic steps of proof we reproduce here for the sake of completeness. In this lemma and in the sequel, the symbols $\langle .,. \rangle$ and $|.|$ stand respectively for the euclidean product and the euclidean norm in the space $\mathbb{R}^n$.

Lemma 2.2. *Assume that h_1 and h_2 are continuous real valued functions defined on I and such that*

$$-\infty < d_a = \min\left\{ \min_{t \in I} h_1(t),\ \min_{t \in I} h_2(t) \right\} \leqslant a$$

and

$$b \leqslant d_b = \max\left\{ \max_{t\in I} h_1(t),\ \max_{t\in I} h_2(t)\right\} < +\infty$$

and $G = \{t \in I\colon h_i(t) = a \text{ or } h_i(t) = b,\ i = 1,2\}$ *is finite.*

Also, let $\hat{x}$ *be a continuous* $\mathbb{R}^n$*-valued function defined on* $[d_a, d_b]$ *which is continuously differentiable on* $[d_a, a]$, I *and* $[b, d_b]$ *and piecewise twice differentiable on* I. *Let* x *be the restriction of* $\hat{x}$ *to* I, *i.e.* $\hat{x}\big|I = x$.

Moreover, assume that there exist positive constants R, $\alpha, \beta, \alpha', \gamma$ *and* γ' *with* $\alpha < 1$, $\alpha' < \frac{1}{8D}(1-\alpha)^2$ *and such that the following relations are valid:*

$$\sup_{t\in I}\big|x(t)\big| \leqslant D, \tag{2.1}$$

$$-\langle x(t), x''(t)\rangle \leqslant \alpha\big|\hat{x}'\big(h_1(t)\big)\big|^2 + \beta,\ t \in I - A \tag{2.2}$$

and

$$\big|\langle x'(t), x''(t)\rangle\big| \leqslant \Big(\alpha'\big|\hat{x}'\big(h_2(t)\big)\big|^2 + \gamma\Big)\big|\hat{x}'\big(h_2(t)\big)\big| + \gamma'\big|x'(t)\big|,\ t \in I - A \tag{2.3}$$

where

$$A = G \cup B \text{ and } B = \{t \in I\colon x''(t-0) \neq x''(t+0)\}.$$

Then there exits a number M *depending only on* $\hat{x}\big|[d_a, a] \cup [b, d_b]$, $b-a$, D, α, β, α', γ, γ' *but not on* x *such that*

$$\max_{t\in I}\big|x'(t)\big| \leqslant M.$$

Proof. We set $M = \max_{t\in I}\big|x'(t)\big| = \big|x'(t_0)\big|$, where $t_0 \in I$. For every piecewise twice differentiable on I function σ, by a Taylor expansion, we have

$$\sigma(t_0+\mu) - \sigma(t_0) = \mu\sigma'(t_0) + \int_{t_0}^{t_0+\mu} \sigma''(s)(t_0+\mu-s)\,\mathrm{d}s$$

provided $t_0 + \mu \in I$. We apply this formula to the function $\sigma(t) = \int_a^t \big|x'(s)\big|^2\,\mathrm{d}s$, $t \in I$ obtaining

$$\int_{t_0}^{t_0+\mu} \big|x'(s)\big|^2\,\mathrm{d}s = \mu\big|x'(t_0)\big|^2 + 2\int_{t_0}^{t_0+\mu} \big\langle x'(s), x''(s)\big\rangle(t_0+\mu-s)\,\mathrm{d}s. \tag{2.4}$$

Integrating by parts and using (2.1), (2.2) we have

$$\begin{aligned}\left|\int_{t_0}^{t_0+\mu} \big|x'(s)\big|^2\,\mathrm{d}s\right| &\leqslant 2DM + \left|\int_{t_0}^{t_0+\mu} \big(\alpha\big|\hat{x}'\big(h_1(s)\big)\big| + \beta\big)\,\mathrm{d}s\right| \\ &\leqslant 2DM + \big(\alpha M_1^2 + \beta\big)\delta\end{aligned} \tag{2.5}$$

where $M_1 = \max\{M, m\}$, $m = \sup_{t\in[d_a,a]\cup[b,d_b]} \left|\hat{x}'(t)\right|$ and $\delta = |\mu|$.

On the other hand, by (2.4), (2.3) and (2.5) we obtain

$$\delta M^2 \leqslant 2\left|\int_{t_0}^{t_0+\mu} \left(\alpha'\left|\hat{x}'\left(h_2(s)\right)\right|^3 + \gamma\left|x'\left(h_2(s)\right)\right| + \gamma'\left|x'(s)\right|\right)\left|t_0+\mu-s\right| \mathrm{d}s\right|$$
$$+ 2DM + \alpha M_1^2\delta + \beta\delta \leqslant \left(\alpha M_1^3 + \beta' M_1\right)\delta^2 + 2DM + \alpha M_1^2\delta + \beta\delta$$

where $\beta' = \gamma + \gamma'$.

Therefore

$$\delta M^2 \leqslant \left(\alpha' M^3 + \beta' M\right)\delta^2 + 2DM + \alpha M^2\delta + \beta\delta, \quad \text{if } M_1 = M$$

or

$$\delta M^2 \leqslant \left(\alpha' m^3 + \beta' m\right)\delta^2 + 2DM + \alpha m^2\delta + \beta\delta, \quad \text{if } M_1 = m$$

from which, following exactly the same arguments as in [4], we obtain

$$M \leqslant \max\left\{\frac{8D}{(1-\alpha)(b-a)}, \frac{(b-a)(1-\alpha)}{4D} \cdot \frac{\beta(1-\alpha)+4D\beta'}{(1-\alpha)^2 - 8D\alpha'}\right\}$$

or

$$M \leqslant \max\left\{\frac{8D}{(1-\alpha)(b-a)}, \frac{M_2}{2D}\right\},$$

respectively, where $M_2 = \frac{1}{4}\left[(\alpha' m^3 + \beta' m)(b-a)^2 + 2\alpha m^2(b-a) + 2\beta(b-a)\right]$.

Therefore, in any case we have that M can be bounded independently of x, which proves the lemma. □

3. Existence results for the solutions of the B.V.P.s (E₁)–(BC) and (E₂)–(BC)

If $J = [a_0, b_0]$ and $\hat{J} = [\hat{a}, \hat{b}]$ we set

$$B_0 = C(J, \mathbb{R}^n)$$

for the space of all $\mathbb{R}^n$-valued continuous functions defined on J and

$$B_1 = C(\hat{J}, \mathbb{R}^n) \cap C^1\left(\hat{E}(a) \cup \hat{E}\,(b), \mathbb{R}^n\right) \cap C^1(I, \mathbb{R}^n)$$

for the space of all $\mathbb{R}^n$-valued continuous functions defined on $\hat{J}$ which have continuous first derivative on $\hat{E}\,(a) \cup \hat{E}(b)$ and are also continuously differentiable on I, endowed with the norms

$$\|x\|_0 = \max_{t\in J}\left|x(t)\right|, \; x \in B_0$$

and

$$\|x\|_1 = \max\left\{ \max_{t\in J} |x(t)|, \max_{t\in \hat{E}(a)\cup\hat{E}(b)} |x'(t)|, \max_{t\in I} |x'(t)| \right\}, \ x \in B_1,$$

respectively. It is well known that B_0 and B_1 are Banach spaces.

For the sake of simplicity, for every function $z \in B_0$ and for every $t \in I$ we set

$$\big(t, z(t), z(\sigma_1(t)), \dots, z(\sigma_k(t))\big) = \big(t, z(t), z[\sigma(t)]\big).$$

Also, for every function $z \in B_1$ and for every $t \in I$ we set

$$\begin{aligned}&\big(t, z(t)\big), z(\sigma_1(t)), \dots, z(\sigma_k(t)), z'(t), z'(g_1(t)), \dots, z'(g_m(t))\\ &= \Big(t, z(t), z[\sigma(t)], z'(t), z'[g(t)]\Big).\end{aligned}$$

The following Theorem 3.1 guarantees the existence of solutions of the B.V.P. (E_1)–(BC) which are bounded by an a priori given function φ.

Theorem 3.1. *Assume that $\varphi\colon I \to (0,\infty)$ is a twice continuously differentiable function such that*

$$\text{(3.1)} \qquad \begin{aligned} -|\alpha_0|\varphi(a) - |\alpha_1|\varphi'(a) &> |q_1(a)|, \quad \text{if } \alpha_1 \neq 0,\\ |\alpha_0|\varphi(a) &> |q_1(a)|, \quad \text{if } \alpha_1 = 0\end{aligned}$$

and

$$\text{(3.2)} \qquad \begin{aligned} -|\beta_0|\varphi(b) + |\beta_1|\varphi'(b) &> |q_2(b)|, \quad \text{if } \beta_1 \neq 0,\\ |\beta_0|\varphi(b) &> |q_2(b)|, \quad \text{if } \beta_1 = 0.\end{aligned}$$

Also, we suppose that

$$\text{(3.3)} \qquad \varphi(t)\varphi''(t) + \Big\langle x(t), f\big(t, x(t), x[\sigma(t)]\big)\Big\rangle \leqslant 0$$

for any $x \in B_0$ with $|x(t)| = \varphi(t)$ and $\langle x(t), x'(t)\rangle = |x(t)|\varphi'(t)$, $t \in I$.

Then the B.V.P. (E_1)–(BC) *has at least one solution x such that $|x(t)| \leqslant \varphi(t)$, $t \in I$.*

Proof. The Green function for the homogeneous B.V.P.

$$\begin{aligned} x''(t) &= 0, \ t \in I,\\ \alpha_0 x(a) + \alpha_1 x'(a) &= 0,\\ \beta_0 x(b) + \beta_1 x'(b) &= 0\end{aligned}$$

is given by the formula

$$G(t,s) = \frac{1}{\ell}\begin{cases} (\beta_0 t - \beta_0 b - \beta_1)(\alpha_0 s - \alpha_0 a - \alpha_1), & s \leqslant t, \\ (\beta_0 s - \beta_0 b - \beta_1)(\alpha_0 t - \alpha_0 a - \alpha_1), & t \leqslant s \end{cases}$$

where $\ell = \alpha_0\beta_0(b-a) + \alpha_0\beta_1 - \beta_0\alpha_1 \neq 0$ because of (1.1) (see Agarwal [1]). Now we define a function $w\colon J \to \mathbb{R}^n$ as

$$w(t) = \begin{cases} \Big\{w(a) + \frac{1}{\alpha_1}\int_a^t q_1(s)\exp\Big(\frac{\alpha_0}{\alpha_1}(s-a)\Big)\,\mathrm{d}s\Big\}\exp\Big(-\frac{\alpha_0}{\alpha_1}(t-a)\Big), \\ \quad \text{if } \alpha_1 \neq 0,\ t < a, \\ \frac{1}{\alpha_0}q_1(t), \ \text{if } \alpha_1 = 0,\ t < a, \\ \frac{1}{\ell}\big[\beta_0(b-t)q_1(a) + \beta_1 q_1(a) - \alpha_1 q_2(b) + \alpha_0(t-a)q_2(b)\big],\ t \in I, \\ \Big\{w(b) + \frac{1}{\beta_1}\int_b^t q_2(s)\exp\Big(\frac{\beta_0}{\beta_1}(s-b)\Big)\,\mathrm{d}s\Big\}\exp\Big(-\frac{\beta_0}{\beta_1}(t-b)\Big), \\ \quad \text{if } \beta_1 \neq 0,\ t > b, \\ \frac{1}{\beta_0}q_2(t),\ \text{if } \beta_1 = 0,\ t > b. \end{cases}$$

It is obvious that $w \in B_0$. Hence the operator T defined on B_0 by the formula

$$Tx(t) = Lx(t) + w(t),\ t \in J,$$

where

$$Lx(t) = \begin{cases} \int_a^b G(t,s)f\big(s, x[\sigma(s)]\big)\,\mathrm{d}s,\ t \in I, \\ \exp\big(\frac{\alpha_0}{\alpha_1}(t-a)\big)Lx(a),\ t < a,\ \alpha_1 \neq 0, \\ 0,\ t < a,\ \alpha_1 = 0, \\ \exp\big(-\frac{\beta_0}{\beta_1}(t-a)\big)Lx(b),\ t > b,\ \beta_1 \neq 0, \\ 0,\ t > b,\ \beta_1 = 0 \end{cases}$$

is a compact operator with values in B_0 (see [9]).

We also define an open set in the space B_0 as

$$\Omega = \big\{x \in B_0\colon \big|x(t)\big| < \varphi(t),\ t \in I\big\}$$

and an operator A on B_0 by the formula

$$Ax(t) = \begin{cases} \int_a^b G(t,s)Kx(s)\,\mathrm{d}s, & t \in I, \\ Ax(a), & t < a, \\ Ax(b), & t > b \end{cases}$$

where K is a constant such that

$$K > \max_{t \in I}\frac{\varphi''(t)}{\varphi(t)}.$$

Obviously, A is a compact operator.

Now, we observe that the operator $I - A$ is one to one. Indeed, let $(I - A)x = (I - A)y$ with x, y in B_0. Then $(I - A)z = 0$, where $z = x - y$. Thus $z = Az$ and hence z must be a solution of the B.V.P.

$$(*) \qquad \begin{aligned} z''(t) &= Kz(t), \\ \alpha_0 z(a) + \alpha_1 z'(a) &= 0, \\ \beta_0 z(b) + \beta_1 z'(b) &= 0. \end{aligned}$$

We shall prove that this B.V.P. has the unique solution $z = 0$.

The general solution of the above equation has the form

$$z(t) = c_1 e^{\sqrt{K}t} + c_2 e^{-\sqrt{K}t}.$$

On account of the above boundary conditions we take

$$\frac{(\alpha_0 + \alpha_1\sqrt{K})(\beta_0 - \beta_1\sqrt{K})}{(\alpha_0 - \alpha_1\sqrt{K})(\beta_0 + \beta_1\sqrt{K})} \neq e^{2(b-a)\sqrt{K}}.$$

Since $e^{2(b-a)\sqrt{K}} > 1$, $K > 0$ the last is true for every $K > 0$ if the left hand side is less than or equal one. But this is clear from (1.1) and (1.2). Therefore $z = 0$ or $x = y$. Moreover, $0 \in (I - A)(\Omega)$ since $0 \in \Omega$ and $(I - A)0 = 0$.

In order to apply Lemma 2.1, it remains to prove that no solutions of the equation

$$x = \lambda Tx + (1 - \lambda)Ax \tag{3.4}$$

belong to $\partial\Omega$.

To this end assume the contrary. Thus, let x be a solution of (3.4) on $\partial\Omega$. Then there exists a $\xi \in [a, b]$ such that the function

$$g(t) = \left|x(t)\right|^2 - \varphi^2(t), \ \ t \in I \tag{3.5}$$

assumes its maximum value, which is zero, for $t = \xi$. Then, if $\xi \in (a, b)$, we have the relations

$$\left|x(\xi)\right| = \varphi(\xi), \tag{3.6}$$

$$\left\langle x(\xi), x'(\xi)\right\rangle = \varphi(\xi)\varphi'(\xi) \tag{3.7}$$

and

$$L \equiv \left\langle x(\xi), x''(\xi)\right\rangle + \left|x'(\xi)\right|^2 - \varphi'(\xi)^2 - \varphi(\xi)\varphi''(\xi) \leqslant 0. \tag{3.8}$$

Now assume that x is a solution of (3.4). Then by (3.3), (3.6) and (3.7) we obtain

$$\begin{aligned} L &\equiv -\lambda\langle x(\xi), f(\xi, x(\xi), x[\sigma(\xi)])\rangle + (1-\lambda)K|x(\xi)|^2 \\ &\quad + |x'(\xi)|^2 - \varphi'(\xi)^2 - \varphi(\xi)\varphi''(\xi) \\ &\geqslant (1-\lambda)[K\varphi(\xi)^2 - \varphi(\xi)\varphi''(\xi)] + |x'(\xi)|^2 - \varphi'(\xi)^2 \\ &\geqslant (1-\lambda)\varphi(\xi)[K\varphi(\xi) - \varphi''(\xi)], \end{aligned}$$

since $|x'(\xi)|^2 - \varphi'(\xi)^2 = |x'(\xi)|^2 - \frac{\langle x(\xi), x'(\xi)\rangle^2}{|x(\xi)|^2} \geqslant 0$, by the Cauchy-Schwarz inequality. Consequently $L > 0$, $\lambda \in [0,1)$, since $K > \frac{\varphi''(t)}{\varphi(t)}$, $t \in (a,b)$, contradicting (3.8).

Next we show that $\xi \neq a$. If $\xi = a$ then the following must hold:

$$g(a) = 0 \text{ and } g'(a) \leqslant 0.$$

Then $|x(a)| = \varphi(a)$ and $-|x'(a)| \leqslant \varphi'(a)$. But, by the first boundary condition, we have

$$|\alpha_1||x'(a)| \leqslant |q_1(a)| + |\alpha_0||x(a)|.$$

Hence

$$-|\alpha_1|\varphi'(a) \leqslant |q_1(a)| + |\alpha_0|\varphi(a), \text{ if } \alpha_1 \neq 0$$

or

$$|\alpha_0|\varphi(a) \leqslant |q_1(a)|, \text{ if } \alpha_1 = 0,$$

which contradicts (3.1). Therefore $\xi \neq a$ as required.

Finally, we show that $\xi \neq b$. If, on the contrary, we assume that $\xi = b$, then

$$g(b) = 0 \text{ and } g'(b) \geqslant 0$$

imply

$$|x(b)| = \varphi(b) \text{ and } \varphi'(b) \leqslant |x'(b)|.$$

From the second boundary condition we obtain

$$|\beta_1||x'(b)| \leqslant |q_2(b)| + |\beta_0||x(b)|.$$

Hence

$$|\beta_1|\varphi'(b) \leqslant |q_2(b)| + |\beta_0|\varphi(b), \text{ if } \beta_1 \neq 0$$

or

$$|\beta_0|\varphi(b) \leqslant |q_2(b)|, \text{ if } \beta_1 = 0,$$

contradicting (3.2).

Hence, by Lemma 1, the operator T has a fixed point in Ω or, otherwise, there exists a solution x of the B.V.P. (E_1)–(BC) such that

$$|x(t)| \leqslant \varphi(t), \ t \in I,$$

completing the proof of the theorem. □

The next Theorem 3.2 gives an analogous result for the B.V.P. (E_2)–(BC). Under appropriate conditions we can obtain solutions x of the B.V.P. (E_2)–(BC) which, as in the previous theorem, are bounded by a function φ and, moreover, the derivative of x is bounded by an a priori given constant.

Theorem 3.2. *Assume that $\varphi\colon I \to (0,\infty)$ is a function satisfying the conditions* (3.1) *and* (3.2). *Also, assume that*

$$\varphi(t)\varphi''(t) + \Big\langle x(t), \hat f\Big(t, x(t), x\big[\sigma(t)\big], x'(t), x'\big[g(t)\big]\Big)\Big\rangle \leqslant 0 \tag{3.9}$$

for any $x \in B_1$ with $|x(t)| = \varphi(t)$ and $\langle x(t), x'(t)\rangle = |x(t)||\varphi'(t)$, $t \in I$.

Moreover, for any $(t, u, u_1, \dots, u_k, v, v_1, \dots, v_m) \in I \times (\mathbb{R}^n)^{k+m+2}$ with $|u| \leqslant \varphi(t)$ and $|u_i| \leqslant \varphi\big(\sigma_i(t)\big)$, $i = 1, 2, \dots, k$, when $\sigma_i(t) \in I$, there are τ and μ in $\{0, 1, \dots, m\}$ with $v_0 = v$ such that

$$\langle u, \hat f(t, u, u_1, \dots, u_k, v, v_1, \dots, v_m)\rangle \leqslant \alpha|v_\tau|^2 + \beta, \tag{3.10}$$

$$\big|\langle v, \hat f(t, u, u_1, \dots, u_k, v, v_1, \dots, v_m)\rangle\big| \leqslant \big(\alpha'|v_\mu|^2 + \gamma\big)|v_\mu| + \gamma'|v| \tag{3.11}$$

where the positive numbers $\alpha, \beta, \alpha', \gamma, \gamma'$ are such that

$$\alpha < 1 \text{ and } \alpha' < \frac{1}{8d}(1-\alpha)^2, \ d = \sup_{t\in I} \varphi(t).$$

Then the B.V.P. (E_2)–(BC) *has at least one solution such that*

$$|x(t)| \leqslant \varphi(t), \ t \in I$$

and

$$|x'(t)| \leqslant \varrho, \ t \in I$$

where ϱ is an appropriate constant non depending on $x|I$.

Proof. For a positive constant K such that $K > \max_{t\in I} \frac{\varphi''(t)}{\varphi(t)}$ and for arbitrary $\lambda \in (0,1)$ we consider the equation

$$x''(t) + \lambda \hat f\Big(t, x(t), x\big[\sigma(t)\big], x'(t), x'\big[g(t)\big]\Big) = (1-\lambda)Kx(t). \tag{3.12}$$

First of all we shall prove, by using Lemma 2.2, that there exists a constant M such that for every $\lambda \in (0,1)$ and every solution of (3.12) we have $|x'(t)| \leqslant M$, $t \in I$.

Indeed, let x be a solution of (3.12). Then, taking into account (3.10), we get

$$\begin{aligned}-\langle x(t), x''(t)\rangle &= \lambda\langle x(t), \hat{f}(t, x(t), x[\sigma(t)], x'(t), x[g(t)])\rangle - (1-\lambda)K|x(t)|^2 \\ &\leqslant \lambda\alpha|x'(g_\tau(t))|^2 + \lambda\beta \\ &< \alpha|x'(g_\tau(t))|^2 + \beta.\end{aligned}$$

Also, by (3.11), using the same argument we obtain

$$\begin{aligned}|\langle x'(t), x''(t)\rangle| &\leqslant (\alpha'|x'(g_\mu(t))|^2 + \gamma)|x'(g_\mu(t))| + \gamma'|x'(t)| + Kd|x'(t)| \\ &\leqslant (\alpha'|x'(g_\mu(t))|^2 + \gamma)|x'(g_\mu(t))| + \hat{\gamma}|x'(t)|\end{aligned}$$

with $\hat{\gamma} = \gamma' + Kd$.

Thus, by Lemma 2.2, there exists M such that

$$|x'(t)| \leqslant M, \ t \in I.$$

Now, we define operators T and A as in the proof of Theorem 3.1 (with $\hat{f}$ in the place of f) and we let Ω_1 be an open subset of B_1 given by

$$\Omega_1 = \{x \in B_1 : |x(t)| < \varphi(t) \text{ and } |x'(t)| < M+1, \ t \in I\}.$$

We observe that T is a compact operator defined on B_1 with values in B_1.

Next, for an arbitrary $\lambda \in (0,1)$ we suppose that x is a solution of the equation (3.4). Then, the following situation occurs:

The equation (3.12) has a solution x satisfying the boundary conditions (BC) and either there exists $\xi \in (a,b)$ such that the function $g(t) = |x(t)|^2 - \varphi^2(t)$ assumes its maximum value 0 at $t = \xi$ (since $\xi \neq a$ and $\xi \neq b$ by (3.1) and (3.2)) or there exists $\xi_1 \in [a,b]$ such that $|x'(\xi_1)| = M+1$. As we have proved in Theorem 3.1 the first of these two cases leads to a contradiction. But, since x is a solution of (3.12) for some $\lambda \in (0,1)$, the computation following (3.12) shows that $|x'(t)| \leqslant M$ and hence $|x'(t)| < M+1$ for every $t \in [a,b]$. Consequently, the second case cannot occur, either.

Hence no solutions of the equation (3.4) belong in $\partial\Omega_1$ and so, by Lemma 2.1, the equation $x = Tx$ has at least one solution in $\partial\overline{\Omega}_1$. Namely, there exists a solution x of the B.V.P. (E_2)–(BC) such that

$$|x(t)| \leqslant \varphi(t) \text{ and } |x'(t)| \leqslant \varrho, \ t \in I$$

with $\varrho = M+1$. Thus the proof of the theorem is complete. □

Remark 3.3. It is obvious from the proof of Theorem 3.1 that the conditions (1.2) on the constants α_i, β_i, $i = 0, 1$, are suggested because of the choice of the operator A. More precisely, the conditions (1.2) are such that the B.V.P. (*) which follows as an equivalent to the equation $z = Az$, $z \in C^2(I, \mathbb{R}^n)$, has the zero solution as its unique solution. Clearly, a different choice of the operator A implies a modification on these conditions.

4. Smooth solutions

The first derivatives of solutions of B.V.P. (E_i)–(BC), $i = 1, 2$ have in general discontinuities at the ends a and b of the interval I. This occurs because the equations (E_i), $i = 1, 2$ are equations with deviating arguments. If we have $x'(a-0) = x'(a+0)$ and $x'(b-0) = x'(b+0)$ (in addition to the obvious relations $x(a-0) = x(a+0)$ and $x(b-0) = x(b+0)$) then this solution x is called *a smooth solution* for the B.V.P. (E_i)–(BC), $i = 1, 2$, otherwise it is called a *non-smooth solution*. Usually, for boundary value problems involving equations with deviating arguments smoothness of solutions at the points a and b is not required. Therefore it is interesting to examine when a B.V.P. with deviating arguments has smooth solutions.

For a discussion concerning such problems we refer to our recent paper [6] and the references given therein.

In the following we give a result in this direction for the B.V.P. (E_i)–(BC), $i = 1, 2$. To this end it is necessary to introduce the following definition.

Definition 4.1. i) A function x is called a smooth solution of the B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) if $x \in C^1(J, \mathbb{R}^n) \cap C^2(I, \mathbb{R}^n)$ (resp. $x \in C^1(\hat{J}, \mathbb{R}^n)$ and x is piecewise twice differentiable on I) and satisfies the equation (E_1) (resp. (E_2)) for $t \in I$ and the boundary conditions (BC) for $t \in E(a) \cup E(b)$ (resp. $t \in \hat{E}(a) \cup \hat{E}(b)$).

ii) A function x is called a left-side smooth solution of B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) if

$$x \in C(J, \mathbb{R}^n) \cap C^1\left([a_0, b], \mathbb{R}^n\right) \cap C^1\left(E\ (a), \mathbb{R}^n\right) \cap C^2(I, \mathbb{R}^n)$$

(resp. $x \in C(\hat{J}, \mathbb{R}^n) \cap C^1\left([\hat{a}, b], \mathbb{R}^n\right) \cap C^1\left(\hat{E}(a), \mathbb{R}^n\right)$ and x is piecewise twice differentiable on I) and satisfies the equation (E_1) (resp. (E_2)) for $t \in I$ and the boundary conditions (BC) for $t \in E(a) \cup E(b)$ (resp. $t \in \hat{E}(a) \cup \hat{E}\ (b)$).

iii) A function x is called a right-side smooth solution of B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) if

$$x \in C(J, \mathbb{R}^n) \cap C^1\left(E(a), \mathbb{R}^n\right) \cap C^1\left([a, b_0], \mathbb{R}^n\right) \cap C^2(I, \mathbb{R}^n)$$

(resp. $x \in C(\hat{J}, \mathbb{R}^n) \cap C^1(\hat{E}(a), \mathbb{R}^n) \cap C^1([a, \hat{b}], \mathbb{R}^n)$ and x is piecewise twice differentiable on I) and satisfies the equation (E_1) (resp. (E_2)) for $t \in I$ and the boundary conditions (BC) for $t \in E(a) \cup E(b)$ (resp. $t \in \hat{E}(a) \cup \hat{E}(b)$).

In the sequel we consider the space $C^1(J, \mathbb{R}^n)$ (resp. $C^1(\hat{J}, \mathbb{R}^n)$) endowed with the norm

$$\|x\| = \max_{t \in J} |x(t)|$$

$$\Big(\text{resp. } \|x\| = \max\Big\{ \max_{t \in \hat{J}} |x(t)|, \max_{t \in \hat{J}} |x'(t)| \Big\}\Big).$$

The main result in this section is the following:

Theorem 4.2. *Assume that the hypotheses of Theorem 3.1 (resp. 3.2) are satisfied. Then, if $\alpha_1 \neq 0 \neq \beta_1$ the B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) has at least one smooth solution x such that*

$$|x(t)| \leqslant \varphi(t), \ t \in I$$

(resp. $|x(t)| \leqslant \varphi(t)$ and $|x'(t)| \leqslant \varrho$, $t \in I$, where ϱ is an appropriate constant not depending on $x|I$).

Proof. The proof can proceed along the established lines of reasoning of the proof of Theorem 3.1 (resp. 3.2). So, we omit the details. It is noteworthy that the restriction $\alpha_1 \neq 0 \neq \beta_1$ guarantees that

$$(Tx)'(a-0) = (Tx)'(a+0)$$

and

$$(Tx)'(b-0) = (Tx)'(b+0).$$

□

As an immediate consequence of the above theorem we have the following corollary, which concerns left or right-side smooth solutions.

Corollary 4.3. *Assume that the hypotheses of Theorem 3.1 (resp. 3.2) are satisfied. Then, if $\alpha_1 \neq 0$ the B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) has at least one left-side smooth solution satisfying the conclusion of Theorem 4.2. Similarly, if $\beta_1 \neq 0$ the B.V.P. (E_1)–(BC) (resp. (E_2)–(BC)) has at least one right-side smooth solution.*

Examples of B.V.P. which have smooth or non-smooth solutions were given in [6].

For a given B.V.P. of the form (E_i)–(BC) $i = 1, 2$, it is important to know about the existence of functions φ for which the B.V.P. has a solution x such that $|x(t)| \leqslant \varphi(t)$, $t \in I$. Much more, we are interested in more information about the properties of φ or about the formula for φ. Since the conditions on φ appearing in Theorems 3.1 and 3.2 are rather complicated, this can be done only for special cases of the equation (E_i), $i = 1, 2$.

Here we suppose that $h\colon I \to I$ is a so called (see [8]) *involution mapping*. That is, h is different from the identity mapping and such that

$$h\big(h(t)\big) = t, \ t \in I.$$

Now, we consider the vector linear equation

$$\text{(L)} \qquad x''(t) + p(t)x(t) + q(t)x\big(h(t)\big) + r(t)x'(t) + s(t) = 0, \ t \in I$$

where p, q and r are continuous real valued functions defined on I and $s\colon I \to \mathbb{R}^n$ is also a continuous function.

Since Range $(h) \subseteq I$, the boundary conditions (BC) yield the boundary conditions

$$\text{(bc)} \qquad \begin{aligned} \alpha_0 x(a) + \alpha_1 x'(a) &= \gamma_1, \\ \beta_0 x(b) + \beta_1 x'(b) &= \gamma_2 \end{aligned}$$

where α_i, β_i, $i = 0, 1$ are real constants satisfying the conditions (1.1), (1.2) and γ_1, γ_2 are constants in $\mathbb{R}^n$.

We set $P = \sup\limits_{t \in I} p(t)$, $Q = \sup\limits_{t \in I} q(t)$, $R = \sup\limits_{t \in I} r(t)$, $S = \sup\limits_{t \in I} |s(t)|$ and formulate the next proposition.

Proposition 5.1. *If there exist real constants m, n with $n \geqslant P$, $m \geqslant \max\{Q, R, S\}$, such that the inequality*

$$\text{(5.1)} \qquad \varphi''(t) + n\varphi(t) + m\Big(|\varphi'(t)| + \varphi\big(h(t)\big) + 1\Big) \leqslant 0$$

has a strictly positive solution φ such that

$$\text{(5.2)} \qquad \begin{aligned} -|\alpha_0|\varphi(a) - |\alpha_1|\varphi'(a) &> |\gamma_1|, \ \textit{if}\ \alpha_1 \neq 0, \\ |\alpha_0|\varphi(a) &> |\gamma_1|, \ \textit{if}\ \alpha_1 = 0 \end{aligned}$$

and

$$(5.3)\qquad -|\beta_0|\varphi(b)+|\beta_1|\varphi'(b)>|\gamma_2|,\ \text{if } \beta_1\neq 0,$$
$$|\beta_0|\varphi(b)>|\gamma_2|,\ \text{if } \beta_1=0$$

then the B.V.P. (L)–(bc) *has at least one solution* x *such that*

$$|x(t)|\leqslant\varphi(t),\ t\in I.$$

Moreover, there exists a real constant ϱ, *nondepending on* x, *such that*

$$|x'(t)|\leqslant\varrho,\ t\in I.$$

Proof. It is enough to check the conditions of Theorem 3.2 for the function

$$f(t,u,w,v)=p(t)u+q(t)w+r(t)v+s(t),\ (t,u,w,v)\in I\times\mathbb{R}^3.$$

Indeed, for every $x\in B_1$ with $|x(t)|=\varphi(t)$ and $\langle x(t),x'(t)\rangle=|x(t)|\varphi'(t)$, $t\in I$, we have

$$\begin{aligned}\langle x(t),f(t,x(t),x(h(t)),x'(t))\rangle &= p(t)|x(t)|^2+q(t)\langle x(t),x(h(t))\rangle\\ &\quad+r(t)\langle x(t),x'(t)\rangle+\langle x(t),s(t)\rangle\\ &\leqslant n|x(t)|^2+m|x(t)||x(h(t))|\\ &\quad+m|x(t)||\varphi'(t)|+m|x(t)|\\ &=n\varphi^2(t)+m\varphi(t)\varphi(h(t))+m\varphi(t)|\varphi'(t)|+m\varphi(t)\\ &=\varphi(t)\big[n\varphi(t)+m(\varphi(h(t))+|\varphi'(t)|+1)\big].\end{aligned}$$

This relation together with (5.1) implies condition (3.9).

Moreover, for every $(t,u,w,v)\in I\times\mathbb{R}^n$ with $|u|\leqslant\varphi(t)$ and $|w|\leqslant\varphi(h(t))$ we have

$$\begin{aligned}\langle u,f(t,u,w,v)\rangle &= p(t)u^2+q(t)\langle u,w\rangle+r(t)\langle u,v\rangle+\langle u,s(t)\rangle\\ &\leqslant P\varphi^2(t)+Q\varphi(t)\varphi(h(t))+R\varphi(t)|v|+S\varphi(t)\\ &\leqslant A+B|v|\end{aligned}$$

where $A=(P+Q)d^2+dS$ and $B=Rd$, $d=\sup\limits_{t\in I}\varphi(t)$.

Now, we observe that if $|v|\geqslant 1$, then we have

$$A+B|v|\leqslant A+B|v|^2$$

and hence the relation (3.10) is satisfied.

If $|v| < 1$, then, for every $B_1 \geqslant 0$, we have

$$A + B|v| = A + B_1|v|^2 + B|v| - B_1|v|^2 \leqslant A + B + B_1|v|^2.$$

Hence the relation (3.10) is satisfied in any case.

From the relation (3.11) we have

$$\begin{aligned}\big|\langle v, f(t,u,w,v)\rangle\big| &= |p(t)|\big|\langle v,u\rangle\big| + |q(t)|\big|\langle v,w\rangle\big| + |r(t)||v|^2 + \big|\langle v, s(t)\rangle\big| \\ &\leqslant |P|d|v| + |Q|d|v| + |R||v|^2 + S|v| \\ &\leqslant \big(|P|d + |Q|d + S\big)|v| + |R||v|^2.\end{aligned}$$

We again consider two cases.

If $|v| \geqslant 1$ then, obviously,

$$\big|\langle v, f(t,u,w,v)\rangle\big| \leqslant \big(|P|d + |Q|d + S\big)|v| + |R||v|^3,$$

i.e. we take (3.11).

If $|v| < 1$, we get

$$\begin{aligned}\big|\langle v, f(t,u,w,v)\rangle\big| &\leqslant C_1|v| + |R||v|^2 \\ &= C_1|v| + |R||v|^2 + N|v|^3 - N|v|^3 \\ &\leqslant \big(C_1 + |R| + N|v|^2\big)|v|\end{aligned}$$

for every $N \geqslant 0$, where $C_1 = |p|d + |Q|d + S$. Hence, we have again (3.11).

We can assume that the conditions $\alpha < 1$ and $\alpha' < \frac{1}{8d}(1-\alpha)^2$ appearing in Theorem 3.2 are fulfilled for an appropriate choice of the constants which are involved in the expressions for α and α'.

Thus, the proof of the proposition is complete. □

Example 5.2. We give an example of a B.V.P. which involves a differential equation with reflection of the arguments, which is a particular case of a functional differential equation whose arguments are involutions. Such equations have applications in the study of differential-difference equations. B.V.P. for such equations were studied for the first time by Wiener and Aftabizadeh in [10].

More precisely, we consider the B.V.P.

$$(\mathrm{L_r}) \qquad x''(t) + p(t)x(t) + q(t)x(-t) + r(t)x'(t) + s(t) = 0,\ t \in [-1,1],$$

$$(\mathrm{bc})_\mathrm{r} \qquad \begin{aligned}\alpha_0 x(-1) + \alpha_1 x'(-1) &= \gamma_1,\\ \beta_0 x(1) + \beta_1 x'(1) &= \gamma_2\end{aligned}$$

where the functions p, q, r and s are as in equation (L) and such that

$$(*) \qquad 2n + 5m + 2 \leqslant 0.$$

In order to apply Proposition 5.1 we must prove that inequality (5.1) has a strictly positive solution satisfying (5.2) and (5.3). It is easy to check that the function $\varphi(t) = t^2 + 1$, $t \in [-1, 1]$ is a solution of the inequality (5.1) (with $h(t) = -t$) because of $(*)$. Thus, if we assume that the constants α_0, α_1, β_0, β_1 are such that

$$-2|\alpha_0| + 2|\alpha_1| > |\gamma_1| \text{ if } \alpha_1 \neq 0,$$
$$2|\alpha_0| > |\gamma_1| \text{ if } \alpha_1 = 0$$

and

$$-2|\beta_0| + 2|\beta_1| > |\gamma_2| \text{ if } \beta_1 \neq 0,$$
$$2|\beta_0| > |\gamma_2| \text{ if } \beta_1 = 0$$

then the B.V.P. $(\mathrm{L_r})$–$(\mathrm{bc})_\mathrm{r}$ has at least one solution x such that

$$|x(t)| \leqslant \varphi(t) = t^2 + 1, \ t \in [-1, 1].$$

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Authors' address: University of Ioannina, Department of Mathematics, Ioannina, Greece.