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AN OPERATOR RELATION OF THE USSOR AND THE JACOBI ITERATION MATRICES OF A p-CYCLIC MATRIX *

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Abstract. Let the Jacobi matrix B associated with the linear system $Ax = b$ be a weakly cyclic matrix, generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ as this is defined by Li and Varga. The same authors derived the corresponding functional equation connecting the eigenvalues λ of the unsymmetric successive overrelaxation (USSOR) iteration matrix $T_{\omega\hat{\omega}}$ and the eigenvalues μ of the Jacobi matrix B extending previous results by Gong and Cai. In this paper, the validity of an analogous matrix relationship connecting the operators $T_{\omega\hat{\omega}}$ and B is proved. Moreover, the "equivalence" of the USSOR method and a certain two-parametric p-step method for the solution of the initial system is established. The tool for the proof of our main result is elementary graph theory.

Key words. USSOR method, p-cyclic matrices, graph theory, matrix relationship

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1. Introduction. Let us consider the matrix $A \in \mathbb{C}^{n,n}$ and let us suppose that it is partitioned into $p \times p$ blocks where its diagonal blocks are square and nonsingular. For the solution of the linear system

$$Ax = b, \tag{1.1}$$

we consider the unsymmetric successive overrelaxation (USSOR) iterative method

$$x^{(m+1)} = T_{\omega\hat{\omega}}x^{(m)} + c, \qquad m = 0, 1, 2, \ldots, \tag{1.2}$$

where $x^{(0)} \in \mathbb{C}^n$ is arbitrary, and ω and $\hat{\omega}$ are the overrelaxation parameters. The iteration matrix $T_{\omega\hat{\omega}}$ is given by

$$T_{\omega\hat{\omega}} = (I - \hat{\omega}U)^{-1}[(1 - \hat{\omega})I + \hat{\omega}L](I - \omega L)^{-1}[(1 - \omega)I + \omega U], \tag{1.3}$$

where L and U are, respectively, the strictly lower and the strictly upper block triangular parts of the block Jacobi matrix B and the vector c is given by

$$c = (\omega + \hat{\omega} - \omega\hat{\omega})(I - \hat{\omega}U)^{-1}(I - \omega L)^{-1}b. \tag{1.4}$$

Let the associated block Jacobi matrix B be a weakly cyclic matrix generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$. This definition given by Li and Varga [9] is as follows.

DEFINITION. *The $p \times p$ block matrix B is a weakly cyclic matrix, generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, if there exists a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ of the integers $\{1, 2, \ldots, p\}$ such that*

$$B_{\sigma_j\sigma_{j+1}} \not\equiv 0,\ j = 1(1)p,\ \text{and } B_{ij} \equiv 0 \text{ otherwise}, \tag{1.5}$$

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where $\sigma_{p+1} = \sigma_1$.

We remark here that the well-known definition for the consistently ordered matrix ([16] and [21]) is derived from the one above with $\sigma = (p, p-1, p-2, \ldots, 1)$, while that of the $(q, p-q)$-generalized consistently ordered $(q, p-q)$-GCO matrix ([2], [7], and [4]) is derived from the permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, where $\sigma_{j+1} = p - q + \sigma_j$ or $\sigma_{j+1} = \sigma_j - q$ such that $1 \leq \sigma_j \leq p, j = 1(1)p$. So, the definition (1.5) is the most general for the family of p-cyclic matrices. It is obvious that the graph of the block matrix B is a cycle as this is also noted in [9].

Li and Varga [9] derived the functional equation

$$
\begin{aligned}
&[\lambda - (1-\omega)(1-\hat{\omega})]^p \\
&\quad = (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} \lambda^k [\lambda\omega + \hat{\omega} - \omega\hat{\omega}]^{|\zeta_L| - k} [\lambda\hat{\omega} + \omega - \omega\hat{\omega}]^{|\zeta_U| - k} \mu^p,
\end{aligned} \tag{1.6}
$$

which couples the nonzero eigenvalues λ of the USSOR iteration matrix $T_{\omega\hat{\omega}}$ with the eigenvalues μ of the Jacobi matrix B. In (1.6) $|\zeta_L|$ and $|\zeta_U|$ are the cardinalities of the sets ζ_L and ζ_U, which are the two disjoint subsets of $P \equiv \{1, 2, \ldots, p\}$ associated with the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$ as these are defined in [9], i.e.,

$$
\zeta_L = \{\sigma_j : \sigma_j > \sigma_{j+1}\}, \qquad \zeta_U = \{\sigma_j : \sigma_j < \sigma_{j+1}\}. \tag{1.7}
$$

The integer k is well defined in [9] as is the number of nonzero block elements of the matrix product LU. Li and Varga gave also the directed graph interpretation of the number k. It is obvious that $\zeta_L \cup \zeta_U = \{1, 2, \ldots, p\}$ and $\zeta_L \cap \zeta_U = \varnothing$, consequently, $|\zeta_L| + |\zeta_U| = p$. In other words $|\zeta_L|$ and $|\zeta_U|$ are the numbers of the nonzero block elements of the matrices L and U, respectively.

Equation (1.6) generalizes the following previous works: (i) The results of Saridakis [12] on the USSOR iteration matrix for consistently ordered weakly p-cyclic matrices; (ii) the ones of Gong and Cai [5] and of Varga, Niethammer, and Cai [17] on the SSOR iteration matrix for p-cyclic matrices; (iii) the well-known results of Young [19, 21] on the SOR matrix for the two-cyclic case; (iv) the well-known results of Varga [15, 16] on the SOR iteration matrix for the consistently ordered weakly p-cyclic Jacobi matrix; and (v) the results of Verner and Bernal [18] on the SOR matrix for the $(q, p-q)$-GCO case. It should be noted that the result in the last case was mentioned for the first time by Varga in [16]. Finally, a relationship similar in character on the modified SOR (MSOR) matrix for the $(q, p-q)$-GCO case, was derived by Taylor [14].

Our main objective in this work is to derive the matrix analogue of the functional equation (1.6). More specifically, we show that the identity

(1.8)

$$
\begin{aligned}
&[T_{\omega\hat{\omega}} - (1-\omega)(1-\hat{\omega})I]^p \\
&\quad = (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} T_{\omega\hat{\omega}}^k [\omega T_{\omega\hat{\omega}} + (\hat{\omega} - \omega\hat{\omega})I]^{|\zeta_L| - k} [\hat{\omega} T_{\omega\hat{\omega}} + (\omega - \omega\hat{\omega})I]^{|\zeta_U| - k} B^p
\end{aligned}
$$

always holds.

It is interesting to mention that the matrix analogues of the functional equations of cases (ii)–(v) were derived by Galanis, Hadjidimos, and Noutsos (see [1–3]), by using elementary graph theory (Harary [8], Varga [16]). The matrix analogue of the equation corresponding to the MSOR case was derived by Young and Kincaid [20] for the special case $(p, q) = (2, 1)$, by Hadjidimos and Yeyios [6] for the cases $(p, q) = (3, 1), (3, 2)$ by the straightforward analytic calculations and by Hadjidimos and Noutsos [7] for all values of p and q, by elementary graph theory.

The proof of (1.8) is given in §2. As will be seen, the main tool will be combinatorics and to guide intuition elementary graph theory will be used. Also by considering special cases of (1.8) with $\hat{\omega} = 0$ or $\omega = 0$, known, other results for the SOR as well as the backward SOR methods will be obtained. In §3 the "equivalence" of the USSOR method and a certain two-parametric p-step method in the sense of Niethammer and Varga [10] is established. Apart from the theoretical interest presented by the identity (1.8), it is also of practical importance, since the problem of determination of "good" or "optimal" parameters ω and $\hat{\omega}$ for the solution of the linear system (1.1), using the USSOR method, is equivalent to that of the determination of the same parameters of a two-parametric p-step iterative method. This problem, however, still remains an open one.

2. Main result and preliminary analysis. The statement of our main result is given in the following theorem.

THEOREM 2.1. *Let B be the weakly cyclic block Jacobi matrix, generated by the cyclic permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_p)$, and $T_{\omega\hat{\omega}}$ in* (1.3) *be the block* USSOR *iteration matrix associated with A in* (1.1). *Then the matrix relationship* (1.8) *holds.*

The proof of Theorem 2.1 will be given later, where a number of other auxiliary statements will be stated and proved. First, the background material on which these proofs are based is developed.

It is noted that (1.8) trivially holds if $\omega = \hat{\omega} = 0$. So we assume that $\omega \neq 0$ and $\hat{\omega} \neq 0$. We will see that this assumption can be made without any loss of generality.

To simplify the proof of Theorem 2.1, we will prove the validity of another simpler relationship which is produced from (1.8) by setting

$$\tilde{T}_{\omega\hat{\omega}} = (I - \hat{\omega}U)T_{\omega\hat{\omega}}(I - \hat{\omega}U)^{-1} \tag{2.1}$$

in the place of $T_{\omega\hat{\omega}}$. We then begin our analysis by introducing the directed graphs.

The directed graph G is a pair (V, E) where $E \subseteq V \times V$ (see [16] or [8]). In our analysis the vertex set $V \equiv P$, following [13] or [7], we identify G with the edge set E. Also for a block partitioned matrix A, the graph of A is defined to be $G(A) = \{(i, j) : A_{ij} \neq 0\}$. So the directed graph $G(B)$ of the Jacobi matrix B will be

$$G(B) = \bigcup_{i=1}^{p} \{(\sigma_i, \sigma_{i+1})\}, \tag{2.2}$$

where $\sigma_{p+1} = \sigma_1$. (In the sequel the node σ_{p+1} will be denoted as σ_1.)

An example for $p = 5$ is given now to demonstrate the analysis. Let

$$B = \begin{pmatrix} 0 & 0 & 0 & B_{14} & 0 \\ 0 & 0 & 0 & 0 & B_{25} \\ 0 & B_{32} & 0 & 0 & 0 \\ 0 & 0 & B_{43} & 0 & 0 \\ B_{51} & 0 & 0 & 0 & 0 \end{pmatrix}$$

be the Jacobi matrix. From the definition we have $\sigma = (2, 5, 1, 4, 3), \zeta_L = \{5, 4, 3\}$, and $\zeta_U = \{2, 1\}$. The graph $G(B)$ is shown in Fig. 1.

From Fig. 1 it is easily seen that $G(B)$ is a cyclic graph. It is also noted that (i) there are exactly k paths which go from a node of ζ_L to a node of ζ_U corresponding to the nonzero blocks of LU. We call these paths "backward" paths. (ii) There are exactly k paths which go from a node of ζ_U to a node of ζ_L corresponding to the

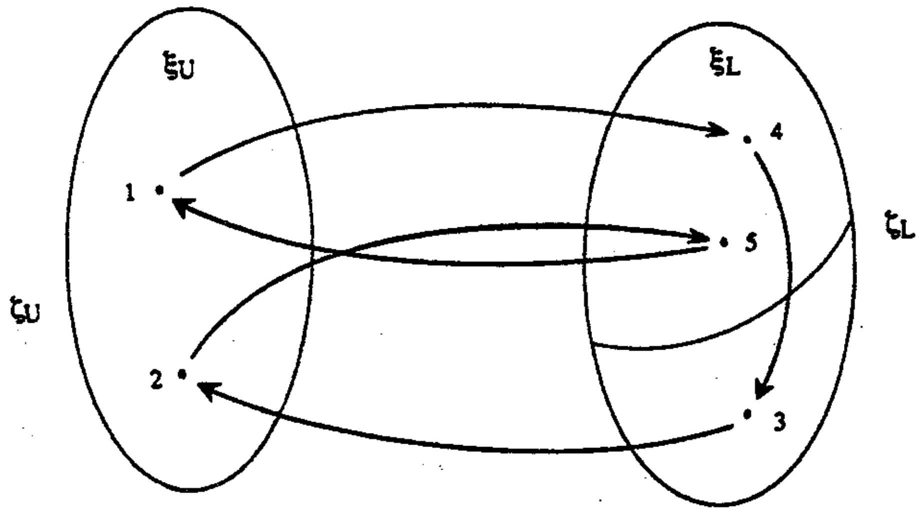

FIG. 1.

nonzero blocks of UL. We call these paths "forward" paths. In our example, $k = 2$ corresponds to the two backward paths $(3, 2)$ and $(5, 1)$ or to the two forward paths $(2, 5)$ and $(1, 4)$.

To derive the graph of the matrix B^p we can observe that starting from the node σ_i, we return to σ_i after p paths of B passing through all the nodes $\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{i-1}$. So B^p is a block diagonal matrix which comes from a sum of products of powers of L's and U's. Each product contains totally a number of $|\zeta_L|, L$'s and $|\zeta_U|, U$'s.

The graph $G(\tilde{T}_{\omega\hat{\omega}})$ of the matrix $\tilde{T}_{\omega\hat{\omega}}$ is now studied. From (1.3) and (2.1) we get that

$$\tilde{T}_{\omega\hat{\omega}} = [(1-\hat{\omega})I + \hat{\omega}L](I - \omega L)^{-1}[(1-\omega)I + \omega U](I - \hat{\omega}U)^{-1}. \tag{2.3}$$

Obviously the following relations hold:

$$(I - \omega L)^{-1} = \sum_{i=0}^{q_L} (\omega L)^i \text{ and } (I - \hat{\omega}U)^{-1} = \sum_{i=0}^{q_U} (\hat{\omega}U)^i, \tag{2.4}$$

where q_L and q_U are the largest integers such that $L^{q_L} \neq 0$ and $U^{q_U} \neq 0$, respectively, (in the above example, $q_L = 2$ and $q_U = 1$). By substituting (2.4) in (2.3) and after simple operations, we obtain that

$$\begin{aligned} \tilde{T}_{\omega\hat{\omega}} &= (1-\omega)(1-\hat{\omega})I + (\omega + \hat{\omega} - \omega\hat{\omega}) \\ &\times \left\{ (1-\omega)\sum_{i=1}^{q_L} \omega^{i-1}L^i + (1-\hat{\omega})\sum_{i=1}^{q_U} \hat{\omega}^{i-1}U^i \right. \\ &\left. + (\omega + \hat{\omega} - \omega\hat{\omega}) \sum_{i=1}^{q_L}\sum_{j=1}^{q_U} \omega^{i-1}L^i\hat{\omega}^{j-1}U^j \right\}. \end{aligned} \tag{2.5}$$

It is noted that ωL and $\hat{\omega}U$ are of exactly the same form as L and U. So ωL and $\hat{\omega}U$ will be denoted from now on by L and U. Thus, after this convention (2.5) can be written as

$$\begin{aligned} \tilde{T}_{\omega\hat{\omega}} &= (1-\omega)(1-\hat{\omega})I + (\omega + \hat{\omega} - \omega\hat{\omega}) \\ &\times \left\{ \frac{1-\omega}{\omega}\sum_{i=1}^{q_L} L^i + \frac{1-\hat{\omega}}{\hat{\omega}}\sum_{i=1}^{q_U} U^i + \frac{\omega + \hat{\omega} - \omega\hat{\omega}}{\omega\hat{\omega}} \sum_{i=1}^{q_L}\sum_{j=1}^{q_U} L^iU^j \right\}. \end{aligned} \tag{2.6}$$

Since in relation (2.6) we have different scalar coefficients for the matrices I, L^i, U^i, and L^iU^j, we introduce the weighted graph of $\tilde{T}_{\omega\hat{\omega}}$. Thus we define (i) the paths weighted by $(\omega+\hat{\omega}-\omega\hat{\omega})\ \frac{1-\omega}{\omega}$ as single-arrowed paths; (ii) the paths weighted by $(\omega+\hat{\omega}-\omega\hat{\omega})\ \frac{1-\hat{\omega}}{\hat{\omega}}$ as double-arrowed paths; (iii) the paths weighted by $\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}$ as triple-arrowed paths; and (iv) the paths weighted by $(1-\omega)(1-\hat{\omega})$ as four-arrowed paths. So from the right-hand side of (2.6) we have the following. The first term of (2.6) gives the four-arrowed identity paths

$$(\overrightarrow{\overrightarrow{\overrightarrow{\overrightarrow{\sigma_i, \quad \sigma_i}}}}), \quad i = 1(1)p. \tag{2.7}$$

The second term, which contains a sum of powers of L, gives the single-arrowed paths

$$(\overrightarrow{\sigma_i, \sigma_{i+j}}), \quad j = 1(1)q_{L,i}, \quad \sigma_i \in \zeta_L, \tag{2.8}$$

where $q_{L,i}$ is an integer such that all the successive nodes $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+q_{L,i}-1}$ belong to ζ_L and $\sigma_{i+q_{L,i}} \in \zeta_U$. (From Fig. 1 we can see that if $\sigma_i = 5$ or $\sigma_i = 3$ then $q_{L,i} = 1$ while if $\sigma_i = 4$ then $q_{L,i} = 2$.) The third term, which contains a sum of powers of U, gives the double-arrowed paths

$$(\overrightarrow{\overrightarrow{\sigma_i, \sigma_{i+j}}}), \quad j = 1(1)q_{U,i}, \quad \sigma_i \in \zeta_U, \tag{2.9}$$

where $q_{U,i}$ is an integer such that all the successive nodes $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+q_{U,i}-1}$ belong to ζ_U and $\sigma_{i+q_{U,i}} \in \zeta_L$. (Figure 1 gives that $q_{U,i} = 1$ for both cases $\sigma_i = 1$ or 2.) Finally the last term, which contains a double sum of products of powers of L and U, gives the triple-arrowed paths

$$(\overrightarrow{\overrightarrow{\overrightarrow{\sigma_i, \sigma_{i+j}}}}), \quad j = q_{L,i} + 1(1)q_{LU,i}, \quad \sigma_i \in \zeta_L, \tag{2.10}$$

where $q_{LU,i} = q_{L,i} + q_{U,i+q_{L,i}}$ (in our example $q_{LU,i} = 3$ for $\sigma_i = 4$, which corresponds to the three successive paths $(4,3), (3,2)$, and $(2,5)$). It is noted here that $\sigma_s := \sigma_{s-p}$ if $s > p$ in (2.8), (2.9), and (2.10). The union of all the paths in (2.7), (2.8), (2.9), and (2.10) gives the graph of $\tilde{T}_{\omega\hat{\omega}}$.

$$\begin{aligned} G(\tilde{T}_{\omega\hat{\omega}}) = {} & \left(\bigcup_{i=1}^{p} \{(\overrightarrow{\overrightarrow{\overrightarrow{\overrightarrow{\sigma_i, \sigma_i}}}})\} \right) \\ & \cup \left(\bigcup_{\sigma_i \in \zeta_L} \left[\bigcup_{j=1}^{q_{L,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \bigcup_{j=q_{L,i}+1}^{q_{LU,i}} \{(\overrightarrow{\overrightarrow{\overrightarrow{\sigma_i, \sigma_{i+j}}}})\} \right] \right) \\ & \cup \left(\bigcup_{\sigma_i \in \zeta_U} \left[\bigcup_{j=1}^{q_{U,i}} \{(\overrightarrow{\overrightarrow{\sigma_i, \sigma_{i+j}}})\} \right] \right). \end{aligned} \tag{2.11}$$

The subgraphs of $G(\tilde{T}_{\omega\hat{\omega}})$ of our example that contain only the paths that have the origin node $4 \in \zeta_L$ or $1 \in \zeta_U$ are illustrated in Fig. 2(a) and Fig. 2(b), respectively.

We distinguish the subset ξ_L of ζ_L which contains the nodes σ_j such that $\sigma_{j-1} \in \zeta_U$ and the subset ξ_U of ζ_U which contains the nodes σ_j such that $\sigma_{j-1} \in \zeta_L$. It is easily seen from Fig. 1 that both ξ_L and ξ_U contain exactly k nodes. In our example we have $\xi_L = \{4, 5\}$ and $\xi_U = \{1, 2\}$.

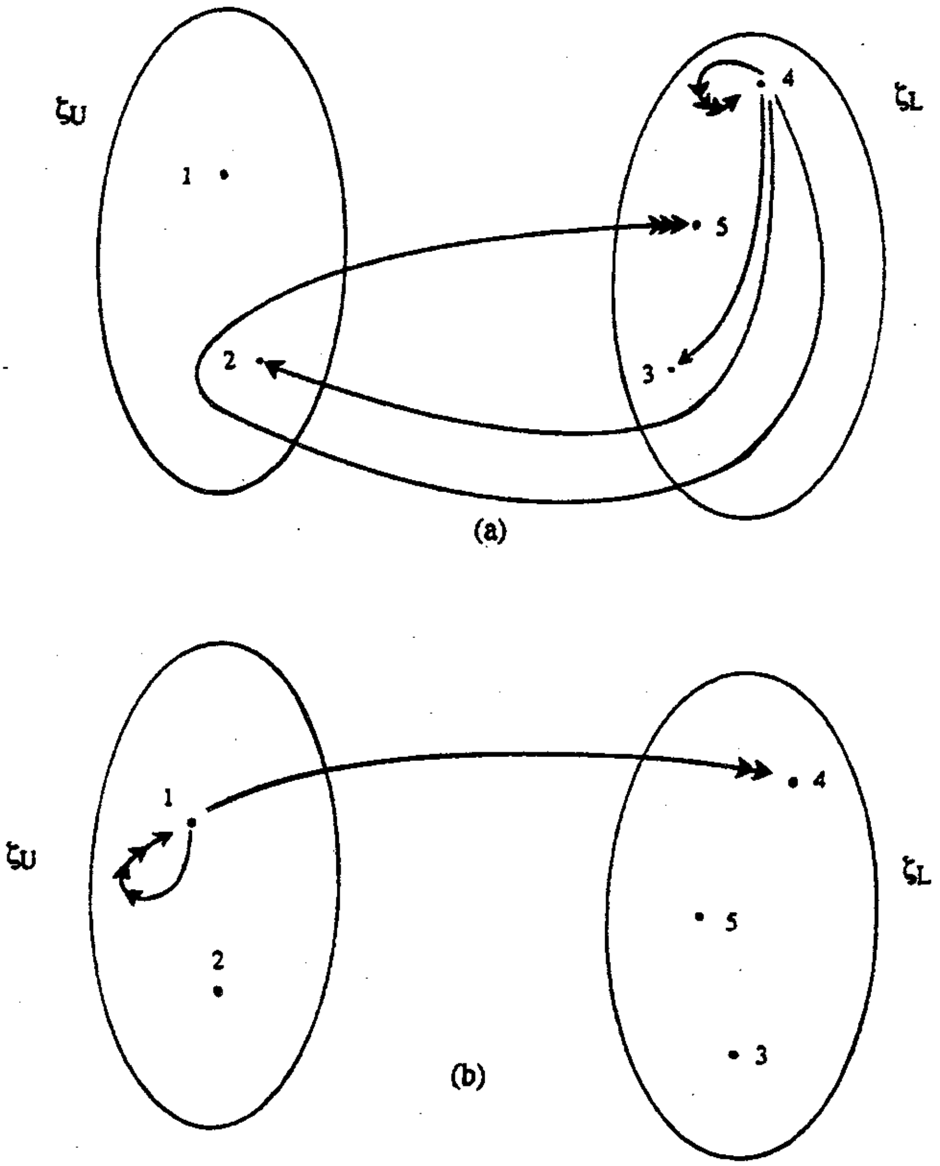

FIG. 2.

After replacing ωL and $\hat{\omega} U$ by L and U, the matrix relationship (1.8) will be equivalent to

(2.12)

$$[\tilde{T}_{\omega\hat{\omega}} - (1-\omega)(1-\hat{\omega})I]^p = \left[\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^k \tilde{T}^k_{\omega\hat{\omega}} \left[\tilde{T}_{\omega\hat{\omega}} + \frac{\hat{\omega}(1-\omega)}{\omega} I\right]^{|\zeta_L|-k} \times \left[\tilde{T}_{\omega\hat{\omega}} + \frac{\omega(1-\hat{\omega})}{\hat{\omega}} I\right]^{|\zeta_U|-k} B^p.$$

It is noted that in (2.12) we have put B^p for $\omega^{|\zeta_L|}\hat{\omega}^{|\zeta_U|}B^p$, since B^p constitutes the sum of products of $|\zeta_L|, L$'s and $|\zeta_U|, U$'s.

From (2.6) we can see that the graph of the matrix $\tilde{T}_{\omega\hat{\omega}} - (1-\omega)(1-\hat{\omega})I$ contains no identity paths. So, from (2.11), we have

(2.13)

$$G(\tilde{T}_{\omega\hat{\omega}} - (1-\omega)(1-\hat{\omega})I) = \left(\bigcup_{\sigma_i \in \zeta_L} \left[\bigcup_{j=1}^{q_{L,i}} \{(\sigma_i, \overrightarrow{\sigma_{i+j}})\} \bigcup_{j=q_{L,i}+1}^{q_{LU,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right) \cup \left(\bigcup_{\sigma_i \in \zeta_U} \left[\bigcup_{j=1}^{q_{U,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right).$$

The graph in (2.13) is derived from $G(\tilde{T}_{\omega\hat{\omega}})$ by simply omitting the identity paths.

It is easily checked from (2.6) that the matrix $\tilde{T}_{\omega\hat{\omega}} + (\hat{\omega}(1-\omega)/\omega)I$ is given by

(2.14)

$$\tilde{T}_{\omega\hat{\omega}} + \frac{\hat{\omega}(1-\omega)}{\omega} I = (\omega + \hat{\omega} - \omega\hat{\omega}) \times \left\{ \frac{1-\omega}{\omega} \sum_{i=0}^{q_L} L^i + \frac{1-\hat{\omega}}{\hat{\omega}} \sum_{i=1}^{q_U} U^i + \frac{\omega + \hat{\omega} - \omega\hat{\omega}}{\omega\hat{\omega}} \sum_{i=1}^{q_L} \sum_{j=1}^{q_U} L^i U^j \right\}.$$

So the identity paths now become single-arrowed paths and the graph of the matrix is given by

$$G\left(\tilde{T}_{\omega\hat{\omega}} + \frac{\hat{\omega}(1-\omega)}{\omega} I\right) = \left(\bigcup_{i=1}^{p} \{(\sigma_i, \overrightarrow{\sigma_i})\} \right) \cup \left(\bigcup_{\sigma_i \in \zeta_L} \left[\bigcup_{j=1}^{q_{L,i}} \{(\sigma_i, \overrightarrow{\sigma_{i+j}})\} \bigcup_{j=q_{L,i}+1}^{q_{LU,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right) \cup \left(\bigcup_{\sigma_i \in \zeta_U} \left[\bigcup_{j=1}^{q_{U,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right). \tag{2.15}$$

This graph is derived from $G(\tilde{T}_{\omega\hat{\omega}})$ by simply replacing the four-arrowed identity paths with single-arrowed paths. Similarly, the matrix $\tilde{T}_{\omega\hat{\omega}} + (\omega(1-\hat{\omega})/\hat{\omega})I$ is given by

(2.16)

$$\tilde{T}_{\omega\hat{\omega}} + \frac{\omega(1-\hat{\omega})}{\hat{\omega}} I = (\omega + \hat{\omega} - \omega\hat{\omega}) \times \left\{ \frac{1-\omega}{\omega} \sum_{i=1}^{q_L} L^i + \frac{1-\hat{\omega}}{\hat{\omega}} \sum_{i=0}^{q_U} U^i + \frac{\omega + \hat{\omega} - \omega\hat{\omega}}{\omega\hat{\omega}} \sum_{i=1}^{q_L} \sum_{j=1}^{q_U} L^i U^j \right\}$$

and its graph by

$$G\left(\tilde{T}_{\omega\hat{\omega}} + \frac{\omega(1-\hat{\omega})}{\hat{\omega}} I\right) = \left(\bigcup_{i=1}^{p} \{(\overrightarrow{\sigma_i, \sigma_i})\} \right) \cup \left(\bigcup_{\sigma_i \in \zeta_L} \left[\bigcup_{j=1}^{q_{L,i}} \{(\sigma_i, \overrightarrow{\sigma_{i+j}})\} \bigcup_{j=q_{L,i}+1}^{q_{LU,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right) \cup \left(\bigcup_{\sigma_i \in \zeta_U} \left[\bigcup_{j=1}^{q_{U,i}} \{(\overrightarrow{\sigma_i, \sigma_{i+j}})\} \right] \right), \tag{2.17}$$

which is derived from $G(\tilde{T}_{\omega\hat{\omega}})$ by simply replacing the four-arrowed identity paths with double-arrowed identity paths.

A lemma is now stated and proved that shows the equivalence of (1.8) and (2.12).

LEMMA 2.2. *If the matrix relationship* (2.12) *holds then so does* (1.8) *and vice versa.*

Proof. We prove the validity of the matrix relationship (2.12) from that of (1.8) by replacing at the same time the ωL's and $\hat{\omega}U$'s by L's and U's, respectively.

By taking the inverse similarity transformation of (2.1) on both sides of (2.12), we have

$$\begin{aligned}(I-U)^{-1}&[\tilde{T}_{\omega\hat{\omega}}-(1-\omega)(1-\hat{\omega})I]^p(I-U)\\ &=(I-U)^{-1}\left[\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^k\tilde{T}^k_{\omega\hat{\omega}}\left[\tilde{T}_{\omega\hat{\omega}}+\frac{\hat{\omega}(1-\omega)}{\omega}\,I\right]^{|\zeta_L|-k}\\ &\quad\times\left[\tilde{T}_{\omega\hat{\omega}}+\frac{\omega(1-\hat{\omega})}{\hat{\omega}}\,I\right]^{|\zeta_U|-k}B^p(I-U)\end{aligned}$$

or from (2.1)

$$\begin{aligned}[T_{\omega\hat{\omega}}-(1-\omega)(1-\hat{\omega})I]^p&=\left[\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^k T^k_{\omega\hat{\omega}}\left[T_{\omega\hat{\omega}}+\frac{\hat{\omega}(1-\omega)}{\omega}\,I\right]^{|\zeta_L|-k}\\ &\quad\times\left[T_{\omega\hat{\omega}}+\frac{\omega(1-\hat{\omega})}{\hat{\omega}}\,I\right]^{|\zeta_U|-k}(I-U)^{-1}B^p(I-U).\end{aligned}\tag{2.18}$$

For (2.18) to hold it must be proved that $(I-U)^{-1}B^p(I-U)=B^p$ or $B^p(I-U)=(I-U)B^p$ or simply that

$$B^pU=UB^p.\tag{2.19}$$

The proof of (2.19) is given by elementary graph theory. Since the graph $G(B^p)$ contains the identity paths $(\sigma_i,\sigma_i), i=1(1)p$ which constitute p successive simple paths of $G(B)$ and the graph $G(U)$ contains the simple paths $(\sigma_i,\sigma_{i+1}), \sigma_i\in\zeta_U$, the graph $G(B^pU)$ contains the paths $(\sigma_i,\sigma_{i+1}), \sigma_i\in\zeta_U$ which constitute $p+1$ successive simple paths of $G(B)$. Similarly, the graph $G(UB^p)$ contains the same paths. So these two graphs describe the graphs of the same matrices and the proof is complete. Moreover, it is noted here that an analogous proof gives that the matrices B^p and L also commute. □

Now we have all the necessary tools to prove our main theorem.

Proof of Theorem 2.1. Let C and D be the matrices denoting the left- and right-hand sides of (2.12), respectively. The proof is due to the following simple idea: Since C and D have been expanded in sums of terms of products of L's and U's, we must prove that if there exists a term of the expansion of C then there exists also such a term of the expansion of D with the same coefficient and vice versa. This means, in graph analogue, that if there exists a path (σ_i,σ_j) of $G(C)$ then there exists also such a path of $G(D)$ weighted with the same weight, for all the pairs σ_i,σ_j and vice versa. Each of these paths consists of consecutive subpaths and represents the graph of a nonidentically zero block of the term in question. Our objective will be accomplished if we show that all paths in $G(C)$ and $G(D)$ from σ_i to σ_j with m backward subpaths $(0\le m\le p)$ coincide and are associated with equal overall weights. It is obvious that any two paths (σ_i,σ_j) of $G(C)$ and (σ_i,σ_j) of $G(D)$ with a particular number m

of backward edges correspond to the same expansion in terms of nonidentically zero products of L's and U's. They differ from each other only because of the different weights of the single-, double-, triple-, or four-arrowed subpaths as they are described above. For example, let $\sigma_i = 4$ and $\sigma_j = 5$. Then Figs. 1 and 2 give that the path $(4, 5)$ of $G(C)$ constitutes paths associated with three different numbers m of backward edges. $m = 1$ corresponds to the matrix product $LLU, m = 3$ corresponds to $LLULULLU$, and $m = 5$ corresponds to $LLULULLULULLU$. The union of all the above paths from σ_i to σ_j with m backward edges will be considered as one path, with which an overall weight will be associated. This overall weight will be equal to the sum of all the weights associated with each individual path. The determination of this weight constitutes the basic key to the proof of our main result.

We try to find the overall weight of $G(C)$ with $k + m$ backward subpaths $(0 \leq m \leq p - k)$. (The number of $k + m$ backward subpaths is taken since the smallest number of backward subpaths of the matrix $C = [\tilde{T}_{\omega\hat{\omega}} - (1 - \omega)(1 - \hat{\omega})I]^p$ is k. This is obtained by considering the path of the smallest possible way, which contains p consecutive subpaths of the form (σ_l, σ_{l+1}) with their weights.) From the graph expression (2.13) of the matrix $\tilde{T}_{\omega\hat{\omega}} - (1 - \omega)(1 - \hat{\omega})I$, from Fig. 2, and from elementary graph theory we can see that this path consists of the union of all possible combinations of p consecutive subpaths of $G(\tilde{T}_{\omega\hat{\omega}} - (1 - \omega)(1 - \hat{\omega})I)$ that go from σ_i to σ_j with $k + m$ backward edges. This remark leads us to the conclusion that to analyze and study the problem at hand, the use of combinatorics theory together with elementary graph theory must be made.

The analysis requires that we distinguish four cases

(i) $\sigma_i, \sigma_j \in \zeta_L$,

(ii) $\sigma_i, \sigma_j \in \zeta_U$,

(iii) $\sigma_i \in \zeta_L$ and $\sigma_j \in \zeta_U$,

(iv) $\sigma_i \in \zeta_U$ and $\sigma_j \in \zeta_L$.

Since the argumentation is quite similar in all the four cases, only the first case is presented in detail. The others can be found in [11].

From Fig. 2(a) we see that there are two types of backward edges: the single-arrowed path with ending node belonging to ξ_U (see path $(\overrightarrow{4, 2})$) and the triple-arrowed path with ending node belonging to $(\zeta_U \backslash \xi_U) \cup \xi_L$ (see path $(\overrightarrow{\overrightarrow{\overrightarrow{4, 5}}})$). If we take r ending nodes of the first type and $k + m - r$ of the second type we have $\binom{k+m}{r}$ cases to consider. Then let t be the number of consecutive nodes in the way from σ_i to σ_j with $k + m$ backward edges, with σ_j being included and t_L and t_U being the number of those nodes, respectively, which belong to ζ_L and $\zeta_U (t_L + t_U = t)$. We consider all possible combinations of t nodes by taking p of them as ending nodes of $G(\tilde{T}_{\omega\hat{\omega}} - (1 - \omega)(1 - \hat{\omega})I)$. In our example, from node 4 to node 5 with three backward edges, we have the consecutive nodes: 3, 2, 5, 1, 4, 3, 2, and 5 as we can see in Fig. 1. So, $t = 8$. Five of these nodes (3, 5, 4, 3, and 5) are taken from ζ_L and three nodes (2, 1, and 2) from ζ_U. So, $t_L = 5$ and $t_U = 3$. This way corresponds to the product of blocks $B_{43}B_{32}B_{25}B_{51}B_{14}B_{43}B_{32}B_{25}$.

Let $\sigma_j \in \xi_L$, as in our example $\sigma_j = 5$. The analysis of the case $\sigma_j \in \zeta_L \backslash \xi_L$ is similar. In the sequel we will see that the way of going from one node of the set ζ_U to one of ζ_L can be given by means of the nodes of ξ_L only. So the $k + m$ nodes of ξ_L will all be taken (the three nodes 5, 4, and 5 of our example). The number of nodes that have been taken so far is $r + k + m (r$ of ξ_U and $k + m$ of $\xi_L)$. From the remaining nodes we take q nodes belonging to $\zeta_U \backslash \xi_U$ and s nodes belonging to $\zeta_L \backslash \xi_L$. So, $r + q + s = p - k - m$. The q nodes are taken from t_U nodes of ζ_U except the

$k+m$ nodes of ξ_U corresponding to the $k+m$ backward edges which were taken before. This gives $\binom{t_U-k-m}{q}$ different ways to consider. In our example we have $t_U = k+m = 3$ since $\zeta_U = \xi_U = \{1,2\}$. This means that we have only one possible way. The s nodes are taken from t_L nodes of ζ_L except the $k+m$ nodes of ξ_L. Similarly this gives a number of $\binom{t_L-k-m}{s}$ different ways to consider. Totally, we have

$$\binom{k+m}{r}\binom{t_U-k-m}{q}\binom{t_L-k-m}{s} \tag{2.20}$$

different ways to consider. The associated weight comes from $k+m-r$ triple-arrowed subpaths, from $q+k+m-(k+m-r) = q+r$ double-arrowed subpaths (q nodes of $\zeta_U\backslash\xi_U$ plus $k+m$ nodes of ξ_L except the $k+m-r$ triple-arrowed subpaths), and from the remaining $r+s$ single-arrowed subpaths. So this weight is

(2.21)

$$\left[(\omega+\hat{\omega}-\omega\hat{\omega})\,\frac{1-\hat{\omega}}{\hat{\omega}}\right]^{q+r}\left[(\omega+\hat{\omega}-\omega\hat{\omega})\,\frac{1-\omega}{\omega}\right]^{r+s}\left[\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^{k+m-r}.$$

By considering all possible values of r, q, and s such that $r+q+s = p-k-m$, we get the total overall weight equal to

$$\begin{aligned} N_C = &(\omega+\hat{\omega}-\omega\hat{\omega})^p \sum_{r+q+s=p-k-m}\binom{k+m}{r}\binom{t_U-k-m}{q}\binom{t_L-k-m}{s} \\ &\times\left[\frac{1-\hat{\omega}}{\hat{\omega}}\right]^{q+r}\left[\frac{1-\omega}{\omega}\right]^{r+s}\left[\frac{\omega+\hat{\omega}-\omega\hat{\omega}}{\omega\hat{\omega}}\right]^{k+m-r}. \end{aligned} \tag{2.22}$$

In our example:

$$N_C = (\omega+\hat{\omega}-\omega\hat{\omega})^5\sum_{r=0}^{2}\binom{3}{r}\binom{2}{2-r}\left[\frac{1-\hat{\omega}}{\hat{\omega}}\right]^r\left[\frac{1-\omega}{\omega}\right]^2\left[\frac{\omega+\hat{\omega}-\omega\hat{\omega}}{\omega\hat{\omega}}\right]^{3-r}.$$

Now we try to show that there exists the same path in $G(D)$ with the same weight. Only the case where $\sigma_i \in \zeta_L$ and $\sigma_j \in \xi_L$ is studied here.

From (2.12) we note that B^p is the last factor of the matrix D. The graph $G(B^p)$ consists of the identity paths (σ_i, σ_i) containing k backward edges. This means that in the graph of D, the last path (σ_j, σ_j) containing k backward edges belongs to the graph of B^p and has no weight. So, we must find the overall weight of the path from σ_i to σ_j with m backward edges of the graph

$$G\left(\tilde{T}^k_{\omega\hat{\omega}}\left[\tilde{T}_{\omega\hat{\omega}}+\frac{\hat{\omega}(1-\omega)}{\omega}\,I\right]^{|\zeta_L|-k}\left[\tilde{T}_{\omega\hat{\omega}}+\frac{\omega(1-\hat{\omega})}{\hat{\omega}}\,I\right]^{|\zeta_U|-k}\right). \tag{2.23}$$

From the graph expressions (2.11), (2.15), (2.17), and Fig. 2 it is easily seen that this path exists since the same nodes are used in the way from σ_i to σ_j. The total number of nodes are $t-p$(p nodes belong to the graph of B^p). In our example we have the nodes 3, 2, and 5. However, $t_U - |\zeta_U|$ of them belong to ζ_U and $t_L - |\zeta_L|$ belong to ζ_L. The main difference from the previous case is that now there are identity paths involved. We can also see that the graph expressions (2.11), (2.15), and (2.17) have the same paths which differ only in the weight of the identity paths. We then must find all the possible combinations by taking the first k consecutive paths from (2.11), the second $|\zeta_L| - k$ consecutive paths from (2.15), and the last $|\zeta_U| - k$ consecutive paths from (2.17).

Let us consider r_1 nodes from ξ_U, q_1 nodes from $\zeta_U \backslash \xi_U$ and q_2 nodes from $\zeta_L \backslash \xi_L$ of the path in the way from σ_i to σ_j with m backward edges. This gives a number of

$$\binom{m}{r_1}\binom{t_U - |\zeta_U| - m}{q_1}\binom{t_L - |\zeta_L| - m}{q_2} \tag{2.24}$$

different ways. Let us also take s_1 four-arrowed identity paths from the k paths of the graph (2.11), s_2 single-arrowed identity paths from the $|\zeta_L| - k$ paths of the graph (2.15), and s_3 double-arrowed identity paths from the $|\zeta_U| - k$ paths of the graph (2.17). So we must first distribute the number of times of the above s_1 identity paths to the $k - s_1 + 1$ nodes (the first σ_i node being included). This gives the number of combinations with repetitions of $k - s_1 + 1$ chosen s_1, that is

$$\binom{k - s_1 + 1 + s_1 - 1}{k - s_1 + 1 - 1} = \binom{k}{k - s_1} = \binom{k}{s_1}. \tag{2.25}$$

Similarly we obtain a number of $\binom{|\zeta_L| - k}{s_2}$ different cases because of the identity paths of (2.15) and a number of $\binom{|\zeta_U| - k}{s_3}$ different cases because of the identity paths of (2.17). After these considerations are made it is obvious that there is a number of

$$\binom{m}{r_1}\binom{t_U - |\zeta_U| - m}{q_1}\binom{t_L - |\zeta_L| - m}{q_2}\binom{k}{s_1}\binom{|\zeta_L| - k}{s_2}\binom{|\zeta_U| - k}{s_3} \tag{2.26}$$

different ways. The associated weight consists of $r_1 + q_2 + s_2$ single-arrowed paths of $r_1 + q_1 + s_3$ double-arrowed paths, of $m - r_1$ triple-arrowed paths, and of s_1 four-arrowed paths. This gives a weight of

$$\begin{aligned} &\left[(\omega + \hat{\omega} - \omega\hat{\omega})\ \frac{1 - \hat{\omega}}{\hat{\omega}}\right]^{r_1 + q_1 + s_3} \left[(\omega + \hat{\omega} - \omega\hat{\omega})\ \frac{1 - \omega}{\omega}\right]^{r_1 + q_2 + s_2} \\ &\times \left[\frac{(\omega + \hat{\omega} - \omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^{m - r_1} [(1 - \omega)(1 - \hat{\omega})]^{s_1}. \end{aligned} \tag{2.27}$$

The total number of subpaths of (2.23) is $k + (|\zeta_L| - k) + (|\zeta_U| - k) = p - k$. Since the m subpaths with ending nodes in ξ_L must be taken, the integers r_1, q_1, q_2, s_1, s_2, and s_3 vary but satisfy the relationship $r_1 + q_1 + q_2 + s_1 + s_2 + s_3 = p - k - m$. From (2.26), and (2.27) we have the total weight of the path of the graph (2.23) from σ_i to σ_j with m backward edges, which is

$$\begin{aligned} &\sum_{r_1 + q_1 + q_2 + s_1 + s_2 + s_3 = p - k - m} \binom{m}{r_1}\binom{t_U - |\zeta_U| - m}{q_1}\binom{t_L - |\zeta_L| - m}{q_2}\binom{k}{s_1} \\ &\times \binom{|\zeta_L| - k}{s_2}\binom{|\zeta_U| - k}{s_3}(\omega + \hat{\omega} - \omega\hat{\omega})^{p-k}\left[\frac{1 - \hat{\omega}}{\hat{\omega}}\right]^{q_1 + r_1 + s_3 + s_1} \\ &\times \left[\frac{1 - \omega}{\omega}\right]^{r_1 + q_2 + s_2 + s_1}\left[\frac{\omega + \hat{\omega} - \omega\hat{\omega}}{\omega\hat{\omega}}\right]^{m - r_1 - s_1}. \end{aligned} \tag{2.28}$$

By considering $q_1+s_3=q, q_2+s_2=s$, and $r_1+s_1=r$, the sum (2.28) takes the form

(2.29)
$$\sum_{r+q+s=p-k-m}\left(\sum_{r_1+s_1=r}\binom{m}{r_1}\binom{k}{s_1}\left[\frac{1-\hat{\omega}}{\hat{\omega}}\right]^{r_1+s_1}\left[\frac{1-\omega}{\omega}\right]^{r_1+s_1}\left[\frac{\omega+\hat{\omega}-\omega\hat{\omega}}{\omega\hat{\omega}}\right]^{m-r_1-s_1}\right)$$
$$\times\left(\sum_{q_1+s_3=q}\binom{t_U-|\zeta_U|-m}{q_1}\binom{|\zeta_U|-k}{s_3}\left[\frac{1-\hat{\omega}}{\hat{\omega}}\right]^{q_1+s_3}\right)$$
$$\times\left(\sum_{q_2+s_2=s}\binom{t_L-|\zeta_L|-m}{q_2}\binom{|\zeta_L|-k}{s_2}\left[\frac{1-\omega}{\omega}\right]^{q_2+s_2}\right)(\omega+\hat{\omega}-\omega\hat{\omega})^{p-k}.$$

By applying combinatorics theory, (2.29) gives

$$(2.30)\quad \begin{aligned}&(\omega+\hat{\omega}-\omega\hat{\omega})^{p-k}\sum_{r+q+s=p-k-m}\binom{k+m}{r}\binom{t_U-k-m}{q}\binom{t_L-k-m}{s}\\&\times\left[\frac{1-\hat{\omega}}{\hat{\omega}}\right]^{q+r}\left[\frac{1-\omega}{\omega}\right]^{r+s}\left[\frac{\omega+\hat{\omega}-\omega\hat{\omega}}{\omega\hat{\omega}}\right]^{m-r}.\end{aligned}$$

The total weight N_D of the path from σ_i to σ_j with $k+m$ backward edges of $G(D)$ is given by multiplying (2.30) with the coefficient

$$\left[\frac{(\omega+\hat{\omega}-\omega\hat{\omega})^2}{\omega\hat{\omega}}\right]^k$$

of the right-hand side of (2.12). This gives exactly the quantity N_C of (2.22). So

$$(2.31)\qquad N_C\equiv N_D.$$

Obviously (2.31) is satisfied for all pairs $(\sigma_i,\sigma_j), i,j=1(1)p$ and the proof of our theorem is complete. □

Based on the analysis so far, it is easy to prove the following statement.

THEOREM 2.3. *Under the assumptions of Theorem* 2.1 *there holds*

$$(2.32)\qquad B^pT_{\omega\hat{\omega}}=T_{\omega\hat{\omega}}B^p,$$

that is, the matrices B^p and $T_{\omega\hat{\omega}}$ commute.

Proof. The proof is obvious from Lemma 2.2, since the matrix B^p is commutative with the matrices U and L. □

The above result gives a more general matrix relationship than (1.8). In fact, it is not necessary that the factor B^p of the right-hand side be put as the last factor of the product. It can be put as its first factor or as any intermediate one.

Based on the main result already obtained, we can obtain some similar results for the SSOR, the SOR, and the backward SOR methods. These are presented in the following corollary.

COROLLARY 2.4. *Let B be the weakly cyclic block Jacobi matrix, generated by the cyclic permutation $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_p)$. Let also S_ω be the block* SSOR, *L_ω be the block*

SOR, and U_ω *be the block backward* SOR *iteration matrices, respectively, associated with* A *in* (1.1). *Then the following matrix relationships*

$$[S_\omega - (1-\omega)^2 I]^p = \omega^p (2-\omega)^{2k} S_\omega^k [S_\omega + (1-\omega)I]^{p-2k} B^p, \tag{2.33}$$

$$[L_\omega - (1-\omega)I]^p = \omega^p L_\omega^{|\zeta_L|} B^p, \tag{2.34}$$

and

$$[\mathsf{U}_\omega - (1-\omega)I]^p = \omega^p\, \mathsf{U}_\omega^{|\zeta_U|}\, B^p \tag{2.35}$$

hold.

Proof. It is easily proved that the analysis of the proof of our main result above holds when $\omega = \hat{\omega}$ or $\omega = 0$ or $\hat{\omega} = 0$; see also [11]. Putting $\omega = \hat{\omega}$ or $\hat{\omega} = 0$ or $\omega = 0$ (and using ω instead of $\hat{\omega}$) in $T_{\omega\hat{\omega}}$ in (1.3) reduces this matrix to the SSOR matrix S_ω, the SOR matrix L_ω, or to the backward SOR matrix U_ω, respectively. Consequently, putting $\omega = \hat{\omega}$ or $\hat{\omega} = 0$ or $\omega = 0$ (and using ω instead of $\hat{\omega}$) in (1.8) reduces the relationship in question to the matrix relationships (2.33), (2.34), or (2.35), respectively. □

The first result generalizes the previous result by Galanis, Hadjidimos, and Noutsos [3] for the p-cyclic consistently ordered case and the second result generalizes the previous one by Galanis, Hadjidimos, and Noutsos [2] for the $(q, p-q)$-generalized consistently ordered case. It is noted here that the proof of Corollary 2.4 can be obtained independently of the result (1.8) by using an analogous analysis and elementary graph theory in each particular case.

3. Equivalence of the USSOR and a two-parametric p-step method. To show that the USSOR method, used for the solution of (1.1), is equivalent to a certain two-parametric p-step method we proceed in a way analogous to that in [1–3]. For this let $x^{(m-p)}$ be the $(m-p)$th iteration of (1.2) with $m = p, p+1, p+2, \ldots$. From (1.8) we have

$$\begin{aligned}[T_{\omega\hat{\omega}} - (1-\omega)(1-\hat{\omega}))I]^p x^{(m-p)} &= (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} T_{\omega\hat{\omega}}^k [\omega T_{\omega\hat{\omega}} + (\hat{\omega} - \omega\hat{\omega})I]^{|\zeta_L|-k} \\ &\quad \times [\hat{\omega} T_{\omega\hat{\omega}} + (\omega - \omega\hat{\omega})I]^{|\zeta_U|-k} B^p x^{(m-p)}.\end{aligned} \tag{3.1}$$

By expanding both sides of (3.1) in terms of $T_{\omega\hat{\omega}}$ and by successively applying (1.2), after some modest amount of algebra takes place (see [11]), we get the following two-parametric p-step iterative scheme:

$$\begin{aligned} x^{(m)} &= -\sum_{j=1}^{p} (-1)^j (1-\omega)^j (1-\hat{\omega})^j \binom{p}{j} x^{(m-j)} \\ &\quad + (\omega + \hat{\omega} - \omega\hat{\omega})^{2k} B^p \sum_{i=0}^{|\zeta_L|-k} \sum_{j=0}^{|\zeta_U|-k} \binom{|\zeta_L|-k}{i} \\ &\quad \times \binom{|\zeta_U|-k}{j} (\hat{\omega} - \omega\hat{\omega})^i (\omega - \omega\hat{\omega})^j \omega^{|\zeta_L|-k-i} \hat{\omega}^{|\zeta_U|-k-j} x^{(m-k-i-j)} \\ &\quad + (\omega + \hat{\omega} - \omega\hat{\omega})^p \left(\sum_{i=0}^{p-1} B^i \right) b, \end{aligned} \tag{3.2}$$

where $x^{(j)} \in \mathbb{C}^n, j = 0(1)p-1$ are arbitrary.

In the sense explained above, the USSOR method (1.2) and (3.2) are equivalent and the study of (1.2) can be made by studying (3.2) and vice versa.

We must remark here that by putting $\omega = \hat{\omega}$ or $\hat{\omega} = 0$ or $\omega = 0$ in (3.2), we recover the monoparametric p-step schemes related to the SSOR, SOR, or backward SOR iterative methods, respectively. These schemes can also be obtained from the matrix relationships (2.33), (2.34), or (2.35), respectively.

One may also observe that because of the special cyclic nature of B, scheme (3.2) can be split into p simpler and smaller-dimension p-step iterative methods provided that all the vectors involved are partitioned in accordance with B. Each of these p simpler p-step methods has the same convergence rate, in the way considered in [10], as that of (3.2). So the solution of any one of these simpler methods provides us with the corresponding vector component of the solution x of (1.1), and from (1.1) all the other components of x. Therefore x itself can be readily obtained.

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