FINITE DIFFERENCE SCHEMES FOR THE 'PARABOLIC' EQUATION IN A VARIABLE DEPTH ENVIRONMENT WITH A RIGID BOTTOM BOUNDARY CONDITION*

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Abstract. We consider a linear, Schrödinger type p.d.e., the 'Parabolic' Equation of underwater acoustics, in a layer of water bounded below by a rigid bottom of variable topography. Using a change of depth variable technique we transform the problem into one with horizontal bottom, for which we establish an *a priori* H^1 estimate and prove an optimal-order error bound in the maximum norm for a Crank-Nicolson type finite difference approximation of its solution. We also consider the same problem with an alternative rigid bottom boundary condition due to Abrahamsson and Kreiss, and prove again *a priori* H^1 estimates and optimal order error bounds for a Crank-Nicolson scheme.

1. Introduction

The linear partial differential equation of Schrödinger type

$$\psi_r = \frac{i}{2k_0}\psi_{zz} + \frac{i}{2}k_0(n^2(z,r)-1)\psi, \tag{PE}$$

known as the (standard) *Parabolic Equation* (PE), [T], [LMc2], is widely used in underwater acoustics as a model for the simulation of one-way, long-range sound propagation near a horizontal plane, in inhomogeneous, weakly range-dependent marine environments. The PE may be derived, cf. [T], [BEHJ], as a narrow-angle paraxial approximation to the Helmholtz equation in cylindrical coordinates in the presence of azimuthal symmetry:

$$\Delta p + k_0^2 n^2(z,r)p = 0. \tag{HE}$$

Here r is the range, i.e. the horizontal distance from a harmonic point source placed in the water and emitting sound at frequency f_0, and $z \geq 0$ is the depth variable increasing downwards. We shall suppose that the medium, consisting for simplicity of a single layer of water of constant density, occupies the region $0 \leq z \leq \ell(r)$, $r \geq 0$, of the (z,r) plane. Here, $z = 0$ is the free surface and $z = \ell(r)$ is a positive smooth function representing the range-dependent topography of the bottom. The function $p = p(z,r)$ denotes the resulting acoustic pressure field, $k_0 := 2\pi f_0/c_0$ is a reference wavenumber, c_0 is a (constant) reference

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sound speed, and n, the index of refraction, is a smooth function of z and r defined as $c_0/c(z,r)$, where c is the speed of sound in the water. The equation (HE) will be supplemented by a pressure release boundary condition $p = 0$ at the surface $z = 0$, and by the idealized, rigid bottom (Neumann) boundary condition $\frac{\partial p}{\partial \nu} = 0$ at $z = \ell(r)$, where ν is the normal vector to the surface $z = \ell(r)$. We shall write this as

(B) $$p_z - \ell'(r)p_r = 0 \quad \text{at} \quad z = \ell(r).$$

If we change the dependent variable in (HE) so that $p(z,r) = \frac{\psi(z,r)}{\sqrt{k_0 r}}\exp(ik_0 r)$ (i.e. remove cylindrical spreading and apply an envelope transformation), we obtain a two-way equation for the complex-valued function ψ, for which the one-way model (PE) may be derived if we assume that $|2ik_0\psi_r| >> |\psi_{rr}|$ (paraxial approximation), and neglect an $O(r^{-2})$ coefficient of ψ (far-field approximation). Originally the PE was derived, [T], to model propagation in domains with a horizontal bottom but subsequently it has been also extensively used for domains with mildly varying bottom topography in the presence of low backscattering. A natural question to study then is the well-posedness of the initial- and boundary-value problem for the PE under various bottom boundary conditions.

In this paper we consider the rigid bottom case. Performing the operations outlined above on (B) we obtain its PE-equivalent

(PB1) $$\psi_z - \ell'(r)\psi_r - ik_0\ell'(r)\psi = 0 \quad \text{at} \quad z = \ell(r).$$

Similar rigid bottom boundary conditions were used quite early for computations with the PE, cf. e.g. [LP1], [LP2], [LBP]. The specific form of (PB1) is consistent with the far-field approximation level of the PE itself. A slightly more general condition is

(PB2) $$\psi_z - \ell'(r)\psi_r - g(r)\ell'(r)\psi = 0 \quad \text{at} \quad z = \ell(r),$$

wherein $g(r)$ can be taken as $(H_0^{(1)}(k_0 r))_r/H_0^{(1)}(k_0 r)$, [LP2], where $H_0^{(1)}$ is the Hankel function of the first kind of order zero, or as its long-range approximation $ik_0 - \frac{1}{2r}$, or simply as ik_0, which yields (PB1).

Imposing at $z = 0$ the pressure release condition $\psi = 0$, we may solve then an initial- and boundary-value problem to determine $\psi(z,r)$ for $0 \le z \le \ell(r)$, $r \ge 0$, given an initial profile $\psi(z,0) = \psi_0(z)$, $0 \le z \le \ell(0)$, simulating the effect of the source at $r = 0$.

This initial- and boundary-value problem may be transformed into a problem in a domain with horizontal bottom by a change of variables. With this aim in mind, we first write the equations in dimensionless form by introducing new variables defined by $x := z/L$, $t := r/L$, $u := \psi/\psi_{\text{ref}}$, where as characteristic length we take $L := 1/k_0$, and put $\psi_{\text{ref}} := \max|\psi_0|$. Then, letting $s(t) := k_0\ell(k_0^{-1}t)$, $g_*(t) := k_0 g(k_0^{-1}t)$, $\beta(x,t) := \frac{1}{2}(n^2(k_0^{-1}x, k_0^{-1}t) - 1)$, we see that (PE) and (PB2) become, respectively,

$$u_t = \tfrac{i}{2}u_{xx} + i\beta(x,t)u, \quad 0 \le x \le s(t), \quad t \ge 0,$$
$$u_x - \dot{s}(t)u_t - g_*(t)\dot{s}(t)u = 0, \quad x = s(t), \quad t \ge 0,$$

where a dot denotes differentiation with respect to t. In addition we require that $u(0,t) = 0$ for $t \ge 0$, and $u(x,0) = u_0(x) := \frac{1}{\psi_{\text{ref}}}\psi_0(k_0^{-1}x)$ for $0 \le x \le s(0)$.

Generalizing slightly (by allowing for a complex index of refraction with an attenuation coefficient as its imaginary part, and adding a forcing term in the right-hand side of the PE) we pose the problem that will be finally considered: Let $T > 0$, suppose $s \in C^1([0,T],\mathbb{R})$ with $s(t) > 0$ for $t \in [0,T]$, define $I(t) := [0, s(t)]$, $P(t) := (s(t),t)$ for $t \in [0,T]$, $\mathcal{D} := \{(x,t) \in \mathbb{R}^2 : t \in [0,T] \text{ and } x \in I(t)\}$, and seek a function $u : \mathcal{D} \longrightarrow \mathbb{C}$ such that

(1.1a) $$u_t = i\alpha u_{xx} + i\big(\beta_R(x,t) + i\beta_I(x,t)\big)u + f(x,t) \quad \text{for} \quad (x,t) \in \mathcal{D},$$

(1.1b) $$u(0,t) = 0 \quad \text{for} \quad t \in [0,T],$$

(1.1c) $$u_x(P(t)) - \dot{s}(t)\big[u_t(P(t)) + g_*(t)u(P(t))\big] = 0 \quad \text{for} \quad t \in [0,T],$$

(1.1d) $$u(x,0) = u_0(x) \quad \text{for} \quad x \in I(0),$$

where α is a nonzero real number, $g_* : [0,T]\longrightarrow\mathbb{C}$, $\beta_R, \beta_I : \mathcal{D}\longrightarrow\mathbb{R}$, $f : \mathcal{D}\longrightarrow\mathbb{C}$ and $u_0 : I(0)\longrightarrow\mathbb{C}$.

This problem may be transformed into an equivalent one posed on a horizontal strip of unit depth by the simple, range-dependent change of variable $y := \frac{x}{s(t)}$. This leads us to consider then the following problem: Let $D := [0,1]\times[0,T]$ and seek $w : D\longrightarrow\mathbb{C}$ such that

$$w_t = i\tfrac{1}{\xi(t)}w_{yy} + y\mu(t)w_y + i\big(\gamma_R(y,t) + i\gamma_I(y,t)\big)w + \zeta(y,t) \quad \text{for} \quad (y,t)\in D, \tag{1.2a}$$

$$w(0,t) = 0 \quad \text{for} \quad t\in[0,T], \tag{1.2b}$$

$$w_y(1,t) = S_1(t)w_t(1,t) + S_2(t)w(1,t) + S_3(t) \quad \text{for} \quad t\in[0,T], \tag{1.2c}$$

$$w(y,0) = w_0(y) \quad \text{for} \quad y\in[0,1], \tag{1.2d}$$

where $\xi : [0,T]\longrightarrow\mathbb{R}-\{0\}$, $\mu : [0,T]\longrightarrow\mathbb{R}$, γ_R, $\gamma_I : D\longrightarrow\mathbb{R}$, $\zeta : D\longrightarrow\mathbb{C}$, $S_1 : [0,T]\longrightarrow\mathbb{R}$, S_2, $S_3 : [0,T]\longrightarrow\mathbb{C}$, and $w_0 : [0,1]\longrightarrow\mathbb{C}$.

It is easily seen that the function $w_* : D\longrightarrow\mathbb{C}$ defined by $w_*(y,t) = u(ys(t),t)$ for $(y,t)\in D$ is a solution of the problem (1.2), with

$$\begin{gathered} w_0(y) = u_0(ys(0)), \qquad \xi(t) = \frac{s^2(t)}{\alpha}, \quad \mu(t) = \frac{\dot{s}(t)}{s(t)}, \\ S_1(t) = \frac{\dot{s}(t)s(t)}{1+(\dot{s}(t))^2}, \quad S_2(t) = g_*(t)S_1(t), \quad S_3(t) = 0, \\ \gamma_R(y,t) = \beta_R(ys(t),t), \quad \gamma_I(y,t) = \beta_I(ys(t),t) \quad \text{and} \quad \zeta(y,t) = f(ys(t),t). \end{gathered} \tag{1.3}$$

Abrahamsson and Kreiss have studied in [AK1] a generalization of (1.2). They first establish an H^1 *a priori* estimate of the solution when $S_1(t) = 0$, i.e. when the bottom is horizontal in the original problem (1.1). Using this intermediate result they prove, under the hypothesis that $|S_1(t)| > 0$ for $t\in[0,T]$ (i.e. when $|\dot{s}(t)|$ is bounded away form zero in (1.1)) an *a priori* estimate in H^2, which enables them to establish existence and uniqueness of solutions. They point out that if $\dot{s}(t)$ varies with t and goes to zero at some point, the question of well-posedness of (1.1) is open.

In Section 2 of the paper at hand we prove an *a priori* H^1 estimate of the solution of (1.2) under the hypothesis that the coefficients μ and S_1 in (1.2a) and (1.2c) satisfy the condition

$$\sup_{t\in[0,T]}\Big\{S_1(t)[2-\mu(t)S_1(t)]\Big\} \le S_* < 0. \tag{Σ}$$

If in the original problem (1.1) we suppose that s is strictly decreasing, i.e. if $\max_{t\in[0,T]}\dot{s}(t) < 0$, and if the coefficients of (1.2) are defined by (1.3), then it is easily seen that (Σ) holds. An H^1 estimate on u, the solution of (1.1), follows therefore from the H^1 estimate on w_*. If however s is strictly increasing, it is shown that the function $\vartheta = \exp(\lambda)w_*$, where $\lambda : D\longrightarrow\mathbb{C}$ is suitably chosen, solves a problem of the type (1.2) with coefficients that satisfy (Σ). Hence, in this case too, an H^1 estimate on u may be established as a consequence of the H^1 estimate on ϑ. By a different energy technique not shown here, it is possible to establish an H^1 estimate on the solution of (1.1) directly, provided $\dot{s}\le 0$ and a sufficiently smooth solution of (1.1) exists in $\mathcal{D}$. (It is trivial to see that an L^2 estimate for the solution of (1.1) holds if $\dot{s}(t) = 0$. Indeed, the problem is L^2-conservative if in addition $\beta_I = f = 0$.)

Our main motivation for showing an H^1 estimate for the solution of (1.2) is using this proof as a guide in obtaining a maximum norm error estimate for a second-order accurate, Crank-Nicolson type finite difference scheme for this problem. In Section 3 we construct such a scheme using uniform meshes with meshlengths h and k, in the y and t directions, respectively, and prove that it possesses an $O(k^2+h^2)$ error bound in the maximum norm under no conditions relating h and k. In the proof we assume that the solution of (1.2) is smooth enough and that either (Σ) holds or that the coefficients of (1.2) are given by (1.3) and $\dot{s}(t)\le 0$ for $t\in[0,T]$. The proof of this optimal-order error estimate is long and technically complicated, since, to begin with, one must somehow overcome the fact that the truncation error and its first t-difference quotient is only of $O(k^2+h+\frac{k^2}{h})$ at the boundary $y=1$. Moreover, the technique of comparing the discrete solution with an 'elliptic' approximation of the solution of (1.2), of the type introduced and used for the PE and its wider angle extension in [AD] and [ADZ] in order to treat similar local reduction of accuracy at interfaces, cannot be applied to the problem at hand due to the presence of the w_t term in the boundary condition (1.2c). The crucial steps needed to achieve optimal-order accuracy are using repeatedly a $H^{-1}-H^1$ type bound of some discrete L^2 inner products involving the error, and estimating in a suitable way a y-difference quotient of the error at the boundary $y=1$. In both cases, one needs a bound of a discrete L^1 norm of the truncation error, which turns out to be of optimal order.

In their paper [AK1] Abrahamsson and Kreiss, in addition to proving existence and uniqueness of the solution for a generalization of (1.2), considered also a constant coefficient analog of (1.1) and proved that there exist downsloping bottom profiles, i.e. profiles with $\ell'(r)>0$, equivalently, $\dot{s}(t)>0$, for which the H^1 norm of ψ with respect to z grows exponentially with r. It would seem then that for some downsloping profiles the term $\ell'(r)\psi_r$ in (PB1) acts as a source of energy at the bottom and increases the amplitude of the pressure field. Abrahamsson and Kreiss continue their study in [AK2] and first point out, by means of a numerical experiment, that a Crank-Nicolson finite difference scheme for a problem of the type (1.2) (with a constant coefficient analog of the (PE) on a straight, downsloping bottom profile and a high depth mode as initial condition at $t=0$), has solutions whose ℓ_2 norm grows fast with t. This evidence motivates them to abandon the physically correct boundary condition (PB1) and derive a new one, the condition (6) in [AK2], which in our notation is

$$\text{(PB1}'\text{)} \qquad \psi_z - ik_0\ell'(r)\psi = 0 \quad \text{at} \quad z=\ell(r).$$

The new condition, a 'paraxialization' of (PB1) in the terminology of [S], may be obtained from (PB1) if one uses (PE) and notes that both terms in the right-hand side of the expression

$$\ell'(r)\psi_r = \tfrac{i}{2k_0}\ell'(r)\psi_{zz} + \tfrac{i}{2}k_0\ell'(r)(n^2(z,r)-1)\psi$$

are small, in view of the fact that the PE models propagation in directions that form a narrow angle with the horizontal, and assumes mild range dependence, i.e. that $\ell(r)$ and $n(z,r)$ vary slowly with r. Writing the problem in nondimensional form we may now, instead of (1.1), consider seeking $u:\mathcal{D}\longrightarrow\mathbb{C}$ such that

$$\text{(1.4a)} \qquad u_t = i\alpha u_{xx} + i\big[\beta_R(x,t)+i\beta_I(x,t)\big]u + f(x,t) \quad \text{for} \quad (x,t)\in\mathcal{D},$$

$$\text{(1.4b)} \qquad u(0,t)=0 \quad \text{for} \quad t\in[0,T],$$

$$\text{(1.4c)} \qquad i2\alpha u_x(P(t)) + \dot{s}(t)u(P(t)) = 0 \quad \text{for} \quad t\in[0,T],$$

$$\text{(1.4d)} \qquad u(x,0)=u_0(x) \quad \text{for} \quad x\in I(0),$$

where the notation of (1.1) has been used. A t-dependent change of scale of the depth and of the dependent variable leads us to the analogous to (1.2) problem of finding $w:D\longrightarrow\mathbb{C}$ such that

$$\text{(1.5a)} \qquad w_t = i\tfrac{1}{\xi(t)}w_{yy} + y\mu(t)w_y + \tfrac{\mu(t)}{2}w + i\big[\gamma_R(y,t)+i\gamma_I(y,t)\big]w + \zeta(y,t) \quad \text{for} \quad (y,t)\in D,$$

$$\text{(1.5b)} \qquad w(0,t)=0 \quad \text{for} \quad t\in[0,T],$$

$$\text{(1.5c)} \qquad w_y(1,t) = i\tfrac{\xi(t)\mu(t)}{2}w(1,t) + g(t) \quad \text{for} \quad t\in[0,T],$$

$$\text{(1.5d)} \qquad w(y,0)=w_0(x) \quad \text{for} \quad y\in[0,1],$$

where $\xi:[0,T]\longrightarrow\mathbb{R}-\{0\}$, $\mu:[0,T]\longrightarrow\mathbb{R}$, γ_R, $\gamma_I:D\longrightarrow\mathbb{R}$, $\zeta:D\longrightarrow\mathbb{C}$, $g:[0,T]\longrightarrow\mathbb{C}$, and $w_0:[0,1]\longrightarrow\mathbb{C}$. We changed variables so that the function $w_*:D\longrightarrow\mathbb{C}$ defined by $w_*(y,t)=\sqrt{s(t)}\,u(ys(t),t)$ for $(y,t)\in D$ is a solution of the problem (1.5), with

$$\text{(1.6)} \qquad \begin{gathered} \xi(t)=\tfrac{s^2(t)}{\alpha}, \quad \mu(t)=\tfrac{\dot{s}(t)}{s(t)}, \quad g(t)=0, \quad \gamma_R(y,t)=\beta_R(ys(t),t), \quad \gamma_I(y,t)=\beta_I(ys(t),t), \\ w_0(y)=\sqrt{s(0)}\,u_0(ys(0)) \quad \text{and} \quad \zeta(y,t)=\sqrt{s(t)}\,f(ys(t),t). \end{gathered}$$

This problem is well-posed with no restrictions on $s(t)$ other than positivity. In fact, an H^1 estimate is obtained in [AK1] for a more general version of (1.5) by means of an exponential transformation and one differentiation of the solution with respect to y. It is also pointed out in [AK2], in the context of the constant coefficient, homogeneous, nondissipative PE considered therein with the new boundary condition, that the L^2 norm of $u(\cdot, r)$ is conserved. In addition, numerical examples in [AK2] show that a Crank-Nicolson type scheme approximates the new problem well. In fact, a numerical experiment shown in [AK2] for a problem with a straight downsloping bottom profile (for which an exact solution of the Helmholtz equation is available), suggests that the new model is a good approximation to the Helmholtz equation over long ranges.

In Sections 4 and 5 of the present paper we complement the results of [AK1] and [AK2] by proving first *a priori* L^2 and H^1 energy estimates for the solution of (1.5). These estimates are obtained directly, without exponential transformations or differentiation with respect to y. We then analyze a Crank-Nicolson type finite difference scheme for (1.5) on uniform meshes and prove that it satisfies, unconditionally, an $O(k^2+h^2)$ error bound in the maximum norm. (This scheme is conservative in a discrete L^2 sense if $\gamma_I = \zeta = 0$ in (1.5a) and $g = 0$ in (1.5c); it thus mimics the analogous L^2 conservation property of the homogeneous analog of (1.5) in the absence of dissipation.) For this scheme the truncation error and its first t-difference quotient is also of $O(k^2 + h + \frac{k^2}{h})$ at the boundary $y = 1$. So, we employ again the same type of discrete $H^{-1} - H^1$ estimation used in the analysis of the Crank-Nicolson scheme for (1.2). However, the proof is less complicated now because of the absence of terms involving the y-difference quotient of the error at $y = 1$.

We close this introductory section with some remarks on related initial- and boundary-value problems for the PE:

(i). The rigid bottom boundary condition makes both the continuous problem and its numerical approximation quite hard to study. Much easier to analyze is the problem with a homogeneous Dirichlet boundary condition $\psi = 0$ at $z = \ell(r)$. (This condition may also be more realistic physically in many instances; indeed an upsloping wedge with zero Dirichlet boundary conditions at the free surface and at the bottom was one of the benchmarks chosen by the Acoustical Society of America in a code comparison exercise for range-dependent two-dimensional problems, [JF].) For such a pressure-released bottom it is easy to see that the initial- and boundary-value problem for (PE) on the variable domain is L^2 conservative. Upon changing the depth variable by $y = \frac{x}{s(t)}$ as usual (this had been done for the linear Schrödinger equation in a different physical context already in [MBFF]), one may check that Crank-Nicolson-type finite difference schemes for the transformed problem may be easily analyzed. Alternatively, one may approximate the problem directly on the variable domain (without a change of the depth variable that is) using Crank-Nicolson type schemes on nonuniform meshes, as was done for the heat equation in [J]. We refer the reader to [AD1] and [Z] for relevant optimal-order error estimates for this type of schemes in the case of the PE.

(ii). Change of the depth variable techniques have been used in recent years in numerical simulations of the PE with finite difference and finite element methods in several more realistic underwater sound propagation problems, including variable interface problems for two-layered media (e.g. water over a penetrable fluid bottom layer), and three-dimensional (i.e. not axisymmetric) problems with variable bottom topography and interfaces. See e.g. [DK], [KS], [SPF], [SFP], [S]. Changing variables has the advantage of avoiding the loss of accuracy due to the staircase approximation of sloping bottoms and interfaces implemented in several PE codes. (For alternative approaches, and examples and discussions of the relevant energy conservation issues cf. e.g. [PJF], [CW], [C], [LMc1], [BTW], [KF].) However, it is not clear if it will be useful in problems with many layers.

2. H^1 stability for problem (1.2)

In this section we shall first prove an *a priori* H^1 estimate for the solution of the initial and boundary value problem (1.2) under the hypothesis that the coefficients S_1 and μ satisfy the condition

$$(\Sigma) \qquad \sup_{t\in[0,T]} \Big\{ S_1(t)\big[2 - \mu(t) S_1(t)\big] \Big\} \le S_* < 0.$$

We shall then show how this result can be used to establish an H^1 estimate for the original PE problem (1.1) when s is strictly monotone.

In the sequel we let $H^1 = H^1(0,1)$, resp. $L^2 = L^2(0,1)$, be the usual (complex) Sobolev space of order one, resp. the space of complex-valued square-integrable functions defined on $(0,1)$. We define $\overset{\circ}{H} := \{\psi \in H^1 : \psi(0) = 0\}$, $\|\psi\| := (\int_0^1 |\psi|^2 dx)^{1/2}$ for $\psi \in L^2$, $\|\psi\|_1 := (\|\psi\|^2 + \|\psi'\|^2)^{1/2}$ for $\psi \in H^1$, and $(\psi_1, \psi_2) := \int_0^1 \psi_1 \overline{\psi_2} dx$ for ψ_1, $\psi_2 \in L^2$, where an overbar denotes complex conjugation. We set $\omega(y) := y$ for $y \in [0,1]$ and note that

$$2\,\mathrm{Re}(\omega\varphi', \varphi) = |\varphi(1)|^2 - \|\varphi\|^2, \quad \forall \varphi \in H^1 \tag{2.1a}$$

and

$$|\varphi(1)| \le \|\varphi'\|, \quad \forall \varphi \in \overset{\circ}{H}. \tag{2.1b}$$

Finally, we put $\|\psi\|_{-1} := \sup\{|(\psi,\varphi)| : \varphi \in H^1 \ \text{ with } \ \|\varphi\|_1 = 1\}$, for $\psi \in L^2$.

Theorem 2.1. *Let w be the solution of the problem (1.2). If (Σ) holds, then there exists a positive constant C such that*

$$\|w(\cdot,t)\|_1^2 \le C\exp(Ct)\Big\{\|w_0\|_1^2 + \sup_{\tau\in[0,t]} \|\zeta(\cdot,\tau)\|_{-1}^2 + \int_0^t \big(|S_3(\tau)|^2 + \|\zeta(\cdot,\tau)\|^2 + \|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2\big)d\tau\Big\} \tag{2.2}$$

for $t \in [0,T]$.

Proof. Multiplying (1.2a) by $(\overline{w_t} - \omega\mu\overline{w_y})$, integrating by parts over $(0,1)$, taking imaginary parts and using (2.1a), we obtain

$$\begin{aligned}\frac{d}{dt}\|w_y(\cdot,t)\|^2 =& 2\,\mathrm{Re}\big[w_y(1,t)\overline{w_t}(1,t)\big] - \mu(t)\Big[|w_y(1,t)|^2 - \|w_y(\cdot,t)\|^2\Big]\\ &- 2\xi(t)\mu(t)\,\mathrm{Re}(\omega\gamma(\cdot,t)w(\cdot,t), w_y(\cdot,t)) + 2\xi(t)\,\mathrm{Re}(\gamma(\cdot,t)w(\cdot,t), w_t(\cdot,t))\\ &+ 2\xi(t)\,\mathrm{Im}(\zeta(\cdot,t), w_t(\cdot,t)) - 2\mu(t)\xi(t)\,\mathrm{Im}(\omega\zeta(\cdot,t), w_y(\cdot,t)), \quad \forall t\in[0,T],\end{aligned}$$

where $\gamma := \gamma_R + i\gamma_I$. From the last relation, using (1.2c) and (2.1b), it follows

$$\begin{aligned}\frac{d}{dt}\|w_y(\cdot,t)\|^2 \le& C\Big(\|w(\cdot,t)\|_1^2 + \|\zeta(\cdot,t)\|^2 + \|\partial_t\zeta(\cdot,t)\|_{-1}^2 + |S_3(t)|^2\Big)\\ &+ 2\xi(t)\,\mathrm{Re}(w_t(\cdot,t), \gamma(\cdot,t)w(\cdot,t)) + S_1(t)\big[2-\mu(t)S_1(t)\big]|w_t(1,t)|^2\\ &+ 2\big[1-\mu(t)S_1(t)\big]\Big\{\mathrm{Re}\big[S_2(t)w(1,t)\overline{w_t}(1,t)\big] + \mathrm{Re}\big[S_3(t)\overline{w_t}(1,t)\big]\Big\}\\ &+ \frac{d}{dt}\big\{\mathrm{Im}(2\zeta(\cdot,t)\xi(t), w(\cdot,t))\big\}, \quad \forall t\in[0,T].\end{aligned} \tag{2.3}$$

(Here, and in the other estimates in the sequel, the symbol C denotes a generic constant, not necessarily the same in any two places, depending on T and on the coefficients of the p.d.e. (1.2a) and of the boundary condition (1.2c). For example, the constant C in (2.3) is a polynomial function of $\|\mu\|_{L^\infty(0,T)}$, $\|\xi\|_{L^\infty(0,T)}$, $\|\dot{\xi}\|_{L^\infty(0,T)}$, $\|S_2\|_{L^\infty(0,T)}$, and $\|\gamma\|_{L^\infty(D)}$.)

Multiplying (1.2a) by $2\xi\overline{\gamma w}$, integrating by parts over $(0,1)$ and taking real parts we get now

$$\begin{aligned}2\xi(t)\,\mathrm{Re}(w_t(\cdot,t), \gamma(\cdot,t)w(\cdot,t)) =& -2\,\mathrm{Im}\big[\overline{\gamma}(1,t)w_y(1,t)\overline{w}(1,t)\big] + 2\,\mathrm{Im}(w_y(\cdot,t), \gamma_y(\cdot,t)w(\cdot,t))\\ &+ 2\,\mathrm{Im}(w_y(\cdot,t), \gamma(\cdot,t)w_y(\cdot,t)) + 2\mu(t)\xi(t)\,\mathrm{Re}(\omega w_y(\cdot,t), \gamma(\cdot,t)w(\cdot,t))\\ &+ 2\xi(t)\,\mathrm{Re}(\zeta(\cdot,t), \gamma(\cdot,t)w(\cdot,t)), \quad \forall t\in[0,T],\end{aligned}$$

which, in view of (1.2c) and (2.1b), yields

$$2\xi(t)\,\mathsf{Re}(w_t(\cdot,t),\gamma(\cdot,t)w(\cdot,t)) \le C\big(\|w(\cdot,t)\|_1^2+\|\zeta(\cdot,t)\|_{-1}^2+|S_3(t)|^2\big) \\ -2S_1(t)\,\mathsf{Im}\big[w_t(1,t)\overline{\gamma}(1,t)\overline{w}(1,t)\big],\quad \forall t\in[0,T]. \tag{2.4}$$

Now, (2.3), (2.4) and (Σ) give

$$\frac{d}{dt}\|w_y(\cdot,t)\|^2 \le C(1+\tfrac{1}{\varepsilon})\big\{\|w(\cdot,t)\|_1^2+\|\zeta(\cdot,t)\|^2+\|\partial_t\zeta(\cdot,t)\|_{-1}^2+|S_3(t)|^2\big\} \\ +\frac{d}{dt}\big\{\mathsf{Im}(2\zeta(\cdot,t)\xi(t),w(\cdot,t))\big\}+(S_*+3\varepsilon)|w_t(1,t)|^2,\quad \forall\varepsilon>0,\quad \forall t\in[0,T]. \tag{2.5}$$

Integrating (2.5) with respect to t and choosing $\varepsilon=-\frac{S_*}{3}$, we obtain

$$\|w_y(\cdot,t)\|^2 \le \|w_0\|_1^2+C\int_0^t\big\{\|w(\cdot,\tau)\|_1^2+\|\zeta(\cdot,\tau)\|^2+\|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2+|S_3(\tau)|^2\big\}d\tau \\ +2|\xi(t)|\|\zeta(\cdot,t)\|_{-1}\|w(\cdot,t)\|_1+2|\xi(0)|\|\zeta(\cdot,0)\|_{-1}\|w_0\|_1,\quad \forall t\in[0,T],$$

from which, using Poincaré's inequality and the arithmetic-geometric mean inequality, we conclude that

$$\|w(\cdot,t)\|_1^2 \le C\Big\{\|w_0\|_1^2+\sup_{\tau\in[0,t]}\|\zeta(\cdot,\tau)\|_{-1}^2+\int_0^t\big(\|\zeta(\cdot,\tau)\|^2+\|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2+|S_3(\tau)|^2\big)d\tau\Big\} \\ +C\int_0^t\|w(\cdot,\tau)\|_1^2d\tau,\quad \forall t\in[0,T]. \tag{2.6}$$

Applying the Gronwall lemma to (2.6) we get (2.2). The constant C in (2.2) depends on T, S_*, the $L^\infty(0,T)$ norms of ξ, $\dot{\xi}$, μ, S_1 and S_2, and the $L^\infty(D)$ norms of γ and γ_y. ■

Consider now the original PE problem (1.1) and suppose that s is strictly decreasing, i.e., that

$$\max_{t\in[0,T]}\dot{s}(t)<0.$$

As pointed out in the Introduction, the function $w_*: D\longrightarrow\mathbb{C}$ defined by the change of variables $w_*(y,t)=u(ys(t),t)$ is a solution of the problem (1.2) with initial value and coefficients given by the formulas (1.3). Hence

$$S_1(t)\big[2-\mu(t)S_1(t)\big]=\dot{s}(t)\;s(t)\;\frac{2+(\dot{s}(t))^2}{(1+(\dot{s}(t))^2)^2}$$

and (Σ) clearly holds. An H^1 estimate on the solution u of (1.1) follows now in view of Theorem 2.1.

If s is strictly increasing, an additional change of the dependent variable is needed. To this end, let $\lambda: D\longrightarrow\mathbb{C}$, and $\vartheta: D\longrightarrow\mathbb{C}$ be defined by $\vartheta=\exp(\lambda)w_*$. Then

$$\exp(\lambda)(w_*)_y=\vartheta_y-\lambda_y\vartheta,\quad \exp(\lambda)(w_*)_t=\vartheta_t-\lambda_t\vartheta\quad\text{and}\quad \exp(\lambda)(w_*)_{yy}=\vartheta_{yy}-2\lambda_y\vartheta_y+\big[(\lambda_y)^2-\lambda_{yy}\big]\vartheta.$$

Hence, ϑ solves the following problem:

$$\begin{aligned}&\vartheta_t=iA(t)\vartheta_{yy}+B(y,t)\vartheta_y+G(y,t)\vartheta+F(y,t)\quad\text{for}\quad(y,t)\in D,\\ &\vartheta(0,t)=0,\quad\forall t\in[0,T],\\ &\vartheta_y(1,t)=R_1(t)\vartheta_t(1,t)+R_2(t)\vartheta(1,t),\quad\forall t\in[0,T],\\ &\vartheta(y,0)=\vartheta_0(y)\quad\text{for}\quad y\in[0,1],\end{aligned} \tag{2.7}$$

where

$$A(t) := \frac{\alpha}{s^2(t)}, \quad B(y,t) := y\frac{\dot{s}(t)}{s(t)} - i\frac{2\alpha}{s^2(t)}\lambda_y(y,t), \quad F(y,t) := \exp(\lambda(y,t))f(ys(t),t),$$

$$G(y,t) := \lambda_t(y,t) - y\frac{\dot{s}(t)}{s(t)}\lambda_y(y,t) + i\beta_R(ys(t),t) - \beta_I(ys(t),t) + i\frac{\alpha}{s^2(t)}\big[(\lambda_y(y,t))^2 - \lambda_{yy}(y,t)\big], \tag{2.8}$$

$$R_1(t) := \frac{\dot{s}(t)s(t)}{1+(\dot{s}(t))^2}, \quad R_2(t) := \big[g_*(t) - \lambda_t(1,t)\big]R_1(t) + \lambda_y(1,t), \quad \vartheta_0(y) := \exp(\lambda(y,0))u_0(ys(0)).$$

We now ask whether it is possible to construct a function λ so that the problem (2.7) is of the type (1.2) and so that the analog of (Σ) is satisfied. The answer is affirmative: Let $\sigma : [0,T] \longrightarrow \mathbb{R}$ be a function to be chosen presently and define

$$\lambda(y,t) := i(\sigma(t)-1)\frac{\dot{s}(t)s(t)}{4\alpha}y^2, \quad \forall (y,t) \in D. \tag{2.9}$$

Then, by (2.8)

$$B(y,t) = y\widetilde{\mu}(t) \quad \text{with} \quad \widetilde{\mu}(t) := \frac{\dot{s}(t)}{s(t)}\sigma(t)\cdot$$

It follows that the choice (2.9) for λ makes the problem (2.7) to be of the type (1.2). We now have

$$R_1(t)\big[2 - \widetilde{\mu}(t)R_1(t)\big] = -\left\{\sigma(t) - 2\,\frac{1+(\dot{s}(t))^2}{(\dot{s}(t))^2}\right\}\frac{s(t)(\dot{s}(t))^3}{[1+(\dot{s}(t))^2]^2}\cdot \tag{2.10}$$

Thus, if we choose, for some $\varepsilon > 0$,

$$\sigma(t) = 2\,\frac{1+(\dot{s}(t))^2}{(\dot{s}(t))^2} + \varepsilon,$$

(2.10) gives

$$R_1(t)\big[2 - \widetilde{\mu}(t)R_1(t)\big] = -\varepsilon\frac{s(t)(\dot{s}(t))^3}{[1+(\dot{s}(t))^2]^2},$$

and hence (Σ) is satisfied, since in the case under consideration $\min\limits_{t\in[0,T]} \dot{s}(t) > 0$. It should be noted that this choice of σ requires assuming that $s \in C^2[0,T]$ so that G, G_y and R_2 are bounded, as needed in the course of the proof of the a priori H^1 estimate. The required H^1 estimate on u is derived from those of λ and w_*.

3. The finite difference scheme for problem (1.2)

In this section we construct and analyze an unconditionally stable finite difference scheme of second-order accuracy in y and t for approximating the solution of the initial and boundary value problem (1.2).

3.1 Notation and preliminaries.

Let $N, J \in \mathbb{N}$. We define a uniform partition of the interval $[0,T]$ with step $k := \frac{T}{N}$, nodes $t^n := nk$ for $n = 0,\dots,N$, and intermediate nodes $t^{n+\frac{1}{2}} := t^n + \frac{k}{2}$ for $n = 0,\dots,N-1$. Also, we set $h := \frac{1}{J+1}$ and consider a uniform partition of the interval $[0,1]$ with nodes $y_j := jh$ for $j = 0,\dots,J+1$.

We introduce the space

$$\mathbb{C}_0^{J+2} := \big\{(w_0,\dots,w_{J+1})^T \in \mathbb{C}^{J+2} : w_0 = 0\big\},$$

and define the discrete operators Δ_h, $\delta_h : \mathbb{C}^{J+2} \longrightarrow \mathbb{C}_0^{J+2}$ by

$$\Delta_h v_j := \begin{cases} \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, & j = 1,\dots,J \\ \dfrac{2(v_J - v_{J+1})}{h^2}, & j = J+1 \end{cases} \quad \text{and} \quad \delta_h v_j := \begin{cases} \dfrac{v_{j+1} - v_{j-1}}{2h}, & j = 1,\dots,J \\ 0, & j = J+1 \end{cases}.$$

On $\mathbb{C}_0^{J+2}$ we define norms $\|\cdot\|_h$ (discrete L^2), $|\cdot|_{1,h}$ (discrete H^1), $|\!|\!|\cdot|\!|\!|_h$ (discrete L^1) and $\|\cdot\|_\infty$ (discrete L^∞), by the formulas

$$\|v\|_h := \Big\{ h\sum_{j=1}^{J} |v_j|^2 + \frac{h}{2}|v_{J+1}|^2 \Big\}^{1/2}, \quad |v|_{1,h} := \Big\{ h\sum_{j=0}^{J} \big|\tfrac{v_{j+1}-v_j}{h}\big|^2 \Big\}^{1/2},$$

$$|\!|\!|v|\!|\!|_h := h\sum_{j=1}^{J} |v_j| + \frac{h}{2}|v_{J+1}| \quad \text{and} \quad \|v\|_\infty := \max_{1\le j\le J+1} |v_j|.$$

The norm $\|\cdot\|_h$ corresponds to the discrete L^2 inner product $(\cdot,\cdot)_h$ defined by

$$(v,\chi)_h := h\sum_{j=1}^{J} v_j\overline{\chi_j} + \frac{h}{2} v_{J+1}\overline{\chi_{J+1}}, \quad \forall v,\chi\in\mathbb{C}_0^{J+2}.$$

It is straightforward to check that

$$\|v\|_h \le \|v\|_\infty \le |v|_{1,h}, \quad \forall v\in\mathbb{C}_0^{J+2}, \tag{3.1.1}$$

and

$$|(v,\chi)_h| \le |\!|\!|v|\!|\!|_h\,\|\chi\|_\infty \le |\!|\!|v|\!|\!|_h\,|\chi|_{1,h}, \quad \forall v,\chi\in\mathbb{C}_0^{J+2}. \tag{3.1.2}$$

Finally, given $\{V^n\}_{n=0}^N \subset \mathbb{C}_0^{J+2}$, we set $\partial V^n := \frac{V^{n+1}-V^n}{k}$ and $V^{n+\frac12} := \frac{V^{n+1}+V^n}{2}$ for $n=0,\dots,N-1$.

3.2 The scheme.

Let w be the solution of (1.2). We set $w^n := (w(y_0,t^n),\dots,w(y_{J+1},t^n))^T \in \mathbb{C}_0^{J+2}$ for $n=0,\dots,N$, and approximate w^n by $W^n\in\mathbb{C}_0^{J+2}$, specified recursively by the formulas

$$W^0 := w^0 \tag{3.2.1}$$

and, for $n=0,\dots,N-1$:

$$\partial W_j^n = i\frac{1}{\xi^{n+\frac12}}\Delta_h W_j^{n+\frac12} + y_j\mu^{n+\frac12}\delta_h W_j^{n+\frac12} + i\gamma_j^{n+\frac12}W_j^{n+\frac12} + \zeta_j^{n+\frac12}, \quad j=1,\dots,J, \tag{3.2.2a}$$

$$\begin{aligned} \partial W_{J+1}^n =& i\frac{1}{\xi^{n+\frac12}}\Big\{\Delta_h W_{J+1}^{n+\frac12} + \frac{2}{h}\Big[S_1^{n+\frac12}\partial W_{J+1}^n + S_2^{n+\frac12}W_{J+1}^{n+\frac12} + S_3^{n+\frac12}\Big]\Big\} \\ &+ y_{J+1}\mu^{n+\frac12}\Big[S_1^{n+\frac12}\partial W_{J+1}^n + S_2^{n+\frac12}W_{J+1}^{n+\frac12} + S_3^{n+\frac12}\Big] \\ &+ i\gamma_{J+1}^{n+\frac12}W_{J+1}^{n+\frac12} + \zeta_{J+1}^{n+\frac12}. \end{aligned} \tag{3.2.2b}$$

Here, $\xi^{n+\frac12} := \xi(t^{n+\frac12})$, $\mu^{n+\frac12} := \mu(t^{n+\frac12})$, $S_m^{n+\frac12} := S_m(t^{n+\frac12})$ for $m=1,2,3$, $\zeta_j^{n+\frac12} := \zeta(y_j,t^{n+\frac12})$, and $\gamma_j^{n+\frac12} := (\gamma_R)_j^{n+\frac12} + i(\gamma_I)_j^{n+\frac12}$ with $(\gamma_R)_j^{n+\frac12} := \gamma_R(y_j,t^{n+\frac12})$ and $(\gamma_I)_j^{n+\frac12} := \gamma_I(y_j,t^{n+\frac12})$. Given W^n, W^{n+1} is computed as the solution of a tridiagonal system of equations. The proof of existence of W^{n+1} is deferred until Corollary 3.1 below.

3.3 Consistency.

For $n = 0, \ldots, N-1$ we define $\eta^n \in \mathbb{C}_0^{J+2}$ by

$$\partial w_j^n = i\frac{1}{\xi^{n+\frac12}}\Delta_h w_j^{n+\frac12} + y_j \mu^{n+\frac12}\delta_h w_j^{n+\frac12} + i\gamma_j^{n+\frac12} w_j^{n+\frac12} + \zeta_j^{n+\frac12} + \eta_j^n, \quad j = 1, \ldots, J, \tag{3.3.1a}$$

and

$$\begin{aligned}\partial w_{J+1}^n =& i\frac{1}{\xi^{n+\frac12}}\Big\{\Delta_h w_{J+1}^{n+\frac12} + \frac{2}{h}\Big[S_1^{n+\frac12}\partial w_{J+1}^n + S_2^{n+\frac12} w_{J+1}^{n+\frac12} + S_3^{n+\frac12}\Big]\Big\}\\ &+ y_{J+1}\mu^{n+\frac12}\Big[S_1^{n+\frac12}\partial w_{J+1}^n + S_2^{n+\frac12} w_{J+1}^{n+\frac12} + S_3^{n+\frac12}\Big]\\ &+ i\gamma_{J+1}^{n+\frac12} w_{J+1}^{n+\frac12} + \zeta_{J+1}^{n+\frac12} + \eta_{J+1}^n.\end{aligned} \tag{3.3.1b}$$

In these formulas by $w_j^{n+\frac12}$ we mean $\frac12(w(y_j, t^{n+1}) + w(y_j, t^n))$, whilst the coefficients ξ, μ etc. are evaluated at $t^{n+\frac12}$ as noted above. Long but straightforward calculations and use of Taylor's formula yield the estimates

$$\max_{0\le n\le N-1}\Big\{\max_{1\le j\le J}|\eta_j^n|\Big\} \le \mathcal{C}_1(k^2 + h^2), \quad \max_{0\le n\le N-1}|\eta_{J+1}^n| \le \mathcal{C}_2\Big(k^2 + h + \frac{k^2}{h}\Big), \tag{3.3.2}$$

$$\max_{0\le n\le N-2}\Big\{\max_{1\le j\le J}|\partial\eta_j^n|\Big\} \le \mathcal{C}_3(k^2 + h^2), \quad \max_{0\le n\le N-2}|\partial\eta_{J+1}^n| \le \mathcal{C}_4\Big(k^2 + h + \frac{k^2}{h}\Big), \tag{3.3.3}$$

where the $\mathcal{C}_m$ are positive constants independent of k and h. To prove these order of accuracy estimates we need to assume, of course, that the coefficients and the solution of (1.2) are sufficiently smooth. For example, the first estimate of (3.3.2) requires that $\partial_y^m \partial_t^j w \in C(D)$ for $0 \le m \le 4$, $0 \le j \le 3$; we write that as $w \in C^{4,3}(D)$. The second estimate of (3.3.2) needs the lower regularity $w \in C^{3,3}(D)$, while the two estimates (3.3.3) for $\partial\eta^n$ require that $w \in C^{4,4}(D)$ and $w \in C^{3,4}(D)$, respectively.

It is important to note that (3.3.2) and (3.3.3) imply that the discrete L^1 norms of η^n and $\partial\eta^n$ are of optimal order, i.e., that

$$\max_{0\le n\le N-1}|\!|\!|\eta^n|\!|\!|_h \le \mathcal{C}(k^2 + h^2) \ \text{ and } \max_{0\le n\le N-2}|\!|\!|\partial\eta^n|\!|\!|_h \le \mathcal{C}(k^2 + h^2). \tag{3.3.4}$$

3.4 Stability and convergence.

Let $e^n := w^n - W^n \in \mathbb{C}_0^{J+2}$ for $n = 0, \ldots, N$. Using (3.2.1) and subtracting the relations (3.2.2) from those of (3.3.1), we get

$$e^0 = 0, \tag{3.4.1}$$

and for $n = 0, \ldots, N-1$:

$$\partial e_j^n = i\frac{1}{\xi^{n+\frac12}}\Delta_h e_j^{n+\frac12} + y_j\mu^{n+\frac12}\delta_h e_j^{n+\frac12} + i\gamma_j^{n+\frac12} e_j^{n+\frac12} + \eta_j^n, \quad j = 1, \ldots, J, \tag{3.4.2a}$$

$$\begin{aligned}\partial e_{J+1}^n =& i\frac{1}{\xi^{n+\frac12}}\Big\{\Delta_h e_{J+1}^{n+\frac12} + \frac{2}{h}\Big[S_1^{n+\frac12}\partial e_{J+1}^n + S_2^{n+\frac12} e_{J+1}^{n+\frac12}\Big]\Big\}\\ &+ y_{J+1}\mu^{n+\frac12}\Big[S_1^{n+\frac12}\partial e_{J+1}^n + S_2^{n+\frac12} e_{J+1}^{n+\frac12}\Big] + i\gamma_{J+1}^{n+\frac12} e_{J+1}^{n+\frac12} + \eta_{J+1}^n.\end{aligned} \tag{3.4.2b}$$

In proving our error estimates, we will consider the following two cases:

$$\text{(A.1)}\qquad \begin{array}{c}\text{The function } w \text{ is a sufficiently smooth solution of the problem (1.2)}\\ \text{with coefficients given by (1.3), with } \dot{s}(t)\le 0 \text{ for } t\in[0,T], \text{ and}\\ h^2 \le \frac{1}{2(2+C_{*,1})}\inf_{t\in[0,T]}\Big\{\frac{(s(t))^2}{[1+(\dot{s}(t))^2]^2}\Big\},\end{array}$$

or

$$\text{(A.2)}\qquad \begin{array}{c}\text{the function } w \text{ is a sufficiently smooth solution of the problem (1.2)}\\ \text{the condition } (\Sigma) \text{ is satisfied and}\\ h^2 \le -\frac{S_*}{4(2+C_{*,2})}.\end{array}$$

The constants $C_{*,1}$ and $C_{*,2}$ will be defined in the course of the proof below (cf. (3.4.15b)). Although, as was explained in Section 2, condition (Σ) is satisfied by the coefficients of a problem of the type (1.2)-(1.3) when $\dot{s}$ is strictly negative, we treat the case (A.1) separately to cover the case $\dot{s}(t)\le 0$ assuming of course existence, uniqueness and smoothness of u.

Fix $n\in\{0,\dots,N-1\}$. Multiply (3.4.2a) by $\overline{\partial e_j^n}-y_j\mu^{n+\frac12}\overline{\delta_h e_j^{n+\frac12}}$ and then sum the resulting equations with respect to j from 1 to J. Also, multiply (3.4.2b) by $\frac12\overline{\partial e_{J+1}^n}$, and then add the resulting relation to the previous sum. Take imaginary parts and multiply by $h\xi^{n+\frac12}$ to obtain

$$-\,\mathsf{Re}(\Delta_h e^{n+\frac12},\partial e^n)_h=\xi^{n+\frac12}\,\mathsf{Im}(\eta^n,\partial e^n)_h+\sum_{j=1}^{7}\Psi_j^n, \tag{3.4.3}$$

where

$$\Psi_1^n:=\frac{h}{2}\mu^{n+\frac12}\xi^{n+\frac12}\,\mathsf{Im}\Big\{S_2^{n+\frac12}e_{J+1}^{n+\frac12}\overline{\partial e_{J+1}^n}\Big\},\quad \Psi_2^n:=\mathsf{Re}\Big\{S_2^{n+\frac12}e_{J+1}^{n+\frac12}\overline{\partial e_{J+1}^n}\Big\}, \tag{3.4.4a}$$

$$\Psi_3^n:=-\mu^{n+\frac12}\xi^{n+\frac12}\,\mathsf{Re}\Big\{h\sum_{j=1}^{J}\gamma_j^{n+\frac12}y_je_j^{n+\frac12}\overline{\delta_h e_j^{n+\frac12}}\Big\}, \tag{3.4.4b}$$

$$\Psi_4^n:=-\mu^{n+\frac12}\xi^{n+\frac12}\,\mathsf{Im}\Big\{h\sum_{j=1}^{J}y_j\eta_j^n\overline{\delta_h e_j^{n+\frac12}}\Big\}, \tag{3.4.4c}$$

$$\Psi_5^n:=\xi^{n+\frac12}\,\mathsf{Re}\Big\{h\sum_{j=1}^{J}\gamma_j^{n+\frac12}e_j^{n+\frac12}\overline{\partial e_j^n}+\frac{h}{2}\gamma_{J+1}^{n+\frac12}e_{J+1}^{n+\frac12}\overline{\partial e_{J+1}^n}\Big\}, \tag{3.4.4d}$$

$$\Psi_6^n:=-\mu^{n+\frac12}\,\mathsf{Re}\Big\{h\sum_{j=1}^{J}y_j\Delta_h e_j^{n+\frac12}\overline{\delta_h e_j^{n+\frac12}}\Big\}\quad\text{and}\quad \Psi_7^n:=S_1^{n+\frac12}|\partial e_{J+1}^n|^2. \tag{3.4.4e}$$

We estimate now the various terms in the right-hand side of (3.4.3).

`Estimation of` Ψ_1^n, Ψ_2^n, Ψ_3^n `and` Ψ_4^n: It is straightforward to check that

$$\Psi_1^n+\Psi_2^n\le\begin{cases}\varepsilon\,|S_1^{n+\frac12}|\,|\partial e_{J+1}^n|^2+\dfrac{C}{\varepsilon}\|e^{n+\frac12}\|_\infty^2,\\[2mm] \varepsilon\,|\partial e_{J+1}^n|^2+\dfrac{C}{\varepsilon}\|e^{n+\frac12}\|_\infty^2,\end{cases}\qquad \forall\varepsilon>0. \tag{3.4.5a}$$

(The two branches in the right-hand side of the inequality (3.4.5a) — and of other similar inequalities that follow — correspond to the two routes in the proof alluded to above. If (A.1) is assumed, we shall make use of the upper estimates; if (A.2) holds, of the lower.)

We also have

$$\Psi_3^n \le C\|e^{n+\frac12}\|_h |e^{n+\frac12}|_{1,h} \tag{3.4.5b}$$

and

$$\Psi_4^n \le C \max_{1\le j\le J} |\eta_j^n| \ \ |e^{n+\frac12}|_{1,h}. \tag{3.4.6}$$

`Estimation of` Ψ_5^n: Using (3.4.2a-b) we obtain

$$\Psi_5^n := \sum_{j=1}^{5} \Upsilon_j^n, \tag{3.4.7}$$

with

$$\Upsilon_1^n := -\,\mathsf{Im}\Big\{ \ h\sum_{j=1}^{J} \overline{\gamma_j^{n+\frac12}}\,\Delta_h e_j^{n+\frac12}\,\overline{e_j^{n+\frac12}} + \frac{h}{2}\overline{\gamma_{J+1}^{n+\frac12}}\,\Delta_h e_{J+1}^{n+\frac12}\,\overline{e_{J+1}^{n+\frac12}} \ \Big\}, \tag{3.4.8a}$$

$$\Upsilon_2^n := \mu^{n+\frac12}\xi^{n+\frac12}\ \mathsf{Re}\Big\{h\sum_{j=1}^{J}\overline{\gamma_j^{n+\frac12}}\,y_j\delta_h e_j^{n+\frac12}\,\overline{e_j^{n+\frac12}}\Big\}, \tag{3.4.8b}$$

$$\Upsilon_3^n := \xi^{n+\frac12}\ \mathsf{Re}\Big\{h\sum_{j=1}^{J}\overline{\gamma_j^{n+\frac12}}\,\eta_j^n\,\overline{e_j^{n+\frac12}} + \frac{h}{2}\overline{\gamma_{J+1}^{n+\frac12}}\,\eta_{J+1}^n\,\overline{e_{J+1}^{n+\frac12}}\Big\}, \tag{3.4.8c}$$

$$\Upsilon_4^n := \mu^{n+\frac12}\xi^{n+\frac12}\frac{h}{2}\ \mathsf{Re}\Big\{S_1^{n+\frac12}\partial e_{J+1}^n\overline{\gamma_{J+1}^{n+\frac12}\,e_{J+1}^{n+\frac12}} + S_2^{n+\frac12}\overline{\gamma_{J+1}^{n+\frac12}}|e_{J+1}^{n+\frac12}|^2\Big\}, \tag{3.4.8d}$$

$$\Upsilon_5^n := -\,\mathsf{Im}\Big\{S_1^{n+\frac12}\partial e_{J+1}^n\overline{\gamma_{J+1}^{n+\frac12}\,e_{J+1}^{n+\frac12}} + S_2^{n+\frac12}\overline{\gamma_{J+1}^{n+\frac12}}|e_{J+1}^{n+\frac12}|^2\Big\}. \tag{3.4.8e}$$

Now, we can easily see that

$$\Upsilon_2^n \le C\|e^{n+\frac12}\|_h\,|e^{n+\frac12}|_{1,h},\quad \Upsilon_3^n \le C|\!|\!|\eta^n|\!|\!|_h\,\|e^{n+\frac12}\|_\infty \tag{3.4.9a}$$

and

$$\Upsilon_4^n + \Upsilon_5^n \le \begin{cases} \varepsilon|S_1^{n+\frac12}|\,|\partial e_{J+1}^n|^2 + C(1+\frac{1}{\varepsilon})\|e^{n+\frac12}\|_\infty^2, \\ \varepsilon|\partial e_{J+1}^n|^2 + C(1+\frac{1}{\varepsilon})\|e^{n+\frac12}\|_\infty^2, \end{cases} \quad \forall\varepsilon>0. \tag{3.4.9b}$$

Also, we have

$$\begin{aligned}\Upsilon_1^n =& -\frac{1}{h}\,\mathsf{Im}\Big\{ \sum_{j=0}^{J}\overline{\gamma_j^{n+\frac12}}(e_{j+1}^{n+\frac12}-e_j^{n+\frac12})\overline{e_j^{n+\frac12}} - \sum_{j=0}^{J}\overline{\gamma_{j+1}^{n+\frac12}}(e_{j+1}^{n+\frac12}-e_j^{n+\frac12})\overline{e_{j+1}^{n+\frac12}} \ \Big\}\\ =& -h\sum_{j=0}^{J}(\gamma_I)_j^{n+\frac12}\Big|\tfrac{e_{j+1}^{n+\frac12}-e_j^{n+\frac12}}{h}\Big|^2 + \mathsf{Im}\Big\{\sum_{j=0}^{J}\frac{\gamma_j^{n+\frac12}-\gamma_{j+1}^{n+\frac12}}{h}\overline{(e_{j+1}^{n+\frac12}-e_j^{n+\frac12})}e_{j+1}^{n+\frac12}\Big\},\end{aligned}$$

which yields

$$\Upsilon_1^n \le C\big\{|e^{n+\frac12}|_{1,h}^2 + \|e^{n+\frac12}\|_h |e^{n+\frac12}|_{1,h}\big\}. \tag{3.4.9c}$$

Using (3.4.7), (3.4.8a-e), (3.4.9a-c) and (3.1.1) , we conclude that

$$\Psi_5^n \le C(1+\tfrac{1}{\varepsilon})\big\{|e^{n+\frac12}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big\} + \varepsilon |S_1^{n+\frac12}|\,|\partial e_{J+1}^n|^2, \quad \forall \varepsilon > 0, \tag{3.4.10a}$$

or

$$\Psi_5^n \le C(1+\tfrac{1}{\varepsilon})\big\{|e^{n+\frac12}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big\} + \varepsilon |\partial e_{J+1}^n|^2, \quad \forall \varepsilon > 0. \tag{3.4.10b}$$

`Estimation of` $\Psi_6^n + \Psi_7^n$: From (3.4.4e), we have

$$\begin{aligned}\Psi_6^n &= -\frac{\mu^{n+\frac12}}{2}\Big\{ \sum_{j=1}^{J} y_j \Big|\tfrac{e_{j+1}^{n+\frac12} - e_j^{n+\frac12}}{h}\Big|^2 - \sum_{j=1}^{J} y_j \Big|\tfrac{e_{j}^{n+\frac12} - e_{j-1}^{n+\frac12}}{h}\Big|^2 \Big\}\\ &= -\frac{\mu^{n+\frac12}}{2}\Big\{ \Big|\tfrac{e_{J+1}^{n+\frac12} - e_J^{n+\frac12}}{h}\Big|^2 - h\sum_{j=0}^{J} \Big|\tfrac{e_{j+1}^{n+\frac12} - e_j^{n+\frac12}}{h}\Big|^2 \Big\},\end{aligned}$$

which yields

$$\Psi_6^n \le -\frac{\mu^{n+\frac12}}{2}\Big|\tfrac{e_{J+1}^{n+\frac12} - e_J^{n+\frac12}}{h}\Big|^2 + C|e^{n+\frac12}|_{1,h}^2. \tag{3.4.11}$$

From (3.4.2b), multiplying by $ih\frac{\xi^{n+\frac12}}{2}$, we obtain

$$\begin{aligned}\tfrac{e_{J+1}^{n+\frac12} - e_J^{n+\frac12}}{h} =& S_1^{n+\frac12}\partial e_{J+1}^n + ih\xi^{n+\frac12}\Big[\tfrac{1-\mu^{n+\frac12}S_1^{n+\frac12}}{2}\Big]\partial e_{J+1}^n\\ &+ \Big[S_2^{n+\frac12} + h\tfrac{\xi^{n+\frac12}\gamma_{J+1}^{n+\frac12}}{2} - ih\tfrac{\mu^{n+\frac12}\xi^{n+\frac12}S_2^{n+\frac12}}{2}\Big]e_{J+1}^{n+\frac12} - ih\tfrac{\xi^{n+\frac12}}{2}\eta_{J+1}^n.\end{aligned} \tag{3.4.12}$$

If we use now the identity $|z_1+z_2+z_3+z_4|^2 = \sum_{j=1}^{4}|z_j|^2 + 2\sum_{j=1}^{3}\sum_{\ell=j+1}^{4}\mathsf{Re}(z_j\overline{z_\ell})$ for $z_1,\ z_2,\ z_3,\ z_4 \in \mathbb{C}$, we conclude from (3.4.12) that

$$-\frac{\mu^{n+\frac12}}{2}\Big|\tfrac{e_{J+1}^{n+\frac12} - e_J^{n+\frac12}}{h}\Big|^2 = \sum_{j=1}^{10} E_j^n, \tag{3.4.13}$$

with

$$\begin{aligned}E_1^n &:= -\tfrac{\mu^{n+\frac12}}{2}(S_1^{n+\frac12})^2|\partial e_{J+1}^n|^2,\\ E_2^n &:= -\tfrac{\mu^{n+\frac12}}{2}h^2(\xi^{n+\frac12})^2\Big[\tfrac{1-\mu^{n+\frac12}S_1^{n+\frac12}}{2}\Big]^2|\partial e_{J+1}^n|^2,\\ E_3^n &:= -\tfrac{\mu^{n+\frac12}}{2}\Big|S_2^{n+\frac12} + h\tfrac{\xi^{n+\frac12}\gamma_{J+1}^{n+\frac12}}{2} - ih\tfrac{\mu^{n+\frac12}\xi^{n+\frac12}S_2^{n+\frac12}}{2}\Big|^2|e_{J+1}^{n+\frac12}|^2,\\ E_4^n &:= -h^2\tfrac{\mu^{n+\frac12}(\xi^{n+\frac12})^2}{8}|\eta_{J+1}^n|^2,\\ E_5^n &:= -\tfrac{\mu^{n+\frac12}}{2}\,\mathsf{Re}\Big\{iS_1^{n+\frac12}\xi^{n+\frac12}h\Big[-1+\mu^{n+\frac12}S_1^{n+\frac12}\Big]\Big\}|\partial e_{J+1}^n|^2 = 0,\end{aligned} \tag{3.4.14a}$$

and

$$
\begin{aligned}
E_6^n &:= -\tfrac{\mu^{n+\frac12}}{2} S_1^{n+\frac12}\,\mathrm{Re}\Big\{\Big[2S_2^{n+\frac12} + h\xi^{n+\frac12}\gamma_{J+1}^{n+\frac12} - ih\mu^{n+\frac12}\xi^{n+\frac12}S_2^{n+\frac12}\Big]e_{J+1}^{n+\frac12}\overline{\partial e_{J+1}^n}\Big\},\\
E_7^n &:= -\tfrac{\mu^{n+\frac12}\xi^{n+\frac12}}{2} S_1^{n+\frac12}\,\mathrm{Im}\Big\{h\eta_{J+1}^n\overline{\partial e_{J+1}^n}\Big\},\\
E_8^n &:= -h\tfrac{\mu^{n+\frac12}}{4}\xi^{n+\frac12}\big[1-\mu^{n+\frac12}S_1^{n+\frac12}\big]\\
&\qquad \mathrm{Im}\Big\{\Big[2S_2^{n+\frac12} + h\xi^{n+\frac12}\gamma_{J+1}^{n+\frac12} - ih\mu^{n+\frac12}\xi^{n+\frac12}S_2^{n+\frac12}\Big]e_{J+1}^{n+\frac12}\overline{\partial e_{J+1}^n}\Big\},\\
E_9^n &:= -h\tfrac{\mu^{n+\frac12}}{4}(\xi^{n+\frac12})^2(\mu^{n+\frac12}S_1^{n+\frac12}-1)\,\mathrm{Re}\Big\{\partial e_{J+1}^n h\overline{\eta_{J+1}^n}\Big\},\\
E_{10}^n &:= -\tfrac{\mu^{n+\frac12}}{4}\xi^{n+\frac12}\,\mathrm{Re}\Big\{i\Big[2S_2^{n+\frac12} + h\xi^{n+\frac12}\gamma_{J+1}^{n+\frac12} - ih\mu^{n+\frac12}\xi^{n+\frac12}S_2^{n+\frac12}\Big]e_{J+1}^{n+\frac12}h\overline{\eta_{J+1}^n}\Big\}.
\end{aligned}
\tag{3.4.14b}
$$

Estimating the terms E_j^n we obtain

$$
E_3^n \le C\|e^{n+\frac12}\|_\infty^2,\quad E_4^n \le C|\!|\!|\eta^n|\!|\!|_h^2,\quad E_{10}^n \le C|\!|\!|\eta^n|\!|\!|_h\|e^{n+\frac12}\|_\infty,
\tag{3.4.15a}
$$

$$
E_2^n \le \begin{cases} C_{*,1}\,h^2\,|\mu^{n+\frac12}|\,|\partial e_{J+1}^n|^2\\ C_{*,2}\,h^2\,|\partial e_{J+1}^n|^2\end{cases},\qquad E_7^n \le \begin{cases}\varepsilon\,|S_1^{n+\frac12}|\,|\partial e_{J+1}^n|^2 + \dfrac{C}{\varepsilon}|\!|\!|\eta^n|\!|\!|_h^2\\ \varepsilon|\partial e_{J+1}^n|^2 + \dfrac{C}{\varepsilon}|\!|\!|\eta^n|\!|\!|_h^2\end{cases},\quad \forall\varepsilon>0,
\tag{3.4.15b}
$$

$$
E_9^n \le \begin{cases} h^2\,|\mu^{n+\frac12}|\,|\partial e_{J+1}^n|^2 + C|\!|\!|\eta^n|\!|\!|_h^2\\ h^2\,|\partial e_{J+1}^n|^2 + C|\!|\!|\eta^n|\!|\!|_h^2\end{cases},\qquad E_8^n \le \begin{cases} h^2\,|\mu^{n+\frac12}|\,|\partial e_{J+1}^n|^2 + C\|e^{n+\frac12}\|_\infty^2\\ h^2\,|\partial e_{J+1}^n|^2 + C\|e^{n+\frac12}\|_\infty^2\end{cases},
\tag{3.4.15c}
$$

and

$$
E_6^n \le \begin{cases}\varepsilon\,|S_1^{n+\frac12}|\,|\partial e_{J+1}^n|^2 + \dfrac{C}{\varepsilon}\|e^{n+\frac12}\|_\infty^2\\ \varepsilon\,|\partial e_{J+1}^n|^2 + \dfrac{C}{\varepsilon}\|e^{n+\frac12}\|_\infty^2\end{cases},\qquad \forall\varepsilon>0.
\tag{3.4.15d}
$$

Finally, using (3.4.4e), (3.4.11), (3.4.13), (3.4.14a-b), (3.4.15a-d) and (3.1.1), we obtain

$$
\begin{aligned}
\Psi_6^n + \Psi_7^n \le& C(1+\tfrac{1}{\varepsilon})\big\{|e^{n+\frac12}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big\}\\
&+\big\{\mathcal{A}^{n+\frac12} + \varepsilon|S_1^{n+\frac12}| + h^2(2+C_{*,1})|\mu^{n+\frac12}|\big\}|\partial e_{J+1}^n|^2,\quad \forall\varepsilon>0,
\end{aligned}
\tag{3.4.16a}
$$

or

$$
\Psi_6^n + \Psi_7^n \le C(1+\tfrac{1}{\varepsilon})\big\{|e^{n+\frac12}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big\} + \big\{\mathcal{A}^{n+\frac12} + \varepsilon + h^2(2+C_{*,2})\big\}|\partial e_{J+1}^n|^2,\quad \forall\varepsilon>0,
\tag{3.4.16b}
$$

with

$$
\mathcal{A}^{n+\frac12} := S_1^{n+\frac12}\Big(1-\frac12\mu^{n+\frac12}S_1^{n+\frac12}\Big).
\tag{3.4.16c}
$$

Putting all these estimates together, we may now prove

Lemma 3.1. *Under the hypotheses* (A.1) *or* (A.2)*, for some constant* C *independent of* h *and* k*, we have*

(3.4.17)
$$
\begin{aligned}
|e^{n+1}|_{1,h}^2 \le & |e^n|_{1,h}^2 + Ck\big\{|e^{n+1}|_{1,h}^2 + |e^n|_{1,h}^2 + (k^2+h^2)^2\big\} \\
& + 2k\xi^{n+\frac{1}{2}}\,\mathrm{Im}(\eta^n, \partial e^n)_h, \quad n = 0, \dots, N-1.
\end{aligned}
$$

Proof. Noting that $(\Delta_h v, \chi)_h = \overline{(\Delta_h \chi, v)_h}$ and $(\Delta_h v, v)_h = -|v|_{1,h}^2$ for $v,\, \chi \in \mathbb{C}_0^{J+2}$, we have

$$
-\mathrm{Re}(\Delta_h e^{n+\frac{1}{2}}, \partial e^n)_h = \frac{1}{2k}\big(|e^{n+1}|_{1,h}^2 - |e^n|_{1,h}^2\big).
$$

Using now (3.4.3), (3.4.4a-e), (3.4.5a-b), (3.4.6), (3.4.10a-b), (3.4.16a-b), (3.3.2), (3.3.4), (3.1.1) and (1.3), we obtain

(3.4.18)
$$
\begin{aligned}
|e^{n+1}|_{1,h}^2 - |e^n|_{1,h}^2 \le & C(1+\tfrac{1}{\varepsilon})k\big(|e^{n+1}|_{1,h}^2 + |e^n|_{1,h}^2\big) + 2k\xi^{n+\frac{1}{2}}\,\mathrm{Im}(\eta^n, \partial e^n)_h \\
& + C(1+\tfrac{1}{\varepsilon})k(k^2+h^2)^2 + 2k\mathcal{B}_\varepsilon|\partial e_{J+1}^n|^2, \quad \forall \varepsilon > 0,
\end{aligned}
$$

where

(3.4.19)
$$
\mathcal{B}_\varepsilon := \begin{cases} \mathcal{A}^{n+\frac{1}{2}} - 3\varepsilon S_1^{n+\frac{1}{2}} - h^2(2+C_{*,1})\mu^{n+\frac{1}{2}} & \text{in the case (A.1)} \\ \frac{S_*}{2} + h^2(2+C_{*,2}) + 3\varepsilon & \text{in the case (A.2)} \end{cases}.
$$

In the case (A.1), choosing $\varepsilon = \frac{1}{6}$ we have

(3.4.20a)
$$
\begin{aligned}
\mathcal{B}_\varepsilon = & \mathcal{A}^{n+\frac{1}{2}} - \tfrac{1}{2}S_1^{n+\frac{1}{2}} - h^2(2+C_{*,1})\mu^{n+\frac{1}{2}} \\
= & \tfrac{\dot{s}(t^{n+\frac{1}{2}})}{s(t^{n+\frac{1}{2}})}\Big\{\tfrac{(s(t^{n+\frac{1}{2}}))^2}{2[1+(\dot{s}(t^{n+\frac{1}{2}}))^2]^2} - h^2(2+C_{*,1})\Big\} \le 0.
\end{aligned}
$$

In the case (A.2), if $\varepsilon = -\frac{S_*}{12}$, it follows that

(3.4.20b)
$$
\mathcal{B}_\varepsilon = \tfrac{S_*}{4} + h^2(2+C_{*,2}) \le 0.
$$

We conclude then, easily, that (3.4.17) holds, by combining (3.4.18), (3.4.19) and (3.4.20a-b). ■

Corollary 3.1. *If the coefficients of the problem (1.2) are given by the formulas (1.3) and we suppose that* $\dot{s}(t) \le 0$ *for* $t \in [0,T]$*, or, alternatively, if the condition* (Σ) *holds on the coefficients of (1.2), the fully discrete scheme (3.2.1)–(3.2.2a-b) has a unique solution* $\{W^n\}_{n=0}^N$*, provided* h *and* k *are sufficiently small.*

Proof. The solution $\{W^n\}_{n=0}^N$ of the fully discrete scheme satisfies the equations (3.2.1) and (3.2.2a-b). Since our problem is linear, the formulas (3.2.2a-b) are identical with the homogeneous error equations obtained by setting $\eta_j^n = 0$, $1 \le j \le J+1$, in (3.4.2a-b). Therefore, the analysis leading to the homogeneous version of (3.4.18) holds, mutatis mutandis, for the W_j^n as well. Under our hypotheses therefore, it follows that for a fixed n we have

$$
|W^{n+1}|_{1,h}^2 - |W^n|_{1,h}^2 \le Ck(|W^{n+1}|_{1,h}^2 + |W^n|_{1,h}^2).
$$

Hence, if $W^n = 0$ we have, for k sufficiently small, $W^{n+1} = 0$ implying uniqueness, and hence existence of the solution of the linear system of equations that defines W^{n+1} in terms of W^n. ■

We are now in position to conclude our error estimation argument:

Theorem 3.1. *Under the hypotheses* (A.1) *or* (A.2), *if k is sufficiently small, we have*

$$\max_{0\le n\le N}\|e^n\|_\infty \le C(k^2+h^2) \tag{3.4.21}$$

for some constant C independent of h and k.

Proof. Using Lemma 3.1, we have

$$\begin{aligned}|e^{n+1}|^2_{1,h} \le & |e^n|^2_{1,h} + Ck\big\{|e^{n+1}|^2_{1,h} + |e^n|^2_{1,h} + (k^2+h^2)^2\big\} + 2k\xi(t^n)\,\mathrm{Im}(\eta^n,\partial e^n)_h\\ & + 2k\big[\xi(t^{n+\frac12})-\xi(t^n)\big]\,\mathrm{Im}(\eta^n,\partial e^n)_h,\quad n=0,\dots,N-1.\end{aligned} \tag{3.4.22}$$

Now, using (3.1.2), (3.1.1) and (3.3.4), we have

$$\begin{aligned}2k\big[\xi(t^{n+\frac12})-\xi(t^n)\big]\,\mathrm{Im}(\eta^n,\partial e^n)_h \le & Ck^2|\!|\!|\eta^n|\!|\!|_h\|\partial e^n\|_\infty\\ \le & Ck\big\{|\!|\!|\eta^n|\!|\!|_h^2 + (|e^{n+1}|_{1,h}+|e^n|_{1,h})^2\big\}\\ \le & Ck\big\{(k^2+h^2)^2 + |e^{n+1}|^2_{1,h} + |e^n|^2_{1,h}\big\}.\end{aligned}$$

Hence, (3.4.21) yields

$$|e^{n+1}|^2_{1,h} \le |e^n|^2_{1,h} + C_*k\big\{|e^{n+1}|^2_{1,h} + |e^n|^2_{1,h} + (k^2+h^2)^2\big\} + 2k\xi(t^n)\,\mathrm{Im}(\eta^n,\partial e^n)_h,\quad n=0,\dots,N-1.$$

Assuming that $kC_*\le\frac13$, we get from the above

$$|e^{n+1}|^2_{1,h} \le \left(\tfrac{1+C_*k}{1-C_*k}\right)|e^n|^2_{1,h} + \tfrac{1}{1-C_*k}k\Big[C_*(k^2+h^2)^2 + 2\xi(t^n)\,\mathrm{Im}(\eta^n,\partial e^n)_h\Big],\quad n=0,\dots,N-1.$$

Then, using a simple induction argument and (3.4.1), we obtain

$$|e^n|^2_{1,h} \le k\sum_{m=0}^{n-1}\left(\tfrac{1+C_*k}{1-C_*k}\right)^{n-1-m}\tfrac{1}{1-C_*k}\Big[C_*(k^2+h^2)^2 + 2\xi(t^m)\,\mathrm{Im}(\eta^m,\partial e^m)_h\Big],\quad n=0,\dots,N. \tag{3.4.23}$$

Let $\delta_*:=\frac{1+C_*k}{1-C_*k}$ and $n\in\{1,\dots,N\}$. Using (3.1.1) in (3.4.23), we have

$$\begin{aligned}|e^n|^2_{1,h} \le & C(k^2+h^2)^2 + \tfrac{2\xi(t^{n-1})}{(1-C_*k)}\,\mathrm{Im}(\eta^{n-1},e^n)_h - \tfrac{2}{(1-C_*k)}\,\mathrm{Im}\Big\{k\sum_{m=1}^{n-1}(\delta_*)^{n-1-m}\xi(t^m)(\partial\eta^{m-1},e^m)_h\\ & + \sum_{m=1}^{n-1}\big[\delta_*^{n-1-m}\big(\xi(t^m)-\xi(t^{m-1})\big) + \big(\delta_*^{n-1-m}-\delta_*^{n-m}\big)\xi(t^{m-1})\big](\eta^{m-1},e^m)_h\Big\},\end{aligned}$$

i.e. finally

$$|e^n|^2_{1,h} \le C\Big\{(k^2+h^2)^2 + \{\max_{0\le m\le N-1}|\!|\!|\eta^m|\!|\!|_h + \max_{0\le m\le N-2}|\!|\!|\partial\eta^m|\!|\!|_h\}\max_{0\le m\le N}|e^m|_{1,h}\Big\}. \tag{3.4.24}$$

Hence, from (3.4.24) and (3.3.4), we get

$$\max_{0\le n\le N}|e^n|^2_{1,h} \le C\big\{(k^2+h^2)^2 + (k^2+h^2)\max_{0\le m\le N}|e^m|_{1,h}\big\},$$

which yields

$$\max_{0\le m\le N}|e^m|_{1,h} \le C(k^2+h^2). \tag{3.4.25}$$

Finally, (3.4.21) follows easily from (3.4.25) and (3.1.1). ■

4. L^2 and H^1 estimates for problem (1.5)

In this section we shall consider the initial and boundary value problem (1.5) and prove *a priori* estimates in L^2 and H^1 for its solution. We shall use for the norms, function spaces etc. the notation introduced in Section 2.

Theorem 4.1. *Let w be the solution of problem (1.5) with $g = 0$. Then, we have*

$$\|w(\cdot,t)\| \le \exp(Ct)\Big\{\|w_0\| + 2\int_0^t \|\zeta(\cdot,\tau)\|d\tau\Big\}, \quad \forall t\in[0,T], \tag{4.1}$$

and

$$\|w(\cdot,t)\|_1^2 \le C\exp(Ct)\Big\{\|w_0\|_1 + \sup_{\tau\in[0,t]}\|\zeta(\cdot,\tau)\|_{-1}^2 + \int_0^t \big(\|\zeta(\cdot,\tau)\|_{-1}^2 + \|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2\big)d\tau\Big\}, \quad \forall t\in[0,T]. \tag{4.2}$$

If, in addition, $\gamma_I = 0$ and $\zeta = 0$, the L^2 norm is conserved, i.e.

$$\|w(\cdot,t)\| = \|w_0\|, \quad \forall t\in[0,T]. \tag{4.3}$$

Proof. We take the $(\cdot,\cdot)$ inner product of (1.5a) with $2w$, and then real parts to obtain

$$\begin{aligned}\frac{d}{dt}\|w(\cdot,t)\|^2 = &-\tfrac{2}{\xi(t)}\,\mathrm{Im}\big[w_y(1,t)\overline{w}(1,t)\big] + 2\mu(t)\,\mathrm{Re}(\omega w_y(\cdot,t),w(\cdot,t)) + \mu(t)\|w(\cdot,t)\|^2\\ &- 2(\gamma_I(\cdot,t)w(\cdot,t),w(\cdot,t)) + 2\,\mathrm{Re}(\zeta(\cdot,t),w(\cdot,t)), \quad \forall t\in[0,T],\end{aligned}$$

which, using (1.5c) and (2.1a), yields

$$\frac{d}{dt}\|w(\cdot,t)\|^2 = -2(\gamma_I(\cdot,t)w(\cdot,t),w(\cdot,t)) + 2\,\mathrm{Re}(\zeta(\cdot,t),w(\cdot,t)), \quad \forall t\in[0,T]. \tag{4.4}$$

Conservation of the L^2 norm, i.e. (4.3), follows immediately if $\gamma_I = \zeta = 0$. Otherwise, integrating (4.4) with respect to t, we obtain

$$\|w(\cdot,t)\| \le \Big\{\|w_0\| + 2\int_0^t \|\zeta(\cdot,\tau)\|d\tau\Big\} + C\int_0^t \|w(\cdot,\tau)\|d\tau, \quad \forall t\in[0,T]. \tag{4.5}$$

Applying the Gronwall lemma to (4.5) we obtain (4.1).

We now take the $(\cdot,\cdot)$ inner product of (1.5a) with w_t and use (1.5c), to obtain

$$\begin{aligned}\|w_t(\cdot,t)\|^2 = &-\tfrac{i}{\xi(t)}(w_y(\cdot,t),w_{yt}(\cdot,t)) + i(\gamma(\cdot,t)w(\cdot,t),w_t(\cdot,t)) + (\zeta(\cdot,t),w_t(\cdot,t))\\ &+ \tfrac{\mu(t)}{2}(\omega w_y(\cdot,t),w_t(\cdot,t)) - \tfrac{\mu(t)}{2}(\omega w(\cdot,t),w_{yt}(\cdot,t), \quad \forall t\in[0,T],\end{aligned} \tag{4.6}$$

where $\gamma := \gamma_R + i\gamma_I$. Taking now imaginary parts in (4.6) and setting $\varrho(t) := \mu(t)\xi(t)$ for $t\in[0,T]$, we have

$$\begin{aligned}\frac{d}{dt}\Big[\|w_y(\cdot,t)\|^2 - \varrho(t)\,\mathrm{Im}(\omega w_y(\cdot,t),w(\cdot,t))\Big] =& 2\xi(t)\,\mathrm{Re}(\gamma(\cdot,t)w(\cdot,t),w_t(\cdot,t))\\ &+ \tfrac{d}{dt}\Big[\mathrm{Im}(2\xi(t)\zeta(\cdot,t),w(\cdot,t))\Big]\\ &- 2\dot{\xi}(t)\,\mathrm{Im}(\zeta(\cdot,t),w(\cdot,t)) - 2\xi(t)\,\mathrm{Im}(\zeta_t(\cdot,t),w(\cdot,t))\\ &- \dot{\varrho}(t)\,\mathrm{Im}(\omega w_y(\cdot,t),w(\cdot,t)), \quad \forall t\in[0,T].\end{aligned} \tag{4.7}$$

Let

$$\nu(t;\varphi) := \|\varphi'\|^2 - \varrho(t)\,\mathrm{Im}(\omega\varphi',\varphi) + \mathcal{M}\|\varphi\|^2, \quad \forall t\in[0,T], \quad \forall \varphi \in \overset{\circ}{H},$$

where $\mathcal{M} := \frac{1}{2}(1+\sup_{t\in[0,T]}|\varrho(t)|^2)$. Then, we can easily see that

$$c_1\|\varphi\|_1^2 \le \nu(t;\varphi) \le c_2\|\varphi\|_1^2, \quad \forall t\in[0,T], \quad \forall \varphi\in\overset{\circ}{H}. \tag{4.8}$$

From (4.4) and (4.7), we obtain that

$$\begin{aligned}\frac{d}{dt}\nu(t;w(\cdot,t)) \le & C\big[\nu(t;w(\cdot,t)) + \|\zeta(\cdot,t)\|_{-1}^2 + \|\zeta_t(\cdot,t)\|_{-1}^2\big] \\ & + \tfrac{d}{dt}\Big[\mathrm{Im}(2\xi(t)\zeta(\cdot,t),w(\cdot,t))\Big] \\ & + 2\xi(t)\,\mathrm{Re}(w_t,\gamma(\cdot,t)w(\cdot,t)), \quad \forall t\in[0,T].\end{aligned} \tag{4.9}$$

Multiplying the differential equation (1.5a) by $2\xi\overline{\gamma w}$, integrating over $(0,1)$ by parts, using (1.5c) and taking real parts we obtain

$$\begin{aligned}2\xi(t)\,\mathrm{Re}(w_t(\cdot,t),\gamma(\cdot,t)w(\cdot,t)) = & 2\xi(t)\mu(t)\,\mathrm{Re}(\omega w_y(\cdot,t),\gamma(\cdot,t)w(\cdot,t)) + 2\xi(t)\,\mathrm{Re}(\zeta(\cdot,t),\gamma(\cdot,t)w(\cdot,t)) \\ & + \xi(t)\mu(t)\,\mathrm{Re}(w(\cdot,t),\gamma(\cdot,t)w(\cdot,t)) \\ & - 2\,\mathrm{Im}(w_y(\cdot,t),\gamma_y(\cdot,t)w(\cdot,t)) - 2\,\mathrm{Im}(w_y(\cdot,t),\gamma(\cdot,t)w_y(\cdot,t)) \\ & - \gamma_R(1,t)\mu(t)\xi(t)|w(1,t)|^2, \quad \forall t\in[0,T].\end{aligned} \tag{4.10}$$

Using (4.9), (4.10) and (2.1b), we arrive at

$$\begin{aligned}\frac{d}{dt}\nu(t;w(\cdot,t)) \le & C\big[\nu(t;w(\cdot,t)) + \|\zeta(\cdot,t)\|_{-1}^2 + \|\zeta_t(\cdot,t)\|_{-1}^2\big] \\ & + \tfrac{d}{dt}\Big[\mathrm{Im}(2\xi(t)\zeta(\cdot,t),w(\cdot,t))\Big], \quad \forall t\in[0,T].\end{aligned} \tag{4.11}$$

Integrating (4.11) with respect to t, we obtain

$$\begin{aligned}\nu(t;w(\cdot,t)) \le & \nu(0;w_0) + C\int_0^t \big\{\nu(\tau;w(\cdot,\tau)) + \|\zeta(\cdot,\tau)\|_{-1}^2 + \|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2\big\}d\tau \\ & + C\|\zeta(\cdot,t)\|_{-1}^2 + \frac{1}{2}\nu(t;w(\cdot,t)) + C\|\zeta(\cdot,0)\|_{-1}^2 + \nu(0;w_0), \quad \forall t\in[0,T],\end{aligned}$$

which, finally, yields

$$\begin{aligned}\nu(t;w(\cdot,t)) \le & C\Big\{\nu(0;w_0) + \sup_{\tau\in[0,t]}\|\zeta(\cdot,\tau)\|_{-1}^2 + \int_0^t\big\{\|\zeta(\cdot,\tau)\|_{-1}^2 + \|\partial_\tau\zeta(\cdot,\tau)\|_{-1}^2\big\}d\tau\Big\} \\ & + C\int_0^t \nu(\tau;w(\cdot,\tau))d\tau, \quad \forall t\in[0,T].\end{aligned} \tag{4.12}$$

Applying the Gronwall lemma to (4.12) and using (4.8) we verify (4.2). ■

5. The finite difference scheme for problem (1.5)

5.1 Notation and preliminaries.

We use the notation introduced in Section 3.1. In addition we will make use of the evaluation operator $I_h : \mathbb{C}^{J+2} \longrightarrow \mathbb{C}_0^{J+2}$ and the discrete space derivative operator $\tilde{\delta}_h : \mathbb{C}^{J+2} \longrightarrow \mathbb{C}_0^{J+2}$ defined by

$$I_h v_j := \begin{cases} \dfrac{v_{i+1}+v_{j-1}}{2}, & j=1,\dots,J \\ v_J, & j=J+1 \end{cases} \qquad \tilde{\delta}_h v_j := \begin{cases} \dfrac{v_{j+1}-v_{j-1}}{2h}, & j=1,\dots,J \\ \dfrac{v_{J+1}-v_J}{h}, & j=J+1 \end{cases}.$$

Also, for v, $\chi\in\mathbb{C}^{J+2}$, we define $v\otimes\chi\in\mathbb{C}_0^{J+2}$ by $(v\otimes\chi)_j := v_j\chi_j$ for $j=1,\dots,J+1$.

Lemma 5.1.

(5.1.1) $$(y\otimes\tilde{\delta}_h v, z)_h = -\overline{(y\otimes\tilde{\delta}_h z, v)_h} + v_{J+1}\overline{z_{J+1}} - (I_h v, z)_h, \quad \forall v, z \in \mathbb{C}_0^{J+2}.$$

Proof. Let v, $z \in \mathbb{C}_0^{J+2}$. Then, we have

$$\begin{aligned}(y\otimes\tilde{\delta}_h v, z)_h =& \tfrac{1}{2}\sum_{j=1}^{J} y_j(v_{j+1}-v_{j-1})\overline{z_j} + \tfrac{1}{2}(v_{J+1}-v_J)\overline{z_{J+1}}\\ =& \tfrac{1}{2}\sum_{j=1}^{J}(y_{j+1}v_{j+1}\overline{z_j} - y_{j-1}v_{j-1}\overline{z_j}) + \tfrac{1}{2}(v_{J+1}-v_J)\overline{z_{J+1}} - h\sum_{j=1}^{J} I_h v_j\overline{z_j}\\ =& -h\sum_{j=1}^{J} y_j v_j\overline{\tilde{\delta}_h z_j} + \tfrac{1}{2}v_{J+1}\overline{z_J} + \tfrac{1-h}{2}v_J\overline{z_{J+1}} + \tfrac{1}{2}(v_{J+1}-v_J)\overline{z_{J+1}} + \tfrac{h}{2}v_J\overline{z_{J+1}} - (I_h v, z)_h\\ =& -h\sum_{j=1}^{J} y_j v_j\overline{\tilde{\delta}_h z_j} - \tfrac{1}{2}v_{J+1}(\overline{z_{J+1}} - \overline{z_J}) + v_{J+1}\overline{z_{J+1}} - (I_h v, z)_h\\ =& -\overline{(y\otimes\tilde{\delta}_h z, v)_h} + v_{J+1}\overline{z_{J+1}} - (I_h v, z)_h. \quad \blacksquare\end{aligned}$$

5.2 The scheme.
We will approximate the solution u of (1.4), by a Crank–Nicolson type finite difference discretization of the solution w of problem (1.5). For $m = 0, \ldots, N$, we approximate w^m (defined as $(w(y_0, t^m), \ldots, w(y_{J+1}, t^m))^T \in \mathbb{C}_0^{J+2}$) by $W^m \in \mathbb{C}_0^{J+2}$. The latter is specified recursively by the formulas

(5.2.1) $$W^0 := w^0$$

and for $n = 0, \ldots, N-1$:

(5.2.2a) $$\partial W_j^n = i\tfrac{1}{\xi^{n+\frac{1}{2}}}\Delta_h W_j^{n+\frac{1}{2}} + y_j\mu^{n+\frac{1}{2}}\tilde{\delta}_h W_j^{n+\frac{1}{2}} + \tfrac{\mu^{n+\frac{1}{2}}}{2}I_h W_j^{n+\frac{1}{2}} + i\gamma_j^{n+\frac{1}{2}}W_j^{n+\frac{1}{2}} + \zeta_j^{n+\frac{1}{2}}, \quad j = 1, \ldots, J,$$

(5.2.2b) $$\begin{aligned}\partial W_{J+1}^n =& i\tfrac{1}{\xi^{n+\frac{1}{2}}}\Big\{\Delta_h W_{J+1}^{n+\frac{1}{2}} + \tfrac{2}{h}\Big[i\tfrac{\mu^{n+\frac{1}{2}}\xi^{n+\frac{1}{2}}}{2}W_{J+1}^{n+\frac{1}{2}} + g^{n+\frac{1}{2}}\Big]\Big\} + y_{J+1}\mu^{n+\frac{1}{2}}\tilde{\delta}_h W_{J+1}^{n+\frac{1}{2}}\\ &+ \tfrac{\mu^{n+\frac{1}{2}}}{2}I_h W_{J+1}^{n+\frac{1}{2}} + i\gamma_{J+1}^{n+\frac{1}{2}}W_{J+1}^{n+\frac{1}{2}} + \zeta_{J+1}^{n+\frac{1}{2}}.\end{aligned}$$

The evaluation of the terms in the formulas has been explained in Section 3.2. Again, computing W^{n+1} requires solving a tridiagonal system of equations, the invertibility of which will be proved in Corollary 5.2 below.

5.3 Consistency.
For $n = 0, \ldots, N-1$ we define $\eta^n \in \mathbb{C}_0^{J+2}$ by

(5.3.1a) $$\partial w_j^n = i\tfrac{1}{\xi^{n+\frac{1}{2}}}\Delta_h w_j^{n+\frac{1}{2}} + y_j\mu^{n+\frac{1}{2}}\tilde{\delta}_h w_j^{n+\frac{1}{2}} + \tfrac{\mu^{n+\frac{1}{2}}}{2}I_h w_j^{n+\frac{1}{2}} + i\gamma_j^{n+\frac{1}{2}}w_j^{n+\frac{1}{2}} + \zeta_j^{n+\frac{1}{2}} + \eta_j^n, \quad j = 1, \ldots, J,$$

and

(5.3.1b) $$\begin{aligned}\partial w_{J+1}^n =& i\tfrac{1}{\xi^{n+\frac{1}{2}}}\Big\{\Delta_h w_{J+1}^{n+\frac{1}{2}} + \tfrac{2}{h}\Big[i\tfrac{\mu^{n+\frac{1}{2}}\xi^{n+\frac{1}{2}}}{2}w_{J+1}^{n+\frac{1}{2}} + g^{n+\frac{1}{2}}\Big]\Big\}\\ &+ y_{J+1}\mu^{n+\frac{1}{2}}\tilde{\delta}_h w_{J+1}^{n+\frac{1}{2}} + \tfrac{\mu^{n+\frac{1}{2}}}{2}I_h w_{J+1}^{n+\frac{1}{2}} + i\gamma_{J+1}^{n+\frac{1}{2}}w_{J+1}^{n+\frac{1}{2}} + \zeta_{J+1}^{n+\frac{1}{2}} + \eta_{J+1}^n.\end{aligned}$$

Using Taylor's formula, we obtain the following estimates after tedious but straightforward computations:

$$\max_{0\le n\le N-1}\Big\{\max_{1\le j\le J}|\eta_j^n|\Big\}\le \mathcal{C}_1(k^2+h^2),\quad \max_{0\le n\le N-1}|\eta_{J+1}^n|\le \mathcal{C}_2\Big(k^2+h+\frac{k^2}{h}\Big), \tag{5.3.2}$$

$$\max_{0\le n\le N-2}\Big\{\max_{1\le j\le J}|\partial\eta_j^n|\Big\}\le \mathcal{C}_3(k^2+h^2),\quad \max_{0\le n\le N-2}|\partial\eta_{J+1}^n|\le \mathcal{C}_4\Big(k^2+h+\frac{k^2}{h}\Big). \tag{5.3.3}$$

Here, as in Section 3.3, the $\mathcal{C}_m$ are constants independent of h and k. Proving (5.3.2) and (5.3.3) requires the same regularity assumptions on the solution of (1.5) as those mentioned after (3.3.3).

Finally, (5.3.2) and (5.3.3) obviously yield

$$\max_{0\le n\le N-1}|\!|\!|\eta^n|\!|\!|_h\le \mathcal{C}(k^2+h^2)\ \text{ and }\ \max_{0\le n\le N-2}|\!|\!|\partial\eta^n|\!|\!|_h\le \mathcal{C}(k^2+h^2). \tag{5.3.4}$$

5.4 Stability and convergence.

Let $e^n:=w^n-W^n\in\mathbb{C}_0^{J+2}$ for $n=0,\dots,N$. Using (5.2.1) and subtracting (5.2.2a-b) from (5.3.1a-b), we get

$$e^0=0. \tag{5.4.1}$$

And, for $n=0,\dots,N-1$,

$$\partial e_j^n=i\tfrac{1}{\xi^{n+\frac12}}\Delta_h e_j^{n+\frac12}+y_j\mu^{n+\frac12}\tilde{\delta}_h e_j^{n+\frac12}+\tfrac{\mu^{n+\frac12}}{2}I_h e_j^{n+\frac12}+i\gamma_j^{n+\frac12}e_j^{n+\frac12}+\eta_j^n,\quad j=1,\dots,J, \tag{5.4.2a}$$

and

$$\begin{aligned}\partial e_{J+1}^n=&i\tfrac{1}{\xi^{n+\frac12}}\Big\{\Delta_h e_{J+1}^{n+\frac12}+\tfrac{2}{h}\big[i\tfrac{\mu^{n+\frac12}\xi^{n+\frac12}}{2}e_{J+1}^{n+\frac12}\big]\Big\}+y_{J+1}\mu^{n+\frac12}\tilde{\delta}_h e_{J+1}^{n+\frac12}\\&+\tfrac{\mu^{n+\frac12}}{2}I_h e_{J+1}^{n+\frac12}+i\gamma_{J+1}^{n+\frac12}e_{J+1}^{n+\frac12}+\eta_{J+1}^n.\end{aligned} \tag{5.4.2b}$$

Fix $n\in\{0,\dots,N-1\}$. Multiply (5.4.2a) by $2hk\overline{e_j^{n+\frac12}}$ and then sum the resulting relations with respect to j from 1 up to J. Also, multiply (5.4.2b) by $kh\overline{e_{J+1}^{n+\frac12}}$, and then add to the previous sum and take real parts to obtain

$$\|e^{n+1}\|_h^2-\|e^n\|_h^2=-\tfrac{2k}{\xi^{n+\frac12}}\,\mathsf{Im}(\Delta_h e^{n+\frac12},e^{n+\frac12})_h+\sum_{j=1}^5\Lambda_j^n, \tag{5.4.3}$$

where

$$\Lambda_1^n:=2k\mu^{n+\frac12}\,\mathsf{Re}(y\otimes\tilde{\delta}_h e^{n+\frac12},e^{n+\frac12})_h, \tag{5.4.4a}$$

$$\Lambda_2^n:=-2k\Big\{h\sum_{j=1}^J(\gamma_I)_j^{n+\frac12}|e_j^{n+\frac12}|^2+\frac{h}{2}(\gamma_I)_{J+1}^{n+\frac12}|e_{J+1}^{n+\frac12}|^2\Big\}, \tag{5.4.4b}$$

$$\Lambda_3^n:=k\mu^{n+\frac12}\,\mathsf{Re}(I_h e^{n+\frac12},e^{n+\frac12})_h,\quad \Lambda_4^n:=-k\mu^{n+\frac12}|e_{J+1}^{n+\frac12}|^2, \tag{5.4.4c}$$

$$\Lambda_5^n:=2k\,\mathsf{Re}(\eta^n,e^{n+\frac12})_h. \tag{5.4.4d}$$

Next, we will estimate the right-hand side of (5.4.3).

`Estimation of` Λ_2^n `and` Λ_5^n: We can easily see that

$$\Lambda_2^n \le Ck\|e^{n+\frac{1}{2}}\|_h^2 \quad\text{and}\quad \Lambda_5^n \le Ck|\!|\!|\eta^n|\!|\!|\, |e^{n+\frac{1}{2}}|_{1,h}. \tag{5.4.5a}$$

`Estimation of` $\Lambda_1^n + \Lambda_3^n + \Lambda_4^n$: Lemma 5.1 yields

$$2\,\mathsf{Re}(y\otimes\tilde{\delta}_h e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h = |e_{J+1}^{n+\frac{1}{2}}|^2 - \mathsf{Re}(I_h e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h$$

which obviously implies

$$\Lambda_1^n + \Lambda_3^n + \Lambda_4^n = 0. \tag{5.4.5b}$$

Since

$$(\Delta_h e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}})_h = -|e^{n+\frac{1}{2}}|_{1,h}^2,$$

from (5.4.3), (5.4.5a-b) and (3.1.1), it follows that

$$\|e^{n+1}\|_h^2 - \|e^n\|_h^2 \le Ck\big(|e^{n+\frac{1}{2}}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big). \tag{5.4.6}$$

Next, multiply (5.4.2a) by $\overline{\partial e_j^n}$ and then sum the resulting equations with respect to j from 1 up to J. Also, multiply (5.4.2b) by $\frac{1}{2}\overline{\partial e_{J+1}^n}$ and then add to the previous sum. Finally, take imaginary parts and multiply by $h\xi^{n+\frac{1}{2}}$ to obtain

$$-\mathsf{Re}(\Delta_h e^{n+\frac{1}{2}}, \partial e^n)_h = \xi^{n+\frac{1}{2}}\,\mathsf{Im}(\eta^n, \partial e^n)_h + \sum_{j=1}^{4}\widetilde{\Psi}_j^n, \tag{5.4.7}$$

where

$$\widetilde{\Psi}_1^n := \tfrac{\xi^{n+\frac{1}{2}}\mu^{n+\frac{1}{2}}}{2}\,\mathsf{Im}(I_h e^{n+\frac{1}{2}}, \partial e^n)_h, \quad \widetilde{\Psi}_2^n := \xi^{n+\frac{1}{2}}\,\mathsf{Re}(\gamma^{n+\frac{1}{2}}\otimes e^{n+\frac{1}{2}}, \partial e^n)_h, \tag{5.4.8a}$$

$$\widetilde{\Psi}_3^n := \xi^{n+\frac{1}{2}}\mu^{n+\frac{1}{2}}\,\mathsf{Im}(y\otimes\tilde{\delta}_h e^{n+\frac{1}{2}}, \partial e^n)_h \quad\text{and}\quad \widetilde{\Psi}_4^n := -\tfrac{\xi^{n+\frac{1}{2}}\mu^{n+\frac{1}{2}}}{2}\,\mathsf{Im}\Big\{e_{J+1}^{n+\frac{1}{2}}\overline{\partial e_{J+1}^n}\Big\}. \tag{5.4.8b}$$

`Estimation of` $\widetilde{\Psi}_1^n + \widetilde{\Psi}_3^n + \widetilde{\Psi}_4^n$: Using Lemma 5.1, we see that

$$\begin{aligned}\widetilde{\Psi}_1^n + \widetilde{\Psi}_3^n + \widetilde{\Psi}_4^n &= \tfrac{\mu^{n+\frac{1}{2}}\xi^{n+\frac{1}{2}}}{2}\Big\{\mathsf{Im}(y\otimes\tilde{\delta}_h e^{n+\frac{1}{2}}, \partial e^n)_h + \mathsf{Im}(y\otimes\tilde{\delta}_h\partial e^n, e^{n+\frac{1}{2}})_h\Big\}\\ &= \tfrac{\mu^{n+\frac{1}{2}}\xi^{n+\frac{1}{2}}}{2k}\Big\{\mathsf{Im}(y\otimes\tilde{\delta}_h e^{n+1}, e^{n+1})_h - \mathsf{Im}(y\otimes\tilde{\delta}_h e^n, e^n)_h\Big\}.\end{aligned} \tag{5.4.9a}$$

`Estimation of` $\widetilde{\Psi}_2^n$: Using (5.4.2a-b), we obtain

$$\widetilde{\Psi}_2^n = \sum_{j=1}^{5}\widetilde{\Upsilon}_j^n, \tag{5.4.10}$$

with

(5.4.11a) $$\widetilde{\Upsilon}_1^n := -\,\mathsf{Im}(\Delta_h e^{n+\frac12}, \gamma^{n+\frac12} \otimes e^{n+\frac12})_h,$$

(5.4.11b) $$\widetilde{\Upsilon}_2^n := \mu^{n+\frac12}\xi^{n+\frac12}\,\mathsf{Re}(y \otimes \tilde{\delta}_h e^{n+\frac12}, \gamma^{n+\frac12} \otimes e^{n+\frac12})_h,$$

(5.4.11c) $$\widetilde{\Upsilon}_3^n := \tfrac{\mu^{n+\frac12}\xi^{n+\frac12}}{2}\,\mathsf{Re}(I_h e^{n+\frac12}, \gamma^{n+\frac12} \otimes e^{n+\frac12})_h, \quad \widetilde{\Upsilon}_4^n := \xi^{n+\frac12}\,\mathsf{Re}(\eta^n, \gamma^{n+\frac12} \otimes e^{n+\frac12})_h,$$

(5.4.11d) $$\widetilde{\Upsilon}_5^n := -\tfrac{\mu^{n+\frac12}\xi^{n+\frac12}}{2}\gamma_R(1, t^{n+\frac12})|e_{J+1}^{n+\frac12}|^2.$$

Now, we can easily see that

(5.4.12a) $$\widetilde{\Upsilon}_2^n \le C\|e^{n+\frac12}\|_h\,|e^{n+\frac12}|_{1,h}, \quad \widetilde{\Upsilon}_3^n \le C\|e^{n+\frac12}\|_h^2, \quad \widetilde{\Upsilon}_4^n \le C|\!|\!|\eta^n|\!|\!|_h\,\|e^{n+\frac12}\|_\infty, \quad \widetilde{\Upsilon}_5^n \le C\|e^{n+\frac12}\|_\infty^2.$$

Also, we have

$$\widetilde{\Upsilon}_1^n = -h\sum_{j=0}^{J}(\gamma_I)_j^{n+\frac12}\Big|\frac{e_{j+1}^{n+\frac12}-e_j^{n+\frac12}}{h}\Big|^2 + \mathsf{Im}\Big\{\sum_{j=0}^{J}\frac{\gamma_j^{n+\frac12}-\gamma_{j+1}^{n+\frac12}}{h}\,\overline{(e_{j+1}^{n+\frac12}-e_j^{n+\frac12})}\,e_{j+1}^{n+\frac12}\Big\},$$

which yields as in (3.4.9c)

(5.4.12b) $$\widetilde{\Upsilon}_1^n \le C\big\{|e^{n+\frac12}|_{1,h}^2 + \|e^{n+\frac12}\|_h|e^{n+\frac12}|_{1,h}\big\}.$$

Using (5.4.10), (5.4.11), (5.4.12a-b) and (3.1.1) , we conclude that

(5.4.9b) $$\widetilde{\Psi}_2^n \le C\big\{|e^{n+\frac12}|_{1,h}^2 + |\!|\!|\eta^n|\!|\!|_h^2\big\}.$$

With these estimates in hand we can now prove the following key result.

Lemma 5.2. *Let* $\varrho := \xi\mu$, $\varrho_* := \frac12(1+\sup_{[0,T]}|\varrho|^2)$ *and*

$$\nu^n := |e^n|_{1,h}^2 - \varrho(t^n)\,\mathsf{Im}(y \otimes \tilde{\delta}_h e^n, e^n)_h + \varrho_*\|e^n\|_h^2, \quad n = 0,\dots,N.$$

Then, if the solution of the problem (1.5) is sufficiently smooth, we have

(5.4.13) $$\nu^{n+1} \le \nu^n + Ck\big\{\nu^{n+1}+\nu^n+(k^2+h^2)^2\big\} + 2k\xi^{n+\frac12}\,\mathsf{Im}(\eta^n, \partial e^n)_h, \quad n = 0,\dots,N-1.$$

Proof. First, we remark that

(5.4.14) $$\nu^n \ge \frac12\big\{\|e^n\|_h^2 + |e^n|_{1,h}^2\big\} \ge 0, \quad n = 0,\dots,N.$$

We multiply by parts (5.4.6) by $\dfrac{\varrho_*}{2k}$ and then add it to (5.4.7). Then, since

$$-\,\mathsf{Re}(\Delta_h e^{n+\frac12}, \partial e^n)_h = \frac{1}{2k}\big(|e^{n+1}|_{1,h}^2 - |e^n|_{1,h}^2\big),$$

we use (5.4.8a-b), (5.4.9a-b), (5.4.14) and (5.3.4) to obtain

(5.4.15) $$\begin{aligned}\nu^{n+1}-\nu^n \le\,& Ck\Big\{\nu^{n+1}+\nu^n+(k^2+h^2)^2\Big\} + 2k\xi^{n+\frac12}\,\mathsf{Im}(\eta^n, \partial e^n)_h\\ &-(\varrho(t^{n+1})-\varrho(t^{n+\frac12}))\,\mathsf{Im}(y \otimes \tilde{\delta}_h e^{n+1}, e^{n+1})_h\\ &+(\varrho(t^n)-\varrho(t^{n+\frac12}))\,\mathsf{Im}(y \otimes \tilde{\delta}_h e^n, e^n)_h.\end{aligned}$$

We conclude that (5.4.13) holds by combining (5.4.15) and (3.1.1). ■

Corollary 5.2. *The fully discrete scheme (5.2.1)-(5.2.2a-b) has a unique solution* W^n *for* $0 \le n \le N$, *provided* k *is sufficiently small. If* $\gamma_I = 0$, $\zeta = 0$ *and* $g = 0$, *the scheme is conservative in the* $\|\cdot\|_h$ *norm and has a unique solution with no restriction on* k.

Proof. Let N^n be defined for $0 \le n \le N$ as ν^n in Lemma 5.2, but with W^n instead of e^n. Then the homogeneous counterpart of (5.4.13) holds for N^n, i.e. we have

$$N^{n+1} \le N^n + ck(N^{n+1} + N^n).$$

If $W^n = 0$, then $N^n = 0$. Hence $N^{n+1} = 0$ for k sufficiently small, by above. This implies that $W^{n+1} = 0$ by the analog of (5.4.14). If $\gamma_I = \zeta = 0$ and $g = 0$, consider the analog of (5.4.3) with $\eta^n = 0$ and W^n instead of e^n. The right-hand side of this equation is zero, in view of the W^n-analogs of (5.4.5b) and of the identity immediately following (5.4.5b). We conclude that $\|W^{n+1}\|_h = \|W^n\|_h$, and unconditional existence–uniqueness of W^{n+1} follows. ■

Finally we have our error estimate:

Theorem 5.1. *If* k *is sufficiently small, and the solution of (1.5) is sufficiently smooth, we have*

$$\max_{0\le n\le N} \|e^n\|_\infty \le C(k^2 + h^2).$$

Proof. The proof follows from the estimate (5.4.13), in view of (5.4.14) and (3.1.1). The last term in the right-hand side of (5.4.13) is treated as in the proof of Theorem 3.1. ■

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