ΠΑΝΕΠΙΣΤΗΜΙΟ ΙΩΑΝΝΙΝΩΝ
ΣΧΟΛΗ ΘΕΤΙΚΩΝ ΕΠΙΣΤΗΜΩΝ
ΤΜΗΜΑ ΦΥΣΙΚΗΣ

ΕΝΟΠΟΙΗΜΕΝΑ ΠΡΟΤΥΠΑ ΣΤΟΙΧΕΙΩΔΩΝ ΣΩΜΑΤΙΔΙΩΝ

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ΔΙΔΑΚΤΟΡΙΚΗ ΔΙΑΤΡΙΒΗ

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UNIVERSITY OF IOANNINA
PHYSICS DEPARTMENT

UNIFIED MODELS OF PARTICLE PHYSICS

Michael A. Paraskevas

PhD Thesis

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ΠΡΑΚΤΙΚΟ
ΔΗΜΟΣΙΑΣ ΠΑΡΟΥΣΙΑΣΗΣ, ΕΞΕΤΑΣΗΣ ΚΑΙ ΑΞΙΟΛΟΓΗΣΗΣ
ΔΙΔΑΚΤΟΡΙΚΗΣ ΔΙΑΤΡΙΒΗΣ

Σήμερα Τρίτη **25-6-2013**, ώρα **12.00** στην αίθουσα **Σεμιναρίων του Τμήματος Φυσικής, κτίριο Φ2,** του Πανεπιστημίου Ιωαννίνων, πραγματοποιήθηκε, σύμφωνα με το άρθρο 12, παρ. 5 του Ν.2083/92, η διαδικασία της δημόσιας παρουσίασης, εξέτασης και αξιολόγησης της εργασίας του υποψήφιου για την απόκτηση Διδακτορικής Διατριβής **κ. Μιχαήλ Παρασκευά.**

Την Επταμελή Εξεταστική Επιτροπή, που συγκροτήθηκε με απόφαση της Γενικής Συνέλευσης Ειδικής Σύνθεσης του Τμήματος Φυσικής (συν. 424/11-6-2013), αποτελούν τα ακόλουθα μέλη:

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7) Αθανάσιος Δέδες, Αναπληρωτής Καθηγητής του Τμήματος Φυσικής του Παν/μίου Ιωαννίνων

Παρόντα ήταν και τα 7 μέλη της εξεταστικής επιτροπής. Το θέμα της διδακτορικής διατριβής που εκπόνησε ο κ. Παρασκευάς και που παρουσίασε σήμερα είναι **«Ενοποιημένα Πρότυπα Στοιχειωδών Σωματιδίων».**

Ο υποψήφιος παρουσίασε και ανάπτυξε το θέμα και απάντησε σε σχετικές ερωτήσεις τόσο των μελών της εξεταστικής επιτροπής όσο και του ακροατηρίου. Στη συνέχεια αποσύρθηκε η εξεταστική επιτροπή και μετά από συζήτηση κατέληξε στα ακόλουθα:

α) Η συγγραφή της διατριβής έγινε με τρόπο που δείχνει ιδιαίτερη μεθοδικότητα και πλήρη ενημέρωση του υποψήφιου πάνω στη σχετική βιβλιογραφία.

β) Η ερευνητική εργασία καταλήγει σε σημαντικά αποτελέσματα τα οποία προάγουν την επιστήμη. Από την εργασία αυτή έχουν προκύψει τρεις εργασίες δημοσιευμένες σε έγκριτα επιστημονικά που έγιναν από τον υποψήφιο.

γ) Η παρουσίαση και ανάπτυξη του θέματος της εργασίας από τον υποψήφιο και οι εύστοχες απαντήσεις στις ερωτήσεις που του τέθηκαν έδειξαν πλήρη γνώση του θέματος και γενικότερων σχετικών θεμάτων Φυσικής Στοιχειωδών Σωματιδίων.

Με βάση τα ανωτέρω, τα μέλη της Επταμελούς Εξεταστικής Επιτροπής εγκρίνουν ομόφωνα την εργασία και εισηγούνται ανεπιφύλακτα την απονομή Διδακτορικού Διπλώματος στον κ. Μιχαήλ Παρασκευά με βαθμό Άριστα.

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Η παρούσα έρευνα έχει συγχρηματοδοτηθεί από την Ευρωπαϊκή Ένωση (Ευρωπαϊκό Κοινωνικό Ταμείο - ΕΚΤ) και από εθνικούς πόρους μέσω του Επιχειρησιακού Προγράμματος «Εκπαίδευση και Δια Βίου Μάθηση» του Εθνικού Στρατηγικού Πλαισίου Αναφοράς (ΕΣΠΑ) ; Ερευνητικό Χρηματοδοτούμενο Έργο : Ηράκλειτος ΙΙ . Επένδυση στην κοινωνία της γνώσης μέσω του Ευρωπαϊκού Κοινωνικού Ταμείου

Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο
ΕΠΙΧΕΙΡΗΣΙΑΚΟ ΠΡΟΓΡΑΜΜΑ
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επένδυση στην κοινωνία της γνώσης
ΥΠΟΥΡΓΕΙΟ ΠΑΙΔΕΙΑΣ & ΘΡΗΣΚΕΥΜΑΤΩΝ, ΠΟΛΙΤΙΣΜΟΥ & ΑΘΛΗΤΙΣΜΟΥ
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ΕΥΡΩΠΑΪΚΟ ΚΟΙΝΩΝΙΚΟ ΤΑΜΕΙΟ
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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Preface

Particle physics is currently established as one of the most active and fruitful fields of research for both theoretical and experimental physicists. Its cornerstone, the Standard Model (SM), seems to offer a deeper understanding of the fundamental laws of nature as these appear through the observed interactions of all known elementary particles. However, such a model, despite being remarkably accurate and concrete, carries along several inherent deficiencies which become more severe at higher energies. These suggest that perhaps another more fundamental theory should also exist that could serve as a completion of the SM.

The scope of this thesis is to examine several issues and questions, the SM and its standard extensions left open, and to offer new proposals and possible solutions. Such solutions are believed to eventually lead us into a more elaborate and perhaps more fundamental theory. In what follows we mainly discuss the SM and two of its standard extensions, commonly referred to as *Supersymmetry* (SUSY) and *Grand Unified Theory* (GUT). This thesis is organized as follows:

In the first chapter after introducing the fundamental concepts of gauge symmetry and renormalizability we proceed to revisit general aspects and properties of the SM. We focus on the basic structure of this model as well as its physical spectrum. The latter consists of all known elementary particles to date. At the end of this chapter we present some of the SM inadequacies which motivate us to search for answers in theories beyond the SM.

Thus, in the second chapter we introduce these "beyond the SM theories". We start with Supersymmetry (SUSY), a general framework which is regarded as a minimal but particularly promising extension of the SM. We avoid an extensive discussion on the vast subject of SUSY by concentrating on selected topics that further enable us to build the Minimal Supersymmetric SM (MSSM). As in the SM we revisit only certain aspects of this particular model, namely only those relevant to our subsequent analysis or research. Next, we introduce the framework of GUTs which is another extension of the SM that may be considered also independently of SUSY. However, we focus on SUSY-GUTs which seem to offer phenomenologically more viable and theoretically more elegant realizations. Thus, we review three distinct SUSY-GUT models, based on the standard $SU(5)$, the "flipped" $SU(5)$ and the $SO(10)$ gauge groups.

Then in the third chapter, we eventually arrive at one aim of our research, namely the fermion masses and mixing puzzle. We examine this problem with two different approaches each of which with its own virtues and attractive aspects. In our first approach we confront the problem of neutrino masses and mixing from the viewpoint of an explicit and consistent non-minimal $SU(5)$ model. In our second approach we examine the masses and mixing puzzle in both the quark and the lepton sector and having developed a useful theoretical tool we apply it in a possible SUSY-GUT realization based on the $SO(10)$ group.

In the fourth chapter we visit another topic of our research, namely the discrete R-symmetries. These may appear within the MSSM and its extensions offering potential escapes to phenomenological difficulties of SUSY models. Thus they are obviously of direct interest to us. We focus on those symmetries arising within the standard supersymmetric extensions of the SM, having both a phenomenological interest and a possible dynamical origin. Finally, in the general overview we present briefly our conclusions as well as potential perspectives of our research.

Our research has produced the three following published articles:

- M. Paraskevas and K. Tamvakis, "Hierarchical neutrino masses and mixing in non minimal-SU(5)," Phys. Rev. D **84**, 013010 (2011) [arXiv:1104.1901 [hep-ph]].
- M. Paraskevas and K. Tamvakis, "Bimaximal mixing from lopsided neutrinos," Phys. Rev. D **85**, 073014 (2012) [arXiv:1202.2812 [hep-ph]].
- M. Paraskevas and K. Tamvakis, "On Discrete R-Symmetries in MSSM and its Extensions," Phys. Rev. D **86**, 015009 (2012) [arXiv:1205.1391 [hep-ph]].

Contents

Chapter 1

The Standard Model

1.1 Introduction.

The Standard Model (SM) of particle physics [1–6, 8–11] has been established as a theory describing the dynamics of all known elementary particles at low energies. Developed in the second half of the 20th century, it incorporates the fundamental concepts of gauge symmetry and renormalizability into a Quantum Field Theory (QFT) with solid predictions. Its wide recognition stems from a remarkable consistency with experimental data extracted from a large number of independent experimental tests. In fact, recently, the discovery of a scalar field at the LHC with seemingly the same properties as the Higgs particle, one of the SM cornerstones, was regarded as another major success for this model. Such an event, if further verified, in practice completes the quest for the elementary particles introduced in the SM and leaves behind a highly functional model of interactions at low energies. Although such a model lacks the ability to embody gravity and therefore by default fails as a fundamental theory of all forces, it possesses all necessary ingredients required for a theoretical *terra firma* on which new models, beyond the SM, can be built and new physics may be discovered. This approach, attractive on its own but also further supported by the Effective Field Theory (EFT) point of view seems to receive an overwhelming appreciation from both theoretical and experimental physicists.

The elementary particles of the SM can be classified in various ways. According to their representation under the Lorentz group they can be divided into three general classes. The first class includes the gauge bosons, twelve spin-1 particles mediating the three fundamental forces. The photon of the electromagnetic interaction and the eight gluons of the strong interactions are massless, while the $W^{\pm}, Z$ of the weak interactions are massive, with a mass around $100\,GeV$. The second class includes the fermions, twelve spin-1/2, particles regarded as the building blocks of the known matter. Out of these twelve particles the six quarks and the three charged leptons exhibit a hierarchical mass spectrum varying between the MeV and the GeV scale. On the other hand the three neutrinos, treated as massive in the current formulation of the SM, have tiny masses probably lying in the sub-eV scale.[1] The third class includes a single particle, the aforementioned Higgs scalar (spin-0) with a possible mass at $126\,GeV$, bearing unique properties which preserve the self-consistency of the model.

As a theoretical model the SM exhibits many appealing features. To begin with, it shares all attractive properties of quantum gauge theories. In such a theory, gauge bosons, the mediators of the fundamental forces, appear inevitably in order to ensure its invariance under a set of local symmetry transformations. These symmetries govern the interactions between particles and, in the case where the gauge theory is also anomaly-free, they further protect it from inconsistencies to all orders in perturbation theory. Another beautiful aspect is the renormalizability of the model. Divergences arising from perturbation theory can be absorbed in a redefinition of well-defined quantities. With the help of a number of techniques, developed alongside the SM, the renormalizability of the theory has been proven, even in the case of spontaneous symmetry breaking(SSB). The elegance of the Higgs mechanism, which is the explicit realisation of the SSB within the SM giving mass to elementary particles, is yet another appealing feature of the model and the list continues further.

In what follows in this section we study in more detail the above, among other, selected topics concerning the theoretical structure of the SM. We then proceed with the explicit construction of the model and its phenomenological implications. Finally, we conclude with problems and inadequacies arising within its context that naturally lead us to search for new physics in theories beyond the SM.

1.2 Quantum Gauge Theories.

It seems at least unexpected that an initially unnoticed mathematical symmetry underlying Maxwell's classical theory of electromagnetism would turn into a principle with profound impact on modern particle physics. Nowadays, gauge symmetries and QFT, yet another theoretical advance developed in the last century, have merged into the very elegant and powerful framework of quantum gauge theories. This general framework offers not only severe technical restrictions for model building but also a

[1] Strictly speaking the exact mass spectrum of the light neutrinos is yet undetermined and a possibility for one massless neutrino is not yet excluded.

deeper understanding of fundamental concepts such as those of particle, charge and force.

1.2.1 Global Symmetries.

An enlightening example of the way symmetries are realized within a field theory is that of a single-scalar theory with a $U(1)$ global symmetry.

The $U(1)$ is the simplest example of a continuous group of transformations. As an abelian group it shares the standard group properties and additionally obeys a commutative multiplication law for its elements, namely $ab = ba \ \forall\, a, b \in U(1)$. Under its action fields transform by a random complex phase $U(1) : \phi(x) \to \phi(x) = e^{iq\theta}\phi(x)$ where q is identified as the $U(1)$ charge of the respective field and θ as the group parameter. In the global symmetry case the group parameter is considered constant in contrast with the local symmetry case where the group parameter is a function of space-time $\theta(x)$. Even though a gauge is in fact a local symmetry a first discussion on global symmetries seems more illustrative for our purposes.

Any symmetry, and in particular the $U(1)$ considered in this example, is realised in a field theory by demanding that the Lagrangian remains invariant under all transformations of the given symmetry group. This is not very restrictive since the terms respecting the condition above turn out to be infinitely many. But when augmented by the principle of renormalizability these terms reduce to very few and the Lagrangian becomes uniquely defined. Leaving a more detailed discussion on the issue of renormalization for a following section we assume for the moment that only terms up to quartic in the fields and quadratic in their derivatives are allowed in the Lagrangian density. Then for any non-trivial values of the $U(1)$ charge $q \neq 0$

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2(\phi^\dagger \phi) - \frac{\lambda}{2}(\phi^\dagger \phi)^2 \tag{1.1}$$

is the most general expression respecting this $U(1)$ symmetry along with the Lorentz symmetry that any relativistic field theory should respect.

Of course, symmetry groups are a far more complicated subject and extend well beyond this trivial example of a $U(1)$ symmetry. In fact, the most interesting of these turn out to be the non-abelian Lie groups. An important property Lie groups share is that their elements depend smoothly on the group parameters. This allows for representations in which group elements can be parameterized exponentially in a compact form as $U(\theta) = e^{i\theta_a T_a}$. The T_a's are the generators, a set of Hermitian defined operators which satisfy the Lie algebra of the group through the relation $[T_a, T_b] =$

$if^{abc}T_c$ with the f^{abc} being the structure constants, unique for a given non-trivial group. In this description one can use successive infinitesimal transformations $U(d\theta) = 1 + i\, d\theta_a T^a$ to span the whole space of group elements.

Noether's Theorem.

In a field theory, an underlying continuous symmetry manifests itself through conserved charges and currents. This picture is better understood when seen through a very powerful theorem proved by E. Noether in 1918, associating symmetries of the action of a physical system with conservation laws. The theorem states that for any continuous symmetry of the action $S = \int Ldt$ there always exist a corresponding conserved quantity called current satisfying $\partial^\mu J_\mu = 0$ and an associated invariant quantity called charge satisfying $\frac{dQ}{dt} = 0$. This statement is general and requires only a continuous symmetry of the action. In the special case of internal symmetries, as the global Lie groups considered here, where the Lagrangian density remains invariant the theorem is easily proved as follows.

Let a general Lagrangian described by

$$L = \int d^3x \; \mathcal{L}(\partial^\mu \phi_i(x), \phi_i(x)) \tag{1.2}$$

with a Lagrangian density $\mathcal{L}$ invariant under the transformation of a global Lie-group G. Then

$$G : \phi_i \to \phi_i' + \delta\phi_i \tag{1.3}$$

under an infinitesimal transformation of the group where $\delta\phi_i = i\theta_a T^a_{ij}\phi_j$ for small θ_a. This corresponds to a variation in the Lagrangian density

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi_i)}\partial^\mu(\delta\phi_i) + \frac{\delta\mathcal{L}}{\delta\phi_i}\delta\phi_i \\
&= \partial^\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi_i)}\delta\phi_i\right) \\
&= \theta_a\partial^\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi_i)}iT^a_{ij}\phi_j\right)
\end{aligned} \tag{1.4}$$

with the equations of motion being silently invoked in the derivation. Obviously if G is an internal symmetry then $\delta\mathcal{L} = 0$ which implies the existence of the conserved currents[2]

$$J_\mu^a = \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi_i)} i T_{ij}^a \phi_j . \tag{1.5}$$

Now since $\partial^\mu J_\mu^a = 0$, then

$$\int d^3x \; \partial^0 J_0^a = -\int d^3x \; \partial^i J_i^a = -\oint dS_i \; J_i^a \tag{1.6}$$

Considering the surface terms, at the right hand side, vanishing at infinity a set of invariant quantities called charges may then be defined through

$$Q^a \equiv \int d^3x \; J_0^a \tag{1.7}$$

satisfying $\frac{dQ^a}{dt} = 0$.

1.2.2 Aspects of symmetries.

Having developed a useful tool that enables us to understand how symmetries are realized in a field theory at a fundamental level we may proceed further with a more detailed discussion on certain aspects of this realization. For our purpose we consider a scalar-theory where all fields belong to the fundamental representation of the non-abelian SU(N) group. Their complex conjugates transform in the anti-fundamental representation. By using the same arguments as in the abelian case we restrict ourselves to the expression

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m^2(\Phi^\dagger \Phi) - \frac{\lambda}{2}(\Phi^\dagger \Phi)^2 \,, \tag{1.8}$$

but with Φ here being an N-dimensional column vector including the fields ϕ_i. Clearly, all ϕ_i share a common mass as it can be directly seen from the quadratic term in the above expression. In fact this is a more general property. Fields belonging to the same irreducible representation of a given symmetry group have degenerate masses as long as the symmetry remains exact.

The associated currents in this theory will be given by (1.5) generalized to include

[2]Currents are uniquely defined up to terms with vanishing divergence, namely $J_\mu^a = J_\mu^a + K_\mu^a$ with $\partial^\mu K_\mu^a = 0$

complex fields. The conserved currents will have the form

$$J_\mu^a = -i(\partial_\mu \phi_i^\dagger \, T_{ij}^a \, \phi_j - \phi_i^\dagger \, T_{ij}^a \, \partial_\mu \phi_j)\,, \tag{1.9}$$

out of which we can associate a set of charges

$$Q^a = -i \int d^3x \; (\partial_0 \phi_i^\dagger \, T_{ij}^a \, \phi_j - \phi_i^\dagger \, T_{ij}^a \, \partial_0 \phi_j)\,. \tag{1.10}$$

The number of different charges or currents as determined by the running of the index a reflects the number of the generators T^a of the given group. For the SU(N) considered here it is $N^2 - 1$.

In a quantum field theory the charges are a representation of the symmetry generators. To see this we may impose the equal time commutation relation of the canonical quantization formalism between a set of fields and their conjugate canonical momentum $\pi_i = \partial_0 \phi_i^\dagger$,

$$[\phi_i(\boldsymbol{x}, t), \pi_j(\boldsymbol{x}', t)] = i\delta_{ij} \, \delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{1.11}$$

Since charges are time-independent operators their commutator will also be time-independent

$$\begin{aligned} [Q^a, Q^b] &= -i \int d^3x \; (\partial_0 \Phi^\dagger \, [T^a, T^b] \, \Phi - \Phi^\dagger \, [T^a, T^b] \, \partial_0 \Phi) \\ &= \; i f^{abc} Q^c \end{aligned} \tag{1.12}$$

reflecting the algebra of the SU(N) group.

The commutation relations of the charges turn out to be more general relations that remain valid, in a time-dependent form, even in the presence of symmetry breaking terms. If one introduces such a term in the Lagrangian the definitions in (1.9)(1.10) may still apply but the current is no longer conserved and the charge is no longer time-independent. Nevertheless, the conjugate canonical momentum is still the same allowing us to use once again (1.11) in the charge commutator to obtain

$$[Q^a(t), Q^b(t)] = i f^{abc} Q^c(t) \tag{1.13}$$

Therefore, even though the charges are not conserved, their commutation relation, commonly known as *charge algebra*, remains intact. Such relations can also be extended to include commutators of charges and currents or only currents forming the *current* *algebra* of the theory.

An important issue we came along in our previous analysis and is worth mentioning is the representation of the fields. Field operators and therefore particles belong to certain representations of the symmetry group which further determines how they transform under the symmetry. To see this recall the action of an infinitesimal transformation on fields that belong to a certain multiplet, as the fundamental of $SU(N)$ we previously encountered. There $\delta\phi_i = i\theta_a T^a_{ij}\phi_j$ with the T^a's considered as $N \times N$ matrices respecting the algebra of the group. If we had considered another representation of the $SU(N)$ group, for example an M-dimensional multiplet, the generators following the dimensionality of the representation could have been expressed as $M \times M$ matrices still respecting the group algebra. It suffices then to say that generators are abstract objects that acquire specific forms once the representation content of the theory is determined. As a result the symmetry transformation itself, as seen through $\delta\phi_i$ or $U(\theta)$, also acquires a specific form acting separately on each representation.

More than that, by following certain rules we may combine different representations to build new ones. Since, for a symmetric Lagrangian, each term should transform trivially under the symmetry group we can use these rules to write down all possible combinations of group representations that produce invariant terms (i.e. singlets). It turns out that this provides severe constraints for model building which eventually determine the possible interactions between particles.

1.2.3 Spontaneous Symmetry Breaking.

As already mentioned the symmetry transformations act separately on the different representations of the theory. Within a given irreducible representation they transform a particle state onto others but they cannot connect states within different representations. Moreover, it can be shown that the energy eigenstates within an irreducible representation will be degenerate as long as the respective group of transformations remains a symmetry of the theory.

Since we discuss energy eigenstates the Hamiltonian formalism seems more appropriate. In this formalism we introduce a Hamiltonian H which we consider invariant under the symmetry. Then a symmetry transformation acting on the Hamiltonian will satisfy

$$UHU^\dagger = H. \tag{1.14}$$

If we now assume that the symmetry also transforms states within an irreducible representation as

$$U|i\rangle = |j\rangle \tag{1.15}$$

the energy eigenvalues of H will be degenerate since

$$E = \langle i|H|i\rangle = \langle i|U^\dagger U H U^\dagger U|i\rangle = \langle j|H|j\rangle \tag{1.16}$$

Our assumption, however, is admissible if the ground state respects the symmetry transformation. That is because physical states are produced by the action of some creation operators on the ground state as

$$\phi_i|0\rangle = |i\rangle\,,\ \ \phi_j|0\rangle = |j\rangle$$

A symmetry transformation satisfying $U\phi_i U^\dagger = \phi_j$ would imply

$$U|i\rangle = (U\phi_i U^\dagger)U|0\rangle = \phi_j U|0\rangle = |j\rangle \tag{1.17}$$

if only $U|0\rangle = |0\rangle$.

In any other case where

$$U|0\rangle \neq |0\rangle \tag{1.18}$$

eq.(1.16) is no longer valid and the energy degeneracy is lifted. For the Lie-groups we consider $U(\theta) = e^{i\theta_a Q^a}$ and hence for an infinitesimal transformation this condition would translate into

$$Q^a|0\rangle \neq 0. \tag{1.19}$$

and if further the commutation relation $[\phi_i, Q^a] = T^a_{ij}\phi_j$ is taken into account then

$$T^a_{ij}\ \langle 0|\phi_j|0\rangle \neq 0 \tag{1.20}$$

which implies that some fields will acquire a non-vanishing Vacuum Expectation Value (VEV). Due to the invariance of the vacuum under the Poincare transformations the VEVs are bound to be space-time independent quantities meaning $\langle 0|\phi_j(x)|0\rangle = \langle 0|\phi_j(0)|0\rangle$.

This symmetry breaking scenario is commonly known as *Spontaneous Symmetry*

Breaking (SSB). In this scenario the symmetry of the spectrum breaks due to the non-invariance of the vacuum but the symmetry relations for operators remain valid as a result of the Hamiltonian (or the Lagrangian) being invariant under the symmetry transformations.

The Goldstone Theorem.

In theories with SSB of a continuous symmetry, massless states appear unavoidably corresponding to the *broken generators* of the symmetry i.e. those generators satisfying eq.(1.19). The scalar bosons identified with these massless states are commonly referred to as the *Nambu-Goldstone Bosons* [1, 2].

To understand the connection between massless bosons and broken generators it should be first mentioned that in the SSB case the charge corresponding to a conserved Noether current is ill-defined. To see this recall the definition in eq.(1.7). Applying the *broken charge* to the vacuum state ($Q^a|0\rangle \neq 0$) the time independent vacuum element

$$\langle 0|Q^a|0\rangle = \int d^3x \langle 0|J_0(\boldsymbol{x})|0\rangle = \int d^3x \langle 0|J_0(0)|0\rangle \tag{1.21}$$

diverges due to the translational invariance of the ground state. On the other hand the commutator required for an infinitesimal symmetry transformation of a field operator through

$$\delta\phi_i = i\theta_a[Q^a, \phi_i] = -i\theta_a T^a_{ij}\phi_j \tag{1.22}$$

is better behaved since its vacuum expectation value can be derived from the well-defined and space-time independent VEVs of the field operators $\langle 0|\phi_j(0)|0\rangle$. For the broken generators the corresponding VEVs will be non-vanishing giving

$$\begin{aligned} 0 \neq c_i = -T^a_{ij}\langle 0|\phi_j(0)|0\rangle = & \ \langle 0|\,[Q^a(t), \phi_i(0)]\,|0\rangle \\ = & \int d^3x\,\langle 0|\,[J^a_0(\boldsymbol{x}, t), \phi_i(0)]\,|0\rangle \\ = & \int d^3x\,\langle 0|\,[e^{iPx}J^a_0(0)e^{-iPx}, \phi_i(0)]\,|0\rangle \\ = & \sum_k (2\pi)^3\delta^3(\boldsymbol{k})\,\big\{\langle 0|J^a_0(0)|k\rangle\langle k|\phi_i(0)|0\rangle\, e^{-iE_kt} \\ & \qquad -\langle 0|\phi_i(0)|k\rangle\langle k|J^a_0(0)|0\rangle\, e^{iE_kt}\big\} \end{aligned} \tag{1.23}$$

The space-time translation acting on the complete set of states $|k\rangle$, as introduced in the final expression, produces opposite frequency terms. Therefore, for the time-dependence to cancel at any time a zero frequency state $|k\rangle$ should exist satisfying not only

$$E_k = 0\,,\ \boldsymbol{k} = 0 \tag{1.24}$$

but also

$$\langle 0|\phi_i(0)|k\rangle \neq 0\ ,\ \ \langle 0|J^a(0)|k\rangle \neq 0\ ,\ \ |k\rangle \neq |0\rangle$$

for the VEV of at least one field operator to be non-vanishing (i.e $c_i \neq 0$). This state due to (1.24) corresponds to a massless particle identified as the Nambu-Goldstone Boson.

It should be remarked though that the presence of massless Goldstone Bosons in a theory with SSB is directly associated with the presence of conserved currents and charges. Noether's theorem implies that these currents emerge from continuous symmetries of the Lagrangian (or the action). In the other case where the symmetry is discrete (i.e the group parameter is discrete valued) SSB may as well be realized but Goldstone bosons should not be expected to appear.

Spontaneous breaking - Abelian example.

Having established the connection between Goldstone Bosons and broken generators of a continuous symmetry we may proceed with an illustrative example of SSB in the simple case of an Abelian symmetry. The Lagrangian density is given by

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2(\phi^\dagger\phi) - \frac{\lambda}{2}(\phi^\dagger\phi)^2 \tag{1.25}$$

where the complex field is assumed to be non-trivially charged under the considered $U(1)$ namely $q_\phi \neq 0$. The infinitesimal symmetry transformation would then induce

$$\delta\phi = i\theta[Q,\phi] = i\theta\, q_\phi \phi$$

The classical potential on the other hand is identified as

$$V(\phi^*,\phi) = m^2(\phi^*\phi) + \frac{\lambda}{2}(\phi^*\phi)^2 \tag{1.26}$$

and has a minimization condition [3]

$$\frac{\delta V}{\delta \phi}\bigg|_{\phi=v} = \frac{\delta V}{\delta \phi^*}\bigg|_{\phi=v} = 0 \tag{1.27}$$

A non-vanishing parameter v corresponding to the ground-state fields may then rise if only a negative squared mass parameter[4] is considered ($m^2 = -\mu^2 < 0$). Then the minimization condition is solved for

$$|v|^2 = \frac{\mu^2}{\lambda} \neq 0 \tag{1.28}$$

which actually implies that the solution given by $v = e^{i\Theta}\sqrt{\frac{\mu^2}{\lambda}}$ for any Θ is not unique. The phase Θ in fact parametrizes the degenerate ground-state fields which can be depicted as a circle of radius $v_r = \sqrt{\frac{\mu^2}{\lambda}}$ in the complex plane. The symmetry breaking condition is then realized by choosing a certain value for the phase $\Theta = \Theta_0$ or equivalently by picking a specific direction in the aforementioned complex plane. Clearly then, the ground-state field v_0 is no longer invariant under the $U(1)$ transformations and the symmetry is broken due to the non-invariance of the vacuum.

In the QFT language classical fields are promoted into operators and ground state fields into VEVs. The symmetry breaking condition is then

$$\langle 0|\phi|0\rangle = v \tag{1.29}$$

which may be expressed in terms of its real field components ($\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$) as

$$\langle 0|\phi_1|0\rangle + i\langle 0|\phi_2|0\rangle = \sqrt{2}\, v \tag{1.30}$$

By picking the direction of symmetry breaking at $\Theta = 0$ meaning

$$v_0 = \langle 0|\phi_1|0\rangle = \left(\frac{2\mu^2}{\lambda}\right)^{1/2} \tag{1.31}$$

$$\langle 0|\phi_2|0\rangle = 0 \tag{1.32}$$

the vacuum of the theory is uniquely determined. We may then define new field operators with respect to the true non-degenerate vacuum as $\phi_1' = \phi_1 - v_0$, $\phi_2' = \phi_2$

[3]Strictly speaking the classical potential is identified as $\mathcal{V} = V + (\nabla\phi)^2$ with the second term being positive and thus not participating in the potential minimization.

[4]A positive coupling constant ($\lambda > 0$) is also implicitly assumed for the scalar potential to be bounded from below.

and express the Lagrangian density in terms of these *shifted fields*

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi_1')^2 + (\partial_\mu \phi_2')^2 - 2\mu^2(\phi_1')^2\right] - \frac{\lambda v_0}{2}\phi_1'({\phi_1'}^2 + {\phi_2'}^2) - \frac{\lambda}{8}({\phi_1'}^2 + {\phi_2'}^2)^2 \quad (1.33)$$

Since the particle spectrum is derived by small oscillations on the vacuum the above Lagrangian density of the shifted fields describes the interactions between one massive scalar with mass $m_{\phi_1'} = \sqrt{2}\mu$ and one massless ϕ_2 which is identified as the Goldstone boson.

According to the Goldstone theorem the presence of one massless boson should come as no surprise in a theory with a single broken generator as the Q of the $U(1)$ considered here. To clarify further the connection between the massless state encountered in this example and the Goldstone Boson theorem, the formal proof demonstrated previously should be recalled.

For the $U(1)$ considered, Noether's theorem implies a time-independent charge in terms of the real component fields

$$Q = q_\phi \int d^3x J_0(x) = q_\phi \int d^3x \left[(\partial_0\phi_2)\phi_1 - (\partial_0\phi_1)\phi_2\right] \quad (1.34)$$

which due to the equal-time commutators (1.11) it also satisfies the commutation relations

$$[Q, \phi_1] = \ i\, q_\phi \phi_2 \quad (1.35)$$

$$[Q, \phi_2] = -i\, q_\phi \phi_1 \quad (1.36)$$

Since (1.36) acquires a non-vanishing VEV the general relation (1.23) takes the form

$$-i\, q_\phi\, v_0 = \langle 0|\,[Q^a(t), \phi_2(0)]\,|0\rangle = \sum_k (2\pi)^3\delta^3(\boldsymbol{k})\left\{\langle 0|J_0(0)|k\rangle\langle k|\phi_2(0)|0\rangle\, e^{-iE_k t} - \langle 0|\phi_2(0)|k\rangle\langle k|J_0(0)|0\rangle\, e^{iE_k t}\right\}$$

In the sum over all states only $|\phi_2(k)\rangle$ will eventually survive and thus we identify this field as the Goldstone boson possessing a zero frequency mode for vanishing momentum and therefore massless.

To conclude, it should be further mentioned that Goldstone bosons derived from the SSB of a continuous symmetry of the full Hamiltonian (or Lagrangian) remain massless to any order in perturbation theory. Of course, to demonstrate that explicitly, would require a more extensive and rather technical discussion on many other topics including renormalization. A rough picture, though, can be obtained if one considers the basic structure of perturbations. In this picture, the interaction terms introduced through perturbative corrections also respect the symmetry and thus the symmetry manifests itself through conserved charges and currents to any order in perturbation theory. The currents being conserved ensure that a zero frequency mode for vanishing momentum will always exist when SSB is realised. This, on the other hand, is not the situation in the presence of explicit symmetry breaking interactions. In such a theory, where the symmetry of the scalar potential is not respected by the rest of the interactions, the scalars share an identical tree-level mass spectrum with those of the symmetric theory. This is due to the fact that symmetry breaking interactions can only appear in the mass terms through radiative corrections. As a result, in the case of SSB the massless states of the symmetric theory, unavoidably present due to the conserved currents of the Lagrangian, will also appear as massless states at tree level in the explicitly broken theory. At higher orders in perturbation theory, though, the latter will receive non-vanishing mass corrections reflecting the fact that the currents there are not actually conserved. These particles are commonly known as pseudo-Goldstone bosons.

1.2.4 Gauge Symmetries

Up to this section we restricted ourselves to global internal symmetries. The symmetry group under which the Lagrangian density was invariant was characterised by constant space-time group parameters. In this context we were able to classify particle states according to their representation under the given symmetry group, extract information about their mass spectrum and also determine their interactions by introducing all possible terms with respect to this symmetry. Following the symmetry invariance certain powerful tools such as currents and charges were developed enabling us to have a deeper understanding of the theory even in the case where the symmetry was broken.

A remarkable thing happens when we promote these global symmetries into local. Certain vector fields emerge as a necessary condition for the theory to remain invariant under the group of the local transformations. These fields are commonly referred to as *gauge bosons* and as will be shown, they follow the number of generators of the symmetry group. They act in a general theory as mediators of the fundamental interactions while in the explicit context of the SM they are identified as the carriers of the electroweak and strong force.

Abelian gauge theories

Perhaps the easiest way to demonstrate how local symmetries manifest themselves in a field theory is by promoting the corresponding global symmetry and demanding that the Lagrangian will change in a consistent manner respecting this new symmetry. We start with the illustrative example of QED described by the Lagrangian of a free electron field invariant under $U(1)$ transformations.

In the global symmetry case the Lagrangian density will be given by

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi \tag{1.37}$$

respecting, besides Lorentz symmetry and the renormalization argument, an internal $U(1)$ symmetry acting on fermions as an Abelian Lie-group through

$$\psi' = e^{i\theta q_{\psi}}\psi \tag{1.38}$$

$$\overline{\psi}\,' = e^{-i\theta q_{\psi}}\overline{\psi} \tag{1.39}$$

In the case where the fermion field is the electron the charge is chosen to be $q_{\psi} = -1$. By promoting the group parameter into a function of spacetime ($\theta \to \theta(x)$) it can be straightforwardly seen that the Lagrangian is no longer invariant due to the non-vanishing commutator

$$[\partial_{\mu}, U(\theta(x))] = [\partial_{\mu}, e^{i\theta(x)q_{\psi}}] \neq 0 \tag{1.40}$$

It is obvious then that the Lorentz covariant derivative should be promoted into an operator that changes non-trivially under local transformations but not necessarily into a form that commutes with them since that would eventually imply a global group parameter. These type of operators are commonly referred to as *gauge covariant derivatives* defined through

$$D_{\mu} \equiv \partial_{\mu} - iq_{\psi}A_{\mu} \tag{1.41}$$

with the vector field A_{μ} identified as the gauge field satisfying a non-trivial transformation under the local symmetry given by

$$A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\theta(x) \tag{1.42}$$

As a result of these definitions the covariant derivative will transform under the gauge group as

$$D'_\mu = U(\theta(x)) D_\mu U^\dagger(\theta(x)) \tag{1.43}$$

The Lagrangian respecting local symmetry transformations will then be given by the expression

$$\mathcal{L}_f = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi \tag{1.44}$$

with the gauge invariance being trivially satisfied due to (1.43). Of course, for the gauge bosons to become physical degrees of freedom and satisfy a non-trivial equation of motion a kinetic term should as well be introduced. Using our knowledge on the classical theory of electromagnetism a closer look on the field strength tensor seems well motivated. The field strength tensor defined through

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.45}$$

turns out to be gauge invariant as can be straightforwardly checked with the use of (1.42). The obvious choice for a (properly normalized) kinetic term is then

$$\mathcal{L}_g = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{1.46}$$

satisfying both Lorentz and gauge invariance. Augmented by the principle of renormalizability as previously explained it eventually turns out that

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_g \tag{1.47}$$

is the most general Lagrangian respecting all these principles. It describes the interactions of a single fermion with respect to a $U(1)$ local symmetry through the presence of a single vector field, the gauge boson. We identify this set of interactions as the electromagnetic force and this gauge boson as the photon, the mediator of electromagnetism.

To conclude our discussion on this abelian example it should be remarked that the photon is bound to be massless since a quadratic term of the gauge boson fields $A^\mu A_\mu$ will be absent from the Lagrangian due to gauge symmetry(1.42).

Non-Abelian gauge theories.

As a next step in our discussion we examine the possibility that certain local symmetries rising in a field theory correspond to non-abelian groups, namely groups whose elements do not commute with each other. We shall demonstrate this possibility in the framework of the Yang-Mills theories [3] based on the presence of the $SU(N)$ symmetry group. Although in these theories such a possibility may rise for both a simple and a semi-simple (i.e. $G_1 \times G_2 \times \ldots$) gauge group we focus on the former case for reasons of simplicity. The generalization to the semi-simple groups is always straightforward as will be discussed in the explicit construction of the SM.

Extending the previous abelian example to include non-abelian groups and using analogous steps in our formalism we first assume that a set of fermion fields transform in the fundamental representation of $SU(N)$ and thus belong to a N-dimensional multiplet. Then, under a gauge transformation, $U(\theta)$ fermions transform according to

$$\Psi' = e^{-i\theta^a T^a}\Psi \tag{1.48}$$

$$\overline{\Psi}' = \overline{\Psi}' e^{i\theta^a T^a} \tag{1.49}$$

with the $N^2 - 1$ Hermitian generators T^a satisfying the Lie-algebra of the group $[T^a, T^b] = if^{abc}T^c$. The gauge covariant derivative is now defined through

$$D_\mu \equiv \partial_\mu - igA^a_\mu T^a \tag{1.50}$$

with the gauge bosons following the number of group generators and transforming non-trivially under the gauge group through

$$\boldsymbol{A}_\mu \equiv A^a_\mu T^a \ , \ \ \boldsymbol{A}'_\mu = U\boldsymbol{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{1.51}$$

Under these definitions it can be checked that the gauge covariant derivative transforms in analogy with the abelian case (1.43) as

$$D'_\mu = UD_\mu U^\dagger = e^{-i\theta^a T^a} D_\mu \, e^{i\theta^a T^a} \tag{1.52}$$

meaning that the fermion part of the invariant Lagrangian density will be given by

$$\mathcal{L}_f = i\,\overline{\Psi}\gamma^\mu D_\mu\Psi - m\overline{\Psi}\Psi \tag{1.53}$$

For the gauge part the situation turns out to be less trivial. The field strength tensor

$F^a_{\mu\nu}$, if naively considered as in the abelian case, transforms non-trivially under the gauge transformation and $F^a_{\mu\nu}F^{a\mu\nu}$ is not invariant. This suggests that the tensor should be promoted into a more suitable form which will further allow the construction of an invariant term in the Lagrangian. To realize this a closer look on (1.45) is required. There, it can be seen that the field strength may also be expressed through the commutator

$$[D_\mu, D_\nu] = -iqF_{\mu\nu} \tag{1.54}$$

Such a relation can be straightforwardly generalized as a definition in the non-abelian case

$$\begin{aligned} -ig\boldsymbol{F}_{\mu\nu} \equiv -igF^a_{\mu\nu}T^a &\equiv [D_\mu, D_\nu] \\ \boldsymbol{F}_{\mu\nu} &= \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu] \qquad (1.55) \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \qquad (1.56) \end{aligned}$$

with this new tensor following the simple transformation law of the gauge covariant derivative(1.52)

$$\boldsymbol{F}'_{\mu\nu} = U\boldsymbol{F}_{\mu\nu}U^\dagger \tag{1.57}$$

$$F^{a'}_{\mu\nu} = F^a_{\mu\nu} + f^{abc}\theta^b F^c_{\mu\nu}\ , \qquad (\theta \to 0) \tag{1.58}$$

The infinitesimal transformation above reveals a certain representation for the field strength tensor. This is the *adjoint representation* defined through $T^a_{bc} = -if^{abc}$, thus being non-trivial for any non-abelian Lie-group. The invariant quadratic term in the Lagrangian will be proportional to $\mathrm{Tr}[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}]$ or equivalently to $F^a_{\mu\nu}F^{a\mu\nu}$ due to the relation $\mathrm{Tr}[T^aT^b] = \kappa\,\delta^{ab}$. Thus, the Lagrangian density for the gauge bosons in the non-abelian case will be given by

$$\mathcal{L}_g = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} = -\frac{1}{4\kappa}\,\mathrm{Tr}[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}] \tag{1.59}$$

Our previous analysis also reveals certain aspects that require a closer examination. For example, the gauge fields as in the abelian case will be massless since a term quadratic on the fields is also forbidden due to gauge symmetry. Furthermore, the coupling parameter emerging from the definition of the gauge covariant derivative is bound to be universal for any given non-abelian simple gauge group. In other words the gauge covariant derivative D_μ necessarily operates on the different representations of the theory with the same g. This condition in fact arises from the non-trivial nature of the group algebra since the generator commutator participates in the definition of the field strength tensor(1.55). In case of semi-simple groups $G_1 \times G_2 \times \ldots$ this condition is relaxed but still one can associate a single gauge coupling for every simple group G_i. Also, in the case of abelian groups such a condition is obviously absent eventually allowing for particles with different charges.

1.3 Basics on Renormalization

In a relativistic QFT certain measurable quantities, when directly evaluated within the corresponding theoretical framework, turn out to produce infinities, eventually rendering the theory itself ill-defined. For a deeper understanding of this problem one should recall the structure of perturbations. In a perturbative QFT after determining the Feynman rules of the corresponding theory at tree level one proceeds to evaluate physical quantities such as masses, couplings or amplitudes to any order in perturbation theory. In practice, this is usually impossible to be realized since it would imply a sum over an infinite number of terms. But to a good approximation one can stop at a certain order of perturbations which in the formalism of the Feynman diagrams corresponds to a certain number of loops involved. The evaluation of diagrams involving loops, though, often corresponds to the evaluation of divergent integrals reflecting that the momenta of the virtual particles involved are allowed to vary up to infinity. If the theory satisfies certain constraints then these divergences, corresponding to the absence of a natural cut-off for the internal momenta, can be collected and absorbed through a series of techniques in a redefinition of well-behaved quantities.

Such a procedure is commonly known as *renormalization* [4–7]. Renormalization is a vast subject including various techniques and followed by an extensive formalism. However, since we are only interested in general aspects, it is sufficient to restrict ourselves to the rather standard strategy, known as *renormalization with counterterms*, as applied in a scalar-fermion theory.

1.3.1 Renormalization with counterterms

The counterterm renormalization method,[5] is a certain renormalization scheme characterized by the introduction of terms in the Lagrangian specifically designed to cancel

[5]also known as the BPH- renormalization scheme [5], as for Bogoliubov-Parasiuk-Hepp who originally proposed it, further developed by Zimmermann and Lowenstein [6].

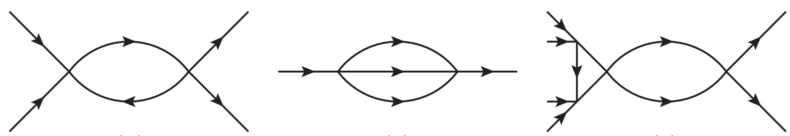

Figure 1.1: Divergent diagrams in ϕ^4 theory. The superficial degree of divergence for (a,b,c) is D=0,2,-2 respectively. Note that although (c) is superficially convergent it is actually divergent due to a subgraph divergence identical to (a).

divergences as those arise in a relativistic, perturbative, QFT. Its concrete mathematical consistency along with its relative simplicity have established it as the standard method for renormalization both for practical calculations as well as theoretical studies.

In order to understand how this scheme is explicitly realized in a QFT we first express a general scalar-fermion Lagrangian in the more convenient form

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \mathcal{L}_i \tag{1.60}$$

where we have separated the free part $\mathcal{L}_0$, from all other possible interaction terms $\mathcal{L}_i$. The Lagrangians we will be interested in are the so-called *renormalizable by power counting*. They consist only of terms with total mass dimension $M_d \leq 4$ as this arises by addition if we assign $1, 1, 3/2$ mass dimensions to derivatives, scalars and fermions respectively. Such an ad hoc choice for the allowed terms in the Lagrangian as well as the field and derivative assignments will be justified in what follows.

Besides the above general considerations, the following definitions will also be useful in our subsequent analysis:

One particle irreducible(1PI) Feynman diagrams. These are distinct diagrams that share a common topological property. They are understood as the connected diagrams which cannot become disconnected through a single cut in any of the internal propagators. With the term "connected" we refer to diagrams with all external propagators linked to each other while with "disconnected" to any other possible case. All diagrams displayed in Fig.1.1 should thus be regarded as 1PI since they immediately comply with the criteria above.

Superficial degree of divergence *D*. This is a particularly useful parameter, associated with the asymptotic behaviour of amplitudes. We define the superficial degree of divergence of a given diagram as the difference between numerator and denominator internal momenta as these may arise from propagators,vertices and integration variables. For example the diagram Fig.1.1(a) for a single scalar ϕ^4 theory with transferred momentum q will give

$$\Gamma^{(4)} \sim \int d^4k \frac{1}{((k+q)^2 - m^2)\,(k^2 - m^2)} \to D = 0 \tag{1.61}$$

Negative values for D indicate superficially convergent integrals while the values 0, 1, 2, etc. indicate logarithmic, linear, quadratic, etc. superficial divergence respectively. The term “superficial” is understood by considering Fig.1.1(c) where while $D = -2$ the diagram is actually divergent due to an inherent subgraph divergence. Nevertheless, as will be discussed, it is the superficial rather than the actual divergence of a diagram that is in practice important for renormalization.

It should be mentioned that for the evaluation of all diagrams in the renormalization procedure, integrals are implicitly assumed a priori regularized with a proper regularization scheme. That is integrals are considered to have become finite[6] without restricting ourselves to a specific regularization scheme and without violating fundamental elements of the theory such as symmetry relations. We also implicitly consider all particles massive so as to avoid a non-analytic behaviour of the integrals around zero momentum. This is a crucial point for our subsequent treatment where we shall consider Taylor series of integral functions around zero external momenta. That is because an expansion of a divergent integral, as in (1.61), in terms of external momenta will give

$$\Gamma(q) = \Gamma(0) + q^2 \left.\frac{\partial \Gamma}{\partial q^2}\right|_{q^2=0} + \ldots \tag{1.62}$$

where it can be checked that only $\Gamma(0)$ is divergent while higher order terms will be convergent ($D < 0$). In this way, we may in general isolate the divergence of a given diagram to a finite number of terms in the Taylor expansion.

The 1PI diagrams of Fig.1.1, displaying the superficial degree of divergence accordingly, may help to understand a generic, useful relation that holds, associating the topology of a connected diagram with the superficial degree of divergence. To demonstrate it we first explain our notation. We denote the number of external scalar(fermion) lines as B(F), of internal scalar(fermion) lines as IB(IF) and of ith

[6] This is either through cut-off or dimensional regularization [7] with the latter scheme being considerably advantageous in the presence of symmetries.

type vertices as n_i corresponding to a certain $\mathcal{L}_i$ term. For a given vertex i we further denote as $b_i(f_i)$ the scalar(fermion) lines attached to it and d_i its possible derivatives. Then the following relations for a general theory with only bosons or only fermions

$$\begin{aligned} B + 2(IB) &= \sum n_i b_i \\ F + 2(IF) &= \sum n_i f_i \end{aligned} \tag{1.63}$$

would immediately hold. A generalization to the case of a scalar-fermion theory, reveals for the superficial degree of divergence

$$D = \sum n_i d_i + 2(IB) + 3(IF) - 4\sum n_i + 4 \tag{1.64}$$

$$= -B - \frac{3}{2}F + 4 + \sum n_i \delta_i \tag{1.65}$$

$$\delta_i \equiv b_i + \frac{3}{2} f_i + d_i - 4 \tag{1.66}$$

where for the second row (1.63) was used. The meaning of (1.64) is rather simple. Every derivative interaction contributes a momentum in the nominator through the corresponding vertex ($\sum n_i d_i$). Every internal boson(fermion) line will contribute two(three) degrees of divergence from the difference between the introduced four-momentum volume of integration and the denominator momenta coming from the propagator $(2(IB), 3(IF))$. Each vertex i imposes a four-momentum delta function which cancels an integration volume ($-4\sum n_i$) except for one delta function which survives corresponding to the four-momentum conservation for the external propagators(+4).

The parameter δ_i as defined in (1.66) is commonly referred to as the *index of divergence* of the ith type vertex or equivalently of the $\mathcal{L}_i$ term. As defined here, it carries the information for the *naive dimension* of the corresponding operator previously considered as "mass dimension" of fields and derivatives. We may now explain further this assignment by dimensional analysis arguments. The Lagrangian density will necessarily carry a mass dimension four for the action to be dimensionless. Since each derivative will have a mass dimension one ($1/L \sim M$) then from the kinetic terms we easily figure out the previously mentioned mass dimensions for scalars and fermions. The term "naive" is irrelevant for our discussion since it applies to the case of vector fields we choose not to consider. We only mention that in this case the naive might be different than the canonical dimension of an operator[7] which is a more appropriate

[7]This is extracted from the asymptotic behaviour of the free-field propagators.

parameter for theories with gauge bosons. In any case, for the scalar-fermion theories we consider, the index of divergence is also given by

$$\delta_i = \mathrm{Dim}\,\mathcal{L}_i - 4 \tag{1.67}$$

We may now describe the counterterm renormalization scheme as the following systematic approach:

- First, a one-loop computation of all 1PI diagrams of the theory is required. We focus on the superficially divergent diagrams satisfying $D \geq 0$. A Taylor expansion of the divergent integrals around the *subtraction point*, here taken at zero external momenta will be used to isolate the divergent part.

- Next, a set of counterterms specifically designed to cancel the terms in the Taylor expansion with $D \geq 0$ is introduced in the Lagrangian.

- The modified Lagrangian is now considered to construct the counterterms for the two-loop diagrams and this procedure reiterates to all orders in perturbation theory.

With this technical treatment all divergences are expected to vanish. We should however point out a certain aspect which also explains the importance of the superficial degree of divergence in this scheme. For that we examine the diagram in Fig.1.1(a) in terms of the ϕ^4 theory with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1.68}$$

Since this is logarithmically divergent we introduce a counterterm as $i\frac{\Gamma(0)}{4!}\phi^4$ which not only cancels the divergence in this diagram but also the subgraph divergence in Fig.1.1(c) which eventually becomes finite. Thus, it is the superficial rather than the actual divergence of a given diagram that implies a counterterm. By systematically treating all superficial divergences in 1PI diagrams of a theory as described above, all divergences are expected to be eliminated.

With all fundamental tools available we may now proceed to examine the principle of renormalizability and the conditions this implies to a general QFT. After all that was our main motivation for this section. The principle states that every physical QFT should be renormalizable. It should be stressed that this is a characteristic of the theory rather than an artifact of the specific renormalization scheme applied. The primary condition for a renormalizable theory is that all operators in the Lagrangian should have a mass dimension $M_d \leq 4$. This immediately implies through (1.67) the condition $\delta_i \leq 0$. As an immediate consequence, the superficial degree of divergence D for such a theory is well-behaved and cannot increase for higher order diagrams as can be checked from (1.65). If that was not the case then one would have to introduce an infinite number of counterterms to treat the increasing number of divergent diagrams as these would unavoidably arise from higher order terms in perturbation theory.

We may generalise this last condition to an essential characteristic of renormalizable theories. That is for such theories only a finite number of counterterms is required to eliminate the divergences to all orders in perturbation theory. However, this "renormalizable by power counting" condition is necessary but not sufficient to render the theory renormalizable. The counterterms in a renormalizable theory should also follow the structure of the original Lagrangian. This property, required for the mathematical consistency of this scheme, is already satisfied for the diagrams of Fig.1.1 which produce a counterterm Lagrangian of the form

$$\mathcal{L}_{ct} = A(\partial_\mu \phi)^2 + B\phi^2 + C\phi^4 \tag{1.69}$$

Only then the *bare* (unrenormalized) Lagrangian can be expressed as

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda}{4!}\phi_0^4 = \mathcal{L} + \mathcal{L}_{ct} \tag{1.70}$$

which implies that we can absorb and redefine all divergences of $\mathcal{L}_0$ into the well behaved $\mathcal{L} + \mathcal{L}_{ct}$.

Of course there are other interesting and non-trivial implications of renormalizability especially when we consider symmetric Lagrangians. Our previous analysis however is adequate to understand the general constraints of renormalization on the structure of a QFT.

1.4 Building the Standard Model

Our previous analysis, focused on certain selected topics within the general framework of quantum gauge theories, may further apply as a guideline in an attempt to explore an explicit gauge model, the SM of the particle physics [10]. Although such a model turns out to exhibit various attractive features, for reasons that will be explained further at the end of this section it should probably be regarded as a highly functional *effective model* of particle interactions at low energies. In this sense various fundamental assumptions as the choice of the gauge group or the representation content of the theory do not imply a deeper motivation other than the remarkably accurate and sometimes elegant description of particle interactions at low energies.

1.4.1 Basic structure of the SM

The SM may be regarded as an explicit realization of the semi-simple symmetry group $\mathcal{G}_{SM} = SU(3) \times SU(2) \times U(1)$ within the framework of a relativistic QFT. In this viewpoint the elementary particles (scalars, fermions, gauge bosons), or strictly speaking the particle states involved can be classified further according to their representation profile below:

$$\begin{aligned} \text{Gauge Bosons}: &\quad G_\mu(8,1,0)\ , &\quad W_\mu(1,3,0)\ , &\quad B_\mu(1,1,0) \\ \text{Fermions}: &\quad Q_L^I(3,2,1/6)\ , &\quad u_R^I(3,1,2/3)\ , &\quad d_R^I(3,1,-1/3) \\ &\quad L_L^I(1,2,-1/2)\ , &\quad e_R^I(1,1,-1)\ , &\quad \nu_R^I(?) \\ \text{Scalars}: &\quad \Phi(1,2,1/2) \end{aligned} \tag{1.71}$$

The first row represent the twelve massless gauge bosons of the unbroken theory corresponding to the twelve generators of the $\mathcal{G}_{SM}$ with G, W, B being the gauge fields of the respective $SU(3), SU(2), U(1)$ subgroups. The second and third row represents the fermion states of the theory with an implicit family replication denoted by the index $I = 1, 2, 3$ and confirmed by experiments. The Q_L, L_L as far as the $SU(2)$ subgroup is concerned [8] are the doublets (u_L, d_L), (ν_L, e_L) respectively. All fermion states are denoted as four-component spinors of certain chirality[9] (left or right-handed) with spinors of opposite chirality being allowed to transform in different representations of the gauge group. When the gauge symmetry will be broken these chiral fields will form together Dirac fermions (four-component spinors) obviously belonging to the same representation of the remnant gauge group. The situation for the neutrinos is more subtle due to the inadequacy of current experimental data as will be discussed extensively in a following chapter. Finally, the last row represents an $SU(2)$ doublet of scalars, charged under $U(1)$, known as the Higgs doublet. It is required for the operation of the Higgs mechanism which will realize in a rather elegant manner the SSB phenomenon within the framework of the SM gauge theory.

Since the Lagrangian of the SM, even in the unbroken case, is rather extensive,

[8] suppressing color-SU(3) indices

[9] These are simply the four-component spinors $\Psi_L \equiv (\psi_L, 0)\,, \Psi_R \equiv (0, \psi_R)$ in a two-component block form.

a decomposition to the different sectors of the theory is required. Thus, using our previous philosophy on gauge theories we may write

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_s \tag{1.72}$$

isolating the gauge, the fermion and the scalar part respectively. Before we examine further each term of the above expression a general remark on the gauge covariant derivative should be made. The gauge covariant derivative can be expressed anywhere in the compact form

$$D_\mu = \partial_\mu - ig_1 Y B_\mu - ig_2 W_\mu^a T_{(2)}^a - ig_3 G_\mu T_{(3)}^a \tag{1.73}$$

Its explicit form will be determined by the respective representation on which it operates each time. The Y, $T_{(2)}$, $T_{(3)}$ are the $U(1)$ hypercharge, the $SU(2)$ and the $SU(3)$ generators with respect. The covariant derivative acting on the Higgs doublet for example would have $Y = -1/2$ and $T_{(2)}$ as the properly normalized generators of the $SU(2)$ fundamental representation[10]. Since this representation is trivial under $SU(3)$ the eight generators $T_{(3)}$ will be vanishing.

With this global definition of the covariant derivative at hand we may proceed to examine the gauge sector. There, we have

$$\mathcal{L}_g = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} W^{\mu\nu\, a} W_{\mu\nu}^a - \frac{1}{4} G^{\mu\nu\, b} G_{\mu\nu}^b \tag{1.74}$$

Obviously, this is the Yang-Mills structure previously discussed in the abelian and non-abelian gauge theories. The definitions for the field strength tensors still apply in the case of the semi-simple group of the SM but with the straightforward generalization that for each subgroup there is a single corresponding gauge coupling involved in the definitions. Thus for example from the definition (1.55) the field strength for the $SU(2)$ gauge bosons $\boldsymbol{W}_{\mu\nu}$ will involve only the gauge coupling g_2 since the covariant derivative (1.73) will be trivial for the g_1, g_3 terms.

In the fermion sector the Lagrangian is more extensive but the structure is rather

[10]These are no other than the well-known Pauli matrices σ_i multiplied by a factor 1/2

simple. We have[11]

$$\begin{aligned}\mathcal{L}_f &= i\overline{Q}_L \not{D} Q_L + i\overline{u}_R \not{D} u_R + i\overline{d}_R \not{D} d_R + i\overline{L}_L \not{D} L_L + i\overline{e}_R \not{D} e_R \\ &+ Y_u^{IJ} \overline{Q}_L^I \tilde{\Phi} u_R^J + Y_d^{IJ} \overline{Q}_L^I \Phi\, d_R^J + Y_e^{IJ} \overline{L}_L^I \Phi\, e_R^J + h.c \\ &\{+i\overline{\nu}_R \not{D} \nu_R + (Y_\nu^{IJ} \overline{L}_L^I \tilde{\Phi} \nu_R^J + m_N^{IJ} \nu_R^I \nu_R^J + h.c.)\}\end{aligned} \tag{1.75}$$

In the first row we observe the kinetic terms which in a gauge theory also include the interactions of fermions with gauge bosons through the explicit form of the covariant derivative. Family indices there are suppressed but are implied diagonal due to a choice of basis (i.e. the gauge basis). In the second row all possible renormalizable interactions of fermions with scalars respecting the symmetry are introduced. The matrices Y in family space, also known as *Yukawa couplings*, are not directly measurable but they give rise to physical quantities such as the fermion masses and mixings when the symmetry is spontaneously broken. The last row corresponds to the rather standard (but not unique) extension of the SM in order to include masses for the neutrinos. These terms were absent in the original theory where the neutrinos were erroneously considered massless and so far experiments have been inconclusive on their explicit structure.

Even from this brief review over the fermion and gauge sector of the unbroken SM the necessity for a symmetry breaking mechanism should have become obvious. Although this model describes adequately particle interactions at higher energies, namely above the respective mass thresholds of the observed particles it falls short when one attempts to explain the origin of masses. To explain further, in the framework previously considered all particles due to gauge symmetry are massless (i.e. there is no mass term allowed in the theory either for fermions or gauge bosons). This nice behaviour of the theory at higher energies, as well as its deficiencies at lower, indicates not only that the theory should be spontaneously broken but also that the mechanism operating should attribute masses to certain particles as these are observed in nature. Such an elegant mechanism, commonly known as the *Higgs mechanism* [8, 9] will be discussed below.

1.4.2 The Higgs mechanism

The scalar part of the Lagrangian was deliberately separated from our previous discussion since not only it describes interactions of the scalar-gauge bosons but under

[11] For economy of notation we employ the Feynman-slash symbol $\gamma^\mu D_\mu \equiv \not{D}$. Also the conjugate scalar doublet is defined as $\tilde{\Phi} = i\sigma_2 \Phi^*$ so as for the proper contraction of gauge indices to be taken into account.

certain circumstances it can give rise to the SSB of the full theory. Since the spontaneous breaking is realized in a similar fashion for both gauge and global symmetries, the $U(1)$ example of a spontaneously broken global symmetry, as discussed in §1.2.3, will be used as a guideline in what follows.

Applying the formalism of the abelian case, the most general renormalizable Lagrangian for the scalar SU(2)-doublet (1.71) of the SM will be given by

$$\mathcal{L}_s = D^\mu \Phi^\dagger D_\mu \Phi + \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2}(\Phi^\dagger \Phi)^2 \tag{1.76}$$

Obviously, for the SSB mechanism to operate both μ^2, λ should be considered positive. The scalar potential of the theory is identified as

$$V(\Phi, \Phi^\dagger) = -\mu^2 \Phi^\dagger \Phi + \frac{\lambda}{2}(\Phi^\dagger \Phi)^2 \ , \quad \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix} \tag{1.77}$$

with a minimization condition satisfied for all ground state fields Φ_{GS} with

$$\Phi^\dagger_{GS} \Phi_{GS} = \frac{\mu^2}{\lambda} \equiv \frac{v^2}{2} \tag{1.78}$$

As in the abelian case, the vacuum will be degenerate and therefore SSB is realized by choosing the scalar doublet to acquire a non-vanishing VEV in a certain direction. We choose

$$\langle 0|\Phi|0\rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \tag{1.79}$$

which in terms of the real scalar fields ϕ_i implies that only $\langle 0|\phi_0|0\rangle$ will be non-vanishing.

The non-degenerate vacuum of the spontaneously broken theory will remain invariant under all $SU(3)$ transformations reflecting the fact that the Higgs weak isodoublet(1.71) is a singlet under $SU(3)$. On the other hand, from the four generators of $SU(2) \times U(1)$ acting on the vacuum only the linear combination

$$Q = T_3 + Y \tag{1.80}$$

remains unbroken satisfying $Q|0\rangle = 0$. This linear combination will correspond to the generator of a new $U(1)$ symmetry belonging to the remnant gauge group. Thus, for the SM we have the pattern of SSB

$$SU(3)_C \times SU(2) \times U(1)_Y \stackrel{\langle 0|\Phi|0\rangle}{\longrightarrow} SU(3)_C \times U(1)_{EM} \tag{1.81}$$

where the surviving abelian subgroup is identified as the symmetry of the electromagnetic interactions ($U(1)_{EM}$).

Introducing the shifted fields with respect to the non-degenerate vacuum of the broken theory as $\phi'_i = \phi_i - v\,\delta_{i0}$ reveals a new remarkable aspect for the SSB mechanism of gauge symmetries. If these symmetries had been considered global three massless states corresponding to the three broken generators of the $SU(2) \times U(1)_Y$ subgroup would have unavoidably appeared as a result of the Goldstone theorem. In gauge theories though these massless states not only can be absorbed through a specific gauge transformation but also provide the theory with the necessary extra degrees of freedom required for the 'broken' gauge bosons[12] to become massive. These massless states of the gauge theory are commonly referred to as the *would-be Goldstones*. The gauge-fixing condition corresponding to this specific gauge transformation is referred to as the *Unitary Gauge*.

In order to demonstrate explicitly this elegant mechanism within the context of the SM we first reexpress the Higgs doublet in terms of the shifted fields. We have

$$\Phi = \Phi' + \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi'_2 + i\phi'_3 \\ (\phi'_0 + v) + i\phi'_1 \end{pmatrix} \tag{1.82}$$

$$\Phi \equiv U^\dagger(x) \begin{pmatrix} 0 \\ \frac{n+v}{\sqrt{2}} \end{pmatrix} = e^{i(\xi_i T_i/v)} \begin{pmatrix} 0 \\ \frac{n+v}{\sqrt{2}} \end{pmatrix} \tag{1.83}$$

where the T_i's are the three broken generators of the theory, conveniently normalized as $\sigma_1, \sigma_2, (Y - \sigma_3/2)$. The consistency of the parametrization in (1.83) can be seen by considering the fields $\xi_i(x)$ as group parameters. Then due to the Lie-group property for infinitesimal transformations

$$\Phi = (1 + i(\xi_i T_i v) + \dots) \begin{pmatrix} 0 \\ \frac{n+v}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_2 + i\xi_1 \\ (n+v) + i\xi_3 \end{pmatrix} + \dots \tag{1.84}$$

the fields n, ξ_i are straightforwardly identified with the shifted fields ϕ'_i when the latter describe small oscillations around the vacuum.

We may use the parametrization of (1.83) to define a new Higgs doublet and new

[12] We use the, rather inaccurate, terminology *" 'broken' gauge bosons "* as an abbreviation in order to describe those gauge bosons corresponding to the broken generators of the theory.

gauge bosons as

$$\Phi^U \equiv U(x)\Phi = \begin{pmatrix} 0 \\ \frac{n(x)+v}{\sqrt{2}} \end{pmatrix} \tag{1.85}$$

$$\boldsymbol{W}_\mu^U \equiv U\boldsymbol{W}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \tag{1.86}$$

$$B_\mu^U \equiv B_\mu \tag{1.87}$$

Clearly, this is a gauge-fixing condition since the new fields, defined through a gauge transformation of the broken generators, are not allowed to transform further under it. To explain in more detail, we should mention that the unbroken theory has $U(x)$ as a symmetry. Thus, one may define the fields in any (legitimate) gauge, namely for any suitable $U'(x)$ including the one of the unitary gauge, and then use the SSB mechanism to derive the Lagrangian of the broken theory. The resulting Lagrangians will describe equivalent theories but they will no longer be connected through the considered gauge transformations since these are not symmetries of the broken theory. By choosing to define the theory in the unitary gauge one is able to absorb the unphysical degrees of freedom ξ_i in the gauge fixing condition described by the above relations. The Lagrangian for the scalar sector will then have the simple structure

$$\mathcal{L}_s = D^\mu \Phi^{U\dagger} D_\mu \Phi^U + \frac{\mu^2}{2}(v+n(x))^2 - \frac{\lambda}{8}(v+n(x))^4 \tag{1.88}$$

From the kinetic term new mass terms quadratic in the 'broken' gauge bosons will emerge when the Higgs will acquire a non-vanishing VEV. But before discussing the explicit mass spectrum as it manifests in the unitary gauge it is worth mentioning that mass terms for the gauge fields also appear for other gauge-fixing conditions. What makes the unitary gauge so special is that the unphysical degrees of freedom which in general would couple to the gauge bosons in bilinear forms are absent, thus rendering the mass spectrum more transparent.

1.4.3 Particle spectrum

The explicit spectrum for the SM particles in the unitary gauge will include:

- A single scalar identified as the Higgs boson with a mass $m_n = \sqrt{2}\mu$ as derived from the n^2 terms in (1.88). It should be remarked that this term will have the correct minus sign in the Lagrangian corresponding to a physical particle with definite mass.

- Six quarks and three charged leptons (neglecting neutrinos) with a mass derived from the Yukawa couplings of (1.75). There, it can be seen that opposite chirality states will form massive Dirac fermions when the Higgs develops a VEV. Due to the family structure of the theory the fermion mass terms can be represented as the 3×3 matrices in family space

$$M_u^{IJ} = -\frac{v}{\sqrt{2}} Y_u^{IJ} \,,\; M_d^{IJ} = -\frac{v}{\sqrt{2}} Y_d^{IJ} \,,\; M_e^{IJ} = -\frac{v}{\sqrt{2}} Y_e^{IJ} \tag{1.89}$$

  The physical masses are then obtained by diagonalizing the above matrices through *bi-unitary transformations*, namely as $U_1^\dagger M U_2$. Fermion mass diagonalization as well as fermion mixing will be discussed in more detail for both charged fermions and neutrinos in a following chapter. It should be mentioned that quarks, contrary to leptons, will carry an extra degree of freedom, the color, due to the fact that they belong to a non-trivial representation of the unbroken $SU(3)_C$.

- Twelve gauge bosons out of which the three, $W^\pm, Z$ corresponding to the broken generators of the $SU(2) \times U(1)_Y$, will become massive through the Higgs kinetic term in (1.88). In fact it turns out that these gauge bosons will acquire their masses in a rather elegant manner which further preserves the renormalizability of the spontaneously broken theory.

In order to illustrate explicitly the mass generation mechanism for the gauge bosons we first focus on the covariant derivative of the Higgs kinetic term. Since the Higgs doublet is trivial under $SU(3)$ the g_3 term of the compact expression (1.73) will be absent. We therefore have

$$D^\mu \Phi^\dagger \, D_\mu \Phi = \left| (\partial_\mu - i g_1 Y_\Phi B_\mu - i g_2 W_\mu^a T^a) \begin{pmatrix} 0 \\ \frac{n(x)+v}{\sqrt{2}} \end{pmatrix} \right|^2 \tag{1.90}$$

with $Y_\Phi = 1/2$, $T^a = \sigma^a/2$. From this we obtain the expression relevant for gauge boson masses

$$\left| \frac{1}{2} \begin{pmatrix} \{g_1 B_\mu + g_2 W_\mu^3\} & g_2(W_\mu^1 - i W_\mu^2) \\ \{g_2(W_\mu^1 + i W_\mu^2)\} & g_1 B_\mu - g_2 W_\mu^3 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \right|^2 \tag{1.91}$$

where obviously the terms in the first column of the above matrix are irrelevant. We next recall that the broken gauge group has a surviving abelian symmetry $U(1)_{EM}$. Hence, all physical particles will transform trivially (neutral) or non-trivially (charged) under the remaining symmetry. The gauge bosons B_μ, W^3_μ are neutral but the fields $W^{1,2}_\mu$ do not transform properly under the electromagnetic transformation (i.e. they are not charge eigenstates.) On the other hand, the linear combinations $W^\pm_\mu \equiv (W^1_\mu \mp i\,W^2_\mu)/\sqrt{2}$ are. They will carry unit charge as can be directly checked from the commutators

$$[Q, T_1 \mp iT_2] \equiv [Q, T_\pm] = \pm T_\pm \tag{1.92}$$

In terms of the charge eigenstates the terms relevant for masses become

$$\frac{g_2^2 v^2}{4}\, W^+_\mu W^{-\mu} + \frac{v^2}{8}(g_1 B_\mu - g_2 W^3_\mu)^2 \tag{1.93}$$

The first term is obviously a mass term with $M_W = g_2 v/2$ for the two charged gauge bosons $W^\pm$. The second contains all possible bilinear mixing of B, W^3. Clearly, a rotation to the mass eigenstate basis is required in order to obtain the mass spectrum. We thus reexpress this term in the more transparent matrix form

$$\frac{v^2}{8}\left(W^{3\mu}\ B^\mu\right)\begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2 \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{1.94}$$

where a zero mass eigenvalue is present due to the vanishing determinant. Diagonalization is obtained through the orthogonal transformation

$$U(\theta_W) \equiv \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\ ,\ \ \tan\theta_W = g_1/g_2 \tag{1.95}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = U(\theta_W)\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{1.96}$$

which will bring (1.94) in the diagonal form

$$\frac{1}{2}\left(Z^\mu\ A^\mu\right)\begin{pmatrix} \frac{v^2}{4}(g_1^2+g_2^2) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \tag{1.97}$$

We straightforwardly obtain a massive neutral gauge boson, the Z-boson, with mass $M_Z = (v/2)(g_1^2 + g_2^2)^{1/2}$ and a massless one, A, which we identify as the photon of the electromagnetic interactions. The $SU(3)_C \times U(1)_{EM}$ symmetry of the spontaneously broken theory ensures that the photon as well as the eight gluons of the strong interactions will be massless, to all orders in perturbation theory. It should also be mentioned that when one considers the mass expressions for the gauge bosons along with the *Weinberg (electroweak) mixing angle* θ_W defined through (1.95) the following relation is obtained

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1 \tag{1.98}$$

It should be mentioned that this relation can be modified only if the Higgs sector is extended to include representations other than the standard weak isodoublets. All experiments to date support such a value for the *ρ-parameter*.

1.5 Limitations of the Standard Model

Although the SM is an extremely accurate and rather elegant gauge model, its fundamental deficiency to incorporate gravity along with other inadequacies that arise within its context, motivate theoretical search for a more concrete framework. Various theories beyond the SM such as *Supersymmetry* (SUSY) or *Grand Unified Theories* (GUTs), discussed here, seem to provide a consistent generalization of the SM with many interesting and appealing features. To better understand the necessity for a more general theoretical framework we present the following major issues that the SM has been insufficient to address.

- *The number of free parameters.* As a gauge theory the SM encompasses a number of free parameters that cannot be determined by any consideration within its context. The value for the Weinberg angle, the hierarchical fermion masses, the mixing angles for the quarks (CKM) and leptons (PMNS)[13] are regarded in the SM as just-so parameters with no deeper physical meaning. But also at a more fundamental level, one is not able to provide with a convincing answer for the choice of the gauge group or the smallness of the electroweak scale as compared with the *Planck scale* where gravity becomes strongly interacting ($M_{Pl} \sim 10^{18} GeV$).

- *The hierarchy problem.* Disregarding the aforementioned issues and keeping only the minimal assumption for a drastically different quantum theory at Planck scale produces a problem of technical nature to the SM itself. Heavy particles, originating from this high-energy theory, naturally drive the light Higgs mass to

[13]Masses and mixing for fermions will be discussed in detail within the more general framework of SUSY-GUTs and as a part of a published paper.

M_{Pl} through renormalization. For this unwanted situation to be absent severe fine-tuning cancellation between M_{Pl} parameters must be realized for an $M_{EW} \sim 10^2 GeV$ Higgs mass to appear. In fact, this is a more general characteristic of *effective gauge theories*, namely theories with a high-energy completion. The larger the hierarchy between the scales, the more the fine-tuning required for the scalars to remain light.

Chapter 2

Theories Beyond the Standard Model

2.1 Supersymmetry

Among various proposed extensions of the SM, supersymmetry (SUSY) [12–14] singles out as a theory that generalizes the SM in a consistent manner and at the same time evades the hierarchy problem. Built on the rather simple idea of a symmetry connecting bosons with fermions, SUSY has developed through several decades into a solid theoretical framework strongly believed to describe particle interactions at scales where the SM is expected to fail. For the moment experiments show no indication for such a symmetry to exist. However, the recent discovery of a light scalar particle at the LHC, believed to be the Higgs particle, has revitalized the interest in supersymmetric extensions of the SM. These seem to provide a more elegant framework for fundamental scalars of "small mass".

2.1.1 Motivations and general properties.

In order to understand how a fermion-boson symmetry may resolve the technical complications of an effective gauge theory with light scalars we examine the radiative corrections of the Higgs in the presence of heavy particles. Using the cut-off regularization scheme, the Higgs coupling to heavy scalars will induce the one-loop mass correction

$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}\Lambda^2 + \dots \tag{2.1}$$

corresponding to a $\lambda_S|\Phi|^2|S|^2$ coupling in the Higgs potential. The parameter Λ is the cut-off scale of the theory, here considered to lie around the scale M_{Pl} where gravity becomes important.

The coupling with heavy Dirac fermions will give an analogous quadratically divergent contribution of the form

$$\Delta m_H^2 = -\frac{|Y_\psi|^2}{8\pi^2}\Lambda^2 + \dots \tag{2.2}$$

corresponding to a (diagonalized) Yukawa coupling expressed in the general form $(Y_\psi \Phi \overline{\Psi}_R \Psi_L + h.c.)$.

These quadratic divergences indicate that the expected mass scale for the Higgs particle should be $\Lambda \sim M_{Pl}$ instead of M_{EW}. This feature turns out to be an intrinsic property of effective theories with scalars rather than an artifact of the regularization scheme followed. If we had used instead the dimensional regularization scheme the quadratic divergences would have been absent but simple poles would appear proportional to the squared masses of the particles in the loops. In both cases, though, one cannot justify the presence of a counterterm specifically tuned to cancel this divergent behaviour of the Higgs mass correction.

Supersymmetry, on the other hand, confronts the hierarchy problem in a rather straightforward manner. From the relations (2.1),(2.2) it becomes obvious that if for every Dirac fermion there are two complex scalars with $\lambda_S = |Y_\psi|^2$, then the minus sign between scalar and fermion loops will ensure the absence of these annoying quadratic divergences. If furthermore, such a cancellation is protected by an underlying symmetry then the idea becomes well established with definite implications on the particle spectrum and interactions.

From our previous experience on symmetry transformations, SUSY is expected to be described by some appropriate generators satisfying a corresponding algebra. But supersymmetric transformations contrary to the internal symmetries previously discussed are special transformations of the spacetime. This should be expected since only then the different spin states (spin-0,1/2,1) of the Lorentz group would be able to mix. Furthermore, SUSY generators should satisfy a very special algebra which evades the severe theoretical restrictions following spacetime symmetries [15]. It turns out that the graded Lie algebra

$$\begin{aligned}
\{Q, Q^\dagger\} &= 2\sigma_\mu P^\mu \\
\{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \\
[Q, P^\mu] &= [Q^\dagger, P^\mu] = 0
\end{aligned} \tag{2.3}$$

with P^μ being the energy-momentum vector, describes the algebra of supersymmetric transformations which for a single pair of Q, $Q^\dagger$ corresponds to the $\mathcal{N} = 1$ SUSY discussed here.

Particle states can be classified according to their transformation property under the SUSY algebra in a manner analogous to the Lie algebras we previously encountered. The irreducible representations of SUSY that accommodate fermion and boson states (*superpartners*) are commonly referred to as *supermultiplets*. Obviously, all superpartners within a given supermultiplet will share an identical mass since $[Q, P^2] = 0$. Furthermore, since $[Q, T^a] = 0$ for any generator of an internal Lie-group, a SUSY transformation will not mix states within an irreducible representation of the gauge group. Therefore the whole supermultiplet will transform as a gauge eigenstate or in other words all component fields within a given supermultiplet will share the same quantum numbers (charge, isospin, colour). In addition, it can be shown that in any given supermultiplet the number of fermion (n_F) and boson (n_B) degrees of freedom will be equal. This can be easily seen if one defines a suitable spin operator $(-1)^{2s}$ anti-commuting with $Q, Q^\dagger$ and with eigenvalues $1(-1)$ when acting on boson (fermion) states. Then it turns out that the trace of this operator over the states of a certain supermultiplet will always vanish as

$$\begin{aligned}
\sum_i \langle i|(-1)^{2s} P^\mu |i\rangle &= \sum_i \langle i|(-1)^{2s}\{Q, Q^\dagger\}|i\rangle = 0 \\
&= p^\mu \sum_i \langle i|(-1)^{2s}|i\rangle
\end{aligned} \tag{2.4}$$

due to the commutation property of the trace and the anticommutation property of the spin operator. The vanishing of $\mathrm{Tr}[(-1)^{2s}]$ subsequently implies the straightforward equality $n_B = n_F$ for the degrees of freedom of the given supermultiplet.

Using this knowledge over the states of a given supermultiplet we may revisit the desired cancellation of the quadratic divergences in the Higgs mass correction. In terms of supersymmetry the Higgs supermultiplet may couple to another supermultiplet which will include the same degrees of freedom for fermions and bosons. The simplest but not unique choice as we will see in more detail is the chiral supermultiplet with a single complex scalar field and a two component Weyl-spinor. Since a Dirac fermion is formed by two different Weyl-spinors the Higgs will necessarily couple to two chiral supermultiplets as $\lambda H\Phi_1\Phi_2$ in order to account for a Higgs-Dirac fermion coupling. Supersymmetry, as will be explained in what follows, guarantees that this coupling not only includes the operators $\lambda h\psi_1\psi_2$ and $|\lambda|^2|h|^2|\phi_1|^2$, $|\lambda|^2|h|^2|\phi_2|^2$ but also imposes the common coupling parameter λ . In addition, the complex scalars ϕ_i will share the same mass with their Weyl-spinor superpartners ψ_i and therefore with the Dirac spinor that they form. As a result, the worrisome quadratic divergences cancel each other without the fine-tuning required due to the hierarchy problem.

2.1.2 Supermultiplets- Introducing the MSSM

The supersymmetric transformations imply a further classification for particle states in terms of the supermultiplets, the irreducible representations of this special graded Lie-algebra (2.3).

The minimal non-trivial choice for a supermultiplet consistent with $n_B = n_F$ is the chiral supermultiplet. It accommodates a massless Weyl two-component spinor ψ and a complex scalar field ϕ as well as an *auxiliary field* F, of mass dimension two, necessary for the supersymmetry algebra to close off-shell. To explain further the purpose this auxiliary field serves, one should recall that a massless Weyl-spinor is described by two complex variables (i.e. four real degrees of freedom). When the equations of motion are invoked, two degrees of freedom are eliminated and the remaining two correspond to the distinct spin polarization states. Since the complex scalars have the same two degrees of freedom both off-shell and on-shell we have to invent a field that carries two extra bosonic degrees which further vanish on-shell. The fields that satisfy such a condition are not true dynamical variables of the system. They appear in the Lagrangian without a corresponding kinetic term thus satisfying a trivial equation of motion. One then may use these equations of motion to express these unphysical degrees of freedom in terms of the physical fields of the theory. For chiral supermultiplets the equation of motion reveals F as a function of the scalar bosons of the theory. Under these considerations we may denote the chiral supermultiplet Φ in the symbolic vector form

$$\Phi = \begin{pmatrix} \phi & \psi & F \end{pmatrix} \tag{2.5}$$

Next, we investigate the possibility for a representation of SUSY that can accommodate the gauge bosons. The *gauge supermultiplets*, as these irreducible representations are commonly referred to, are obviously required to be present in any realistic extension of the SM. They accommodate besides the gauge boson A^μ, the *gaugino*, which is a two component Weyl-spinor denoted as λ and once again an auxiliary field D. The auxiliary field is introduced as previously to account for the mismatch between boson and fermion degrees of freedom off-shell. To explain further we should recall that a massless gauge boson is described by three real parameters due to the existence of the gauge symmetry. Thus to account for the four fermionic degrees of freedom of the two component Weyl-spinor we have to introduce an extra bosonic degree of freedom. Since on shell both gauge-bosons and Weyl-spinors satisfy $n_B = n_F = 2$ this extra degree of freedom should be introduced through the presence of an auxiliary field. As before, this field can in general be expressed in terms of the physical scalar bosons of the theory when the equations of motion are taken into account. We may express the gauge supermultiplet in the symbolic vector form

$$V = \begin{pmatrix} \lambda & A^\mu & D \end{pmatrix} \tag{2.6}$$

There is an additional interesting property for gauge supermultiplets that is worth sharing. The gauginos, following the symmetry properties of the gauge bosons inside the supermultiplet will unavoidably transform in the adjoint representation of the corresponding gauge group. Since the adjoint is a real representation it cannot accommodate chiral fermions, namely fermions with different gauge properties for their opposite chirality states. Then, in a SUSY theory with gauge symmetry breaking, the gauginos corresponding to the unbroken gauge generators will not be able to form Dirac spinors since both $\lambda, \lambda^\dagger$ will always transform in the adjoint representation of the broken (remnant) gauge group. These spinors, however, will correspond to physical fermions through the more restrictive Majorana field equation. It should be further mentioned that this equation allows for massive fermions also, through quadratic self-couplings, a property which will prove crucial for the phenomenological consistency of the MSSM.

Of course there are other possibilities for supermultiplets that may arise within the context of the $\mathcal{N} = 1$ SUSY discussed here. But chiral and gauge supermultiplets are sufficient in order to explore the minimal extension of the SM that incorporates supersymmetry, the commonly known *Minimal Supersymmetric Standard Model*(MSSM) [16]. The particle content of the MSSM will include all SM particle states in distinct supermultiplets along with their superpartners transforming together within irreducible representations of the $\mathcal{G}_{\mathcal{SM}} = SU(3)_C \times SU(2) \times U(1)_Y$ gauge group.

Therefore we have the classification

$$\begin{array}{rlll} \text{Gauge Supermultiplets}: & G(8,1,0) & W(1,3,0) & B(1,1,0) \\ \text{Chiral Matter Supermultiplets}: & Q_I(3,2,1/6) & u_I^c(\overline{3},1,-2/3) & d_I^c(\overline{3},1,1/3) \\ & L_I(1,2,-1/2) & e_I^c(1,1,1) & N_I^c(?) \\ \text{Chiral Higgs Supermultiplets}: & H_d(1,2,-1/2) & H_u(1,2,1/2) & \end{array} \tag{2.7}$$

The gauge supermultiplets will include, besides the SM gauge bosons and neglecting unphysical auxiliary fields, the corresponding gauginos bearing the same gauge properties. These are the eight *gluinos*, the three *winos* and the *bino*. The situation for the chiral supermultiplets will be analogous. For every quark or lepton state there will be a corresponding scalar called *squark* or *slepton* with the same gauge properties. It should be mentioned though, that contrary to our previous notation employed for the SM, here all denoted fermion states , are two-component Weyl-spinors of certain chirality, considered by convention left-handed. This is for later convenience since such a notation will be useful for the construction of the supersymmetric Lagrangian of the MSSM. In this sense, we denote by Q_I the $SU(2)$ doublet of the left-handed Weyl-spinors $(u_L^I\ d_L^I)$ and in an analogous manner the lepton doublet L_I. All matter supermultiplets have an implicit repetitive form as denoted by the index I following the family replication of the SM. The supermultiplets denoted in the f^c form also contain left-handed Weyl-spinors but only those corresponding to the right-handed fermion states of the SM. The Weyl-spinors they contain are given by the general expression[1] $f^c = f_R^\dagger$ which corresponds to a left-handed spinor constructed by the complex conjugate of the right-handed f_R.

The Higgs and the matter supermultiplets were deliberately separated although as far as SUSY is concerned they are both chiral supermultiplets containing as physical fields a scalar boson and a fermion. The main reason for this extra categorization is the existence of a discrete symmetry imposed on the MSSM known as *R-parity.* The existence of this symmetry turns out of major importance for the viable phenomenology of the model. Leaving a more detailed discussion for a following section, for the moment we only mention that the Higgs and the matter supermultiplets transform differently under this discrete symmetry.

The extra symmetry of the MSSM, namely SUSY, seems to have nearly doubled the particles of the SM. For every supermultiplet there is a corresponding SM field

[1]For economy in notation we use the same symbol for the supermultiplets and the SM particle state they contain.

along with its superpartner but with one exception. The Higgs supermultiplets, are introduced as a pair in the MSSM while the SM only required a single isodoublet. One reason for this choice is due to the *gauge anomalies*[2]. The SM is an anomaly free model in the sense that its fermions miraculously satisfy, among others, the conditions[3] $\mathrm{Tr}[T_3 Y^2] = \mathrm{Tr}[Y^3] = 0$. New fermion states due to SUSY may come either from the gauginos or the superpartners of the Higgs particles that could in principle spoil these important relations. Gauginos cannot affect the relevant traces since the adjoint representation is neutral under weak hypercharge($Y = 0$). On the other hand a Higgs isodoublet can, unless it comes as a pair with its conjugate. But there is an additional reason for the introduction of a Higgs pair of supermultiplets which is rather obvious when one considers SUSY Lagrangians. Due to a property of $\mathcal{N} = 1$ SUSY two Higgs doublets are required for both up and down quarks (also for charged leptons and neutrinos) to acquire masses. This property is in fact closely related to our previous choice to define chiral multiplets with Weyl spinors of certain chirality and will be revisited in more detail in the following section.

2.1.3 General Structure of Supersymmetric Lagrangians

Supersymmetric Lagrangians not only introduce new particles as superpartners in a gauge theory but also exhibit an interesting and rather elegant structure which further determines the interactions between all particles. In what follows in this section we start with two illustrative examples, that of a Lagrangian including first only chiral and then only gauge supermultiplets which eventually enable us to construct the coveted MSSM.

Lagrangians with chiral supermultiplets- The superpotential.

First, we define the variations of the component fields within a given chiral supermultiplet with respect to an infinitesimal supersymmetric transformation. These will be given by

$$\delta\phi = \epsilon\psi \tag{2.8}$$

$$\delta\psi_a = i(\sigma^\mu \epsilon^\dagger)_a \partial_\mu \phi + \epsilon_a F \tag{2.9}$$

$$\delta F = i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi \tag{2.10}$$

[2]These are also commonly referred to as triangular or *ABJ* anomalies. Their absence is essential for the consistency of the model since they ensure gauge invariance being manifest at any loop order.

[3]Such conditions turn out to be rather trivial through the viewpoint of Grand Unified Theories.

with ϵ being an anticommuting two component Weyl-spinor parametrizing the infinitesimal supersymmetry transformation in a way that resembles the group parameters of the Lie-algebras[4]. From these relations one can straightforwardly check that the canonical kinetic terms of the form

$$\mathcal{L}_{kin} = \partial^\mu \phi^* \partial_\mu \phi + i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi + F^* F \tag{2.11}$$

will be invariant under SUSY transformations[5] . To account for all renormalizable dynamical interactions between these fields one has to introduce a useful object, known as the *superpotential*.

The interaction Lagrangian for chiral supermultiplets can in principle be expressed in the general form

$$\mathcal{L}_{int} = -\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + c^{ij} F_i F_j + c.c + U(\phi, \phi^*) \tag{2.12}$$

where by power counting c^{ij}, W^{ij}, W^i are considered scalar functions of mass dimension up to $0, 1, 2$ respectively. From the relations in (2.10) it becomes obvious that U should vanish since no function of scalars can be invariant under SUSY and furthermore no other term in (2.12) may produce a variation that could in principle lead to a respective cancellation. The same arguments stand for the c^{ij} term and therefore it should also be absent from the interaction Lagrangian. Hence, the expression (2.12) reduces to the form

$$\mathcal{L}_{int} = -\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + c.c. \tag{2.13}$$

We can continue in this philosophy and further restrict the scalar functions W^{ij}. By focusing on the variation of W^{ij} in terms of its scalar fields ϕ, ϕ^* we obtain

$$\delta \mathcal{L}_{W^{ij}} = -\frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k} (\epsilon \psi_k)(\psi_i \psi_j) - \frac{1}{2} \frac{\delta W^{ij}}{\delta \phi_k^*} (\epsilon^\dagger \psi_k^\dagger)(\psi_i \psi_j) + c.c. \tag{2.14}$$

out of which the first term may only cancel by itself through the Fierz identity

$$(\epsilon \psi_k)(\psi_i \psi_j) = -\{i, j, k\} - \{j, k, i\}$$

as long as i, j, k are symmetric in all interchanges. Since there is no analogous relation

[4]Conventions and identities for Weyl-spinor algebra follow those of [17]

[5]Obviously $|F|^2$ is not a kinetic term but is required to make the Lagrangian invariant under supersymmetry

for the second term then W^{ij} must be a symmetric holomorphic function of ϕ_i for (2.14) to vanish. We may now introduce the superpotential as the general function

$$W = K^i \phi_i + \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{3!} Y^{ijk} \phi_i \phi_j \phi_k \tag{2.15}$$

whose second derivative $\delta W / (\delta \phi_i \delta \phi_j)$ can be identified as W^{ij}. It will be the most general symmetric rank-2 tensor with mass dimension one, analytic in the fields ϕ_i , hence satisfying (2.14) as requested.

For the cancellation to be exact all remaining variations of fields in (2.13) should also vanish, namely

$$-\frac{1}{2} W^{ij} \delta(\psi_i \psi_j) + \delta(W^i F_i) = 0 \tag{2.16}$$

For $W^i = \delta W / \delta \phi_i$ it can be shown that this relation holds since all terms cancel with each other or vanish as surface terms in the Lagrangian i.e. with the use of $W^{ij} \partial_\mu \phi_j = \partial_\mu (\delta W / \delta \phi_i)$.

The above analysis proves that a supersymmetric Lagrangian of chiral supermultiplets should have the general form

$$\begin{aligned}\mathcal{L}_{chiral} = \partial^\mu \phi_i^* \partial_\mu \phi_i + i \psi_i^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i + F_i^* F_i \\ -\frac{1}{2} W^{ij} \psi_i \psi_j + W^i F_i + c.c.\end{aligned} \tag{2.17}$$

depending on a holomorphic scalar function W called the superpotential as defined in (2.15). This expression is reduced further by invoking the equations of motion for the auxiliary fields $W_i = -F_i^*$, $(c.c)$. Thus for the physical fields it will have the form

$$\begin{aligned}\mathcal{L}_{chiral} = &\ \partial^\mu \phi_i^* \partial_\mu \phi_i + i \psi_i^\dagger \overline{\sigma}^\mu \partial_\mu \psi_i - W_i^* W_i \\ &- \frac{1}{2} W^{ij} \psi_i \psi_j + c.c\end{aligned} \tag{2.18}$$

The dynamics of the system for this type of Lagrangians, where gauge interactions are absent, will, in practice, depend only on the explicit form of the superpotential. Thus, determining the coefficients in (2.15) will unambiguously determine all interactions for models with only chiral supermultiplets.

Lagrangians with gauge supermultiplets.

As already mentioned the superpartners of the gauge bosons will necessarily transform in the adjoint representation of the respective gauge group. Thus we have the following general variations under gauge transformations for the component fields of a gauge supermultiplet

$$\delta_g A^a_\mu = f^{abc}\theta^b A^c_\mu - \frac{1}{g}\partial_\mu \theta^a \tag{2.19}$$

$$\delta_g \lambda^a = f^{abc}\theta^b \lambda^c \tag{2.20}$$

$$\delta_g D^a = f^{abc}\theta^b D^c \tag{2.21}$$

The first row corresponds to the infinitesimal form of a general gauge boson transformation derived previously in (1.51). The second and the third are simply the corresponding variations for a fermion and a scalar field that belong to the adjoint.

We may now proceed to examine the variations of these fields under a SUSY transformation. We have

$$\delta A^a_\mu = -\frac{1}{\sqrt{2}}(\epsilon^\dagger \overline{\sigma}^\mu \lambda^a + \lambda^{\dagger a}\overline{\sigma}^\mu \epsilon) \tag{2.22}$$

$$\delta \lambda^a = -\frac{i}{2\sqrt{2}}(\sigma^\mu \overline{\sigma}^\nu \epsilon)F^a_{\mu\nu} + \frac{1}{\sqrt{2}}\epsilon D^a \tag{2.23}$$

$$\delta D^a = \frac{i}{\sqrt{2}}(\epsilon^\dagger \overline{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a}\overline{\sigma}^\mu \epsilon) \tag{2.24}$$

where, of course, the gauge covariant derivatives have appeared as a result of the underlying gauge symmetry. We may construct the Lagrangian for the kinetic terms of the gauge supermultiplets as

$$\mathcal{L}_{gauge} = -\frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} + i\lambda^{\dagger a}\overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2}D^a D^a \tag{2.25}$$

where of course gauge interactions are also implied due to the presence of D_μ. The first term is the usual Yang-Mills term of a gauge theory. The second is the kinetic term of the gauginos properly introduced with a gauge covariant derivative so as to be invariant under gauge transformations. The term quadratic in the auxiliary fields is required as previously to render the Lagrangian invariant under SUSY transformations off-shell. In the case where no chiral supermultiplet is present, those auxiliary fields have a vanishing equation of motion ($D^a = 0$). This situation however is not realized in the general and more interesting case of interacting chiral and gauge supermultiplets as we will see in what follows.

General Lagrangians.

Having demonstrated the structure for Lagrangians with either chiral or gauge supermultiplets we may proceed to generalize it to the case where both are present. Since we are interested in promoting gauge theories as the SM into supersymmetric ones, all Lorentz covariant derivatives ∂_μ should be replaced by the gauge covariant ones D_μ. This implies that the most general Lagrangian will have the structure

$$\mathcal{L} = \mathcal{L}_{chiral} + \mathcal{L}_{gauge} + \mathcal{L}_{gint} \tag{2.26}$$

where as already mentioned the first term will correspond to the properly modified (2.17) the second will correspond to (2.25) with an implicit replication in case of a semi-simple gauge group, while the third will correspond to all possible gauge invariant interactions between chiral and gauge supermultiplets that respect supersymmetry. These are

$$\mathcal{L}_{gint} = -\sqrt{2} g_a (\phi_i^* T_{ij}^a \psi_j)\lambda^a - \sqrt{2} g_a \lambda^{\dagger a} (\psi_i^\dagger T_{ij}^a \phi_j) + g_a (\phi_i^* T_{ij}^a \phi_j) D^a \tag{2.27}$$

and no other term is allowed in this general SUSY gauge invariant Lagrangian.

The preceding discussion can be used as a guideline in order to build the minimal supersymmetric extension of the SM but before that some final general remarks should be made. The superpotential which appears in the Lagrangian through its first and second order functional derivatives should be gauge invariant for $\mathcal{L}_{chiral}$ to be also invariant. Its first derivative will satisfy $W_i T_{ij}^a \phi_j \sim \delta_g W = 0$ which suggests that W_i transforms as the conjugate supermultiplet Φ_i^* and also reveals the contraction $\Phi_i^* V^a T_{ij}^a \Phi_j$ as a gauge singlet. The latter is actually required in order to demonstrate explicitly that $\mathcal{L}_{gint}$ is also gauge invariant. The scalar potential of the theory will have the general form

$$\begin{aligned} V(\phi, \phi^*) &= -F^{i*} F_i - (W^i F_i + c.c.) - \frac{1}{2} D^a D^a - g_a (\phi_i^* T_{ij}^a \phi_j) D^a \\ &= W_i^* W^i + \frac{1}{2} g_a^2 (\phi_i^* T_{ij}^a \phi_j)^2 \\ &= F^{i*} F_i + \frac{1}{2} D^a D^a \end{aligned} \tag{2.28}$$

where for the derivation of the second and third row the equations of motion

$$\begin{aligned} F^i &= -W^{i*} \ , \ F^{i*} = -W^i \\ D^a &= -g_a(\phi_i^* T_{ij}^a \phi_j) \end{aligned} \tag{2.29}$$

were invoked. The corresponding equation for the auxiliary field D will be non vanishing, as promised, if there exist scalars belonging to chiral supermultiplets that share these gauge interactions i.e. transform non-trivially under the relevant gauge group.

Under all the above considerations the construction of the MSSM becomes an admittedly difficult but rather straightforward task.

2.1.4 The general structure of the MSSM

In order to build the Lagrangian density for the MSSM we recall the classification of (2.7). For the superpotential, which is a holomorphic function of scalars, we denote all scalars with the symbol of their corresponding supermultiplets. The MSSM superpotential will have the structure determined not only by the gauge properties of the supermultiplets but also by a further discrete symmetry, the aforementioned R-Parity.

This discrete symmetry is in fact a special Z_2 symmetry acting differently on the component fields of a given supermultiplet. It can be represented by the operator $R_P = (-1)^{3(B-L)+2s}$ acting on the particle states with B, L being the baryon and lepton number respectively. Since for any coupling in the theory the spin is always conserved we may reduce this symmetry to the equivalent and more useful *matter-parity* represented as $M_P = (-1)^{3(B-L)}$ which has the same eigenvalues for all components of a given supermultiplet. Thus, Higgs and gauge supermultiplets will have the same eigenvalue (+1) since they carry vanishing baryon and lepton numbers. On the other hand, matter supermultiplets will have a (−1) eigenvalue which differentiates between chiral supermultiplets carrying the SM fermions and those carrying the Higgs particles.

Under the imposition of this extra discrete symmetry the most general, gauge invariant and renormalizable superpotential with the particle content of (2.7) is

$$\begin{aligned} W = & Y_u^{IJ} u_I^c Q_J H_u + Y_d^{IJ} d_I^c Q_J H_d + Y_e^{IJ} e_I^c L_J H_d + \mu H_u H_d \\ & (+Y_\nu^{IJ} N_I^c L_J H_u + M_N^{IJ} N_I^c N_J^c) \end{aligned} \tag{2.30}$$

where, as in the SM case, we have extended the particle spectrum in the standard fashion, namely with three gauge singlets to account for the right-handed(RH) neutrino supermultiplets so that the neutrinos eventually acquire masses. Due to matter parity a term linear in a gauge singlet as the RH-neutrino here is forbidden in the MSSM.

At this point, where both supersymmetry and gauge invariance are manifest, the model describes massless particles for all supermultiplets besides the Higgs and perhaps the RH-neutrinos (if the latter are indeed present) which are massive. All particle interactions can be obtained by applying the above superpotential in (2.26). We can make some useful remarks on this model with the use of the general expressions we derived in the previous section, even without displaying explicitly the corresponding Lagrangian in full detail . The $\mathcal{L}_{gauge}$ part for the semi-simple $\mathcal{G}_{SM}$ group will include (2.25) in a repetitive form to account for the $SU(3)_C, SU(2), U(1)_Y$ subgroups with the respective relevant gauge couplings. Besides the SM Yang-Mills terms it will include gauge kinetic terms for the massless gauginos and the quadratic terms in the auxiliary fields D_a. The Yukawa couplings of the SM on the other hand will emerge from $\mathcal{L}_{chiral}$ and in fact from the W^{ij} terms. This part of the Lagrangian will also contain, among others, the gauge kinetic terms for SM fermions and sfermions as well as those of Higgs and Higgsinos.

It should be further noticed that our initial choice to represent all supermultiplets in terms of the LH-fields in addition to the holomorphic property of the superpotential has enabled us to express W in the compact form (2.30). Then, it becomes rather obvious that the Yukawa couplings for the up quarks (neutrinos) will necessarily include H_u while those of the down quarks (charged leptons) will include H_d. Any other case is forbidden since $H_u, H_u^\dagger$ or $H_d, H_d^\dagger$ cannot be simultaneously present in the analytic superpotential. The imposition of the matter parity not only allows the suitable SM couplings to be present in the superpotential but also forbids other dangerous couplings that could in principle have phenomenological inconsistencies. The gauge invariant renormalizable terms

$$\begin{aligned} W_{\Delta B} &= Y_B u^c d^c d^c \\ W_{\Delta L} &= Y_{L_1} e^c LL + Y_{L_2} d^c QL + Y_{L_3} LH_u \end{aligned} \tag{2.31}$$

suppressing family indices, correspond to couplings that violate matter parity. If they were present in the superpotential they would introduce baryon and lepton number violating interactions and eventually, among other unobservable processes, would also induce rapid proton decay.

Of course, for this model to be a realistic extension of the SM, supersymmetry must be somehow broken at some scale $M_{SUSY} > M_{EW}$. If this was not the case then all superpartners would share the same masses with the corresponding SM particles and that would contradict all experimental evidence to date. In addition, the scalar potential of the MSSM would not allow for the electroweak SSB to be realized.

Soft SUSY-breaking.

From our previous experience on gauge symmetry breaking, a mechanism for SUSY breaking is expected to emerge within the general considerations of SSB. However, for the case of global supersymmetry we consider, spontaneous SUSY breaking would imply the presence of unwanted Goldstone fermions (*Goldstinos*). On the other hand, Goldstinos can be absorbed in the framework of local SUSY, commonly referred to as *Supergravity*(SUGRA) through a supersymmetric generalization of the Higgs mechanism. In fact, within this attractive scenario, SUGRA breaks down to global SUSY with the presence of soft SUSY-breaking terms [18].

In any case the exact framework of SUSY breaking is yet undetermined since all proposed models introduce more or less a certain amount of arbitrariness through new fields and parameters beyond the MSSM physical content. Nevertheless, we can still study the broken theory as an explicitly broken symmetry whose arbitrary parameters in fact reflect our ignorance on the exact symmetry breaking mechanism.

Under these considerations any renormalizable term that is gauge invariant and non-supersymmetric is a candidate term for this explicitly broken theory. But, fortunately, we may further restrict these terms by simply recalling our initial motivations on quadratic divergences. We may thus require that these symmetry breaking interactions do not induce quadratic divergences to the scalar masses of the theory [19]. It turns out that they will have the form

$$\begin{aligned}\mathcal{L}_{Soft} =& -\frac{1}{2}M^{(a)}\lambda^a\lambda^a - k^i\phi_i - \frac{1}{2}(m_A^2)^{ij}\phi_i\phi_j - \frac{1}{3!}A^{ijk}\phi_i\phi_j\phi_k + c.c. \\ & - (m_{NA}^2)_i^j\phi^{i*}\phi_j \end{aligned} \tag{2.32}$$

$$\mathcal{L}_{Soft?} = -\frac{1}{2}A_i^{\prime jk}\phi^{i*}\phi_j\phi_k \tag{2.33}$$

where $\mathcal{L}_{Soft?}$ is usually neglected since it may induce quadratic divergences under the presence of gauge singlets in the theory. The gaugino mass terms are always present due to the fact that the adjoint representation has a singlet quadratic contraction. On the other hand, the k, m_A^2, A terms follow the gauge properties of the superpotential and therefore are present if the corresponding superpotential terms are also present. The non-analytic terms m_{NA}^2 are not only present for quadratic self-couplings of scalar fields with their conjugates but are also allowed between different scalars as long as gauge symmetry permits it.

The choice to include only soft breaking interactions becomes more clear when we revisit our analysis on Higgs mass corrections. The presence of a coupling that induces quadratic divergences implies that the mass corrections to the light scalar mass become

$$\Delta m_H^2 = \frac{1}{8\pi^2}(\lambda_S - |Y_\psi|^2)\Lambda^2 + \dots \tag{2.34}$$

If this coupling is soft then the quadratic divergence will be absent and the scalar mass correction will be given by

$$\Delta m_H^2 \sim m_{Soft}^2\{\ \log\left(\frac{\Lambda}{m_{Soft}}\right) + \dots\} \tag{2.35}$$

which goes to zero for $m_{soft} \to 0$. This nice UV behaviour though would not have been realized in case the couplings were not soft and quadratic divergences appeared.

The soft couplings of the MSSM following the above considerations will introduce mass terms for gauginos, analytic (mixing) mass term for the Higgs ($m_A^2 H^u H^d$) and non-analytic masses for all Higgs and sfermions. Trilinear couplings will also appear respecting the gauge properties of the fields in the superpotential and thus following an analogous structure.

The Particle Spectrum.

We may investigate the particle spectrum of the broken MSSM even though we have chosen not to present explicitly the full Lagrangian in terms of component fields. With the use of the general expression (2.26) for the MSSM superpotential along with the explicit symmetry breaking terms we may study the mass eigenstates of the theory after electroweak SSB without explicitly demonstrating how these are obtained.

First we focus on the gauge supermultiplets. Before electroweak breaking gauge bosons are massless but all gauginos acquire masses through explicit soft mass terms. These mass terms respecting the unbroken gauge symmetry are unable to split the masses for gauginos of the same subgroup and thus the masses for all gluinos and winos at this point are seperately degenerate. After SSB the gauge bosons will acquire masses in a fashion analogous to the original framework of the SM but with the difference that now there are two VEVs contributing to the mass expressions due to the two Higgs doublets. Electroweak symmetry breaking will unavoidably affect the masses for the gauginos except for the gluinos which remain degenerate since they are singlets under $SU(2) \times U(1)$.

Next, we focus on the matter supermultiplets. Due to the non-analytic soft mass terms sfermions decouple from the SM fermions and become massive. The latter will have the Yukawa structure of the SM and thus will acquire masses due to the Higgs VEVs in practically the same fashion as in the SM. The only difference with the non-supersymmetric model is that in the MSSM there are two different Higgs doublets coupling seperately to up and down quarks (neutrinos and charged leptons also). As such the Yukawa couplings for these two models will be proportional since in order to produce the observed fermion mixing and mass hierarchies they can only differ by an overall scale. The latter is completely fixed once the ratio $\tan\beta = v_2/v_1$ is determined. The situation for sfermions on the other hand is rather more complicated. Since for every weyl-spinor there is a scalar superpartner for every Dirac fermion there will be two corresponding scalars. Thus the sfermion mass matrices will be 6×6 matrices whose structure will depend on the Yukawa couplings of the fermions, the non-analytic masses and the trilinear couplings.

The situation for the Higgs supermultiplets is a lot more complicated mainly due to the rather extensive potential. Nevertheless the potential has a minimum for an undetermined $\tan\beta$ with the VEVs necessarily satisfying

$$\langle H_u^0\rangle^2 + \langle H_d^0\rangle^2 = v_2^2 + v_1^2 = v_{SM}^2 \tag{2.36}$$

These neutral Higgs states will also correspond to the two physical Higgs particles by the usual procedure of shifting the fields and obtaining the mass eigenstates. The remaining particle states within these doublets include two neutral CP-odd and four charged states. One CP-odd and two charged states will correspond to the unphysical would-be Goldstones which in the unitary gauge are absent. The other three orthogonal states will obtain masses corresponding to two oppositely charged scalars with the same mass and a massive neutral pseudoscalar. Their superpartners, namely the four Higgsinos, combine with the SU(2) gauginos and form new fermion mass eigenstates. The two charged winos combine with the two charged higgsinos and form two Dirac-fermions commonly reffered to as the *charginos*. The remaining two neutral higgsinos will combine with the bino and the neutral wino and form four Majorana particles commonly referred to as the *neutralinos*.

To summarize, the physical particle spectrum of the MSSM will include:

- *Standard model particles*: All particles introduced in the SM will also be present in the MSSM unaltered, with the exception of the Higgs doublet which is necessarily replaced by the two Higgs doublets H_u, H_d. The SM predictions for gauge boson and fermion masses are also reproduced in this supersymmetric framework.

- *Extra Higgs particles*: Besides the two physical neutral Higgs particles, one massive neutral pseudoscalar and two oppositely charged particles with the same mass, will also appear.
- *Gauginos:* Only the eight massive gluinos will have degenerate masses since they are not affected by the electroweak breaking. The other four gauginos mix with the higgsinos to form mass eigenstates.
- *Sfermions:* There will be six up squarks, down squarks, sleptons, sneutrinos (three if N^c are absent) all massive.
- *Charginos:* These are two massive Dirac fermions with unit charge corresponding to charged wino-bino-higgsino mixing.
- *Neutralinos:* These are four massive Majorana fermions corresponding to the neutral wino-higgsino mixing.

2.1.5 Spontaneous breaking in supersymmetric gauge theories-A prelude to SUSY-GUTs.

As previously argued, we will be interested in theories with explicit SUSY-breaking through soft terms. Nevertheless, a short general introduction to the various spontaneous breaking patterns of a SUSY gauge theory will turn out particularly useful. In this way, a deeper insight on these theories is obtained and the general strategy employed in a wide, particularly interesting class of models, commonly referred to as *SUSY-GUTs*, is revealed. Since the mechanisms leading to SUSY and gauge symmetry breaking are not necessarily connected, we may proceed presenting separately the general aspects of each.

In order to illustrate the general aspects of spontaneous supersymmetry breaking the anticommutator $\{Q_a, Q^\dagger_{\dot{a}}\} = 2\sigma^\mu_{a\dot{a}} P_\mu$ is required. The Hamiltonian operator can be straightforwardly obtained by taking a trace over the spinor states in the anticommutator giving

$$H = \frac{2\,\mathrm{Tr}[\sigma^0 P_0]}{4} = \frac{1}{4}\left(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2\right) \tag{2.37}$$

If SUSY is an exact symmetry of the theory then necessarily the vacuum state will also respect it giving $Q|0\rangle = 0$ and $Q^\dagger|0\rangle = 0$. Clearly, the energy of the vacuum state will be vanishing since then $\langle 0|H|0\rangle = 0$. Thus, as a general property, the vacuum of a supersymmetric theory will always have zero energy and any departure from this property would signal SUSY breaking. As in gauge symmetries we characterize the symmetry breaking due to the non-invariance of the ground state, spontaneous.

Now, a second look on (2.37) further reveals that the vacuum energy for a spontaneously broken supersymmetry will be positive since

$$\langle 0|H|0\rangle = \frac{1}{4}\left(\|Q_1^\dagger|0\rangle\|^2 + \|Q_1|0\rangle\|^2 + \|Q_2^\dagger|0\rangle\|^2 + \|Q_2|0\rangle\|^2\right) > 0 \tag{2.38}$$

should hold for any Hilbert state with a positive norm. This suggests $\langle 0|V|0\rangle > 0$, which for a general potential of the form (2.28) subsequently implies that F or D or both will be non vanishing. Such a condition indicates not only a non-zero VEV solution for the trivial equations of motion of the auxiliary fields but also that a zero solution is not possible. If the latter was not the case then there would exist a minimum in the theory satisfying $\langle 0|H|0\rangle = 0$ thus being supersymmetric. Then this minimum would essentially correspond to the true vacuum of the theory since the local minima satisfying (2.38) would have more energy. Therefore, the condition for spontaneous supersymmetry breaking necessarily reduces to the constraint for the absence of a zero solution for the equations of motion of the auxiliary fields. If further the VEVs employed break the gauge group, then the pattern of simultaneous SUSY-gauge breaking is realized. Avoiding a more detailed discussion on this issue, we only mention that numerous models and variations exist in the literature using the F-breaking and D-breaking patterns described previously. However, none of them seem to offer an adequate description for a spontaneous breaking of the MSSM down to the SM.

Our preceding analysis may now be used to explore the orthogonal case where only gauge symmetry is spontaneously broken while the theory remains supersymmetric. This will turn out particularly useful and valuable in a following discussion on the explicit realization of supersymmetry within the general framework of GUTs. It should also be mentioned that this symmetry breaking pattern is not expected to arise in the low energy theory but is rather more appropriate for MSSM extensions with a different gauge structure at higher energies. That is because the MSSM with the presence of soft mass terms is a non-supersymmetric gauge theory. Thus, such a pattern by default may only apply above energies where SUSY is effectively restored, namely above the SUSY breaking scale.

Gauge symmetry breaking in a supersymmetric theory can be realized as usual for a non-vanishing VEV of a scalar field. For supersymmetry to be also preserved those VEVs should further satisfy a zero equation of motion for all auxiliary fields ($F^i = D^a = 0$) since only then $\langle 0|H|0\rangle = 0$ would hold. In order to illustrate the basic aspects of this symmetry breaking realization we consider a SUSY model with an underlying $U(1)$ symmetry. The superpotential for two oppositely charged chiral supermultiplets and one neutral will be[6]

$$W = \kappa\phi_0 + \frac{1}{2}\mu_0\phi_0^2 + \mu\,\phi_+\phi_- + g\,\phi_0\phi_+\phi_- \tag{2.39}$$

Then the equations of motion for the F-terms ($F_i^* = -W_i$) exhibit a zero solution for the VEVs ($F_i^* = 0$), taking the explicit form

$$0 = \kappa + \mu_0 v_0 + g v_+ v_- \tag{2.40}$$

$$0 = \mu v_- + g v_0 v_- \tag{2.41}$$

$$0 = \mu v_+ + g v_0 v_+ \tag{2.42}$$

implying that supersymmetry is unbroken. Now, there are two classes of solutions for the above set of equations, each with a different physical meaning, given by

$$\text{Unbroken U(1)}: \quad v_0 = -\frac{\kappa}{\mu_0}\,, \quad v_\pm = 0 \tag{2.43}$$

$$\text{Broken U(1)} \quad : \quad v_0 = -\frac{\mu}{g}\,, \; v_+v_- = \frac{1}{g}\left(\frac{\mu_0\mu}{g} - \kappa\right) \tag{2.44}$$

These solutions suggest that there are two degenerate vacua with zero energy corresponding to two inequivalent SUSY theories with different gauge structure as well as particle spectrum. Vacuum degeneracy is obviously an unattractive but not necessarily disastrous property of SUSY-GUT models. As a matter of fact the fewer the inequivalent vacua are, the more elegant the model should be regarded. In addition, in some models, one may even consider soft breaking terms being sufficiently small so as for the theory to remain effectively supersymmetric while the vacua degeneracy is slightly lifted in the preferred direction.

In any case we may focus on the more interesting pattern of (2.44) which breaks the gauge symmetry spontaneously. A closer look reveals that there is a larger symmetry for the VEVs than that implied by the $U(1)$ symmetry since the minimization condition

[6] We have neglected, for convenience, a ϕ_0^3 term that would unnecessarily complicate the corresponding equations of motion.

can be satisfied for

$$v_{\pm} = e^{\pm(a+i\theta)} \sqrt{\frac{1}{g}\left(\frac{\mu_0\mu}{g} - \kappa\right)} \equiv e^{\pm\lambda} v_c \tag{2.45}$$

with λ being a random complex number instead of the $U(1)$ complex phase. This can be traced back to the structure of the superpotential which by being holomorphic, also respects the complex non-unitary extension of the gauge symmetry group. As in the non-supersymmetric case we may choose $\theta = \theta_0$ to break the gauge group in what in SUSY models is called, an *F-flat* direction ($F_i = 0$). Of course for the theory to remain supersymmetric the VEVs should also respect the analogous zero equation of motion for the D-terms ($D^a = 0$) , or in other words, symmetry breaking should be realized in a *D-flat* direction. Due to the D-relevant part of the Lagrangian they would have to satisfy

$$\mathcal{L}_D = \frac{1}{2} D^a D^a + g(|\phi_+|^2 - |\phi_-|^2) D^a + \rho D \tag{2.46}$$

$$0 = \frac{\delta \mathcal{L}}{\delta D} = D + g\left(|v_+|^2 - |v_-|^2\right) + \rho \tag{2.47}$$

$$\overset{(D=0)}{=} g\left(|v_+|^2 - |v_-|^2\right) + \rho \tag{2.48}$$

where a linear term ρD was introduced in the full Lagrangian since it is both supersymmetric and gauge invariant[7]. Such a term in principle would induce SUSY breaking [20] unless (2.48) is satisfied which in our case can be done by using the freedom to define the VEV parameter a of (2.45). Therefore for any value of ρ we simply require

$$(e^{2a} - e^{-2a}) v_c^2 = -\frac{\rho}{g} \tag{2.49}$$

which is always satisfied for a suitable a.

Eventually, gauge symmetry is completely broken but the theory due to the F-, D- flatness remains supersymmetric. Thus, for every gauge boson acquiring a mass through the SSB of a gauge symmetry, the corresponding gaugino will also acquire the same mass as a superpartner of the unbroken vector supermultiplet. This is essentially the realization of the Higgs mechanism within the supersymmetric framework .

[7]This is the famous Fayet-Iliopoulos D-term.

2.2 Grand Unified Theories

2.2.1 Introduction

An interesting aspect of gauge theories with semi-simple groups is the existence of separate gauge couplings each associated with the respective subgroups. The SM, in particular, will include three independent gauge couplings g_i corresponding to $SU(3)_C$, $SU(2)$, $U(1)_Y$ and the same stands for its supersymmetric extension, the MSSM. Due to renormalization group equations one may evolve the measured low-energy values for these parameters up to the scale where a new quantum theory is expected to appear. Then it is found that the three gauge couplings tend to unify for the SM or completely unify for the MSSM[8] at a scale $M_G \sim 10^{16}\, GeV$.

A suitable framework that may explain the gauge coupling unification in a rather elegant manner is that of *Grand Unified Theories*(GUTs) [21–25]. It is based on the assumption that there is a larger, usually simple[9] group that contains the $\mathcal{G}_{SM}$ as a subgroup. The particle states of the SM (or MSSM) fall into irreducible representations of the unified group and the gauge covariant derivative is expressed through a single unified gauge coupling. When this gauge group breaks down to the SM the three gauge couplings appear and the GUT representations decompose into the irreducible representations of the SM. However, as a result of the unification at high energies, namely above the GUT scale, certain relations associating parameters of the SM emerge. Thus, the independent Yukawa couplings or the Weinberg angle which in the context of the SM were considered just-so parameters, within GUTs are usually constrained or even predicted.

In addition to these attractive properties it should be remarked that the pattern of GUT symmetry breaking essentially follows the principles of the SM electroweak SSB. Thus, it becomes rather obvious why such an idea has motivated a wide theoretical search over the last decades. Unfortunately, along with many successful predictions in the minimal GUT models there also exist relations that contradict experimental data. Realistic models, on the other hand, that may as well reproduce the observed fermion masses and mixing introduce new fields and non-renormalizable operators and hence, clearly, a certain amount of arbitrariness. Nevertheless, the attractive aspects of this framework seem overwhelming, strongly suggesting that GUTs are at some point

[8]Strictly speaking the 1-loop MSSM, with $1\, TeV$ superpartners, predicts exact unification within experimental bounds, while thresholds and two-loop effects deviate gauge couplings by 1%.

[9]The unified group may as well be considered a semi-simple group which can still predict unification in case it can be embedded in a larger simple group.

related with the theory that completes the SM(MSSM) at high energies.

2.2.2 Minimal SUSY-$SU(5)$ as a prototype GUT

In order to study the general GUT framework we first focus on a specific SUSY model based on the simple gauge group SU(5), proposed by Dimopoulos and Georgi in 1981 [26]. This minimal model not only reveals certain general properties shared by other common GUTs but also serves as a guideline to the mathematical formalism that follows this type of theories.

The group SU(5) is a simple Lie-group of rank-4 with its Lie-algebra $[T^a, T^b] = if^{abc}T^c$ being satisfied by a set of 24 linearly independent generators. It is minimal not only in its representation content but also in the sense that it is the smallest simple group that may embed $\mathcal{G}_{SM}$ which is also rank-4 (i.e. four generators of the algebra commute with each other). Out of these generators the SM hypercharge is identified as

$$Y(\overline{5}) = \text{ diag}(1/3, 1/3, 1/3, -1/2, -1/2) \tag{2.50}$$

with all generators of the algebra properly normalized as $\text{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$. The electromagnetic charge will be given as in the SM by $Q = T_3 + Y$ where T_3 is the third component of the weak isospin. As a linear combination of traceless generators it will necessarily obey $\text{Tr}(Q) = 0$ for all representations of $SU(5)$ out of which charge quantization will straightforwardly emerge.

The representations employed in this model are the simplest lowest dimensional irreducible representations of $SU(5)$ that can accommodate the chiral as well as the gauge supermultiplets of the MSSM. Gauge supermultiplets necessarily transform in the 24-dimensional adjoint representation of $SU(5)$, here denoted as V. Matter supermultiplets transform in the $10, \overline{5}\,(F, f^c)$ while the Higgs sector will be rather enriched in order to account for the SSB of $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$. It will include besides a $5, \overline{5}\,(H, H^c)$ accommodating the Higgs doublets H_u, H_d an adjoint Σ, of chiral supermultiplets, which will eventually acquire a non-vanishing VEV in the $\mathcal{G}_{SM}$ singlet direction. The decomposition of the $SU(5)$ representations under $\mathcal{G}_{SM}$ follows the pattern

$$V, \Sigma\ :\ 24 = (8,1)_0 + (1,3)_0 + (1,1)_0 + (3,2)_{-\frac{5}{6}} + (\overline{3},2)_{\frac{5}{6}} \tag{2.51}$$

$$F_I\ :\ 10 = (3,2)_{\frac{1}{6}} + (\overline{3},1)_{-\frac{2}{3}} + (1,1)_1 \tag{2.52}$$

$$\{f^c_I, H^c\}, \{H\}:\ \ \overline{5} = (\overline{3},1)_{\frac{1}{3}} + (1,2)_{-\frac{1}{2}}\,,\ 5 = (3,1)_{-\frac{1}{3}} + (1,2)_{\frac{1}{2}} \tag{2.53}$$

From the above decomposition we may straightforwardly identify the MSSM fields. The matter 10, $\overline{5}$ will accommodate $F(Q, u^c, e^c)$, $f^c(d^c, L)$ respectively and will be introduced in three distinct copies so as to account for family structure. Each Higgs doublet $(H_u)H_d$, necessarily accompanied by an (anti-)triplet will belong to the (anti-)fundamental representation of $SU(5)$, namely $(H^c)H$. Finally, the MSSM gauge supermultiplets will correspond to the first three terms in (2.51). Clearly, as a result of unification, new fields beyond the context of the MSSM are introduced. These are besides the aforementioned Higgs (anti)-triplets in $(H^c)H$, the two coloured isodoublets in V and all Higgs supermultiplets within Σ. Their existence can and will have an important phenomenological impact on the model even at low energies.

The superpotential for this model will have to respect not only $SU(5)$ gauge symmetry but also the imposition of the usual matter parity assignment on supermultiplets i.e. matter fields F_I, f^c_I will transform with matter-parity (-1). That is because only then the matter parity of the MSSM can be reproduced protecting the low-energy theory from the previously discussed phenomenological complications. Under these considerations the most general superpotential will be

$$\begin{aligned} W = &\frac{M}{2}\,\mathrm{Tr}[\Sigma^2] + \frac{\lambda_\Sigma}{3}\,\mathrm{Tr}[\Sigma^3] + \mu\, H H^c + \lambda_H H \Sigma H^c \\ &+ Y^u_{IJ}\, H (F_I F_J)_S + Y^d_{IJ}\, H^c\, F_I f^c_J \end{aligned} \tag{2.54}$$

where gauge contraction, implicit in the suppressed gauge indices, follows the rules of representation theory. For example $(F_I F_J)_S$ denotes the symmetric contraction $\epsilon_{abcde} F^{bc}_I F^{de}_J$ which transforms as $\overline{5}$ following the property $(10 \times 10)_S = \overline{5}$. As a result, the coupling Y^u_{IJ} is bound to be symmetric in family indices at the scale where $SU(5)$ is a good symmetry, namely above M_G. Actually, symmetry and antisymmetry constraints are quite common features of GUT models and as will be explained shortly not the only or the most restrictive ones.

Minimization of the potential reveals three F-, D- flat directions corresponding to the three degenerate supersymmetric vacua of the theory, each with a distinct gauge structure and particle spectrum. The F- flatness condition reveals

$$< H >=< H^c >=< F_I >=< f_I^c >= 0 \tag{2.55}$$

$$M \ < \Sigma > + \lambda_\Sigma < \Sigma >^2 + \zeta I = 0 \tag{2.56}$$

with ζ being a Lagrange multiplier in order to account for $\mathrm{Tr}[\Sigma] = 0$. The three breaking directions with the respective unbroken subgroups are then

$$< \Sigma >= \begin{cases} 0 & , \ SU(5) \\ \frac{M}{3\lambda_\Sigma}\,\mathrm{diag}\begin{pmatrix} 1 & 1 & 1 & 1 & -4 \end{pmatrix} & , \ SU(4)\times U(1) \\ \frac{M}{\lambda_\Sigma}\,\mathrm{diag}\begin{pmatrix} 2 & 2 & 2 & -3 & -3 \end{pmatrix} & , \ SU(3)\times SU(2)\times U(1) \end{cases} \tag{2.57}$$

out of which only the third row corresponds to the desired breaking down to the $\mathcal{G}_{SM}$. The first row corresponds to the trivial case of a vanishing VEV, while the second to the unrealistic breaking to a subgroup in which the SM cannot be embedded. Focusing on the third, phenomenologically meaningful, pattern we may proceed to demonstrate how the Higgs mechanism is explicitly realized in this model. This also serves the purpose of an instructive example so as for the analogous mechanism in other SUSY-GUT models to be understood.

Since the adjoint Higgs supermultiplet Σ acquires a non-vanishing VEV in the SM singlet direction, the MSSM gauge supermultiplets remain massless i.e. the first three terms in (2.51). That should be expected since these fields correspond to the gauge bosons of the unbroken generators and their superpartners. Of course this can be also verified from the gauge kinetic term $|D_\mu\Sigma|^2$ where terms quadratic in the MSSM gauge bosons are essentially absent due to

$$[T^a_{SM}, < \Sigma >] \sim [T^a_{SM}, Y] = 0 \tag{2.58}$$

as well as the gaugino mixing mass terms (2.27)[10]. The remaining vector supermultiplets transforming as $(3,2)_{-\frac{5}{6}}$, $(\overline{3},2)_{\frac{5}{6}}$ will obtain a mass of order M_G. In the matrix representation of (2.50) they correspond to generators forming the non-diagonal (3,2) blocks, thus having non trivial commutators with $< \Sigma >\sim Y$. In order to obtain mass, as in the non-supersymmetric case, they will absorb the corresponding Goldstone boson modes from Σ and eventually form twelve massive gauge bosons. Since Σ is complex the orthogonal combination of the would-be Goldstones will remain unabsorbed but

[10]Soft SUSY-breaking will become important only at the TeV scale and thus may be regarded as irrelevant to our discussion here.

will eventually acquire mass through the VEVs in the D-term $\frac{1}{2}(\Sigma_i^* T_{ij}^a \Sigma_j)^2$. The rest of the adjoint Higgs scalars, namely $(8,1)_0$, $(1,3)_0$, $(1,1)_0$ will obtain a mass through the F-terms originating from the $\mathrm{Tr}[\Sigma^2]$, $\mathrm{Tr}[\Sigma^3]$ self-couplings in the superpotential.

Clearly, the Higgs mechanism is realized in this minimal model in an elegant manner. All non-MSSM fields from V, Σ have become superheavy by coupling with the adjoint Higgs VEV $< \Sigma > \sim M_G$, while all gauge and matter supermultiplets, accommodating the MSSM fields only, remain massless as desired by the low-energy theory.

However, this minimal GUT-model as well as several variations and extensions of it suffer from an intrinsic structural deficiency. This appears in the Higgs sector and is commonly referred to as the infamous *doublet-triplet splitting* problem. It appears through the relevant terms in the superpotential

$$\mu \, H H^c + \lambda_H H \Sigma H^c \tag{2.59}$$

when the adjoint Higgs acquires the $\mathcal{G_{SM}}$ preserving VEV of (2.57). Then the Higgs fundamental and anti-fundamental H, H^c decompose into the weak isodoublets H_u, H_d and coloured triplets H_u^C, H_d^C giving the mass terms in the superpotential

$$W_H = \mu H_u H_d - 3\frac{M}{\lambda_\Sigma} H_u H_d \tag{2.60}$$

$$W_{H^C} = \mu H_u^C H_d^C + 2\frac{M}{\lambda_\Sigma} H_u^C H_d^C \tag{2.61}$$

From these relations it becomes obvious that both doublets and triplets will have a mass of the same order, naturally of order M_G, unless some kind of miraculous cancellation between the different scales is realized. Unfortunately this is what should happen for the model to exhibit a viable phenomenology. To explain further, the coloured isotriplets mediate baryon number violating processes and therefore the lighter they are the faster the proton decay they would induce. On the other hand the doublets should be of order electroweak for the MSSM phenomenology to be reproduced. Nevertheless, for the unattractive fine-tuned choice of parameters satisfying $\mu - \frac{3M}{\lambda_\Sigma} \sim 100 GeV$ the model exhibits at low-energy the content of the MSSM while the dangerous Higgs triplets become superheavy of order M_G. However, the SUSY version of this GUT model has an apparent technical advantage over its non-supersymmetric analogue, associated with this fine-tuning. In SUSY-$SU(5)$ one has to impose this relation only at tree-level and radiative corrections will not affect it. That is due to an inherent fundamental property of SUSY theories in general, described by the *non-renormalization theorems*. For the $\mathcal{N} = 1$ we consider, this property manifests as the invariance of the superpotential under renormalization.

Even in this case however, where an unjustified fine-tuned relation is considered, the most minimal version of the model without non-renormalizable operators and extra fields is in practice ruled out. That is because the masses of the dangerous coloured triplets should satisfy the proton decay limits as well as the gauge coupling unification condition at $M_G \approx 10^{16}\,GeV$ which cannot be simultaneously realized[11]. Clearly then if this model is to be realistic an extended version should be considered.

In addition to the aforementioned inadequacies, the unification relations in the Yukawa sector also favour a non-minimal extension. To better understand the situation for the Yukawa couplings we may focus on the relevant terms in the superpotential

$$Y^u_{IJ}\,H(F_I F_J)_S + Y^d_{IJ}\,H^c\,f^c_I F_J$$

which due to gauge symmetry breaking and by neglecting terms with coloured Higgs triplets decompose into the MSSM couplings

$$W_Y = Y_u^{IJ} u^c_I Q_J H_u + Y_d^{IJ} d^c_I Q_J H_d + Y_e^{IJ} e^c_I L_J H_d \tag{2.62}$$

These Yukawa couplings then should not only satisfy the low-energy conditions, consistent with the observed fermion masses and mixing, but also the non-trivial M_G relation

$$Y^d = Y_d = Y_e^{\top}\;\big|_{M_G} \tag{2.63}$$

along with the aforementioned symmetry structure for Y_u. Clearly this non-trivial condition relating the charged lepton with the down quark matrix will have the definite M_G prediction of equal masses for each family. Such a prediction though is partially successful as the *Georgi-Jarlskog mass relations* imply. These are

$$m_e \approx \frac{1}{3} m_d\,,\;\; m_\mu \approx 3 m_s\,,\;\; m_\tau \approx m_b \tag{2.64}$$

which should instead hold at GUT scale so as for the low-energy masses of the observed fermions to be reproduced.

Even disregarding the inconsistency with the observed proton decay rate, the minimal SUSY-SU(5) model still falls short both technically and phenomenologically. Var-

[11]Strictly speaking there exists a tiny parameter space for the soft breaking sector that could still correspond to a phenomenologically viable model. However it is so severely constrained that such a realization is very unlikely.

ious proposals, although separately treating these major problems sufficiently, seem to be inadequate to provide with a combined elegant solution. For example, one may extend the model with the more suitable $50, \overline{50}, 75$ Higgs representations to keep the Higgs doublets massless [27]. In fact, this is a more general approach, commonly referred to as the *missing partner mechanism*, with various realizations that may also apply to other GUT groups. On the other hand, one may introduce a 45 so as for the desired factors of 3 in (2.64) to appear. Then the desired mass relations would be satisfied at GUT scale. However, models that try to confront both problems simultaneously introduce a certain amount of arbitrariness as well as phenomenological inconsistencies. Nevertheless, the numerous attractive features of the minimal SUSY-SU(5) have established it as the standard paradigm of GUT theories in general. It explains the gauge coupling unification, implied by the MSSM, as the unification of the semi-simple $\mathcal{G}_{SM}$ into a simple group ($SU(5)$) with a single gauge coupling. Quantization of hypercharge as in (2.50) also unavoidably appears. That is because the abelian $U(1)_Y$ is embedded in a larger non-abelian group, hence following a certain normalization for its generators. Furthermore the model is free from gauge anomalies exactly as in the SM or the MSSM for the minimal representation content of (2.53).

In this viewpoint a generalization of the GUT approach into other simple or semi-simple groups seems at least a worth exploring idea. In what follows we will be interested in GUT models that offer possible remedies to the standard GUT problems, as those encountered in the previous $SU(5)$ example. These models should be regarded as improved and more realistic proposals although none of them seems to offer a complete, realistic extension of the MSSM to higher energies.

2.2.3 Other standard unified models.

In this section we briefly review the general structure of two distinct models based on the gauge groups $SU(5) \times U(1)$ and $SO(10)$ with the minimal representation content required for an MSSM embedding. These models by respecting a symmetry group that includes the SU(5) as a subgroup, inherit analogous attractive properties as well as, fortunately milder, deficiencies. Their main characteristic is that both of them offer a possible solution on the doublet-triplet splitting problem and that they may also potentially survive the current nucleon decay constraints.

Flipped SU(5)

The *flipped* $SU(5)$ is a GUT-model based on the semi-simple group $SU(5) \times U(1)_X$ [28–30]. Its gauge group can be embedded in the larger simple $SO(10)$ symmetry group and as any realistic candidate unified model has the $\mathcal{G}_{SM}$ as a subgroup. The rather characteristic property of this model is the non-trivial embedding of the hypercharge $U(1)_Y$ generator, identified as a linear combination of the abelian generator $U(1)_Z$, within $SU(5)$, and the external $U(1)_X$. The weak hypercharge will then be given by

$$Y = \frac{1}{5}(Z + X) \tag{2.65}$$

$$Z \equiv \text{diag}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \tag{2.66}$$

where obviously the $U(1)_Z$ is the properly normalized generator previously identified as the hypercharge of the standard $SU(5)$ model (2.50).

Following an analogous notation as in (2.51)-(2.53) the representations of the SUSY $SU(5) \times U(1)$ involved will decompose under $\mathcal{G}_{\mathcal{SM}}$ as

$$V : \quad 24_0 = (8,1)_0 + (1,3)_0 + (1,1)_0 + (3,2)_{\frac{1}{6}} + (\overline{3},2)_{-\frac{1}{6}} \tag{2.67}$$

$$H, F_I : \quad 10_1 = (3,2)_{\frac{1}{6}} + (\overline{3},1)_{\frac{1}{3}} + (1,1)_0 \tag{2.68}$$

$$\overline{H} : \quad \overline{10}_{-1} = (\overline{3},2)_{-\frac{1}{6}} + (3,1)_{-\frac{1}{3}} + (1,1)_0 \tag{2.69}$$

$$f_I^c : \quad \overline{5}_{-3} = (\overline{3},1)_{-\frac{2}{3}} + (1,2)_{-\frac{1}{2}} \tag{2.70}$$

$$\{h\}, \{h^c\} : \quad 5_{-2} = (\overline{3},1)_{-\frac{2}{3}} + (1,2)_{-\frac{1}{2}} \ , \ \ \overline{5}_2 = (\overline{3},1)_{-\frac{2}{3}} + (1,2)_{-\frac{1}{2}} \tag{2.71}$$

$$e^c : \quad 1_5 = (1,1)_1 \tag{2.72}$$

where we kept the minimal representation content[12]. It should be further remarked that the standard variations of this model usually introduce additional singlets to account for realistic neutrino masses. This in practice has small effect on the basic structure of the theory that we are interested in so we conveniently neglect them.

The trademark of this model originating from the non-trivial embedding of the weak hypercharge(2.65) is the analogous but distinct decomposition of the matter supermultiplets F, f^c as compared with the standard $SU(5)$ theory. This allows to identify in this model u^c instead of d^c within f^c , (N^c, d^c) instead of (e^c, u^c) within F and e^c as an $SU(5)$ singlet (N^c) thus justifying the descriptive term "flipped" for this special $SU(5) \times U(1)_X$ symmetry. The remaining matter supermultiplets Q, L will belong to

[12]Furthermore the $U(1)_X$ assignment implicitly follows the proper embedding of all representations into the irreducible representations of the $SO(10)$ and E_6 gauge groups

F, f^c respectively in both models and therefore the MSSM matter supermultiplets will belong in

$$10_1,\ \overline{5}_{-3},\ 1_5 : F(Q, d^c, N^c),\ f^c(u^c, L),\ e^c \tag{2.73}$$

For the viable phenomenology of the theory in this minimal version it is necessary to impose besides the standard matter parity assignment an extra Z_2 symmetry under which only H transforms non-trivially (-1). Then the most general superpotential will be

$$\begin{aligned} W = & Y^d_{IJ}\, F_I F_J h + Y^u_{IJ}\, F_I f^c_J h^c + Y^e_{IJ}\, e^c_I f^c_J h \\ & + \lambda H H h + \lambda' \overline{H}\, \overline{H} h^c + \mu\, h h^c \end{aligned} \tag{2.74}$$

with the MSSM desired μ term also allowed to be present. Due to the extra Z_2 symmetry a term HH^c is absent thus allowing for the gauge breaking down to $\mathcal{G}_{SM}$ in the F-, D- flat direction

$$< H >=< \overline{H} >=< N^c_H >\equiv M_G \tag{2.75}$$

The most elegant feature of this model is the technically natural doublet-triplet splitting that is realized. For the non-vanishing VEVs above the terms $\lambda HHh, \lambda'\overline{H}\overline{H}h^c$ in the superpotential produce superheavy masses for the coloured triplets while leaving doublets to obtain an electroweak mass from the $\mu-$term. That is because only the coloured triplets in h, h^c will couple with $(\overline{3},1)_{\frac{1}{3}}$, $(3,1)_{-\frac{1}{3}}$ of $H, \overline{H}$ to form mass terms while the doublets cannot couple quadratically to any available field. This is actually the explicit realization of the previously mentioned missing partner mechanism for doublet-triplet splitting within flipped $SU(5)$. The rest of the non-MSSM supermultiplets, besides a linear combination of singlets $(N^c_H - \overline{N^c_H})$, will acquire heavy masses through a super-Higgs mechanism in a manner analogous to the preceeding standard $SU(5)$ example.

It should be remarked that alongside the above technical advantages there are various interesting aspects that also emerge within the context of this model. First, it should be noted that the successful gauge coupling unification predicted in the standard SU(5) may also be directly derived if $SU(5) \times U(1)_X$ is embedded into a larger group like $SO(10)$ or E_6. Furthermore, contrary to other standard GUT-models, the flipped SU(5) realizes gauge symmetry breaking without the presence of adjoint Higgs, a characteristic most appreciated by superstring constructions. However, the natural suppression of dangerous dimension-5 operators relaxing significantly the proton decay constraints, should be regarded as the most attractive feature of this model. Finally, the both successful and problematic Yukawa unification relations met in most unified models are absent or strictly speaking replaced with analogous conditions for up quarks and neutrinos. Since neutrino masses, yet undetermined, most likely appear through a different mass generation mechanism than that of the other charged fermions the flipped Yukawa unification condition cannot in practice impose a restrictive fermion mass condition. We delay a more detailed discussion on this subject for the next chapter.

SO(10) models.

We next focus on a larger symmetry-group that seems particularly interesting not only for model building but predominantly as a general framework in which properties of various GUT models can be understood. This stems from the fact that $SO(10)$ is a larger simple symmetry group that encompasses as maximal subgroups $SU(5)\times U(1)$ and $SU(4)\times SU(2)\times SU(2)$ both independently giving rise to interesting (SUSY)-GUT realizations [31, 33, 59, 60]. The former pathway has already been discussed to some extend through the standard and flipped $SU(5)$ minimal models. The latter includes variations of the so-called left-right symmetric models among which the more famous and rather instructive Pati-Salam model [22]. For our purposes, some interesting features arising from this alternative class of models will be investigated only through the $SO(10)$ viewpoint.

The simple-group $SO(10)$ [34, 35] offers the possibility of several realizations each of which with different advantages as well as flaws. In this sense one cannot uniquely designate as previously a specific model as the minimal version and additionally realistic models exhibit a rather extensive structure. However, in the most standard proposed models, besides an adjoint gauge supermultiplet, matter, with the usual family replication, transforms in the minimal spinorial representation of the symmetry group and at least a part of the electroweak Higgs doublets in the vectorial. We have the decomposition under $SU(5)\times U(1)$

$$
\begin{aligned}
45_V &= 24_0 + 10_{-4} + \overline{10}_4 + 1_0 \\
16_F &= 10_1 + \overline{5}_{-3} + 1_5 \\
10_H &= 5_{-2} + \overline{5}_2
\end{aligned}
\tag{2.76}
$$

Clearly, this is the $U(1)_X$ assignment followed in the previous flipped-$SU(5)$ example which suggests that many of the interesting properties there, as well as in standard $SU(5)$, appear due to the possible embedding within an $SO(10)$ symmetry.

Various symmetry breaking patterns may arise by choosing different representations for the heavy Higgs. These will subsequently develop a non-vanishing VEV in the F- and D- flat directions available eventually breaking $SO(10)$ either directly to the $\mathcal{G}_{SM}$ subgroup or in a stepwise fashion through its respective subgroups. We will avoid an investigation on all these distinct models followed by their explicit superpotentials and instead discuss more general aspects of $SO(10)$ realizations.

In this sense, we may focus on the MSSM matter supermultiplets which will couple to the electroweak Higgs doublets through a term in the superpotential $Y_{IJ}10_H 16_F^I 16_F^J$. As an immediate consequence the Yukawa coupling unification conditions imply the M_G relation

$$Y_u = Y_d = Y_e = Y_\nu \ \ |_{M_G} \tag{2.77}$$

with Y_ν corresponding to the $h_u L N^c$ coupling responsible for neutrino Dirac-mass terms. Even neglecting the neutrino matrix, the Yukawa matrices for charged fermions, here necessarill symmetric at M_G, suggest vanishing quark mixing and common mass ratios between families i.e. $m_u/m_c = m_d/m_s = m_e/m_\mu$. Since both consequences are experimentally excluded, one necessarily has to extend the Higgs content of the theory or take into account higher dimensional operators. Fortunately, the latter are expected to be in any case important, at least to some extend, due to the proximity of the GUT to the Planck scale where new physics is expected to arise.

The usual strategy in $SO(10)$ models, which also exhibits a large variety of distinct realizations, is followed by introducing additional Higgs fields in the representations $16_H, \overline{16}_H$, 45_H. In these models, if the spinorial Higgs acquires a non-vanishing VEV, it will necessarily lie in the only $\mathcal{G}_{SM}$ singlet direction $< 16_H > = < \overline{16}_H > = < N_H^c >$. On the other hand, the adjoint representation may acquire a non-vanishing VEV in different directions leaving each time a different subgroup invariant. Out of these directions the most interesting in a technical sense seem to be

$$< 45_H > \sim \left.\begin{array}{ll} SU(4) \times SU(2)_R \times SU(2)_L & : \mathrm{diag}(0,0,1,1,1) \\ SU(4) \times SU(2)_L \times U(1)_R & : \mathrm{diag}(1,1,0,0,0) \end{array}\right\} \times i\sigma_2$$

where the respective unbroken subgroup is denoted. The VEV in the first row corresponds to the *Dimopoulos-Wilczek mechanism* for doublet-triplet splitting without tuning in SO(10) models. When acting on the light Higgs through a term $10_H < 45_H > 10'_H$

it will obviously attribute a GUT scale mass to the coloured triplets while leaving weak doublets massless. As an alternative one may obtain doublet-triplet splitting with a VEV as in the second row [33]. A VEV in this direction can keep doublets in spinorial representations of $SO(10)$ massless through a term $\overline{16}_H \langle 45_H \rangle 16_H$. This can be easily understood from the $SU(4) \times SU(2)_R \times SU(2)_L$ decomposition

$$16_H = (4,1,2) + (\overline{4},2,1) \tag{2.78}$$

$$\langle 45_H \rangle = \langle (1,3,1) \rangle \tag{2.79}$$

which shows that the quadratic term $(\overline{4},1,2)(4,1,2)$ will remain massless. These massless fields will transform as the h_u, h_d and $Q, \overline{Q}$ of the $\mathcal{G}_{SM}$. Since the two pairs belong to $5, \overline{5}$,10, $\overline{10}$ of $SU(5)$ respectively, another VEV in the $SU(5)$ singlet direction could split their masses. This can be realized by introducing a new set of $\overline{16'}_H, 16'_H, 45'_H$ fields in the superpotential as

$$\langle \overline{16'}_H \rangle 45'_H 16_H + \overline{16}_H 45'_H \langle 16'_H \rangle + 45'_H 45'_H \tag{2.80}$$

This is essentially the missing partner mechanism since $\overline{5}_{16}, 5_{\overline{16}}$ cannot form a quadratic mass term, in contrast to $10_{16}, \overline{10}_{\overline{16}}$ as can be seen from (2.76). One may even use other, rather more complicated, superpotentials with $16_H 10_H$ mixing to realize the doublet-triplet splitting and produce Higgs massless eigenstates coupling to matter through the standard renormalizable $10_H 16_F 16_F$ term. In practice, both mechanisms for doublet-triplet splitting without fine tuning are rather unstable and difficult to realize.

Admittedly, even the so-called minimal models of $SO(10)$ are quite cumbersome mainly due to their extensive superpotential. This, among other issues should simultaneously allow gauge breaking in the preferred direction, produce the doublet-triplet splitting, predict grand unification and break the unwanted Yukawa unification conditions. Nevertheless, models that allow a fit with current experimental data exist, however introducing a significant amount of arbitrariness through new fields, ad hoc symmetries and non-renormalizable operators. In any case a study on $SO(10)$ reveals aspects for GUTs that are not transparent in other "less unified" models. In particular, each family of known matter fits exactly the spinorial representation of $SO(10)$, in a family respecting manner, along with the yet unverified right-handed neutrino. Since the $SO(2N)$, $(N \neq 6)$ groups are free from gauge anomalies, $SO(10)$ models will automatically share this property as well as certain models for the $SO(10)$ subgroups. The latter will be those models whose non-trivial representations fit exactly the $SO(10)$ irreducible representations. Hence, among others, this explains why minimal-SU(5) and flipped-$SU(5)$, with that charge assignments, are both anomaly free. In addition, the Yukawa coupling unification conditions for these models are in practice milder versions of the rather restrictive condition (2.77). Although in $SO(10)$ this necessarily implies the presence of non-renormalizable operators such a fact eventually turns out to be less unattractive. The reason behind this is the strong symmetry which constrains significantly the structure of the non-renormalizable terms. As will be discussed in the following chapter this even allows for predictions whose validity is not necessarily attributed to a specific model but may also indicate the direction to a more elaborate GUT.

Chapter 3

Fermion Masses and Mixing

3.1 Charged fermions within the SM

An unanswered question that unavoidably arises within the context of the SM is the origin of the observed pattern for fermion masses and mixing. That is to be contrasted with the gauge boson sector where the Yang-Mills theory augmented by the Higgs mechanism provides a rather elegant answer for the relevant mass scales. There, massless gauge bosons are a result of the unbroken symmetry subgroup $SU(3)_C \times U(1)_{em}$ while the three massive gauge bosons ($W^{\pm}$, Z) of the broken symmetry will have masses determined in practice by a single free parameter i.e. the Weinberg angle θ_W.

Clearly, the situation in the fermion sector is far more arbitrary. Although fermion masses also arise when the Higgs acquires a non-vanishing VEV, the terms involved now originate from the Yukawa sector of (1.75). We recall that this contributes the charged fermion terms

$$\mathcal{L}_Y = Y_u^{IJ}\overline{Q}_L^I\tilde{\Phi}u_R^J + Y_d^{IJ}\overline{Q}_L^I\Phi\, d_R^J + Y_e^{IJ}\overline{L}_L^I\Phi\, e_R^J + h.c \tag{3.1}$$

out of which when $\langle\Phi\rangle = v$ the following *fermion mass terms* are produced

$$\frac{v}{\sqrt{2}}Y_u^{IJ}\overline{u}_L^I u_R^J + \frac{v}{\sqrt{2}}Y_d^{IJ}\overline{d}_L^I\, d_R^J + \frac{v}{\sqrt{2}}Y_e^{IJ}\overline{e}_L^I\, e_R^J + h.c \tag{3.2}$$

The coefficients of these bilinear terms are the *fermion mass matrices*, previously denoted in (1.89). They will be responsible not only for the physical masses of fermions but also, as will be discussed shortly, for an experimentally verified fundamental property called *fermion mixing*. The Yukawa couplings, which in practice determine these coefficients, are regarded as three dimensional matrices in family space. These are not

Charged Fermion Masses			
Gen.	1	2	3
u_i	$2.3^{+0.7}_{-0.5}\,MeV$	$1.275 \pm 0.025\,GeV$	$173.5 \pm 1.4\,GeV$
d_i	$4.8^{+0.7}_{-0.3}\,MeV$	$95 \pm 5\,MeV$	$4.18 \pm 0.03\,GeV$
e_i	$0.511\,MeV$	$105.66\,MeV$	$1.777\,GeV$

CKM parameters (Standard Par.)			
$\sin\theta_{12}$	$\sin\theta_{23}$	$\sin\theta_{13}$	δ
0.225	0.0412	0.00350	1.20

Table 3.1: Current values for charged fermion masses and CKM mixing parameters [36]

constrained by any theoretical consideration within the context of the SM. In fact, the only consideration that actually restrains their form, while being unable to completely determine them, comes from fitting to the experimental data.

Experimental evidence is rather definite on the mass spectrum for charged fermions as well as on the four parameters which characterize quark mixing. Current values are listed in Tab.3.1 for reference.

3.1.1 Biunitary transformations.

As already mentioned, the mass terms for fermions will appear through the bilinear forms of (3.2). The physical masses, as usual, will appear through a rotation to the mass eigenstate basis. For Dirac fermions this is realized by two, in general unrelated, unitary matrices acting on each side of the mass matrix bringing it to a diagonal form. The diagonal elements obtained through this *biunitary transformation*, will correspond to the mass eigenvalues up to complex phases. The latter can always be absorbed by a redefinition of the fields.

To describe this diagonalization procedure in more detail we employ a mass term as those of (3.2) in the form

$$\overline{\psi}^I_L M^{IJ} \psi^J_R + h.c. \ , \quad M^{IJ} \equiv -\frac{v}{\sqrt{2}} Y^{IJ} \tag{3.3}$$

By assumption we take M to be a 3×3 general complex matrix so as to account for the unconstrained family structure of the SM. It can be shown that a suitable biunitary transformation may always diagonalize M through

$$U_L^\dagger M U_R = M_D \tag{3.4}$$

with M_D being a diagonal non-negative matrix.

To illustrate this, we first mention that since $MM^\dagger$ is Hermitian it will have non-negative eigenvalues obtained through a unitary transformation. Then

$$U_L^\dagger M M^\dagger U_L = M_D^2 \tag{3.5}$$

with U_L being unique up to a diagonal phase matrix P since $U_L P$ may also satisfy the above relation. This property actually ensures that if M_D is diagonal in (3.4) then it can always be brought to the desired real non-negative form so as to account for the physical fermion masses. Now, we may define the following useful Hermitian matrix

$$H \equiv U_L M_D U_L^\dagger \tag{3.6}$$

which implies that the matrix $V \equiv H^{-1} M$ will be unitary. That is because

$$\begin{aligned} VV^\dagger &= H^{-1} M M^\dagger H^{-1} \\ &= U_L M_D^{-1} U_L^\dagger M M^\dagger U_L M_D^{-1} U_L^\dagger \\ &= 1 \end{aligned} \tag{3.7}$$

due to (3.5) and (3.6). Then we may express M_D in terms of the Hermitian H and the unitary V as

$$M_D = U_L^\dagger H U_L = U_L^\dagger M V^\dagger U_L \tag{3.8}$$

where we have used the unitarity condition $V^\dagger = V^{-1} = M^{-1} H$. We may then define $U_R \equiv V^\dagger U_L$ in the above equation which finally reproduces (3.4) as desired.[1]

3.1.2 The CKM matrix.

The previously described biunitary transformation not only diagonalizes the fermion mass matrices but also gives rise to fermion mixing phenomena with certain, exper-

[1]In the above derivation we have implicitly assumed that M is invertible as is the case for all SM charged fermions. However a zero eigenvalue can always be treated by projecting it out and proceed as above in the non-trivial subspace.

imentally verified, implications. For the moment, we shall restrict ourselves only to terms relevant for quark masses and mixing.

By applying the biunitary transformation to the up quark mass term we obtain

$$\begin{aligned} \overline{u}_L M_u u_R &= (\overline{u}_L U_{uL})(U^\dagger_{uL} M_u U_{uR})(U^\dagger_{uR} u_R) \\ &= \overline{u}'_L M_{uD} u'_R \end{aligned} \tag{3.9}$$

where we have defined the primed fields, corresponding to the *mass eigenstates*, as

$$u'_L \equiv U^\dagger_{uL} u_L \tag{3.10}$$

$$u'_R \equiv U^\dagger_{uR} u_R \tag{3.11}$$

An analogous situation will be realized in the down quark sector for the corresponding mass matrix M_d. Keeping the same conventions, the rotation from the initial (*gauge eigenstate*) basis to the mass eigenstate basis, will be realized there by another pair of suitable unitary matrices, denoted as U_{dL}, U_{dR}.

For the SM, the unitary matrices corresponding to rotations of the right-handed fields will leave no physical trace since they can be fully absorbed by the respective field redefinitions. This, on the other hand, will not be the case for U_{uL}, U_{dL} which will eventually give rise to the *CKM matrix* of quark mixing [37, 38].

This is understood if we recall the quark kinetic terms of the SM which are given by

$$i\overline{Q}_L \not{D} Q_L + i\overline{u}_R \not{D} u_R + i\overline{d}_R \not{D} d_R \tag{3.12}$$

Since the right-handed fermions of the SM are singlets under $SU(2)$ the field redefinitions of (3.11) will give the trivial relations

$$\overline{u}_R \not{D} u_R = \overline{u}'_R \not{D} u'_R \tag{3.13}$$

$$\overline{d}_R \not{D} d_R = \overline{d}'_R \not{D} d'_R \tag{3.14}$$

However, for the left-handed fields the $SU(2)$ coupling in the covariant derivative will be non-trivial giving rise to the terms

$$\frac{g_2}{\sqrt{2}} \overline{u}_L \gamma^\mu W^+_\mu d_L + \frac{g_2}{\sqrt{2}} \overline{d}_L \gamma^\mu W^-_\mu u_L \tag{3.15}$$

Since the above operators are related by hermitian conjugation we may focus only on the first term. Then, the rotation to the mass eigenstate basis as in (3.10) will give rise to the previously mentioned non-trivial CKM matrix through

$$\frac{g_2}{\sqrt{2}}\overline{u}_L\gamma^\mu W^+_\mu d_L \sim \overline{u}'_L\gamma^\mu(U^\dagger_{uL}U_{dL})d'_L \tag{3.16}$$

out of which we may define the unitary matrix describing quark mixing phenomena as

$$V_{CKM} \equiv U^\dagger_{uL}U_{dL} \tag{3.17}$$

In principle the CKM matrix is a $U(3)$ transformation which can be described by three real angles and six complex phases. But in the charged currents $J^\mu_- \equiv \overline{u}'_L\gamma^\mu V_{CKM}d'_L$ we may redefine

$$\begin{gathered} u''_L = P_u u'_L \ , \ \ d''_L = P_d\, d'_L \\ P_{u,d} \sim \mathrm{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}) \end{gathered} \tag{3.18}$$

to absorb five complex phases. It is also trivial to check that such an operation cannot affect the non-negative eigenvalues of the quark diagonal matrices M_{uD}, M_{dD} given by (3.9). Therefore we are left with the physical parameters of the CKM matrix, namely with three real angles and one CP-violating phase.

The standard parametrization [36, 39] is

$$\begin{aligned} V_{CKM} &\equiv U_{23}(0)U_{13}(\delta)U_{12}(0) \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \end{aligned} \tag{3.19}$$

where $U_{ij}(\delta)$ imply rotations in the respective family subspaces in analogy to

$$U_{13}(\delta) \equiv \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \tag{3.20}$$

There are various noteworthy aspects emerging from our previous treatment. To begin with, we may recall the observed CKM parameters of Tab.3.1. Then, a closer look on (3.17) reveals that there should exist two distinct rotation matrices $U_{uL} \neq U_{dL}$ operating on the left-handed fields. This necessarily implies a mismatch between gauge and mass eigenstates at least for some of the left-handed quarks. For the right-handed rotations, on the other hand, we cannot extract any information. This obviously stems from the fact that all right-handed fermion states of the SM are singlets under $SU(2)_L$, thus implying the absence of analogous charged currents. Another aspect that is worth mentioning, is the origin of the physical CKM parameters along with the quark mass spectrum. Within the context of the SM these will necessarily arise from the Yukawa couplings. As a result the 18 complex parameters, introduced through Y_u, Y_d will have to account for only six quark masses and four mixing physical parameters. Clearly, a substantial amount of arbitrariness will be present which will further increase due to a similar situation in the lepton sector as discussed in what follows.

Finally we should mention that although this general treatment of fermion masses and mixing was realized within the context of the SM, a generalization to the extensions we consider, such as the MSSM and SUSY-GUTs, is always straightforward. That is mainly due to the fact that the soft SUSY-breaking, considered as the standard method to decouple the SM spectrum from its superpartners, does not introduce mass terms for the SM fermions. As a result fermion masses may receive only loop-suppressed corrections from the unknown soft-breaking sector. Typically, such insertions, also bounded by other considerations, cannot affect drastically the Yukawa structure. In any case, in our following considerations, we will implicitly restrict ourselves to the case where SUSY-breaking is irrelevant for fermion masses [40, 41].

3.2 Neutrino masses and mixing

The situation for the neutrinos seems to be significantly different from that of the charged fermions [42]. Current experimental evidence [43], although yet inconclusive, favours a three generation scenario of very light neutrinos. Furthermore, even though light neutrino masses are not uniquely determined, any realistic neutrino mass pattern should always be consistent with the experimental values of Tab.3.2. In addition to this, a combined analysis from cosmological data [36] indicates an upper bound on the overall neutrino mass scale through the relation $\sum m_i \lesssim 1eV$.

In more detail, the squared mass differences $\delta m_{ij}^2 \equiv |m_i^2 - m_j^2|$ extracted from the neutrino oscillation phenomena indicate three different patterns depending on the mass of the lightest neutrino. These are

- *Normal Hierarchy(NH).* $m_1 \ll m_2 \ll m_3$
  The squared mass differences then imply a strongly hierarchical spectrum with $m_2 \approx \sqrt{\delta m_{12}^2} \ll m_3 \approx \sqrt{\delta m_{23}^2}$

- *Inverse Hierarchy(IH).* $m_3 \ll m_1 \approx m_2$
  The squared mass differences then imply a partially degenerate spectrum with $m_1 \approx m_2 \approx \sqrt{\delta m_{23}^2}$

- *Quasi-Degeneracy(QD).* $m_3 \approx m_2 \approx m_1 \sim M$
  This case arises for an overall neutrino mass scale satisfying $M \gg \sqrt{\delta m_{23}^2}$. If the cosmological constraints are taken into account then this would correspond to a rather small portion of the allowed parameter space with $M \lesssim 0.3\, eV$

Clearly, if the neutrino masses lie in the proximity of $(\delta m_{ij}^2)^{\frac{1}{2}}$ the answer for the pattern is less definite. However, for any allowed mass spectrum, in the three neutrino case we consider, there will always be a lower bound for the mass of the heavier neutrino. That is

$$\max(m_i) \geq \sqrt{\delta m_{23}^2} = 0.0485\, eV \tag{3.21}$$

which simply reflects the fact that a squared difference can never be greater than the maximum of the respective squared parameters. On the other hand, the mass of the lightest neutrino is not practically restricted by any consideration (other than the possible upper bound on the overall mass scale) and the possibility for one massless neutrino cannot be excluded.

Although, as mentioned, experimental data cannot uniquely determine the neutrino mass spectrum, the NH case, which will be followed in our investigated models, distinguishes as the most probable scenario. That is due to the fact that only within this context a hierarchical spectrum is obtained. Since a mass hierarchy is realized for all other fermions, it is naturally expected to arise also for neutrinos. Particularly for neutrinos arising within the framework of the see-saw mechanism, discussed below, this is even more likely to occur. That is because a hierarchy originating from any of the related mass terms will typically propagate to the spectrum for the light neutrino masses. But even beyond that, the NH case is also supported by the experimental data

Neutrino Masses and Mixing				
Squared Mass Differences (eV^2)		Known PMNS parameters		
δm_{12}^2	δm_{23}^2	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$	$\sin^2\theta_{13}$
$7.58^{+0.22}_{-0.26} \times 10^{-5}$	$2.35^{+0.12}_{-0.09} \times 10^{-3}$	$0.306^{+0.018}_{-0.015}$	$0.42^{+0.08}_{-0.03}$	$0.021^{+0.007}_{-0.008}$

Table 3.2: Current values for neutrino squared mass differences as well as lepton mixing angles. One or three CP-violating phases, in case of Dirac or Majorana fermions, are still undetermined.

themselves. If one considers a hierarchical mass spectrum then the hierarchy for the squared mass differences immediately arises. In contrast, the other cases of partial or complete degeneracy appear as fine-tuned. There, one would have to adequately justify not only the presence of two small mass splittings but also the hierarchy between them.

3.2.1 The see-saw mechanism.

The tiny masses of the neutrinos suggest that a different mass generation mechanism should operate in the neutrino sector which distinguishes them from the other fermions of the theory. The standard proposal which seems to offer an elegant answer on this issue without departing from the general considerations of the SM is the *see-saw mechanism* [44]. Due to its various possible realizations, developed over the years, the see-saw is now established as the general framework within which, explicit models for neutrino masses and mixing are realized.

The fundamental idea behind all see-saw models is the introduction of heavy fields in the theory that couple directly or indirectly to the lepton doublets. When heavy degrees of freedom are integrated out the symmetric effective operator

$$\frac{1}{2} M_\nu^{IJ} \nu_I \nu_J \tag{3.22}$$

is produced. Eventually, this Majorana mass term will account for the light neutrino fields ν_I.

Since the presence of heavy fields is required for all explicit realizations of the see-saw idea, the SUSY framework regarded as the high energy completion of the SM seems more appropriate[2]. Then, the supersymmetric versions of the three standard realizations, commonly referred to as *see-saw types*, are:

Type-I (Standard) seesaw. *(Singlet matter supermultiplets)*

This type of seesaw is realized by introducing matter supermultiplets transforming as singlets under $\mathcal{G}_{SM}$ to account for the right-handed neutrinos N_I^c. Although two singlets are sufficient for this realization to be consistent with experimental data, in the usual treatment, one introduces three N_I^c as the family replication of the SM implies for all other fermion states. Due to its obvious simplicity, this scenario has been established as the *standard see-saw* type.

[2] In any case the transition to the non-supersymmetric case is always straightforward.

In particular, in this approach, one introduces three singlets N_I^c which will allow for the terms in the superpotential

$$Y_\nu^{IJ} H_u L_I N_J^c + \frac{1}{2} M_R^{IJ} N_I^c N_J^c \tag{3.23}$$

When the Higgs acquires a non-zero VEV, and suppressing family indices, the mass terms for the neutrinos appear as[3]

$$v_u \, Y_\nu \, \nu N^c + \frac{1}{2} M_R N^c N^c = \frac{1}{2} \begin{pmatrix} \nu & N^c \end{pmatrix} \begin{pmatrix} 0 & v_u Y_\nu \\ v_u Y_\nu^\top & M_R \end{pmatrix} \begin{pmatrix} \nu \\ N^c \end{pmatrix} \tag{3.24}$$

where hermitian conjugation is always implied. Now if we parametrize $M_R^{IJ} = m_R Y_R^{IJ}$ and assume for the scales $m_R \gg v_u \sim M_{EW}$ one obtains a split spectrum for the neutrino masses. This is now given by the terms

$$\frac{1}{2} \nu'^\top M_\nu \nu' + \frac{1}{2} N^{c'\top} M_N N^{c'} \tag{3.25}$$

where the corresponding symmetrical mass matrices will be given approximately by the following expressions

$$M_\nu \approx -\frac{v_u^2}{m_R} \; Y_\nu Y_R^{-1} Y_\nu^\top \tag{3.26}$$

$$M_N \approx M_R \tag{3.27}$$

Clearly in the above treatment, the mass spectrum will be described by two distinct three-dimensional mass matrices with different overall mass scales. The physical masses for the light (or heavy) neutrinos will be obtained by a diagonalization procedure analogous, but not identical to the one previously used for quarks as will be discussed when lepton mixing is investigated. In any case, the mass eigenstates will be obtained from a rotation in the respective primed field subspace with the phenomenologically more interesting light neutrino eigenstates formed by linear combination of the ν' fields.

As (3.26) suggests, a typical value of $v_u^2/m_R \sim 0.1\, eV$ would imply a heavy neutrino (seesaw) mass scale at $m_R \sim 10^{14}\, GeV$. For such a large hierarchy between these two scales the primed fields which are linear combinations of ν, N^c will be equal to the unprimed ones to a good approximation. Thus the light neutrinos will be predominantly

[3] A 3×3 block form is of course implicit in the matrix of the right hand side so as to account for family structure. Also, irrelevant numerical factors are conveniently absorbed in a redefinition of the parameters (e.g. $v_u \equiv \langle H_u \rangle$).

left-handed $\nu' \approx \nu$ which is consistent with all current experimental data.

Type-II seesaw. *(Charged Higgs isotriplets)*

In this type of see-saw, one introduces a Higgs supermultiplet T transforming as (1,3,1) under the $\mathcal{G}_{SM}$ along with its conjugate T^c. These fields will allow for the terms in the superpotential

$$\frac{1}{2}Y_\nu^{IJ} L_I T L_J + \lambda_d H_d T H_d + \lambda_u H_u T^c H_u + m_T T T^c \tag{3.28}$$

where we may represent the isotriplet in the matrix form

$$T = \begin{pmatrix} T^+ & T^{++} \\ T^0 & -T^+ \end{pmatrix} \tag{3.29}$$

and a proper contraction of gauge indices is understood. When the Higgs doublets acquire a non-vanishing VEV, the F- and D- flatness conditions imply $\langle T^0 \rangle \approx -\frac{\lambda_u v_u^2}{m_T}$. This corresponds to a symmetric neutrino mass term with a suppressed overall mass scale given by

$$\frac{1}{2}Y_\nu L \langle T \rangle L = -\frac{1}{2}\frac{\lambda_u v_u^2}{m_T} Y_\nu \, \nu\nu \tag{3.30}$$

where we have again suppressed family indices for simplicity in notation.

The overall mass scale is essentially of the see-saw type suppressed by a heavy mass m_T as desired. The Higgs supermultiplets T, T^c will thus be superheavy, effectively decoupling from the low energy spectrum. This type of seesaw is fundamentally different from the previous one since the small neutrino masses arise from a suppressed VEV in the minimization of the potential. As a result no right-handed neutrinos are required here and additionally the light mass eigenstates are purely left-handed.

Type-III seesaw. *(Neutral matter isotriplets)*

There is always the option to use the neutral component of a weak isotriplet as the right-handed neutrino. Therefore, in this type of see-saw, one introduces, usually three, matter supermultiplets T transforming as $(1, 3, 0)$. The corresponding terms in the superpotential will be

$$Y_\nu^{IJ} L_I T_J H_u + \frac{1}{2} M_R T_I T_J \tag{3.31}$$

where the neutral isotriplet can be represented in the matrix form

$$T = \begin{pmatrix} T^0/\sqrt{2} & T^+ \\ T^- & -T^0/\sqrt{2} \end{pmatrix} \tag{3.32}$$

Since the structure of the superpotential is analogous to the Type-I case(3.23) at least for the neutral components, the see-saw formula for the light (and heavy) neutrino mass matrix will be essentially the same. Thus, when the Higgs acquires a VEV, the corresponding mass terms will be[4]

$$v_u Y_\nu \, \nu T^0 + \frac{1}{2} M_R T^0 T^0 \tag{3.33}$$

producing the split spectrum for neutrinos described by

$$M_\nu \approx -\frac{v_u^2}{m_R} \, Y_\nu Y_R^{-1} Y_\nu^\top \tag{3.34}$$

$$M_N \approx M_R \tag{3.35}$$

with an identical structure to (3.26) and (3.27). The charged components within the isotriplet, namely $T_\pm$, will obtain a superheavy mass of order m_R, thus decoupling from the low-energy theory.

At first sight, the phenomenological implications for this type of see-saw seem redundant when compared with the Type-I, at least as far as the low-energy theory is concerned. Nevertheless such a framework will prove particularly useful when we discuss explicit realizations of see-saw within SUSY-GUTS. That is due to the fact that the $\mathcal{G}_{SM}$ singlets and neutral isotriplets tend to appear together in many irreducible representations of the GUT groups we consider. As we shall see this will eventually restrain severely the family structure of the respective Yukawa couplings even allowing in some cases for certain predictions.

3.2.2 Lepton mixing.

As already mentioned, in the original formulation of the SM the neutrinos were erroneously considered massless. This misconception, however, changed drastically with the discovery of the neutrino oscillation phenomena. Such an effect, verified by a number of experiments, not only suggests that neutrinos are massive (at least two of them)

[4] A proper gauge contraction with suitable normalization factors is always implied. Thus, here, for a consistent derivation from (3.31) the $T_I T_J$ term would correspond to $\mathrm{Tr}(T_I T_J)$

but also that they mix non-trivially. The latter fact in particular, indicates a mismatch between gauge and mass eigenstates in the lepton sector in an obvious analogy to the previously discussed situation for the quarks.

Lepton mixing, also referred to as *neutrino mixing*, is described by the PMNS matrix [56] which is essentially the analogue of the CKM matrix for the leptons. To derive its general form we follow the same procedure as for quark mixing. First, we focus on the terms relevant for leptons using the effective seesaw operator of 3.22. We thus obtain the mass terms from the superpotential

$$M_e^{IJ} e_I^c e_J + \frac{1}{2} M_\nu^{IJ} \nu_I \nu_J \tag{3.36}$$

where we have identified $M_e = -\frac{v_d}{\sqrt{2}} Y_e$ following (3.2). Diagonalization proceeds as usual for the charged leptons with the biunitary transformation

$$e^{c'} = e^c U_{eR} \ , \ \ e' = U_{eL}^\dagger e \tag{3.37}$$

$$M_{eD} = U_{eR}^\dagger M_e U_e \tag{3.38}$$

The diagonalization of the light neutrinos however proceeds with a single unitary transformation as

$$\nu' = U_\nu^\dagger \nu \tag{3.39}$$

$$M_{\nu D} = U_\nu^\top M_\nu U_\nu \tag{3.40}$$

It can be shown that a single unitary transformation is sufficient not only to diagonalize the symmetric complex matrix M_ν but also to ensure its real non positive eigenvalues [63].

The PMNS matrix will originate from the charged currents and in particular from

$$J_\mu^+ = \bar{e}_L \gamma_\mu \nu_L = e^\dagger \sigma_\mu \nu = e^{\dagger\prime} \sigma_\mu (U_{eL}^\dagger U_\nu) \nu' \tag{3.41}$$

$$V_{PMNS} \equiv U_{eL}^\dagger U_\nu \tag{3.42}$$

We should mention that the physical parameters of the PMNS matrix are in general three real angles and one or three CP-violating phases depending on whether the neutrinos have Dirac or Majorana masses respectively. The former case is understood as an exact analogue of the CKM matrix in the lepton sector. The latter case however, which is more interesting due to the see-saw mechanism, exhibits two extra physical CP-violating phases. This is due to the fact that a Majorana mass term as that of

(3.36) is not invariant under phase redefinitions. As a result one can absorb only the three out of the six complex phases of V_{PMNS} by a phase redefinition of the charged leptons.

3.3 Neutrinos within SUSY-GUTs

As previously discussed, one of the deficiencies of the SM also shared by the MSSM is the unconstrained family structure for the Yukawa couplings. As a result, the hierarchy of the fermion mass spectrum as well as the mixing patterns observed in nature cannot be justified by any consideration within those frameworks. It seems then a rather natural choice to search for a SM(MSSM) extension that can explain or even better predict current experimental data for fermions.

A minimal and rather obvious extension would be to consider a new symmetry group for families. Such an approach, at first sight seems well motivated by the existing gauge symmetries which describe sufficiently the dynamics of the fundamental particles. However a gauged family symmetry would immediately imply the presence of new gauge bosons none of which has yet been observed. On the other hand, a global continuous symmetry would come together with the undesired presence of Goldstone bosons. Perhaps the most interesting approach in this field of research seems to be family symmetries based on the discrete symmetry groups and in particular the non-abelian ones [45, 46]. Such a realization, would immediately avoid both preceding problems and in addition could combine well with another possible minimal extension of the MSSM, namely SUSY-GUTs [47].

In fact, this scenario appears as a very attractive possibility since SUSY-GUTs seem to offer a suitable and more general framework for model building with various interesting properties, as discussed previously. Among these properties is the prediction for Yukawa unification which restrains severely the structure of Yukawa couplings. Therefore in our search for a more complete and elaborate theory it seems at least illuminating to investigate SUSY-GUT models that may fit current experimental data on fermion masses and mixing, and examine their implications.

A non-minimal SU(5) model for neutrinos.

Following the above considerations we have investigated the possibility of extending the minimal SUSY-SU(5) with new field representations in a unified model with hierarchical neutrino masses [48].

Although, admittedly, the particle content introduced in this model is rather extensive, there are many virtues that follow this realization. To begin with, it avoids the proton decay constraints that follow the minimal SUSY-SU(5) model. Such constraints are now considered rather disastrous for the phenomenology of the minimal model, eventually rendering it a non-realistic GUT. In any case, this is done by achieving unification at a larger scale than the problematic $M_G \sim 10^{16}\,GeV$ of the original model. Despite the fact that a certain amount of fine-tuning is required to achieve successful unification, we obtain a prediction for the seesaw scale in the phenomenologically preferred region. Within this context we examine the possibility of a hierarchical light neutrino mass spectrum as this is implied from the NH case for the neutrinos and the lepton mixing pattern this may potentially suggest. We find that one of the light neutrinos is necessarily massless at tree level due to a particular Yukawa unification condition shared by a more general class of GUT models. Finally, we should remark that the model is renormalizable and any discussion on non-renormalizable operators is for completeness and for establishing that these would give subdominant effects. In fact, we should emphasize this point since we have chosen to remain at the renormalizable level at the expense of a more complicated structure.

It should be mentioned that for potentially interesting constraints on the scale and structure of neutrino masses, the sector of heavy fields has to partake in the GUT. This can be realized in other GUTs [49], such as $SO(10)$ and *flipped-*$SU(5)$ [30], or by extending the gauge non-singlet field content of $SU(5)$. We recall that the realization of the so called type-I see-saw mechanism in the SM introduces right-handed neutrinos as gauge singlet fields. In contrast, in the type-III right-handed neutrinos are non trivially introduced as the neutral components of isotriplet fields [50]. This can be promoted to extended versions of $SU(5)$ that feature additional chiral superfields in the $\underline{\mathbf{24}}$ representation, each containing two suitable right-handed neutrino candidates[5]. A mixed "type-I+III" see-saw mechanism can then be realized with an extra $\underline{\mathbf{24}}$ [52], while the most appealing three generation scenario with three right-handed neutrinos requires additional $\underline{\mathbf{24}}$'s or $\underline{\mathbf{1}}$'s.

In this model we consider a version of supersymmetric $SU(5)$ extended through the introduction of extra chiral superfields $\mathcal{S}(\mathbf{1})$, $\mathcal{T}(\mathbf{24})$, $\mathcal{T}'(\mathbf{24})$, which provide us with three right-handed neutrino candidates. Our basic assumption is that these right-handed neutrino fields obtain a Majorana mass at a high but still intermediate scale a

[5] Fermions in a single $\underline{\mathbf{24}}$ representation have been introduced in the framework of non-supersymmetric $SU(5)$ in [51], where the see-saw mechanism was realized with two right-handed neutrinos at a predicted low energy scale.

few orders of magnitude below the unification scale. This assumption is supported by a renormalization group analysis, incorporating proton lifetime constraints [53], and allows for an intermediate scale in the vicinity of $(10^{13}\text{-}10^{14})\,GeV$. Not all of the scales involved in the right-handed neutrino Majorana mass matrix are constrained by the renormalization group. Depending on assumptions, several possibilities emerge leading to a different dependence of the resulting light neutrino masses on these scales. Furthermore, the fact that two of the right-handed neutrinos are members of the same $SU(5)$ representation leads to a particular *rank 2* structure of the resulting light neutrino mass matrix that is accompanied by a massless eigenvalue. Although this fact is modified by non-renormalizable terms, there is a definite prediction for one superlight neutrino, not in conflict with observations. Next, we examine the possibility of a hierarchical light neutrino mass spectrum $m_\nu^{(3)} > m_\nu^{(2)} \gg m_\nu^{(1)}$. This can be achieved in a variety of ways depending on assumptions either for the mass scales involved or for the hierarchy of the Yukawa-type couplings. We also consider whether the observed large neutrino mixing can be accommodated in the framework of the model [54]. We conclude that hierarchical mixing patterns with $\theta_{13} \ll \theta_{12} \sim \theta_{23}$ can be obtained with generic choices of Yukawa couplings exhibiting certain structure.

The Model

The renormalizable part of the minimal $SU(5)$ superpotential, in terms of the chiral superfields $\mathcal{Q}_i^c(10), \mathcal{Q}_i(\overline{5}), \mathcal{H}(\overline{5}), \mathcal{H}^c(5), \Sigma(24)$, is

$$\mathcal{W}_0 = \mathcal{Y}_{ij}^u \mathcal{Q}_i^c \mathcal{Q}_j^c \mathcal{H}^c + \mathcal{Y}_{ij}^d \mathcal{Q}_i \mathcal{Q}_j^c \mathcal{H} + \frac{M}{2} Tr(\Sigma^2) + \frac{\lambda}{3!} Tr(\Sigma^3) + \lambda' \mathcal{H}^c \Sigma \mathcal{H} + M' \mathcal{H}^c \mathcal{H}$$

where we have suppressed $SU(5)$-indices and display only the family indices i, j. Let us now introduce extra matter supermultiplets $\mathcal{S}(1), \mathcal{T}(24)$, $\mathcal{T}'(24)$ with the standard matter parity assignment[6]. An extra $\mathcal{Z}_2$ discrete symmetry, under which only $\mathcal{T}'(24)$ changes sign differentiates between them so that $\mathcal{T}'$ does not couple to standard matter fields. The renormalizable contributions of the new fields to the superpotential are

$$\begin{aligned} \mathcal{W}_1 = \mathcal{Y}_i^{\mathcal{S}} \mathcal{Q}_i \mathcal{H}^c \mathcal{S} + \mathcal{Y}_i^{\mathcal{T}} \mathcal{Q}_i \mathcal{H}^c \mathcal{T} + \frac{\mu}{2} \mathcal{S}^2 + \frac{\mu'}{2} Tr(\mathcal{T}^2) + \frac{\mu''}{2} Tr(\mathcal{T}'^2) \\ + \, f\, Tr(\mathcal{T}^2 \Sigma) + f'\, Tr(\Sigma \mathcal{T}) \mathcal{S} + f'' Tr(\mathcal{T}'^2 \Sigma) \; . \end{aligned} \tag{3.43}$$

[6]We have $\mathcal{Q}$, $\mathcal{Q}^c$, $\mathcal{S}$, $\mathcal{T} \to -1$, while Σ, $\mathcal{H}$, $\mathcal{H}^c \to 1$.

The decomposition of the new matter multiplet $\mathcal{T}(24)$ is

$$\mathcal{T}(24) = B(1,1,0) + T(1,3,0) + O(8,1,0) + \mathcal{X}(3,2,-5/6) + \mathcal{X}^c(\overline{3},2,5/6)\,,$$

where the $SU(3) \times SU(2) \times U(1)$ identification of each component is self-explanatory. Analogous is the decomposition of the primed field $\mathcal{T}'(24)$. Denoting by T^0 the neutral component of the isotriplet $T(1,3,0)$, we can identify the three right-handed neutrino candidates as $N_i^c = (\mathcal{S}, B, T^0)$.

Symmetry breaking of SU(5) down to $SU(3) \times SU(2) \times U(1)$ is realized in the standard fashion through a non-zero VEV of Σ in the direction $< \Sigma >= \frac{V}{\sqrt{30}} diag(2,2,2,-3,-3)$. Note that the absence of cubic terms for the new fields, due to their parity assignment, does not allow them to acquire a non-zero VEV and, thus, symmetry breaking proceeds exactly as in the minimal case. All components of Σ are either *higgsed away* or obtain masses of the order of the GUT scale. The splitting between the masses of the Higgs isodoublets H_d, H_u and the Higgs coloured triplets D, D^c contained in $\mathcal{H} = (H_d,\, D^c)$ and $\mathcal{H}^c = (H_u,\, D)$ is produced by the usual fine-tuning $M' = \frac{3\lambda' V}{\sqrt{30}}$, resulting in massless doublets and superheavy triplets. Then, the effective superpotential relevant for masses below the unification scale M_G reads

$$\begin{aligned}\mathcal{W}_{eff} &= Y^u_{ij} u^c_i Q_j H_u + Y^d_{ij} d^c_i Q_j H_d + Y^e_{ij} e^c_i L_j H_d + Y^{\mathcal{S}}_i L_i \mathcal{S} H_u + Y^B_i L_i B H_u \\ &+ Y^T_i L_i T H_u + Y^{\mathcal{X}}_i d^c_i \mathcal{X} H_u + \frac{M_{\mathcal{S}}}{2}\mathcal{S}^2 + \frac{M_B}{2}B^2 + M_{\mathcal{S}B}\mathcal{S}B \\ &+ \frac{M_T}{2}Tr(T^2) + M_{\mathcal{X}}\mathcal{X}\mathcal{X}^c + \frac{M_O}{2}Tr(O^2) + \frac{M_{T'}}{2}Tr(T'^2) + \frac{M_{O'}}{2}Tr(O'^2) \\ &+ M_{\mathcal{X}'}\mathcal{X}'\mathcal{X}'^c + \frac{M_{B'}}{2}B'^2\,. \end{aligned} \tag{3.44}$$

Matching the effective and the $SU(5)$-*symmetric* theory at M_G leads to the following relations for the Yukawa couplings

$$\begin{aligned} Y^u = 2\mathcal{Y}^u\,,\ (Y^e)^{\perp} = Y^d = \mathcal{Y}^d\,,\ Y^{\mathcal{S}} = \mathcal{Y}^{\mathcal{S}} \\ Y^{\mathcal{X}} = Y^T = \frac{\sqrt{30}}{3} Y^B = \mathcal{Y}^{\mathcal{T}}\,, \end{aligned} \tag{3.45}$$

while for the mass parameters we get

$$M_{\mathcal{S}} = \mu\,, \qquad M_B = \mu' - \frac{2fV}{\sqrt{30}}\,, \quad M_T = \mu' - \frac{6fV}{\sqrt{30}}\,, \tag{3.46}$$

$$M_O = \mu' + \frac{4fV}{\sqrt{30}}\,, \qquad M_{\mathcal{X}} = \mu' - \frac{fV}{\sqrt{30}}\,, \quad M_{\mathcal{S}B} = -f'V \tag{3.47}$$

and

$$\begin{aligned} M_{T'} &= \mu'' - \frac{6f''V}{\sqrt{30}},\ M_{B'} = \mu'' - \frac{2f''V}{\sqrt{30}}, \\ M_{O'} &= \mu'' + \frac{4f''V}{\sqrt{30}},\ M_{\mathcal{X}'} = \mu'' - \frac{f''V}{\sqrt{30}}\,. \end{aligned} \tag{3.48}$$

The see-saw scale is the scale of the right-handed neutrino mass matrix expressed in terms of the parameters $M_{\mathcal{S}}$, M_B, M_T and $M_{\mathcal{S}B}$, related through the four parameters μ, μ', fV and f'/f. The allowed range for these parameters will be strongly constrained by the requirements of unification at a sufficiently high scale. This will follow shortly from a renormalization group analysis.

In addition to the renormalizable contributions above, non-renormalizable contributions to the superpotential

$$\mathcal{W}_{NR} = \frac{\lambda_{IJKL}}{M_P}\,\Phi_I\Phi_J\Phi_K\Phi_L + O(1/M_P^2) + \ldots.$$

can, in principle, affect masses, especially whenever we have mass-degeneracies. We have denoted the scale of non-renormalizable interactions generically by M_P, expecting their scale to be the Planck scale. The lowest order terms in $\mathcal{W}_{NR}$ are

$$\begin{aligned} &\mathcal{Q}\mathcal{T}\Sigma\mathcal{H}^c + \mathcal{Q}\Sigma\mathcal{H}^c\mathcal{S} + \mathcal{T}\mathcal{Q}^c\mathcal{H}\mathcal{H} + \mathcal{Q}^c\mathcal{Q}^c\Sigma\mathcal{H}^c + \Sigma\mathcal{Q}^c\mathcal{H}\mathcal{Q} + \mathcal{H}^c\mathcal{Q}\mathcal{Q}\mathcal{H}^c + \mathcal{T}\mathcal{Q}^c\mathcal{Q}\mathcal{Q} \\ &+ \mathcal{Q}^c\mathcal{Q}\mathcal{Q}\mathcal{S} + \mathcal{Q}^c\mathcal{Q}^c\mathcal{Q}^c\mathcal{Q} + \mathcal{T}^2\Sigma^2 + \Sigma^2\mathcal{T}\mathcal{S} + \mathcal{H}\mathcal{T}^2\mathcal{H}^c + \Sigma^2\mathcal{S}^2 + \mathcal{H}\mathcal{T}\mathcal{H}^c\mathcal{S} \\ &+ \mathcal{H}\mathcal{H}^c\mathcal{S}^2 + \mathcal{T}^4 + \mathcal{T}^3\mathcal{S} + \mathcal{T}^2\mathcal{S}^2 + \mathcal{S}^4 + \Sigma^4 + \mathcal{H}\Sigma^2\mathcal{H}^c + \mathcal{H}\mathcal{H}^c\mathcal{H}\mathcal{H}^c + \mathcal{T}'^2\Sigma^2 \\ &+ \mathcal{H}\mathcal{T}'^2\mathcal{H}^c + \mathcal{T}'^4 + \mathcal{T}'^2\mathcal{T}^2 + \mathcal{T}\mathcal{T}'^2\mathcal{S} + \mathcal{T}'^2\mathcal{S}^2\,, \end{aligned} \tag{3.49}$$

suppressing the factor $1/M_P$ and the dimensionless couplings in front of each term, all assumed to be of the same order. Among these terms, those relevant for neutrino masses are the terms $\mathcal{H}^c\mathcal{Q}\mathcal{Q}\mathcal{H}^c$, leading to (tiny) Majorana masses for left-handed neutrinos, the terms $\mathcal{Q}\mathcal{T}\Sigma\mathcal{H}^c$, $\mathcal{Q}\Sigma\mathcal{H}^c\mathcal{S}$, contributing to Dirac masses, and the terms $\mathcal{T}^2\Sigma^2$, $\Sigma^2\mathcal{T}\mathcal{S}$, $\Sigma^2\mathcal{S}^2$, contributing to Majorana masses for the right-handed neutrinos.

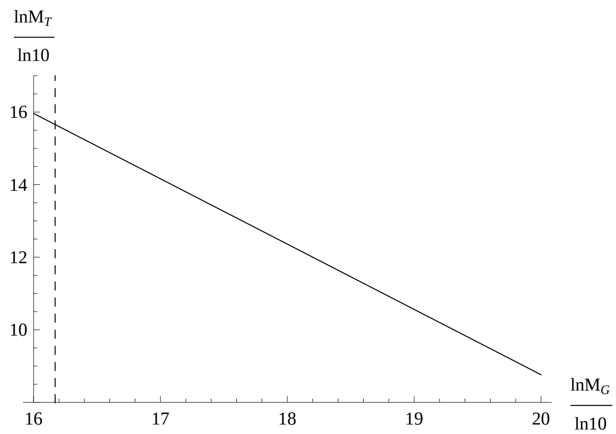

Figure 3.1: Isotriplet mass M_T vs the unification scale M_G. The octet mass satisfying $M_{O'} \leqslant M_G$ sets a lower bound for unification at $M_G \approx 1.5 \times 10^{16}\, GeV$.

Energy Scales

The sector of additional superfields $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{S}$ carries with it a set of extra parameters, namely the mass parameters μ, μ', μ'' and the couplings f, f', f''. A basic assumption of the model is that the Majorana mass of right handed neutrinos is at a high but still intermediate scale, a few orders of magnitude below M_G. Thus, we shall assume that the isotriplet component of $\mathcal{T}$ remains lighter than M_G. In addition, proton lifetime constraints translated to a high enough M_G require the presence of an additional light color octet. These requirements correspond to new fine tunings of parameters, presumably, not worse than the standard GUT fine tunings. As a working set of choices, we take ($M_G^2 = \frac{5g^2}{12}V^2$)

$$\begin{aligned} \mu' = (3\, - \epsilon)M_G/2\,,\ \mu'' \ &= \ (2 + 3\epsilon')M_G/5, \\ f = \frac{5g}{4\sqrt{2}}\,(1 - \epsilon),\quad f'' \ &= \ -\frac{g}{2\sqrt{2}}\,(1\, - \epsilon') \end{aligned} \tag{3.50}$$

where $\epsilon \sim \epsilon' \ll 1$. These choices result in

$$M_T = \epsilon\, M_G\,,\ M_{O'} = \epsilon'\, M_G \tag{3.51}$$

while the rest of the masses are M_O, $M_{\mathcal{X}}$, $M_{\mathcal{X}'}$, $M_{T'}\ \sim\ O(M_G)$.

Thus, we assume that, apart from the MSSM fields and the color octet and isotriplet superfields that have intermediate masses $M_{O'}$ and M_T, all extra superfields decouple at M_G. In addition, we assume that supersymmetry is broken at an approximately common energy scale of $m_S \sim 1\, TeV$ at which all superpartners decouple. From the

M_G	$M_{O'}$	M_T	α_G
3×10^{16}	3.1×10^{15}	1.3×10^{15}	0.04023
5×10^{16}	1.0×10^{15}	5.2×10^{14}	0.04112
8×10^{16}	3.6×10^{14}	2.3×10^{14}	0.04197
1×10^{17}	2.2×10^{14}	1.5×10^{14}	0.04239
3×10^{17}	2.0×10^{13}	2.1×10^{13}	0.04457
5×10^{17}	6.4×10^{12}	8.3×10^{12}	0.04566
8×10^{17}	2.3×10^{12}	3.6×10^{12}	0.04671
1×10^{18}	1.4×10^{12}	2.4×10^{12}	0.04723
3×10^{18}	1.2×10^{11}	3.3×10^{11}	0.04996

Table 3.3: Values (GeV) for the unification scale M_G, the colored octet mass $M_{O'}$ and the weak isotriplet mass M_T. The corresponding unified coupling a_G remains within the perturbative limit.

one-loop renormalization group equations for the three $SU(3) \times SU(2) \times U(1)$ gauge couplings[7], with the intermediate octet and isotriplet mass scales inserted, we obtain the following expressions for these couplings at M_Z

$$\begin{aligned}
\frac{2\pi}{\alpha_3(M_Z)} &= \frac{2\pi}{\alpha_G} - 3\ln\left(\frac{M_G}{M_Z}\right) - 4\ln\left(\frac{m_S}{M_Z}\right) + 3\ln\left(\frac{M_G}{M_{O'}}\right) \\
\frac{2\pi}{\alpha_2(M_Z)} &= \frac{2\pi}{\alpha_G} + \ln\left(\frac{M_G}{M_Z}\right) - \frac{25}{6}\ln\left(\frac{m_S}{M_Z}\right) + 2\ln\left(\frac{M_G}{M_T}\right) \\
\frac{2\pi}{\alpha_1(M_Z)} &= \frac{2\pi}{\alpha_G} + \frac{33}{5}\ln\left(\frac{M_G}{M_Z}\right) - \frac{5}{2}\ln\left(\frac{m_S}{M_Z}\right)
\end{aligned} \tag{3.52}$$

where α_G is the common value of the three couplings at the unification scale M_G. Inserting the existing recent data [36] for $\alpha_3(M_Z)$, $\alpha_2(M_Z)$, $\alpha_1(M_Z)$, we obtain M_G and α_G, as well as the octet mass $M_{O'}$ for various choices of the isotriplet mass treated as input. An octet mass below M_G sets a lower bound of $1.5 \times 10^{16}\,GeV$ for the unification scale. In Fig.3.1 we show the values of M_G obtained in terms of M_T. These values are tabulated in Tab.3.3 together with the corresponding values of $M_{O'}$ and α_G. Note that the values of $M_{O'}$ follow M_T within a close range, indicating an approximately common intermediate scale. The values for M_T in the proximity of $10^{14}\,GeV$, corresponding to a safe $M_G \sim 10^{17}\,GeV$, have the correct order of magnitude required for the seesaw scale, since $(10^2)^2/10^{14} \sim 0.1\,eV$.

[7]The triplet-octet splitting has been previously studied for SU(5) models at one and two loops in [55]

Neutrino Masses

The terms relevant for neutrino masses can be easily singled out from the renormalizable part of the superpotential (3.44).[8] These terms are

$$Y_i^{\mathcal{S}} L_i \mathcal{S} H_u + Y_i^B L_i B H_u + Y_i^T L_i T H_u + \frac{M_{\mathcal{S}}}{2}\mathcal{S}^2 + \frac{M_B}{2}B^2 + M_{\mathcal{S}B}\mathcal{S}B + \frac{M_T}{2}T^2$$

or

$$v_u \left(Y_i^{\mathcal{S}}\, \mathcal{S} \,+\, Y_i^B B \,-\, \frac{Y_i^T}{\sqrt{2}}\,\tau_0 \right) \nu_i \,+\, \frac{M_{\mathcal{S}}}{2}\mathcal{S}^2 \,+\, \frac{M_B}{2}B^2 \,+\, M_{\mathcal{S}B}\mathcal{S}B \,+\, \frac{M_T}{2}\tau_0^2\,.$$

The corresponding terms for charged fermion masses are $M_T\, \tau_+ \tau_- \,-\, v_u Y_i^T\, e_i \tau_+$. The full neutrino mass matrix, in an $(\nu_i,\, \mathcal{S},\, B,\, \tau_0)$-basis, is

$$\mathcal{M}_N \,=\, \begin{pmatrix} 0 & \mathcal{M}_D \\ \\ \mathcal{M}_D^{\perp} & \mathcal{M}_R \end{pmatrix}\,, \tag{3.53}$$

where

$$\mathcal{M}_D \,=\, v_u \begin{pmatrix} Y_1^{\mathcal{S}} & Y_1^B & -\frac{1}{\sqrt{2}}Y_1^T \\ \\ Y_2^{\mathcal{S}} & Y_2^B & -\frac{1}{\sqrt{2}}Y_2^T \\ \\ Y_3^{\mathcal{S}} & Y_3^B & -\frac{1}{\sqrt{2}}Y_3^T \end{pmatrix}, \qquad \mathcal{M}_R \,=\, \begin{pmatrix} M_{\mathcal{S}} & M_{\mathcal{S}B} & 0 \\ \\ M_{\mathcal{S}B} & M_B & 0 \\ \\ 0 & 0 & M_T \end{pmatrix}.$$

Note that $Y_i^B \,=\, \frac{3}{\sqrt{30}} Y_i^T$. The constraints on μ' and f imply that $M_B \,\approx\, M_G$, while $M_{\mathcal{S}}$ and $M_{\mathcal{S}B}$ remain undetermined.

The light neutrino mass matrix will be

$$\mathcal{M}_\nu \,\approx\, -\mathcal{M}_D\, \mathcal{M}_R^{-1}\, \mathcal{M}_D^{\perp}\,. \tag{3.54}$$

[8]For simplicity, in our treatment of masses and mixings we neglect CP-violation

The inverse right-handed neutrino Majorana mass is

$$\mathcal{M}_R^{-1} = \frac{1}{\Delta}\begin{pmatrix} -M_B & M_{\mathcal{S}B} & 0 \\ M_{\mathcal{S}B} & -M_{\mathcal{S}} & 0 \\ 0 & 0 & \frac{\Delta}{M_T} \end{pmatrix}, \tag{3.55}$$

with $\Delta = M_{\mathcal{S}B}^2 - M_{\mathcal{S}} M_B$.

The determinant of $\mathcal{M}_D$ vanishes due to the $SU(5)$ relation $Y_i^T = \frac{\sqrt{30}}{3} Y_i^B$. This propagates to $\mathcal{M}_\nu$ resulting in one massless left-handed neutrino. Such a feature is shared by a wider class of models in which two right-handed neutrinos or more belong to the same GUT representation.

The resulting light neutrino mass matrix can be put in the form

$$(\mathcal{M}_\nu)_{ij} = \frac{v_u^2}{\Delta}\left(A\, Y_i Y_j + B\left(Y_i Y_j' + Y_i' Y_j\right) + C\, Y_i' Y_j'\right). \tag{3.56}$$

where

$$A = M_B,\ B = -\frac{3}{\sqrt{30}} M_{\mathcal{S}B},\quad C = \frac{3}{10} M_{\mathcal{S}} - \frac{\Delta}{2M_T}. \tag{3.57}$$

We have simplified the notation by denoting $Y_i^{\mathcal{S}} = Y_i$ and $Y_i^T = Y_i'$.

By going to the orthogonal basis in flavor space

$$\hat{X}^{(1)} = \frac{\vec{Y}\,' \times \vec{Y}}{|\vec{Y}\,' \times \vec{Y}|},\ \hat{X}^{(2)} = \frac{\vec{Y}}{\sqrt{Y^2}},\quad \hat{X}^{(3)} = \hat{X}^{(1)} \times \hat{X}^{(2)} \tag{3.58}$$

where $\hat{X}^{(1)}$ is the massless eigenvector, we can set the neutrino matrix in the form

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{23} & M_{33} \end{pmatrix}, \tag{3.59}$$

with

$$\begin{aligned}
M_{22} &= \frac{v_u^2}{\Delta}\left(M_B\, Y^2 \,-\, \frac{6}{\sqrt{30}} M_{\mathcal{S}B}(Y\cdot Y') \,+\, \frac{3}{10} M_{\mathcal{S}}\frac{(Y\cdot Y')^2}{Y^2}\right) \,-\, \frac{v_u^2}{2M_T}\frac{(Y\cdot Y')^2}{Y^2} \\
M_{23} &= \sqrt{Y^2Y'^2 - (Y\cdot Y')^2} \\
&\quad \left\{ -\frac{v_u^2}{\Delta}\left(-\frac{3}{\sqrt{30}}M_{\mathcal{S}B} \,+\, \frac{3}{10}M_{\mathcal{S}}\frac{(Y\cdot Y')}{Y^2}\right) \,+\, \frac{v_u^2}{2M_T}\frac{(Y\cdot Y')}{Y^2}\right\} \\
M_{33} &= \frac{1}{Y^2}\left(Y'^2\, Y^2 \,-\, (Y\cdot Y')^2\right)\left\{ -\frac{v_u^2}{2M_T} \,+\, \frac{3v_u^2}{10}\frac{M_{\mathcal{S}}}{\Delta}\right\}. \qquad (3.60)
\end{aligned}$$

Before we extract the light neutrino eigenvalues from this matrix, we must consider the scales involved in these expressions. For the mass scale M_B we have already made the choice $M_B = M_G$. The other two scales $M_{\mathcal{S}}$, $M_{\mathcal{S}B}$, associated with the singlet $\mathcal{S}$, are not constrained.

1st Approach: We shall assume that these two scales are also of the order of M_G. Thus, the dominant entry in the neutrino matrix elements M_{ab} will be the term $-\frac{v_u^2}{M_T}$ contained in C of (3.57), while the rest of the contributions will all be of the order of $\frac{v_u^2}{M_G}$, which is three orders of magnitude smaller. We may write[9]

$$\begin{aligned}
M_{22} &= \frac{v_u^2}{\sqrt{|\Delta|}}\hat{M}_{22} \,-\, \frac{v_u^2}{2M_T}\frac{(Y\cdot Y')^2}{Y^2} \\
M_{23} &= \frac{v_u^2}{\sqrt{|\Delta|}}\hat{M}_{23} \,+\, \frac{v_u^2}{2M_T}\frac{(Y\cdot Y')}{Y^2}\sqrt{Y^2Y'^2 \,-\, (Y\cdot Y')^2} \\
M_{33} &= \frac{v_u^2}{\sqrt{|\Delta|}}\hat{M}_{33} \,-\, \frac{v_u^2}{2M_T}\frac{1}{Y^2}\left(Y'^2\, Y^2 \,-\, (Y\cdot Y')^2\right) \qquad (3.61)
\end{aligned}$$

[9]We have set

$$\begin{aligned}
\hat{M}_{22} &= \frac{1}{\sqrt{|\Delta|}}\left(M_B Y^2 - \frac{6M_{\mathcal{S}B}}{\sqrt{30}}(Y\cdot Y') \,+\, \frac{3M_{\mathcal{S}}}{10}\frac{(Y\cdot Y')^2}{Y^2}\right) \\
\hat{M}_{23} &= \frac{1}{\sqrt{|\Delta|}}\sqrt{Y^2Y'^2 - (Y\cdot Y')^2}\left(\frac{3M_{\mathcal{S}B}}{\sqrt{30}} \,-\, \frac{3M_{\mathcal{S}}}{10}\frac{(Y\cdot Y')}{Y^2}\right) \\
\hat{M}_{33} &= \frac{3M_{\mathcal{S}}}{10\sqrt{|\Delta|}}\left(Y'^2 \,-\, \frac{(Y\cdot Y')^2}{Y^2}\right)
\end{aligned}$$

The resulting light neutrino mass eigenvalues are

$$\begin{aligned} m_\nu^{(3)} &\approx -\frac{v_u^2}{2M_T} Y'^2 \\ m_\nu^{(2)} &\approx \frac{v_u^2}{\sqrt{|\Delta|}} \left\{ \hat{M}_{22} \left(1 - \frac{(Y\cdot Y')^2}{Y^2 Y'^2} \right) + \hat{M}_{33} \frac{(Y\cdot Y')^2}{Y^2 Y'^2} \right. \\ &\left. \qquad + 2\hat{M}_{23} \frac{(Y\cdot Y')}{Y^2 Y'^2} \sqrt{Y^2 Y'^2 - (Y\cdot Y')^2} \right\} \end{aligned} \tag{3.62}$$

As it stands, for $|Y| \sim |Y'|$, the mass hierarchy is

$$m_\nu^{(2)}/m_\nu^{(3)} \sim \frac{v^2}{M_G} O(Y^2) / \frac{v^2}{M_T} O(Y^2) \sim M_T/M_G \sim \epsilon\,,$$

which is too strong a hierarchy to satisfy the data, without any other adjustment of parameters. On the other hand, if the overall scale of the determinant $\sqrt{|\Delta|} = \sqrt{|M_{\mathcal{S}B}^2 - M_{\mathcal{S}} M_B|}$ is set to be $\sqrt{|\Delta|} \sim \lambda\, M_G$, with $\lambda \sim \mathcal{O}(10^{-1})$, the relation $v^2/M_T \gg v^2 \hat{M}_{ab}/\sqrt{|\Delta|}$ still holds and, thus, we obtain

$$m_\nu^{(2)} \sim \frac{v_u^2}{\lambda^2 M_G} O(Y^2), \quad m_\nu^{(3)} \sim \frac{v_u^2}{M_T} O(Y^2) \tag{3.63}$$

This can give the correct overall scale of the neutrino masses and a suitable hierarchy

$$\frac{m_\nu^{(2)}}{m_\nu^{(3)}} \sim \frac{\epsilon}{\lambda^2}\,. \tag{3.64}$$

2nd Approach: An alternative assumption is to assume that the scales associated with the singlet $\mathcal{S}$ are of the same intermediate order as M_T, namely

$$M_{\mathcal{S}} \sim M_{\mathcal{S}B} \sim M_T \tag{3.65}$$

and, thus, $\Delta \approx -M_{\mathcal{S}} M_B$. Despite naturalness objections, this assumption is technically feasible. In this case, we have to leading order

$$m_\nu^{(2,3)} \approx -\frac{v_u^2}{4M_T} \left\{ (Y')^2 + \lambda' Y^2 \pm \sqrt{\mathcal{R}} \right\}, \tag{3.66}$$

where

$$\mathcal{R} \equiv \left(\lambda' Y^2 - Y'^2 \right)^2 + 4\lambda' (Y\cdot Y')^2 \tag{3.67}$$

and $\lambda' \equiv 2M_T/M_{\mathcal{S}}$, a number of $O(1)$ by assumption. Note that a hierarchy can also arise in this approach in the case $Y^2 >> Y'^2$, namely

$$\frac{m_\nu^{(2)}}{m_\nu^{(3)}} \approx \frac{\left(Y^2 Y'^2 - (Y \cdot Y')^2\right)}{\lambda' Y^4} = \frac{Y'^2 \sin^2\alpha}{\lambda' Y^2}\,. \tag{3.68}$$

We have denoted by α the angle $\cos^{-1}(\hat{Y} \cdot \hat{Y}')$. Similar results can also be obtained for $Y'^2 \gg Y^2$ but with

$$\frac{m_\nu^{(2)}}{m_\nu^{(3)}} \approx \frac{\lambda' Y^2 \sin^2\alpha}{Y'^2}\,. \tag{3.69}$$

In this approach there is also another possibility for the existence of a mass-hierarchy, namely, the possibility of almost parallel couplings in generation space ($\alpha \approx 0$)

$$Y \cdot Y' = \sqrt{Y^2}\,\sqrt{Y'^2}\,\cos\alpha \approx \sqrt{Y^2}\,\sqrt{Y'^2}\left(1 - \frac{\alpha^2}{2}\right)\,.$$

In this case, keeping $Y \sim Y'$, we obtain

$$\frac{m_\nu^{(2)}}{m_\nu^{(3)}} \approx \frac{\lambda' Y^2 Y'^2 \alpha^2}{(Y'^2 + \lambda' Y^2)^2}\,. \tag{3.70}$$

Finally, in this approach, there is a third possibility for a hierarchy if we assume that there is a small hierarchy in the scales $M_S : M_T$ corresponding to $\lambda' \sim 0.1$. In this case we get the same expression for the mass ratio as in (3.69) but with the desired hierarchy now originating from λ' instead of Y^2/Y'^2.

The above conclusions rely only on the renormalizable part of the superpotential. There are however some contributions to neutrino masses from various lowest order non-renormalizable terms in (3.49). These are:

Left-handed neutrino Majorana masses from the term

$$\mathcal{H}^c \mathcal{Q}\mathcal{Q}\mathcal{H}^c \sim \lambda_{ij}\frac{v_u^2}{M_P}\,\nu_i\,\nu_j\,. \tag{3.71}$$

These masses are tiny ($10^{-5}\,eV$ or less, depending on the couplings involved) but they remove the massless state arising from the previous analysis giving a lower bound for light neutrino masses.

Right-handed neutrino Majorana masses from the terms

$$\mathcal{T}^2\Sigma^2 + \Sigma^2\mathcal{T}\mathcal{S} + \mathcal{S}^2\Sigma^2 \sim \lambda'_{ij}\,\frac{V^2}{M_P}\,N_j^c N_j^c\,. \tag{3.72}$$

These terms could very well be of the same order of magnitude as the intermediate scale M_T or even larger but become subdominant for relatively small couplings, meaning $\lambda' < 10^{-2}$. In addition to these terms, negligible right-handed Majorana mass contributions $O(v^2/M_P)$ arise from the terms $\mathcal{H}\,(\mathcal{T}^2,\, \mathcal{T}\mathcal{S},\, \mathcal{S}^2)\,\mathcal{H}^c$.
Dirac neutrino masses from the terms

$$\mathcal{Q}\mathcal{T}\Sigma\mathcal{H}^c \,+\, \mathcal{Q}\Sigma\mathcal{H}^c\mathcal{S} \,\sim\, \lambda'_{ij}\,\frac{v_u V}{M_P}\,\nu_i\,N^c_j\,. \tag{3.73}$$

These contributions, suppressed by the factor V/M_P in comparison with renormalizable contributions, can remove massless states that arise due to the symmetries encountered in the renormalizable part of the Dirac neutrino mass matrix $\mathcal{M}_D$. To be specific, the operator $\mathcal{Q}\mathcal{T}\Sigma\mathcal{H}^c$ representing the invariants $\mathcal{Q}_i\mathcal{H}^c Tr(\mathcal{T}\Sigma)$, $\mathcal{Q}_i\mathcal{T}\Sigma\mathcal{H}^c$, $\mathcal{Q}_i\Sigma\mathcal{T}\mathcal{H}^c$ contributes to the superpotential as

$$\lambda''_{1i}\frac{v_u V}{M_P}\nu_i B \,+\, (\lambda''_{2i} \,+\, \lambda''_{3i})\,\frac{v_u V}{M_P}\left(\frac{3}{10}\nu_i B \,-\, \sqrt{\frac{3}{20}}\,\nu_i\,\tau_0\right)\,. \tag{3.74}$$

The presence of these terms modifies the structure of $\mathcal{M}_D$ and removes the massless state. The resulting from the seesaw mechanism light neutrino mass will be suppressed at least by a factor of $(\lambda'' V/M_P)^2 \,<\, 10^{-2}$ compared to the lightest massive neutrino.

Neutrino Mixing

The charged lepton and neutrino mass terms $M_{(\ell)}\,\ell\,\ell^c + \frac{1}{2}M_{(\nu)}\,\nu\,\nu$ can be diagonalized in terms of three unitary matrices $\mathbf{U}_{(\ell)}$, $\mathbf{V}_{(\ell^c)}$ and $\mathbf{U}_{(\nu)}$. These matrices rotate the above *gauge eigenstates* into *mass eigenstates*. If we express the neutrino charge current $J_\mu \propto \ell^\dagger\sigma_\mu\nu$ in terms of mass eigenstates, a combination of two of these matrices will appear $\ell^{\dagger'}\,\sigma_\mu\,\mathcal{U}_{PMNS}\,\nu'$, known as the *Pontecorvo-Maki-Nakagawa-Sakata* [56] matrix

$$\mathcal{U}_{PMNS} \,\equiv\, \mathbf{U}^\dagger_{(\ell)}\,\mathbf{U}_{(\nu)}\,. \tag{3.75}$$

In what follows we shall concentrate on $\mathbf{U}_{(\nu)}$ and put aside the charged lepton mixing matrix, for which, in any case very little is known.

The overall neutrino mixing matrix

$$\mathbf{U}_{(\nu)} \,=\, \mathbf{U}_1\,\mathbf{U}_2, \qquad \Big((\mathbf{U}_1)_{ij} \,=\, \hat{X}^{(i)}_j\Big) \tag{3.76}$$

is composed of the unitary matrix $\mathbf{U}_1$ that rotates the neutrino mass matrix (3.56) into (3.59) and a unitary matrix

$$\mathbf{U}_2 \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \tag{3.77}$$

that diagonalizes (3.59). The rotation angle β is related to the matrix entries through

$$\beta \;\equiv\; \frac{1}{2}\cot^{-1}\left(\frac{M_{22}-M_{33}}{2M_{23}}\right)\,. \tag{3.78}$$

Note that the mass eigenvalues are just

$$m_\nu^{(2,3)} \;=\; \frac{1}{2}\left(M_{22}+M_{33}\,\pm\,\sqrt{(M_{22}-M_{33})^2+4M_{23}^2}\right)\,.$$

The overall diagonalizing matrix is

$$\mathbf{U}_\nu \;=\; \mathbf{U}_1\,\mathbf{U}_2 \;=\; \begin{pmatrix} \hat{X}_1^1 & \cos\beta\,\hat{X}_2^1+\sin\beta\,\hat{X}_3^1 & -\sin\beta\,\hat{X}_2^1\;+\;\cos\beta\,\hat{X}_3^1 \\ \hat{X}_1^2 & \cos\beta\,\hat{X}_2^2+\sin\beta\,\hat{X}_3^2 & -\sin\beta\,\hat{X}_2^2\;+\;\cos\beta\,\hat{X}_3^2 \\ \hat{X}_1^3 & \cos\beta\,\hat{X}_2^3+\sin\beta\,\hat{X}_3^3 & -\sin\beta\,\hat{X}_2^3\;+\;\cos\beta\,\hat{X}_3^3 \end{pmatrix}$$

In order to obtain the corresponding relations between the $\hat{X}^{(a)}$ and the original Yukawa couplings Y_i and Y_i', we note that, as a result of the definitions (3.58), we may write

$$\vec{Y}\,' \;=\; Y'\left(\cos\alpha\,\hat{X}_2\;-\sin\alpha\,\hat{X}_3\right)\,,$$

where $\alpha \;\equiv\; \cos^{-1}\left(\hat{Y}\cdot\hat{Y}\,'\right)$. Substituting, we obtain

$$\mathbf{U}_\nu = (\sin\alpha)^{-1}\begin{pmatrix} \hat{Y}_3\hat{Y}_2'-\hat{Y}_2\hat{Y}_3' & \sin(\alpha{+}\beta)\hat{Y}_1-\sin\beta\hat{Y}_1' & \cos(\alpha{+}\beta)\hat{Y}_1-\cos\beta\hat{Y}_1' \\ \hat{Y}_3'\hat{Y}_1-\hat{Y}_1'\hat{Y}_3 & \sin(\alpha{+}\beta)\hat{Y}_2-\sin\beta\hat{Y}_2' & \cos(\alpha{+}\beta)\hat{Y}_2-\cos\beta\hat{Y}_2' \\ \hat{Y}_2\hat{Y}_1'-\hat{Y}_1\hat{Y}_2' & \sin(\alpha{+}\beta)\hat{Y}_3-\sin\beta\hat{Y}_3' & \cos(\alpha{+}\beta)\hat{Y}_3-\cos\beta\hat{Y}_3' \end{pmatrix}. \tag{3.79}$$

Equating this matrix with the standard parametrization we obtain the relations between the standard mixing angles θ_{23}, θ_{12}, θ_{13} and the above parameters. It is clear that, as long as we have not imposed any additional constraints on the Yukawa coupling directions in family space, we have no predictive restrictions on the mixing angles. In the particular case that we are close to *bimaximal mixing*

$$\theta_{23} \approx \frac{\pi}{4} + \epsilon_{23}, \qquad \theta_{12} \approx \frac{\pi}{4} + \epsilon_{12}, \qquad \theta_{13} \approx \epsilon_{13}\,,$$

from the standard parametrization we obtain

$$\mathbf{U}_{(\nu)} \approx \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{\epsilon_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{\epsilon_{12}}{\sqrt{2}} & \epsilon_{13} \\ -\frac{1}{2} - \frac{\epsilon_{12}}{2} + \frac{\epsilon_{23}}{2} - \frac{\epsilon_{13}}{2} & \frac{1}{2} - \frac{\epsilon_{12}}{2} - \frac{\epsilon_{23}}{2} - \frac{\epsilon_{13}}{2} & \frac{1}{\sqrt{2}} + \frac{\epsilon_{23}}{\sqrt{2}} \\ \frac{1}{2} + \frac{\epsilon_{12}}{2} + \frac{\epsilon_{23}}{2} - \frac{\epsilon_{13}}{2} & -\frac{1}{2} + \frac{\epsilon_{12}}{2} - \frac{\epsilon_{23}}{2} - \frac{\epsilon_{13}}{2} & \frac{1}{\sqrt{2}} - \frac{\epsilon_{23}}{\sqrt{2}} \end{pmatrix}$$

Equating this expression to (3.79), we obtain

$$\hat{Y} = \begin{bmatrix} \frac{\cos\beta}{\sqrt{2}} \\ \frac{\cos\beta}{2} - \frac{\sin\beta}{\sqrt{2}} \\ -\frac{\cos\beta}{2} - \frac{\sin\beta}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \epsilon_{12}\frac{\cos\beta}{\sqrt{2}} - \epsilon_{13}\sin\beta \\ -\left(\epsilon_{12} + \epsilon_{23} + \epsilon_{13}\right)\frac{\cos\beta}{2} - \epsilon_{23}\frac{\sin\beta}{\sqrt{2}} \\ -\left(-\epsilon_{12} + \epsilon_{23} + \epsilon_{13}\right)\frac{\cos\beta}{2} + \epsilon_{23}\frac{\sin\beta}{\sqrt{2}} \end{bmatrix}$$

and

$$\hat{Y}' = \begin{bmatrix} \frac{\cos(\alpha+\beta)}{\sqrt{2}} \\ \frac{\cos(\alpha+\beta)}{2} - \frac{\sin(\alpha+\beta)}{\sqrt{2}} \\ -\frac{\cos(\alpha+\beta)}{2} - \frac{\sin(\alpha+\beta)}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \epsilon_{12}\frac{\cos(\alpha+\beta)}{\sqrt{2}} - \epsilon_{13}\sin(\alpha+\beta) \\ -\left(\epsilon_{12} + \epsilon_{23} + \epsilon_{13}\right)\frac{\cos(\alpha+\beta)}{2} - \epsilon_{23}\frac{\sin(\alpha+\beta)}{\sqrt{2}} \\ -\left(-\epsilon_{12} + \epsilon_{23} + \epsilon_{13}\right)\frac{\cos(\alpha+\beta)}{2} + \epsilon_{23}\frac{\sin(\alpha+\beta)}{\sqrt{2}} \end{bmatrix}$$

Closing this section we note that the range of values for variables $\alpha, \beta, |Y|, |Y'|$, which determine the Yukawa couplings, depends on the mass hierarchy approach followed. Among the different options, the small angle scenario of the *2nd approach* exhibits the most restrictive structure with $\beta \sim \alpha$, while by assumption $|Y| \sim |Y'|$.

An overview of the model

In summary, we have studied a realization of the see-saw mechanism in the framework of an extended renormalizable version of the supersymmetric $SU(5)$ model. The right-handed neutrino fields were introduced as members of chiral $\mathbf{24} + \mathbf{1}$ superfields. In particular, two $\mathbf{24}$ superfields were introduced, out of which, due to different discrete symmetry charges, only one couples to matter and its neutral singlet and isotriplet components are identified as two of the right-handed neutrinos. Our basic assumption is that right-handed neutrinos survive below the grand unification scale having an intermediate mass in the neighborhood of $10^{13} - 10^{14}\,GeV$, a scale suitable to generate, through the see-saw mechanism, a light neutrino mass of the observed mass value of

$O(0.1\, GeV)$. The assumption of an isotriplet of an intermediate mass scale is supported by renormalization group analysis incorporating proton stability constraints. In addition, the model requires a color octet of neighboring mass, which, however, does not couple to ordinary matter. The right-handed neutrino mass matrix, then, depends on the constrained isotriplet scale M_T as well as the free, from renormalization group, scales $M_B, M_{\mathcal{S}}, M_{\mathcal{S}B}$ associated with the SM singlets of $\underline{\mathbf{1}}, \underline{\mathbf{24}}$. If these scales are of $O(M_G)$, an extra fine tuning is required in order to obtain a light neutrino mass hierarchy in agreement with data (*1st approach*). The alternative assumption according to which the scales $M_{\mathcal{S}}$, $M_{\mathcal{S}B}$ are of $O(M_T)$ is also possible (*2nd approach*). In this approach a phenomenologically acceptable neutrino mass hierarchy is possible as a result of the Yukawa hierarchy $Y' \ll Y$ or $Y' \gg Y$, where Y and Y' are the overall scales of the neutrino couplings $\langle H_u \rangle \nu\ (Y\,\mathbf{1} + Y'\,\mathbf{24})$. A second possibility of a hierarchy within this approach arises also when the angle between the Yukawa coupling vectors in family space Y_i and Y_i' is small. Nevertheless, the limiting case of aligned Yukawas is excluded, since it corresponds to two massless neutrinos. Alternatively, the required neutrino mass hierarchy can also arise as a result of a slight hierarchy of the scales $M_S : M_T$. However, in all these approaches, one very light neutrino is always present as a result of the structure of the neutrino mass matrix. Finally, we also find that a hierarchical mixing angle structure $\theta_{23} \sim \theta_{12} \gg \theta_{13}$ can be easily accommodated within the free parameter structure of the model.

3.4 Lopsided Models

3.4.1 A puzzling situation for leptons.

Besides the inefficiency of the SM or of the MSSM to impose any restriction on the fermion masses and mixing, a closer look on the observed experimental data in Tab.3.1 and Tab.3.2 further reveals a puzzling situation. In the quark sector there is a strong hierarchy for both the mass spectrum and the CKM mixing parameters. In the lepton sector, however, while charged leptons and possibly neutrinos[10] display hierarchical masses, the PMNS matrix exhibits a *bilarge mixing* with $\theta_{13} \ll \theta_{12} \lesssim \theta_{23} \approx \pi/4$. In addition to the rather unattractive distinction between the two fermion sectors, a difficulty of technical nature emerges for leptons when one attempts to reconcile large mass hierarchies with large mixings.

In order to identify the source of this problem we introduce two distinct mass

[10] Recall our previous discussion in favour of the NH scenario in §3.2.

matrices for the light neutrinos. For illustrative purposes and without any loss of generality we may restrict ourselves to the two-family case. The mass matrices are

$$M_1 = \begin{pmatrix} 1 & \lambda \\ \lambda & a\lambda^2 \end{pmatrix} \; , \; M_2 = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{3.80}$$

where the symmetric structure is always understood due to (3.22). The matrix M_1, for $\lambda \ll 1$ corresponds to a hierarchical structure with approximate eigenvalues $1, (a-1)\lambda^2$. The unitary rotation required for the diagonalization, realized as in (3.40), will be described by a small mixing angle satisfying $\tan\theta \approx \lambda$. On the other hand, the M_2 matrix for $\mathcal{O}(1)$ elements will typically produce two masses of the same order with a large mixing angle satisfying $\tan\theta = 2b/(a-c) \sim \mathcal{O}(1)$. These two limiting cases already reveal the problem. Hierarchical matrices produce small mixing angles while the desired large mixing can be straightforwardly obtained from non-hierarchical structures. The latter case however, although typically producing same order eigenvalues, for a fine-tuned structure it could also accommodate a hierarchical mass spectrum. For the M_2 considered here that would correspond to a condition for vanishing determinant at dominant level, namely $ac - b^2 \approx 0$. Also, by multiplying the elements of M_1, M_2 with random $\mathcal{O}(1)$ factors we may generalize these considerations beyond the case of symmetric matrices.

From the above analysis, both the hierarchy of quark masses and the small CKM mixing could be understood as rising through hierarchical Yukawa matrices. However, a model with large PMNS mixing angles along with hierarchical lepton masses seems to require either an M_2 structure with a considerable amount of fine tuning or another more elegant pattern.

An interesting proposal that has been put forward and evades the problem of large masses-large mixing is that of the *lopsided* (asymmetric) structure for the Yukawa couplings. Such an approach well motivated by GUT considerations is also general enough to apply even beyond them. The idea, as demonstrated below, will have two distinct manifestations out of which the latter will have a natural embedding within the see-saw mechanism.

3.4.2 The Standard Lopsided approach.

In order to illustrate the main features of the standard lopsided models [59,60] we may consider a general matrix $Y_{ij} = C_{ij}\,\epsilon_j$

$$Y = \begin{pmatrix} C_{11}\epsilon_1 & C_{12}\epsilon_2 & C_{13} \\ C_{21}\epsilon_1 & C_{22}\epsilon_2 & C_{23} \\ C_{31}\epsilon_1 & C_{32}\epsilon_2 & C_{33} \end{pmatrix} \tag{3.81}$$

with a hierarchy of the form

$$\epsilon_1 \ll \epsilon_2 \ll \epsilon_3 \equiv 1 \tag{3.82}$$

and random $\mathcal{O}(1)$ coefficients C_{ij}, taken real for simplicity. Diagonalization proceeds as usual with the biorthogonal transformation[11]

$$Y_{\mathcal{D}} = U_1^{\perp}\, Y\, U_2 = U_2^{\perp}\, Y^{\perp}\, U_1 \tag{3.83}$$

but with U_1 including large $\mathcal{O}(1)$ rotation angles, while those of U_2 are small. Depending on the explicit hierarchical form of the matrix the largest rotation angle inside U_2 may be $\mathcal{O}(\epsilon_2)$ or $\mathcal{O}(\epsilon_1/\epsilon_2)$.

It is easy to understand that such a pattern can be easily embedded within $SU(5)$ models that predict the minimal Yukawa unification relation. There, if we identify the Yukawa couplings as arising from the operator $H^c(f^c Y F)$, unification implies $Y \equiv Y^{(d)} = (Y^{(e)})^{\perp}$. Thus, we immediately identify

$$U_1 = U_{eL}^{\perp} = U_{dR} \tag{3.84}$$

$$U_2 = U_{dL} = U_{eR}^{\perp} \tag{3.85}$$

Therefore, in this lopsided realization, U_1 will participate in the PMNS matrix while U_2 will participate in the CKM. This will eventually attribute large angles to the former and small to the latter as suggested by fermion mixing data. We should also mention that this realization is not strictly restricted to $SU(5)$ models with minimal representation content. It may also be considered within other GUTs, such as $SO(10)$,

[11] A generalization to the case of complex coefficients inducing biunitary instead of biorthogonal transformations is always straightforward.

as long as the above Yukawa unification relation is respected.

3.4.3 Bimaximal mixing from lopsided neutrinos- A hidden lopsided structure.

This interesting idea has another distinct manifestation that may be regarded as orthogonal to the aforementioned standard approach. Although such a possibility had been previously considered by other authors in explicit models [61], in our more general approach [62] we have established the underlying connection between large lepton mixing and a hidden lopsided structure in the symmetric neutrino matrix.

By default a symmetric (or antisymmetric) matrix as the neutrino mass matrix cannot accommodate the lopsided form of (3.81). However, as we shall demonstrate, the see-saw mechanism offers the possibility for an underlying lopsided structure. As a result, the symmetric matrix for light neutrinos will be non-hierarchical but it will immediately satisfy that special relation required for hierarchical eigenvalues without any fine-tuning. Furthermore, in this scheme, a transposition relation analogous to the previous $SU(5)$ unification condition will be obtained through the see-saw formula itself. We should mention that such a property is necessarily required to reproduce the observed large-small mixing of the PMNS-CKM matrices. Thus, this alternative approach should be regarded as well-motivated even beyond GUT considerations.

In what follows we examine analytically a number of lopsided ansatze for the lepton sector that can potentially fit current low energy data. Large lepton mixing is raised from both the charged leptons and the neutrinos or through the neutrino sector exclusively, as in the case of a particularly simple ansatz, which is investigated thoroughly. We also explore the possibility of embedding this pattern within a class of $SO(10)$ models with realistic fermion masses and mixings. Our discussion is organised as follows. First, we illustrate the general features of these alternative lopsided models and their relation to large mixing. Then, we discuss briefly the standard Type-I seesaw framework within the lopsided approach and present our conventions. Next, we consider and study a number of lopsided patterns that lead to the observed lepton mixing. Next, we concentrate on a particularly simple ansatz that leads to lepton mixing exclusively through the neutrino sector, and finally we consider the embedding of the above in a class of $SO(10)$ models.

Symmetric matrices with lopsided substructure.

An attractive aspect of lopsided matrices is that they can produce symmetric matrices that can both accommodate a hierarchical spectrum and large mixing angles in a natural way. Since

$$Y_{\mathcal{D}}^2 \,=\, U_1^{\perp}\, Y\, Y^{\perp}\, U_1 \,=\, U_2^{\perp}\, Y^{\perp}\, Y\, U_2 \tag{3.86}$$

both symmetric matrices $YY^{\perp}$ and $Y^{\perp}Y$ share the same eigenvalues [63]. In fact, it is much easier to extract the mass eigenvalues from $Y^{\perp}Y$ which diagonalizes with small angles due to its hierarchical form. On the other hand $YY^{\perp}$ can be reexpressed as

$$YY^{\perp} = \mathcal{A} + \epsilon_2^2 \mathcal{B} + \epsilon_1^2 \mathcal{C}\,, \tag{3.87}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are symmetric rank-1 matrices. First we diagonalize $\mathcal{A}$ with $U_{\mathcal{A}} = U_{12}U_{23}$ where[12]

$$\tan_{12} = C_{13}/C_{23},\ \ \tan_{23} \,=\, \frac{(C_{13}^2 + C_{23}^2)^{1/2}}{C_{33}}$$

and, thus,

$$U_{\mathcal{A}}^{\perp}\, Y\, Y^{\perp}\, U_{\mathcal{A}} \,=\, \mathcal{A}_{\mathcal{D}} \,+\, \epsilon_2^2 \mathcal{B}' \,+\, \epsilon_1^2\, \mathcal{C}',\ \ \mathcal{A}_{\mathcal{D}} \,=\, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sum_k C_{k3}^2 \end{pmatrix}. \tag{3.88}$$

We should note that there is no reason for the rotated $\mathcal{B}'$ or $\mathcal{C}'$ to be diagonal. In fact, such a tuned case would correspond to proportional coefficients inside Y and thus imply a rank-2 or even a rank-1 form. Next, we rotate with $U_{\mathcal{B}'} \,=\, U_{12}'$, where now $\tan_{12}' \,=\, C_{12}'/C_{22}'$, and obtain[13]

$$U_{\mathcal{B}'}^{\perp}\, U_{\mathcal{A}}^{\perp}\, Y\, Y^{\perp}\, U_{\mathcal{A}}\, U_{\mathcal{B}'} \,=\, \mathcal{A}_{\mathcal{D}} \,+\, \epsilon_2^2 \mathcal{B}'' \,+\, \epsilon_1^2 \mathcal{C}'', \tag{3.89}$$

$$\mathcal{B}'' \,=\, \begin{pmatrix} 0 & 0 & 0 \\ 0 & {C_{12}'}^2 + {C_{22}'}^2 & C_{32}'({C_{12}'}^2 + {C_{22}'}^2)^{1/2} \\ 0 & C_{32}'({C_{12}'}^2 + {C_{22}'}^2)^{1/2} & {C_{32}'}^2 \end{pmatrix}. \tag{3.90}$$

[12] We use the notation $\tan_{ij} \equiv \tan\theta_{ij}$ for trigonometric functions where subscripts indicate rotations in the respective planes of family space.

[13] Primed coefficients correspond to the elements of the rotated matrices. The explicit expressions are

$$C_{12}' \,=\, C_{12}\, \cos_{12}\, - C_{22}\sin_{12}, \qquad C_{22}' \,=\, (C_{12}\sin_{12} + C_{22}\cos_{12})\cos_{23}\, - C_{32}\sin_{23}\ .$$

The full rotation matrix can be approximated by $U_1 \approx U_{\mathcal{A}} U_{\mathcal{B}'} = U_{12} U_{23} U'_{12}$ at dominant level which results to a form

$$U_1^{\perp} Y Y^{\perp} U_1 \sim \begin{pmatrix} \epsilon_1^2 & \epsilon_1^2 & \epsilon_1^2 \\ \epsilon_1^2 & \epsilon_2^2 & \epsilon_2^2 \\ \epsilon_1^2 & \epsilon_2^2 & 1 \end{pmatrix} \tag{3.91}$$

Diagonalization is then completed with subdominant rotations of $\mathcal{O}(\epsilon_2^2)$ or $\mathcal{O}(\epsilon_1^2/\epsilon_2^2)$.

In the above analysis we considered real coefficients C_{ij} for the elements of the Yukawa matrices allowing for an analytic treatment of the rotation matrices and eigenvalues. In the general case of complex parameters equation (3.86) is no longer valid but the main properties of lopsided matrices still hold. For example, $Y\,Y^{\perp}$ and $Y^{\perp}\,Y$, which now have different eigenvalues, are diagonalized by U'_1 and U'_2 respectively. These are different from the U_1, U_2 that diagonalize Y directly through $Y_{\mathcal{D}} = U_1^{\dagger}\,Y\,U_2$. Both symmetric matrices though, in general, share a similar hierarchical spectrum and equations (3.87-3.91) still hold for analogous complex rotations.

Mass scales and Seesaw.

Choosing as a general framework the two-doublets SM, electroweak symmetry breaking is realized through a non-vanishing VEV for $\mathcal{H}_u, \mathcal{H}_d$ in the direction of their neutral component. Thus, we obtain the mass terms for the charged fermions

$$v_d Y_{ij}^{(d)}\, d_i^c\, d_j \,+\, v_u Y_{ij}^{(u)}\, u_i^c\, u_j \,+\, v_d Y_{ij}^{(e)}\, e_i^c\, e_j \tag{3.92}$$

and for the neutrinos

$$v_u Y_{ij}^{(N)}\, \nu_i \mathcal{N}_j^c \,+\, \frac{1}{2} M_R Y_{ij}^{(R)}\, \mathcal{N}_i^c \mathcal{N}_j^c\,. \tag{3.93}$$

For a right handed neutrino mass scale in the neighborhood of $M_R \sim 10^{14}\,GeV$, a standard seesaw mechanism can be realized leading to the effective light neutrino mass

$$M_\nu \approx -\frac{v_u^2}{M_R}\, Y^{(N)}\, Y^{(R)^{-1}}\, Y^{(N)^{\perp}}\,. \tag{3.94}$$

Then, the resulting overall mass scale (v_u^2/M_R) comes out roughly as $\sim 10^{-1}\,eV$, in agreement with present data. Of course, for the above formula to be valid, $v_u Y_{\mathcal{D}}^{(N)} \ll M_R Y_{\mathcal{D}}^{(R)}$ should in general hold for the eigenvalues. If this is not the case, then, heavy $\mathcal{O}(M_W)$ Dirac-like masses would be produced reducing the number of light neutrinos. Under these considerations and neglecting the overall mass scale, eqn.(3.94) can be reexpressed in the more convenient form as

$$Y^{(\nu)} \approx Y Y^{\perp} \,, \tag{3.95}$$

where

$$Y \equiv Y^{(N)} \left(Y_{\mathcal{D}}^{(R)} \right)^{-1/2} . \tag{3.96}$$

This allows us to manipulate neutrino masses and mixings as in eqn.(3.87)-(3.91). All Yukawa matrices for fermions are expressed in a basis where the right handed neutrino mass matrix is diagonal with real and positive entries. The definition in eqn.(3.96) is then straightforward.

A lopsided structure, along with the desired hierarchy, may arise in various ways. For example, if $Y_{\mathcal{D}}^{(R)}$ is a diagonal matrix (possibly with a suitable hierarchy), $Y^{(N)}$ can be responsible for the lopsided form of Y and an associated hierarchy, a possibility well motivated by GUT considerations. Alternatively, if one assumes a generic $Y^{(N)}$ with $\mathcal{O}(1)$ matrix elements and a hierarchical $Y_{\mathcal{D}}^{(R)}$, an analogous lopsided Y can be obtained but in this case a lower bound for the mass of the lightest neutrino is also inherited[14]. In what follows, we will be interested in the explicit form of Y with the remark that the examined patterns can be obtained from the more fundamental matrices $Y^{(R)}$, $Y^{(N)}$.

Lopsided Lepton Patterns.

Next we proceed with the examination of possible lopsided patterns for the matrix Y defined in eqn.(3.96) that can contribute large mixing angles to U_{PMNS} through the neutrino sector. Three working examples are the following Y_1, Y_2, Y_3

$$\begin{pmatrix} C_{11}\lambda^2 & C_{12}\lambda^{1/2} & C_{13} \\ C_{21}\lambda^2 & C_{22}\lambda^{1/2} & C_{23} \\ C_{31}\lambda^2 & C_{32}\lambda^{1/2} & C_{33} \end{pmatrix}, \begin{pmatrix} C_{11}\lambda^2 & C_{12}\lambda^{1/2} & \dots \\ C_{21}\lambda^2 & C_{22}\lambda^{1/2} & \dots \\ C_{31}\lambda^2 & C_{32}\lambda^{1/2} & C_{33} \end{pmatrix}, \begin{pmatrix} C_{11}\lambda^2 & C_{12}\lambda^{1/2} & \dots \\ C_{21}\lambda^2 & C_{22}\lambda^{1/2} & \dots \\ \dots & \dots & C_{33} \end{pmatrix}$$

where the dots signify entries smaller than the ones explicitly shown which we can safely neglect, i.e. $\dots \ll O(\lambda^2)$. One should not be alarmed by the half-integer powers of the bookkeeping small parameter λ, since these matrices correspond, through the see-saw formula, to couplings with integer powers of λ, as could be expected to arise in various flavour-symmetry breaking schemes. All Y_i's correspond to a typical spectrum

[14] A typical hierarchy $\lambda^4 : \lambda : 1$ for the light neutrinos, parametrized by $\lambda \equiv (\delta m_{12}^2/\delta m_{23}^2)^{1/2} \approx 0.18$, would in general require an inverse hierarchy $\lambda^{-4} : \lambda^{-1} : 1$ for the right handed neutrinos, implying a mass eigenvalue $M_R\lambda^{-4}$ close to the physical cutoff of the theory whether this is the GUT, String or Planck scale.

$\lambda^4 : \lambda : 1$ in the NH case of the neutrinos, although they can be easily modified to accommodate a smaller value for the mass of the lightest neutrino.

The associated charged lepton matrices $Y_1^{(e)}, Y_2^{(e)}, Y_3^{(e)}$ are

$$\begin{pmatrix} \tilde{C}_{11}\kappa^3 & \tilde{C}_{12}\kappa^3 & \ldots \\ C_{13}\kappa & C_{23}\kappa & \ldots \\ \ldots & \ldots & \tilde{C}_{33} \end{pmatrix}, \begin{pmatrix} \tilde{C}_{11}\kappa^3 & \ldots & \ldots \\ \ldots & \tilde{C}_{22}\kappa & \tilde{C}_{23}\kappa \\ \ldots & \tilde{C}_{32} & \tilde{C}_{33} \end{pmatrix}, \begin{pmatrix} \tilde{C}_{11}\kappa^3 & \ldots & \ldots \\ \tilde{\kappa} & \tilde{C}_{22}\kappa & \tilde{C}_{23}\kappa \\ \ldots & \tilde{C}_{32} & \tilde{C}_{33} \end{pmatrix},$$

all corresponding to the mass hierarchy $\kappa^3 : \kappa : 1$ parametrized by the small parameter $\kappa = m_\mu/m_\tau$ in a manner consistent with current low energy data.

Y_1 has been previously used in Section 2 as an example where an arbitrary hierarchy $\epsilon_1^2 : \epsilon_2^2 : 1$ was assigned to $YY^\perp$. By substituting ϵ_1^2, ϵ_2^2 with λ^4, λ respectively we obtain the desired neutrino hierarchy and the unitary transformation $U_\nu \approx U_{12}U_{23}U'_{12}U'_{23}$. Among these unitary matrices only U'_{23} is the subdominant rotation of $\mathcal{O}(\lambda)$ needed to complete diagonalization (up to negligible corrections) and all other are $\mathcal{O}(1)$. The large mixing angles are explicitly given by the same expressions as before, namely

$$\begin{aligned} \tan_{12} &= \frac{C_{13}}{C_{23}}, \ \tan_{23} = \frac{(C_{13}^2 + C_{23}^2)^{1/2}}{C_{33}} \\ \tan'_{12} &= \frac{(C_{12}\cos_{12} - C_{22}\sin_{12})}{(C_{12}\sin_{12} + C_{22}\cos_{12})\cos_{23} - C_{32}\sin_{23}}\,. \end{aligned} \tag{3.97}$$

If we neglect the contribution from the charged lepton sector, a direct comparison with the standard parametrization $U_{PMNS} = \mathcal{U}_{23}\mathcal{U}_{13}\mathcal{U}_{12}$ results in three $\mathcal{O}(1)$ angles and therefore a trimaximal scheme in disagreement with present observations. Then, in order to fit the mixing angles, perhaps the easiest way is to assume a large contribution from the charged leptons along with a certain amount of fine tuning through the relation $U_e = U_{12}$ with $\tan_{12} \approx C_{13}/C_{23}$. In this sense $Y_1^{(e)}$ has what is required to obtain $U_{PMNS} \approx U_{23}U'_{12}U'_{23}$ that fits better the observed mixing pattern.

Using again the formalism developed in eqn.(3.87-3.91) for Y_2 we obtain the unitary transformation $U_\nu \approx U_{12}U'_{23}$, where now the neutrino mixing angles are given by the expressions

$$\tan_{12} = \frac{C_{12}}{C_{22}}, \qquad \tan'_{23} \approx \lambda\frac{C_{32}}{C_{33}^2}(C_{12}^2 + C_{22}^2)^{1/2}\,.$$

Since only one large angle is obtained in this way, the contribution of the charged leptons is again required but with the apparent advantage that no fine-tuning has to be imposed. Then, $Y_2^{(e)}$ is diagonalized by a rotation of the left-handed fields $U_e = U_{23}$ with an $\mathcal{O}(1)$ mixing angle given by $\tan_{23} \approx \tilde{C}_{32}/\tilde{C}_{33}$. The resulting unitary transformation describing lepton mixing will then be $U_{PMNS} \approx U_{23}^{\dagger} U_{12} U_{23}'$, which can easily fit lepton mixing data.

There is a third pattern that can be seen as a variation of the previous one with the difference that now the observed small angle of the lepton mixing originates from the charged lepton sector. Assuming Y_3 for the neutrinos, we obtain $U_\nu = U_{12}$ with $\tan_{12} \approx C_{12}/C_{22}$. On the other hand from $Y_3^{(e)}$, with the additional choice for the new scale $\tilde{\kappa} \sim \lambda\kappa$ we obtain $U_e \approx U_{23} U_{12}'$ up to $\mathcal{O}(\kappa^2)$ corrections with $\tan_{23} = \tilde{C}_{32}/\tilde{C}_{33}$, $\tan_{12}' = \tilde{\kappa}/[(\tilde{C}_{22}\cos_{23} - \tilde{C}_{23}\sin_{23})\kappa] \sim \lambda$. Consequently, lepton mixing is now described by $U_{PMNS} \approx U_{12}'^{\dagger} U_{23}^{\dagger} U_{12}$ an expression also consistent with present data.

A Lopsided Neutrino Pattern.

There is an interesting and attractive possibility that the lepton mixing pattern observed in nature originates solely from the neutrino sector. In this section we shall explore this possibility in the general case of complex $\mathcal{O}(1)$ coefficients $C_{ij} = |C_{ij}|e^{i\phi_{ij}}$. The related unitary transformations can be parametrized in terms of a real angle and a complex phase. For example, a unitary complex rotation in the $\{12\}$ plane can be described by [15]

$$U_{12} = \begin{pmatrix} \cos_{12} & \sin_{12} e^{-i\delta_{12}} & 0 \\ -\sin_{12} e^{i\delta_{12}} & \cos_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.98}$$

Let us consider

$$Y = \begin{pmatrix} C_{11}\lambda^2 & C_{12}\lambda^{1/2} & \dots \\ C_{21}\lambda^2 & C_{22}\lambda^{1/2} & C_{23} \\ C_{31}\lambda^2 & C_{32}\lambda^{1/2} & C_{33} \end{pmatrix} \tag{3.99}$$

for the matrix of the neutrinos defined in (3.96), which, as before, corresponds to a typical hierarchy $\lambda^4 : \lambda : 1$ of the NH case. Furthermore, we assume a negligible contribution to lepton mixing from the charged lepton sector, an assumption motivated by the large mass hierarchy of the charged leptons. This covers a large variety of distinct realistic patterns for the $Y^{(e)}$'s. In this sense we can have to a good approximation $U_e \approx I$ and, as a result, the useful property that diagonal phase matrices commute with U_e.

Diagonalization of the neutrino matrix proceeds as usual through the formalism developed in (3.87)-(3.91). Note that by a field redefinition we can absorb the complex

[15] Lepton mixing can be described by various equivalent parametrizations [58]. Nevertheless, the symmetrical parametrization $\mathcal{U}_{23}(\hat{\theta}_{23}; \hat{\delta}_{23})\mathcal{U}_{13}(\hat{\theta}_{13}; \hat{\delta}_{13})\mathcal{U}_{12}(\hat{\theta}_{12}; \hat{\delta}_{12})$ in the presence of CP-violating phases seems more attractive for model building purposes [57].

phases of C_{12}, C_{23}, C_{33}. Diagonalization then begins with the unitary transformation $U_{23}U_{12}$ describing two successive rotations. The corresponding large rotation angles and the complex phases are given by

$$\tan_{23} = C_{23}/C_{33}, \;\; \delta_{23} = 0 \;, \;\; \tan_{12} = C_{12}/|C'_{22}| \,, \delta_{12} = -\phi'_{22} \tag{3.100}$$

with the complex primed coefficients

$$C'_{22} \;=\; C_{22}\cos_{23} - C_{32}\sin_{23}, \quad C'_{32} \;=\; C_{22}\sin_{23} + C_{32}\cos_{23} \;. \tag{3.101}$$

Thus, the neutrino mass matrix $YY^{\perp}$ is brought into the hierarchical form

$$\begin{pmatrix} C'^2_{11}\lambda^4 & (\ldots)\lambda^4 & (\ldots)\lambda^4 \\ (\ldots)\lambda^4 & e^{2i\phi'_{22}}\left(C_{12}^2 + |C'_{22}|^2\right)\lambda & e^{i(\phi'_{22}+\phi'_{32})}|C'_{32}|\left(C_{12}^2 + |C'_{22}|^2\right)^{1/2}\lambda \\ (\ldots)\lambda^4 & e^{i(\phi'_{22}+\phi'_{32})}|C'_{32}|\left(C_{12}^2 + |C'_{22}|^2\right)^{1/2}\lambda & C_{23}^2 + C_{33}^2 + \lambda {C'_{32}}^2 \end{pmatrix} .$$

The coefficients denoted by dots and multiplying the λ^4 elements are irrelevant since only C'_{11} is in practice associated with the lightest neutrino mass and a contribution to the CP-violating phases[16]. A subsequent small complex rotation U'_{23}, with

$$\tan'_{23} = \lambda|C'_{32}|\frac{(C_{12}^2 + |C'_{22}|^2)^{1/2}}{(C_{23}^2 + C_{33}^2)} + \mathcal{O}(\lambda^2), \qquad \delta'_{23} = \phi'_{22} + \phi'_{32} + \mathcal{O}(\lambda) \,,$$

along with negligible $\mathcal{O}(\lambda^3)$ rotations, will finally bring the neutrino matrix to the diagonal form

$$Y_{\mathcal{D}}^{(\nu)} \;\approx\; \begin{pmatrix} {C'_{11}}^2\lambda^4 & 0 & 0 \\ 0 & e^{2i\phi'_{22}}\,(C_{12}^2 + |C'_{22}|^2)\,\lambda & 0 \\ 0 & 0 & C_{23}^2 + C_{33}^2 + \lambda {C'_{32}}^2 \end{pmatrix}$$

Summarizing, lepton mixing in this model is described by the unitary transformation

$$U_{23}(\theta_{23},\,0)\,U_{12}(\theta_{12}, -\phi'_{22})\,U'_{23}(\theta'_{23},\,\phi'_{22} + \phi'_{32})\;\cdot\;\mathcal{P}, \tag{3.102}$$

with

$$\mathcal{P} \;\approx\; diag\left(e^{-i\phi'_{11}},\, e^{-i\phi'_{22}},\, 1\right) \;. \tag{3.103}$$

$\mathcal{P}$ guarantees the real positive mass eigenvalues. We already notice the predictive power

[16] C'_{11} is given explicitly by $C'_{11} = C_{11}\cos_{12} - (C_{21}\cos_{23} - C_{31}\sin_{23})\sin_{12} e^{-i\phi'_{22}}$.

of this pattern. Starting from a general complex matrix Y for the neutrinos, with 8 complex parameters, and assuming a lopsided structure, consistent with a typical hierarchical spectrum, we obtained an one to one fit between C_{23}, C_{33}, $C_{12}, |C_{22}|, |C_{32}|, \phi_{22}, \phi_{32}$ and the two heavier neutrino masses, the three rotation angles and the two (out of three) CP-violating phases. Furthermore, the two rotation angles are predicted $\mathcal{O}(1)$, while the third is $\mathcal{O}(\lambda)$ as a consequence of the neutrino mass hierarchy $\lambda : 1$.

In order to exhibit the explicit relations of observables, we first note that the expression (3.102) is unique up to a left-multiplication by an arbitrary diagonal phase matrix. By a field redefinition of the left-handed charged leptons, having assumed that $U_e \approx I$, we obtain

$$U_{PMNS} \,\approx\, \mathcal{P}^{-1}\, U_{23}(\theta_{23}; 0)\, U_{12}(\theta_{12}; -\phi'_{22})\, U'_{23}(\theta'_{23};\, \phi'_{22} + \phi'_{32})\, \mathcal{P}\,.$$

A direct comparison with the symmetrical parametrization of the physical quantities results in the following relations

$$\tan\theta_{sol} \,\equiv\, \tan_{\hat{12}} \,\approx\, \tan_{12} \tag{3.104}$$

$$\tan\theta_{atm} \,\equiv\, \tan_{\hat{23}} \,\approx\, |\tan_{23} + e^{-i(\phi'_{22}+\phi'_{32})}\cos_{12}\, \tan'_{23}| \tag{3.105}$$

$$|U_{e3}| \,\equiv\, \sin_{\hat{13}} \,\approx\, \sin_{12}\, \sin'_{23} \tag{3.106}$$

$$\hat{\delta}_{12} \,\approx\, -\phi'_{11}, \quad \hat{\delta}_{13} \,\approx\, \phi'_{32} - \phi'_{11}, \quad \hat{\delta}_{23} \,\approx\, -\phi'_{22}\,. \tag{3.107}$$

where the relations for the phases hold up to $\mathcal{O}(\lambda)$ corrections. The Dirac CP-phase of the standard parametrization responsible for CP-violation in neutrino oscillations is identified as

$$\delta_D^{lep} \,\equiv\, \hat{\delta}_{13} - \hat{\delta}_{12} - \hat{\delta}_{23} \,\approx\, \phi'_{22} + \phi'_{32}\,.$$

Our initial choice of same order parametrization coefficients, so that $C_{ij} \sim \mathcal{O}(1)$, is well justified by fitting the current experimental data from neutrino oscillation phenomena. Nevertheless, the Dirac CP-phase is required for a more accurate fit between the three observed mixing angles, the two heavier neutrino masses, and the subset of the parameters $\{C_{23}, C_{33}, C_{12}, |C'_{22}|, |C'_{32}|\}$. A more conclusive test for this model, including the complex phases ϕ_{22}, ϕ_{32}, would further require the measurement of any existing physical Majorana phases. Even at this stage however, taking at face value $\sin'_{23} \approx \lambda$, we arrive at the interesting estimate

$$|U_{e3}| \,\approx\, \lambda \sin\theta_{sol} \,\approx\, \sin 5.9^o\,. \tag{3.108}$$

Concluding our discussion, we note some of the general characteristics and perspectives of this pattern. If any or all of the C_{i1}'s in (3.99) are substituted by texture zeros (or smaller entries) the same relations are obtained up to a different complex phase contribution ϕ'_{11} and a different corresponding light neutrino spectrum of the form $(< \lambda^4) : \lambda : 1$, something still consistent with observations. If on the other hand, either C_{22} or C_{32} (but not both) are replaced with texture zeros, two additional predictive relations are obtained. By taking C_{32} zero we obtain a straightforward relation for the complex phases in (3.107) since $\phi'_{22} = \phi'_{32} = \phi_{22}$ and the relation for the mixing angles

$$\tan\theta_{atm} \approx \frac{|U_{e3}|}{\tan\theta_{sol}} \left| \frac{(m_{\nu_3} - m_{\nu_2}\cos\delta_D^{lep})}{m_{\nu_2}\cos^2\theta_{sol}} + \cos\delta_D^{lep} \right| . \tag{3.109}$$

Using current best-fit values for $\tan\theta_{atm}, \tan\theta_{sol}, m_{\nu_2}/m_{\nu_3}$, the small angle $\hat{\theta}_{13}$ is predicted in the $(4^o\text{-}6^o)$ region. For a vanishing C_{22} an analogous relation can be obtained.

3.5 Embedding in GUTs.

In the previous section we showed how a lopsided structure in the neutrino sector may lead to the observed lepton mixing angles. An interesting feature of this approach is that a similar lopsided structure may account for the small mixing in the quark sector [61]. Such a possibility, apart from its obvious simplicity, is also well motivated by GUT considerations. In what follows we consider as a framework a class of $SO(10)$ models [59, 60] with the realistic mass matrices

$$Y^{(u)} = \begin{pmatrix} 0 & k' & 0 \\ 0 & k & b \\ 0 & 0 & a \end{pmatrix} m_u\,, \quad Y^{(N)} = \begin{pmatrix} 0 & k' & 0 \\ 0 & k & b \\ 0 & 0 & a \end{pmatrix} m_u\,, \tag{3.110}$$

$$Y^{(d)} = \begin{pmatrix} 0 & \delta' - k' & \delta \\ \delta' & -k & \epsilon' - b \\ \delta & \epsilon & a \end{pmatrix} m_d\,, \quad Y^{(e)} = \begin{pmatrix} 0 & \delta' - k' & \delta \\ \delta' & -k & \epsilon - b \\ \delta & \epsilon' & a \end{pmatrix} m_d\,. \tag{3.111}$$

Only the (common) (33) entry of these matrices, denoted by a, is assumed to arise from the standard renormalizable term $\mathbf{16}_3\mathbf{16}_3\mathbf{10}_H$. All other mass entries arise from effective non-renormalizable operators involving additional Higgs fields $\mathbf{16}_H$, $\mathbf{16}'_H$, $\mathbf{45}_H$. These contributions are subdominant and are denoted by a number of small parameters (k, k', δ, δ', ϵ, ϵ'), with the exception of the contribution to the (23) entry, which is assumed to be of the same order as the renormalizable contribution and denoted by the parameter b. The small elements ϵ, ϵ' arise from a non-renormalizable operator $\{\mathbf{16}_i\mathbf{16}_H\}\{\mathbf{16}_j\mathbf{16}'_H\}$. The vev $\langle\mathbf{16}'_H\rangle \sim \langle N^c_H{}'\rangle \sim M_G$ breaks $SO(10)$ to $SU(5)$, while the vev $\langle\mathbf{16}_H\rangle \sim \langle H^0_d\rangle \sim M_W$ breaks it down to $SU(3)_c \times U(1)_{em}$. Only down quarks and charged leptons get contributions from this term. The relevant Yukawa couplings of this operator respect the $SU(5)$ relation $Y^{(e)} = \left(Y^{(d)}\right)^{\perp}$, which has been associated with a lopsided structure in the charged lepton sector [59]. The symmetric elements δ, δ' arise from a different contraction of the same representations, namely $\{\mathbf{16}_i\mathbf{16}_j\}\{\mathbf{16}_H\mathbf{16}'_H\}$, appearing again only in $Y^{(d)}$ and $Y^{(e)}$. A common lopsided structure in the quark and lepton mass matrices arises from the operator $\{\mathbf{16}_i\mathbf{10}_H\}_{16}\{\mathbf{16}_j\mathbf{45}_H\}_{\overline{16}}$ through the elements k, k', b. The vev $\langle\mathbf{45}_H\rangle \sim M_G$ lies in the right-handed isospin direction I_{3R}, responsible for the breaking of the $SU(2)_R$ subgroup of $SO(10)$, while $\langle\mathbf{10}_H\rangle \sim M_W$ is the standard vev in the electroweak breaking direction. The contraction employed allows for general Yukawa textures that respect the relation $Y^{(u)} = Y^{(N)} = -Y^{(d)} = -Y^{(e)}$, the minus sign arising from the different I_{3R} charge of the respective fields.

We proceed by assuming that [59] δ, δ', $k, k' \ll \epsilon$, $\epsilon' \ll a$, b. Note that out of these parameters one can be absorbed in an overall scale redefinition. Equivalently, here we shall impose the simplifying $b^2 + a^2 \equiv 1$. Next, by a field redefinition of the down quarks and charged leptons we restrict the complex phases to the (21),(22) and (13) elements, leaving the rest real and positive. Then, without loss of generality, we express (21) and (12) entries in both the down quarks and charged lepton matrices as $Y_{21} \equiv \delta z$, $Y_{12} \equiv |\delta' - k'|$. Furthermore, assuming $\epsilon \sim \epsilon' \ll b$, we approximate the (23) entry as $Y_{23} \approx b$. Thus all parameters besides z, δ', k, k' are now real in both $Y^{(e)}, Y^{(d)}$. Neglecting the overall mass scales, we obtain in this redefined notation (at M_G)

$$m_b \approx m_\tau \approx (b^2 + a^2)^{1/2} \equiv 1, \tag{3.112}$$

$$m_s/m_b \approx \epsilon b\,, \quad m_\mu/m_\tau \approx \epsilon' b, \tag{3.113}$$

$$|\det Y^{(d)}| \approx |\det Y^{(e)}| \approx |\delta' - k'|\,|za - b|\,\delta\,. \tag{3.114}$$

The model by construction is consistent with the $b - \tau$ unification as a result of the common b, a entries. This is a favourable prediction common in $SO(10)$ and $SU(5)$ models and consistent with the low energy data. To fit the masses m_s, m_μ of the down quarks and charged leptons we notice that these are controlled by the elements $b\epsilon$, $b\epsilon'$. Then the relation $|\det Y^{(d)}| \approx |\det Y^{(e)}|$, along with $m_b \approx m_\tau$, results in

$m_d/m_e \approx m_\mu/m_s$ (at M_G), which is in general agreement with the expected relevant mass ratios at the unification scale. By taking $\epsilon'/\epsilon \approx 3$, the Georgi-Jarlskog factors can be obtained [64].

For the up quark masses we have

$$m_c/m_t \approx \left(|k'|^2 + |ak|^2\right)^{1/2}, \; m_u \approx 0\,. \tag{3.115}$$

The prediction for a massless up quark is, of course, wrong but, since $m_u/m_t \sim 10^{-5}$, a tiny mass for the up quark can always arise from a non-renormalizable operator. Such a small entry in the mass matrices cannot in practice affect the rest of the relations. Furthermore, the parameter $k'(\sim k)$ which appears in both the mass ratio m_c/m_t and the expression for $|\det Y^{(e)}|$ will allow for a relation between the respective scales.

Next, we notice that since the M_G-relation $m_c/m_t \ll m_s/m_b$ is expected to hold, the diagonalization of the up quark matrix will contribute only small corrections to CKM and therefore we can safely consider, in this scheme, quark mixing originating from the down quark matrix. Then, we have the relations

$$V_{cb} \approx \epsilon a\,, \quad V_{ub} \approx \delta(z^* b + a), \tag{3.116}$$

$$V_{us} \approx \frac{\delta\epsilon - V_{cb}V_{ub}}{(m_s/m_b)^2}, \quad -\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} \approx \frac{b^2(zb+a)}{a - a^2(zb+a)}, \tag{3.117}$$

where we can easily fit all mixing angles and the CP-violating phase of the quark sector. Using current best-fit values and the expected scale $|\det Y^{(e)}| \sim 2\cdot 10^{-5}$, we obtain the rough estimate

$$m_c/m_t|_{M_G} \approx \left(|k'|^2 + |ak|^2\right)^{1/2} \sim |\delta' - k'| \approx 4\cdot 10^{-3}\,, \tag{3.118}$$

within the expected allowed range. An additional important relation is also derived from the quark sector, namely

$$b/a \approx \frac{m_s/m_b}{|V_{cb}|}\,. \tag{3.119}$$

We are going to see shortly that this ratio will appear as the dominant contribution to $\tan\theta_{atm}$ of the neutrino mixing.

Let us now proceed assuming[17] a diagonal Majorana mass matrix for the right-handed neutrinos $Y_{\mathcal{D}}^{(R)} \equiv diag(1,\, \Lambda,\, 1)\, M_R$. The new scale Λ is introduced to counter-

[17]Considering Majorana masses that arise from $\mathbf{16_i}\,\overline{\mathbf{126}}_\mathbf{H}\,\mathbf{16_j}$ or the effective operator $\mathbf{16_i}\,\mathbf{16_j}\,\overline{\mathbf{16}}_\mathbf{H}\,\overline{\mathbf{16}}_\mathbf{H}$, all Yukawa couplings, without loss of generality, can be expressed in the basis where $Y_{ij}^{(R)}$ is diagonal.

act the large mass hierarchy inherited from the up quark sector to the Dirac neutrino matrix through the relation $Y^{(u)} = Y^{(N)}$. If this were not the case, the neutrino spectrum would be inconsistent with the observed squared mass differences. The (11) element, taken unity for convenience, is in practice arbitrary as long as the mass ratio m_{ν_1}/m_{ν_3} for the light neutrinos, obtained through the seesaw mechanism, is comparable or smaller than λ^4. We can then manipulate neutrino masses and mixing as previously. Neglecting the overall mass scales, the neutrino matrix defined in (3.96) is now

$$Y \approx \begin{pmatrix} 0 & d' & 0 \\ 0 & de^{i\delta_d} & b \\ 0 & 0 & a \end{pmatrix} . \tag{3.120}$$

Since the charged lepton matrix is diagonalized with small rotations, in contrast to the large ones observed in neutrino oscillations, we may consider $U_e \approx I$ to a good approximation. A diagonal phase matrix can then be used to absorb all complex phases (besides the (22) element) and bring the matrix Y into this form. This form is a special case of the *"Lopsided Neutrino Pattern"* we previously examined and, thus, using the same treatment we obtain the following relations

$$\begin{aligned} U_{PMNS} &\approx U_{23}(\theta_{23}, 0)\, U_{12}(\theta_{12}, -\delta_d)\, U'_{23}(\theta'_{23}, 2\delta_d) \cdot \mathcal{P}, \\ \mathcal{P} &= diag\left(e^{-i\delta_1},\, e^{-i\delta_d},\, 1\right) \\ \tan_{23} \approx b/a, \;\; \tan_{12} \approx d'/(d\cos_{23}), \;\; \tan'_{23} &\approx \frac{d\sin_{23}}{(b^2+a^2)}\left(d'^2 + (d\cos_{23})^2\right)^{1/2} \\ m_{\nu_3} &\approx |b^2 + a^2 + (d\sin_{23})^2 e^{2i\delta_d}| \equiv 1 \\ m_{\nu_2}/m_{\nu_3} &\approx d'^2 + (d\cos_{23})^2 \equiv \lambda \end{aligned} \tag{3.121}$$

For the diagonal $Y_{\mathcal{D}}^{(R)}$ we obtain through the seesaw formula $d'/d = k'/k$. This ratio will allow for a direct fit of the solar angle. We have for the physical parameters

$$\tan\theta_{sol} \approx \tan_{12} \tag{3.122}$$

$$\tan\theta_{atm} \approx \left|\tan_{23} + e^{-2i\delta_d}\cos_{12}\tan'_{23}\right| \tag{3.123}$$

$$|U_{e3}| \approx \sin_{12}\sin'_{23} \tag{3.124}$$

$$\hat{\delta}_{12} \approx -\delta_1\,, \;\; \hat{\delta}_{13} \approx \delta_d - \delta_1\,, \;\; \hat{\delta}_{23} \approx -\delta_d\,, \;\; \delta_D^{lep} \cong 2\delta_d \tag{3.125}$$

From these relations we directly obtain the prediction for the complex phases of the symmetrical parametrization $\hat{\delta}_{23} + \hat{\delta}_{13} - \hat{\delta}_{12} \approx 0$. By fitting the best-fit value for the ratio $\tan_{23} \approx b/a \approx 0.6$ we notice a significant deviation from the observed atmospheric angle $\tan\theta_{atm} \approx 1$, which cannot be accounted for by the subleading term in (3.123). An exact fit would require $m_s/m_b \to |V_{cb}|$ at M_G and, perhaps, a smaller value $\theta_{atm} \approx 40^o$ still within current experimental bounds[18]. Subleading corrections can also have significant effect on this ratio (especially of our initial working assumption $|\epsilon' - b| \approx b$). In any case, δ_d will be close to zero in this model, giving small CP-violation in neutrino oscillation phenomena but also $\hat{\delta}_{13} \approx \hat{\delta}_{12}$. The small mixing angle will obey the relation (3.109) for the corresponding M_G values of the relevant parameters .

Overview.

Summarizing our previous analysis, we have shown how a lopsided structure hidden within the symmetric light neutrino matrix may account partially or completely for the large lepton mixing angles observed in neutrino oscillation phenomena. Although this idea has been previously considered in other models, here, the assumption of a very light neutrino mass ($m_{\nu_1}/m_{\nu_3} \leq \lambda^4$) allows for an analytic treatment of neutrino masses and mixing. An attractive feature of the formalism developed is that approximations enter only at the stage where the matrix has already been brought to a hierarchical form, thus allowing for exact expressions of the large mixing angles. Among the four instructive lepton patterns considered, which can potentially fit current lepton mixing data, the *"Lopsided neutrino pattern"*, has a number of appealing features. Specifically, in this model the magnitudes of the lepton mixing angles are predicted within current experimental bounds and the smallness of the θ_{13} angle is associated with the neutrino mass ratio of the NH case $m_{\nu_2}/m_{\nu_3} \equiv \lambda$. Furthermore, since an analogous lopsided form for the quarks may account for the observed small mixing in the CKM, we also explored the possibility of a common lopsided structure within an $SO(10)$ model with realistic masses and mixing.

[18] Alternatively, by assuming a large but subleading contribution from $\{\mathbf{16}_3\mathbf{10}_H\}_{16}\{\mathbf{16}_3\mathbf{45}_H\}_{\overline{16}}$ in the (33) entries, the prediction for $b-\tau$ unification is preserved but with the corresponding lepton rotation angle $\tan_{23} = b/a'$ with $a' \sim a$, thus allowing for a direct fit of θ_{atm}.

Chapter 4

Discrete R-Symmetries

Among the various attractive features of SUSY theories is the possibility for a new type of an underlying symmetry, commonly referred to as an *R-symmetry*. It can potentially explain the origin of the matter-parity in the MSSM and in addition offers phenomenologically interesting restrictions for model building. In particular, the discrete R-symmetries, which we shall consider here, have the apparent advantage over the global, continuous ones that they are free of massless Goldstones. Furthermore, they provide us with more options for realizations that avoid many of the phenomenological difficulties of the MSSM or its extensions while not being totally unconstrained.

Any symmetry, discrete or continuous, with a possible phenomenological interest should be embeddable in a gauge symmetry. It has been argued that any symmetry, discrete or continuous, can survive to low energies without being violated by quantum gravitational effects [65] if it can be embedded into a gauge symmetry. Then, since any gauge theory should be free from the standard gauge-anomalies, restrictions that emerge from this condition can impose non-trivial constraints to the low energy discrete symmetry charges. In fact this is the case, as we shall discuss, for a $U(1)$ breaking to a discrete $\mathbb{Z}_N$ group. The $\mathbb{Z}_N$ charges will eventually have to obey certain anomaly cancellation conditions.

4.1 Discrete abelian symmetries

4.1.1 The $U(1) \to \mathbb{Z}_N$ breaking.

As an illustrative example for the breaking of a continuous symmetry into a discrete, we may consider the breaking of an abelian $U(1)$ into a $\mathbb{Z}_N$. In any case, this will be the only pattern concerning our following analysis.

Now let a theory with two complex scalars transforming non-trivially under a $U(1)$ gauge symmetry as

$$\phi_1' = e^{iq_1\theta(x)}\phi_1$$
$$\phi_2' = e^{iq_2\theta(x)}\phi_2$$

We may also assume that the respective $U(1)$ charges are quantized (i.e. $q_1, q_2 \in \mathbb{Z}$) as observed in the SM or as expected in its supersymmetric and unified extensions. Then, if the ϕ_1, for example, acquires a non-vanishing VEV then a $\mathbb{Z}_{q_1}$ subgroup will be left invariant. That is because the vacuum will be left invariant for discrete values of the group parameter, namely

$$e^{iq_1\theta}\langle\phi_1\rangle = \langle\phi_1\rangle \;\rightarrow\; \theta = \frac{2\pi k}{q_1}\,,\;\; k \in \mathbb{Z} \tag{4.1}$$

Under this new symmetry ϕ_2 will satisfy

$$\phi_2 = e^{iq_2\theta}\phi_2 = e^{iq_2\frac{2\pi k}{q_1}}\phi_2 \tag{4.2}$$

which suggests that this field is charged with q_2 under this $\mathbb{Z}_{q_1}$ symmetry. For $q_2 \geq 0$ these charges are redundant and can always be *modulo shifted* to the fundamental set of charges $\{0, \ldots q_1 - 1\}$. For example a $q = 4$ charge of a $\mathbb{Z}_3$ symmetry will be redundant and equivalent to the $q' = 1$ since $q = q' \mod 3$.

It should be further remarked that other, non-trivial redundancies may also appear through this breaking of a continuous to a discrete abelian symmetry, depending on the field content of the theory. These redundancies, absent in the case of continuous groups, originate from the fractional form of the $\mathbb{Z}_N$ transformations as in (4.2). There, the fraction (q_2/q_1) can display the same value for distinct pairs of charge assignments, thus potentially allowing for the realization of equivalent symmetry breaking patterns.

To explain this in more detail, we may consider a model, as previously, with $q_1 = 4$, $q_2 = 2$. A non-vanishing VEV for ϕ_1 would straightforwardly induce the breaking $U(1) \rightarrow \mathbb{Z}_4$. In the absence of any other fields however, the corresponding field transformations are not unique, since

$$\phi_1 = e^{i4\frac{2\pi k}{4}}\phi_1 \quad = e^{i2\frac{2\pi k}{2}}\phi_1 \tag{4.3}$$
$$\phi_2 = e^{i2\frac{2\pi k}{4}}\phi_2 \quad = e^{i\frac{2\pi k}{2}}\phi_1 \tag{4.4}$$

As a result, an equivalent breaking pattern $U(1) \rightarrow \mathbb{Z}_2$ with $q_1 = 2$ and $q_2 = 1$ is always possible which suggests that the previous symmetry is actually redundant.

4.1.2 R-symmetries

In $\mathcal{N} = 1$ supersymmetric theories, a new type of an abelian symmetry is always possible acting seperately on the different components of a given supermultiplet. This symmetry which does not commute with the generators of supersymmetry has a rather simple interpretation when one introduces the concepts of superspace and superfield. Even though a detailed discussion on these theoretical tools is beyond the scope of our discussion, we may introduce a few elements relevant to our following analysis.

According to this approach one introduces a chiral supermultiplet in the chiral superfield description as

$$\Phi = \phi(y) + \theta\psi(y) + \theta^2 F(y) \tag{4.5}$$

$$= \phi(x) + i\theta\sigma^\mu\overline{\theta}\,\partial_\mu\phi(x) + \frac{1}{4}\theta^2\overline{\theta}^2\,\partial_\mu\partial^\mu\phi(x)$$

$$+ \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\overline{\theta} + \theta^2 F(x) \tag{4.6}$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$. The θ^a's are Grassmann variables, namely anticommuting parameters with $a = 1, 2$, properly contracted to a self quadratic form or to the Weyl-spinors as $\theta^a\theta_a$ and $\theta^a\psi_a$ respectively.

In this description the superpotential, responsible for the non-gauge interactions in a supersymmetric theory, has an identical form when expressed in terms of superfields. In other words, we may promote all scalar couplings to superfield couplings without changing its structure. Therefore, a general superpotential expressed as

$$W = K^i\Phi_i + \frac{1}{2}M^{ij}\Phi_i\Phi_j + \frac{1}{3}Y^{ijk}\Phi_i\Phi_j\Phi_k \tag{4.7}$$

can have the conventional description of the scalar components for the Φ_i or alternatively these fields may be regarded as superfields. In this latter description, the most general supersymmetric Lagrangian including only chiral superfields has the simple, compact form

$$\mathcal{L} = \Phi^\dagger(x)\Phi(x)\,\Big|_{\theta^2\overline{\theta}^2} + [\ W(y)\Big|_{\theta^2} + c.c.\ ] \tag{4.8}$$

One can generalize this description to include vector (gauge) supermultiplets by introducing the vector superfields and finally construct the most general renormalizable

Lagrangian in terms of superfields only. A general vector superfield is given by

$$V = -\theta\sigma^{\mu}\bar{\theta}A_{\mu} + i\theta^2\bar{\theta}\,\bar{\lambda} - i\bar{\theta}^{\,2}\theta\lambda + \frac{1}{2}\bar{\theta}^{\,2}\theta^2 D \tag{4.9}$$

In this context, R-symmetries have a rather simple interpretation. They are ordinary, abelian symmetries, continuous or discrete, imposed on the superfields, instead of the supermultiplets. Under these symmetries, both component fields and Grassmann variables transform accordingly so as for the respective superfields to carry definite charges of this abelian symmetry.

For a $\mathbb{Z}_N^R$ symmetry we may take the Grassmann variables carrying unit charge, namely transforming as $\theta \to e^{i\frac{2\pi k}{N}}\theta$. Then for a chiral superfield transforming as

$$\Phi \to e^{iq_{\Phi}\frac{2\pi k}{N}}\Phi \tag{4.10}$$

the corresponding component fields will transform accordingly. They will carry the $\mathbb{Z}_N^R$ charges $q_\phi = q_\Phi$, $q_\psi = q_\Phi - 1$, $q_F = q_\Phi - 2$ which can be understood from (4.5). For the Lagrangian to be invariant under this symmetry the superpotential should transform as

$$W = e^{i2\frac{2\pi k}{N}}W \;\to\; q_W = 2 \tag{4.11}$$

as implied from (4.8). It should be remarked that this charge assignment for the superpotential is convention dependent and would be different for another charge assignment of the θ variable. However independent of the convention is the fact that all allowed terms in the superpotential will share the same overall charge q_W.

4.1.3 Anomalies

A major issue of quantum gauge theories is the possible presence of gauge anomalies for fermions. These, unavoidably arise within a general Lagrangian when the fermion representations are not suitably chosen to satisfy a set of restrictive, *anomaly cancellation* conditions. If this is the case, certain symmetry relations, known as *Ward identities*, and vital for the renormalizability of the theory within the context of perturbation theory, fail to reproduce themselves. Or equivalently, the symmetries that may appear at tree (classical) level in the theory will be necessarily violated by the radiative corrections.

A rather compact description on this issue is given by Fujikawa's approach [66]. This relies on the observation that the path integral measure for fermions is not invariant under a general gauge transformation $\mathcal{G}$, but instead transforms as

$$[d\Psi][d\overline{\Psi}] \to e^{i\int d^4x A(x;\theta)}[d\Psi][d\overline{\Psi}] \tag{4.12}$$

where Ψ stands for the fermions of the theory, and θ is the group parameter. Then, one identifies the $A(x;\theta)$ as the anomaly function. In case this is trivial the anomaly will be certainly absent. This function is given by the sum of the gauge and the gravity parts which can be expressed in the general form

$$A_{gauge} = \frac{1}{32\pi^2}\,\mathrm{Tr}(\boldsymbol{\theta}\boldsymbol{F}\tilde{\boldsymbol{F}}) \tag{4.13}$$

$$A_{grav} = -\frac{1}{384\pi^2}R\tilde{R}\,\mathrm{Tr}(\boldsymbol{\theta}) \tag{4.14}$$

where we have adopted a bold notation for implicit contraction with the gauge generators e.g. $\boldsymbol{\theta} \equiv \theta^a T^a$. The field strength tensor and its dual (antisymmetric) form are represented as F, $\tilde{F}$ respectively while $R, \tilde{R}$ stand for the Riemann curvature tensors.

The condition for the absence of the anomaly functions imposes severe constraints on the irreducible representations for the fermions of the theory. To see how these arise from (4.13) and (4.14) we may restrict ourselves to the case $\mathcal{G} = G_S \times U(1)$, where G_S will account for a simple non-abelian group factor. A generalization to the case $G_{S_1} \times \ldots U(1) \times \ldots$ is always straightforward.

For the considered case, the gauge part of the anomaly cancellation conditions will arise from (4.13) satisfying $A_{gauge} \propto \mathrm{Tr}(T_a T_b T_c)$. Then, the trace evaluated over all fermion states will produce the cancellation conditions

$$A_{G_S-G_S-G_S} \to \sum_f \mathrm{Tr}\left(T^a_f\{T^b_f, T^c_f\}\right) = 0 \tag{4.15}$$

$$A_{G_S-G_S-U(1)} \to \sum_f Q_f\,\ell(R_f) = 0 \tag{4.16}$$

$$A_{U(1)-U(1)-U(1)} \to \sum_i Q_i^3 = 0 \tag{4.17}$$

where (i) runs over all fermion states, the sum in (f) runs over all irreducible fermion representations and $\ell(R_f)$ is the Dynkin index of the irreducible representation R_f. Since the generators of all simple Lie-groups are necessarily traceless the $U(1)-U(1)-G_S$ cancellation condition is always satisfied. This also implies that only the constraint for $(grav)^2 - U(1)$ is non-trivial, giving

$$A_{\,grav^2-U(1)} \to \sum_i Q_i = 0 \tag{4.18}$$

Anomalies for $\mathbb{Z}_N$ and $\mathbb{Z}_N^R$ symmetries

If one considers that a $\mathbb{Z}_N$ symmetry should be embeddable in an abelian gauge symmetry, as previously argued, then analogous constraints applying on these new charges will unavoidably arise. In analogy with our previous analysis we now consider $\mathcal{G} = G_S \times U(1) \times \mathbb{Z}_N$. We also recall the action of a $\mathbb{Z}_N$ transformation on a fermion state with a q_f charge, given by

$$\psi \to e^{iq_f \frac{2\pi k}{N}} \psi \tag{4.19}$$

The fact that this transformation is invariant under modulo shifts of the $\mathbb{Z}_N$-charges will produce the anomaly cancellation conditions in analogy with (4.16-4.18) as

$$A_{G_S-G_S-\mathbb{Z}_N} \to \sum_f q_f\, \ell(R_f) = 0 \mod \eta \tag{4.20}$$

$$A_{U(1)-U(1)-\mathbb{Z}_N} \to \sum_i q_i Q_i^2 = 0 \mod \eta \tag{4.21}$$

$$A_{\,grav^2-\mathbb{Z}_N} \to \sum_i q_i = 0 \mod \eta \tag{4.22}$$

In the above, we have introduced a parameter $\eta = N,\ N/2$ for odd or even values of N respectively and all abelian charges were considered quantized as in the SM (MSSM) and its unified extensions.

In the case where this abelian discrete symmetry is an R-symmetry the above equations change accordingly. Our previous analysis in §4.1.2 has shown that fermions within chiral superfields charged with q_Φ will carry a charge $q_\psi = q_\phi - 1$. The other fermions of the SUSY theories we consider are the gauginos. They belong to the vector supermultiplets and they will transform with charge $q_\lambda = q_\theta \equiv 1$ due to our conventional normalization. This can be seen from (4.9) for a vector superfield neutral under this $\mathbb{Z}_N^R$ symmetry.

Under these considerations for the R-charge assignments the respective anomaly cancellation equations in this case will have the form

$$A_{G_S-G_S-\mathbb{Z}_N^R} \to \sum_f \ell(R_f)(q_f-1) + \ell(adj) = 0 \tag{4.23}$$

$$A_{U(1)-U(1)-\mathbb{Z}_N^R} \to \sum_i (q_i-1)Q_i^2 = 0 \mod \eta \tag{4.24}$$

$$A_{grav^2-\mathbb{Z}_N^R} \to -21 + \sum \ell(adj) + \#(U(1)) + \sum_i (q_i-1) = 0 \tag{4.25}$$

In the first equation the two terms represent the contribution of the chiral fermions and gauginos respectively. The first term is the straightforward generalization of (4.20) while the second reflects the property that gauginos contributing R-charge $q_\lambda = 1$ transform in the adjoint representation of the gauge group. It should be remarked that the latter was absent in the previous $\mathbb{Z}_N$ case since, there, gauginos necessarily follow the trivial charge of the vector supermultiplets. In the third equation the (-21) originates from the contribution of the gravitino, necessarily present in SUGRA theories[1]. The second and third term represent the contribution of gauginos, equal to the number of gauginos and, thus, always equal to the dimension of the full gauge group $\dim \mathcal{G}$. Finally, the last term represents the contribution of the chiral fermions to the gravitational anomaly in analogy to (4.22).

We should also mention that the expressions for the abelian gauge factors in both $\mathbb{Z}_N, \mathbb{Z}_N^R$ cases can be absorbed to the non-abelian expressions (4.20), (4.23) respectively when the $U(1)$ generators belong to simple gauge groups. This will turn out particularly useful in our following analysis on the phenomenology of the MSSM and its extensions within the framework of anomaly free discrete R-symmetries [67]. As we will also discuss there, anomaly cancellation does not necessarily imply the absence of the anomaly functions. Other mechanisms can also operate which may eventually render the theory free from anomalies.

4.2 Discrete R-Symmetries within extensions of the MSSM

Supersymmetric unified models, such as MSSM or extensions of it, have been proposed mainly as a framework that eliminates, at least, the technical aspects of the hierarchy problem. Apart from that and various other attractive ingredients, such as the unifica-

[1]In $\mathbb{Z}_N$ symmetries such a term is absent since the graviton superfield, where the gravitino belongs, is neutral.

tion of gauge couplings and the, by default, existence of dark matter candidates, they have a number of problems of their own. These are the so-called μ-problem and the need to eliminate dangerous $D \leq 5$ Baryon/Lepton violating operators. *Discrete symmetries* have been introduced as a means to control these potential problems [68–72]. In addition to that, discrete symmetries, and in particular *discrete R-symmetries* might play a role with respect to the scale of supersymmetry breaking, parametrized by the gravitino mass, since this can be controlled by existing discrete R-symmetries of the superpotential. Therefore, if supersymmetry is going to be interesting for low energy physics, it should be important to investigate the role of such discrete R-symmetries. Such a discrete symmetry has been shown to avoid the μ-problem and protect the electroweak scale in singlet extensions of MSSM [73]. Nevertheless, there exist convincing arguments that a discrete symmetry should be embeddable in an anomaly-free gauge symmetry, otherwise quantum gravitational effects would violate it [74, 75]. Thus, an anomaly-free discrete R-symmetry should be employed. Cancellation of anomalies may be direct [76] or proceed through the *Green-Schwarz mechanism (GS)*. Such a discrete R-symmetry, commuting with $SO(10)$, was shown to avoid the μ-problem in MSSM and proton decay through $D = 5$ operators [77] [78].

In our following analysis we extend the investigation of possible anomaly-free discrete R-symmetries that avoid the μ-problem and the dangerous $D \leq 5$ operators by considering charge assignments that do not commute with the traditional grand unifying simple groups, such as $SU(5)$ or $SO(10)$, but commute instead with the so-called flipped-$SU(5)$, with or without the operation of the GS mechanism. We stay within the framework of $MSSM$, discussing briefly possible singlet or multidoublet extensions. We find that, in the anomaly free case, the symmetries $\mathcal{Z}_3^{(R)}$, $\mathcal{Z}_6^{(R)}$ with flipped assignments are possible and present sets of phenomenologically acceptable charges. In the case of GS-anomaly cancellation, we recover $\mathcal{Z}_N^{(R)}$ symmetries with flipped assignments for $N = 3, 4, 6, 8, 12, 24$ and present sets of phenomenologically acceptable charges. In addition, we discuss a singlet extension of $MSSM$ with $\mathcal{Z}_5^{(R)}$ symmetry as an example of non-unified charge assignments. Finally, we confront the issue of constructing a $4D$ grand unified theory endowed with such symmetry and arrive at an extended flipped $SU(5) \times U(1)_X$ model endowed with an anomaly free (through GS) symmetry $\mathcal{Z}_N^{(R)}$ for $N = 7$ and $N \geq 9$. Sets of phenomenologically allowed charge assignments are listed.

Next, we proceed to review briefly the general framework. Consider an R-symmetry $\mathcal{Z}_N$, under which chiral superfields transform as $\Phi \to e^{i\frac{2\pi q}{N}}\,\Phi$, where the integer q is the *charge* of Φ under it. We are free to choose the superpotential charge to have a specific value and we take $q_{\mathcal{W}} = 2$. The *anomaly coefficients* corresponding to $(gauge)^2\mathcal{Z}_N$ and $(grav)^2\mathcal{Z}_N$ anomalies, denoted respectively by $\mathcal{A}$ and $\mathcal{A}_0$, are

$$\begin{aligned} \mathcal{A} &= \sum_f \ell_f(q_f - 1) \, + \, \ell(adj) \\ \mathcal{A}_0 &= -21 + \sum \ell(adj) + \#(U(1)) + \sum_a (q_a - 1) \end{aligned} \tag{4.26}$$

where, in $\mathcal{A}$, ℓ_f is the Dynkin index of the fermion representation f with $\mathcal{Z}_N$ charge q_f, $\ell(adj)$ is the contribution of the gauginos and, in $\mathcal{A}_0$, -21 is the contribution of the gravitino, $\sum \ell(adj) \, + \, \#(U(1))$ is the contribution of the gauginos and q_a sums over all remaining fermion charges. The $\mathcal{Z}_N^3$ anomaly coefficient is not considered, since the corresponding condition can be interpreted as an *embedding condition*. In the case of the MSSM, we have

$$\begin{aligned} \mathcal{A}_3 &= 3 + \frac{N_f}{2}\,(2q_Q + q_{d^c} + q_{u^c} - 4) \\ \mathcal{A}_2 &= 2 + \frac{N_f}{2}\,(\,3q_Q + q_\ell - 4) + \frac{1}{2}(q_{h_u} + q_{h_d} - 2) \\ \mathcal{A}_1 &= \frac{N_f}{10}\,(\,q_Q + 2q_{d^c} + 8q_{u^c} + 3q_\ell + 6q_{e^c} \, - 20) + \frac{3}{10}(q_{h_u} + q_{h_d} - 2) \\ \mathcal{A}_0 &= -9 + (-1) + N_f\,(6q_Q + 3q_{u^c} + 3q_{d^c} + 2q_\ell + q_{e^c} + q_{N^c} - 16) \\ &\quad + 2(q_{h_u} + q_{h_d} \, - 2) \, + \sum_s (q_s - 1) \end{aligned} \tag{4.27}$$

In the gravitational coefficient we have included the contribution of the *right-handed neutrino* $N_f(q_{N^c} - 1)$ and the contribution of the *dilatino/axino* (-1). We have also left room for the contribution of an unspecified number of singlets with charges q_s.

For $\mathcal{Z}_N$-charges commuting with $SU(5)$, the coefficients are

$$\begin{aligned} \mathcal{A}_3 &= 3 + \frac{N_f}{2}\,(3q_{10} + q_{\overline{5}} - 4) \\ \mathcal{A}_2 &= 2 + \frac{N_f}{2}\,(3q_{10} + \, q_{\overline{5}} - 4) + \frac{1}{2}(q_{h_u} + q_{h_d} - 2) \\ \mathcal{A}_1 &= \frac{N_f}{2}\,(3q_{10} + q_{\overline{5}} - 4) + \frac{3}{10}(q_{h_u} + q_{h_d} - 2) \\ \mathcal{A}_0 &= -9 + (-1) + N_f\,(10q_{10} + 5q_{\overline{5}} + q_1 - 16) \\ &\quad + 2(q_{h_u} + q_{h_d} - 2) + \sum_s (q_s - 1) \end{aligned} \tag{4.28}$$

4.2.1 Non-anomalous Discrete R-Symmetries

One possibility to obtain an anomaly-free theory in the MSSM is to impose

$$\mathcal{A}_3 = \mathcal{A}_2 = \mathcal{A}_1 = 0 \ mod(\eta)\,, \tag{4.29}$$

where the condition on $\mathcal{A}_1$ is required to be met by at least on shifted set of charges. The parameter η is $\eta = N/2$ for $N = 2k$ and $\eta = N$ for $N = 2k+1$. These are considered independently of the gravitational coefficient which, in general, depends to an unknown set of gauge singlets.

The standard case. The conditions, in the case of $SU(5)$-invariant charges (4.28), are equivalent to

$$\begin{aligned} \mathcal{A}_3 &= 0 \ mod(\eta) \\ \mathcal{A}_2 - \mathcal{A}_3 &= \frac{1}{2}(q_{h_u} + q_{h_d} - 4) = 0 \ mod(\eta) \\ \mathcal{A}_1 - \mathcal{A}_3 &= \frac{3}{10}(q_{h_u} + q_{h_d} - 4) - \frac{12}{5} = 0 \ mod(\eta) \end{aligned} \tag{4.30}$$

Imposing as a constraint the existence of the Yukawa couplings necessary for the fermion masses[2],

$$2q_{10} + q_{h_u} = q_{10} + q_{\overline{5}} + q_{h_d} = q_{\overline{5}} + q_1 + q_{h_u} = 2 \ mod(N), \tag{4.31}$$

the conditions (4.30) are equivalent to

$$\begin{aligned} 3 - 2N_f &= 0 \ mod(\eta) \\ (q_{h_u} + q_{h_d} - 4) &= 0 \ mod(N) \\ 12 &= 0 \ mod(\eta) \end{aligned} \tag{4.32}$$

For $N_f = 3$, this restricts the possible $\mathcal{Z}_N$ R-symmetries to

$$N = 3, 6\,. \tag{4.33}$$

[2]These can be *"solved"* $mod(N/2)$ as

$$q_{10} = 1 - q_{h_u}/2,\ \ q_{\overline{5}} = 1 + q_{h_u}/2 - q_{h_d},\ \ q_1 = 1 - 3q_{h_u}/2 + q_{h_d}\,.$$

Any allowed sets of charges should obey these relations.

q_{h_u}	q_{h_d}	q_{10}	$q_{\bar{5}}$	q_1	$\tilde{q}_{h_u}$	$\tilde{q}_{h_d}$	$(N^c)^2$
0	4	1	3	5	12	10	$-$
0	4	4	0	2	12	10	$-$
4	0	5	3	1	10	12	$+$
2	2	3	3	3	14	8	$-$

Table 4.1: Allowed $SU(5)$-Invariant charges for $\mathcal{Z}_6^{(R)}$

Note that the μ-term is absent[3], since $q_u + q_d = 4 \neq 2 \ mod(N)$. Dangerous $D = 5$ Baryon/Lepton violating operators are also absent, since $QQQ\ell$, $u^c u^c e^c d^c \rightarrow 3q_{10} + q_{\bar{5}} = 4 - q_{h_u} - q_{h_d} = 0 \ mod(N)$ and $u^c d^c d^c N^c \rightarrow 2q_{\bar{5}} + q_{10} + q_1 = 4 - q_{h_u} - q_{h_d} = 0 \ mod(N)$. Nevertheless, the potentially dangerous $D = 3, 4$ operators transform as $\ell\, h_u \rightarrow q_{\bar{5}} + q_{h_u}$, $Q\ell\, d^c$, $d^c d^c u^c \rightarrow q_{10} + 2q_{\bar{5}}$ and a choice of charge assignments is required.

In the $N = 3$ case, since $q_{h_u} + q_{h_d} = 1 \ mod(3)$, for $q_{h_u} = 0$ and $q_{h_d} = 1$, we can have $q_{10} = 1$, $q_{\bar{5}} = 0$ and $q_1 = 2$. Then, the $D = 3, 4$ matter-parity violating operators cannot be present. Note however that the right-handed neutrino Majorana mass is not possible. The vanishing of the coefficient $\mathcal{A}_1$ can be met for the shifted charges $\tilde{q}_{h_u} = 12$, $\tilde{q}_{h_d} = 10$, $\tilde{q}_{10} = 4$.

In the $N = 6$ case, we have $q_{h_u} + q_{h_d} = 4 \ mod(6)$, which can be met only with even charges due to the mass relations (4.31). The choice $q_{h_d} = q_{h_u} = 2$ is $SO(10)$ invariant. The phenomenologically acceptable charges, for which the above $D = 3, 4$ operators are absent, are listed in the relevant table (Note that in Tab.4.1 $\tilde{q}_a$ stands for shifted charges and the last entry refers to the right-handed neutrino Majorana mass operator.)

The flipped case. In the above search for a non-anomalous $\mathcal{Z}_N$ R-symmetry it has been assumed that it commutes with $SU(5)$ [76]. This need not be necessarily the case. Let's consider the following *"flipped"* assignments

$$q_Q = q_{d^c} = q_{N^c} = q_{10}, \ \ q_\ell = q_{u^c} = q_{\bar{5}}, \ \ q_{e^c} = q_1 \,, \tag{4.34}$$

which commute with the so-called *"flipped"* $SU(5) \times U(1)$. The corresponding gauge anomaly coefficients are identical to those of (4.28) with the exception of

$$\mathcal{A}_1 = \frac{N_f}{10}\left(3q_{10} + 11q_{\bar{5}} + 6q_1 - 20\right) + \frac{3}{10}(q_{h_u} + q_{h_d} - 2)\,. \tag{4.35}$$

[3]Here, as in all subsequent cases, it is assumed that a μ-term of the correct order of magnitude is generated through non-perturbative effects [79] or through another indirect mechanism [80].

Note however that, when we impose the mass relations

$$2q_{10} + q_{h_d} \;=\; q_{10} + q_{\overline{5}} + q_{h_u} \;=\; q_{\overline{5}} + q_1 + q_{h_d} \;=\; 2 \;\; mod(N)\,, \tag{4.36}$$

we can prove[4]

$$3q_{10} \;+\; 11q_{\overline{5}} \;+\; 6q_1 \;=\; 5\,(3q_{10} \;+\; q_{\overline{5}})\;,$$

leading to

$$\mathcal{A}_1 \;=\; \frac{N_f}{2}\,(\,3q_{10} + q_{\overline{5}} - 4)\;+\;\frac{3}{10}(q_{h_u} + q_{h_d} - 2)\,, \tag{4.37}$$

which is identical to the corresponding expression in (4.28). Thus, in both cases, of straight and flipped charge assignments, all gauge anomaly coefficients are identical, provided we invoke the conditions for the existence of mass Yukawa couplings. Nevertheless, it should be reminded that the anomaly cancellation condition on $\mathcal{A}_1$ rests on the existence of at least one set of shifted charges that satisfy the corresponding condition. Note also, even in the case of non-unified assignments that will follow, we assume that the $U(1)_Y$ hypercharge factor is normalized in the standard E_6 fashion.

The allowed cases for a non-anomalous discrete R-symmetry are still $N \;=\; 3, 6$. Nevertheless, the resulting models need not be the same. The μ-term is still absent, since $q_{h_u} \;+\; q_{h_d} \;=\; 4 \;\neq\; 2 \;\; mod(N)$. Similarly, for the $D = 5$ Baryon/Lepton violating operators, we have $QQQ\ell$, $d^cd^cN^cu^c \;\rightarrow\; 3q_{10} + q_{\overline{5}} \;=\; 4 - (q_{h_u} + q_{h_d}) \;=\; 0 \;\neq\; 2 \;\; mod(N)$ and $u^cu^cd^ce^c \;\rightarrow\; 2q_{\overline{5}} + q_{10} + q_1 \;=\; 4 \;-\; (q_{h_u} + q_{h_d}) \;=\; 0 \;\neq\; 2 \;\; mod(N)$. Thus, these operators are absent. In order to conclude whether the $D = 3, 4$ dangerous operators are allowed, we must proceed further with the charge assignments for each value of N.

In the $N = 3$ case, we can take $q_{h_d} = 1$, $q_{h_u} = 0$. Then, we can have $q_{10} = 2$, $q_{\overline{5}} \;=\; 0$, $\;q_1 \;=\; 1$. The $D = 3$ operator $\ell h_u \rightarrow\; q_{\overline{5}} + q_{h_u} \;=\; 0$ cannot be present. The $D = 4$ operators $Q\ell d^c$, $d^cd^cu^c \;\rightarrow\; 2q_{10} + q_{\overline{5}} \;=\; 1$ cannot be present either. Thus, the $\mathcal{Z}_3$ R-symmetry in this case of flipped assignments is also phenomenologically feasible. The $N = 6$ case proceeds in an analogous fashion. Taking $q_{h_d} = 4$, $q_{h_u} = 0$, we are led to $q_{10} = 5$, $q_{\overline{5}} = 3$, $q_1 = 1$, which disallows ℓh_u and $Q\ell d^c$, $d^cd^cu^c$. Similarly, for the rest of the flipped charge assignments shown in Tab.4.2. Shifted charges are denoted as $\tilde{q}$.

MSSM singlet extensions. If the model is extended beyond the $MSSM$ by the introduction of a singlet S coupled to the Higgses through a term Sh_uh_d, the required

[4] These are now *"solved"* by $q_{10} \;=\; 1 - q_{h_d}/2, q_{\overline{5}} \;=\; 1 + q_{h_d}/2 - q_{h_u}$ and $q_1 \;=\; 1 - 3q_{h_d}/2 + q_{h_u}$.

$\mathcal{Z}_N$	q_Q	q_{d^c}	q_{N^c}	q_ℓ	q_{u^c}	q_{e^c}	q_{h_u}	q_{h_d}	$\tilde{q}_{e^c}$	$\tilde{q}_{h_u}$	$\tilde{q}_{h_d}$
$N=3$	2	2	2	0	0	1	0	1	4	12	10
$N=6$	5	5	5	3	3	1	0	4	7	12	10
$N=6$	2	2	2	0	0	4	0	4	4	12	10
$N=6$	1	1	1	3	3	5	4	0	-1	10	12
$N=6$	3	3	3	3	3	3	2	2	3	14	8

Table 4.2: Allowed Flipped Charges for $\mathcal{Z}_3^{(R)}$ and $\mathcal{Z}_6^{(R)}$

charge of the new field has to satisfy

$$q_S = 2 - q_{h_u} - q_{h_d} = -2 \;\; mod(N)$$

and thus $q_S = 1$ for $\mathcal{Z}_3$ or $q_S = 4$ for $\mathcal{Z}_6$. Nevertheless, in both cases the term S^2 is allowed, corresponding to a large mass for the singlet and making this extension phenomenologically uninteresting.

$\tilde{\mathbf{N}} > 1$ pairs of isodoublets. Another possible extension of the MSSM to be considered is the case of extra Higgs isodoublets with identical charges. The anomaly coefficients for $\tilde{N} \geq 1$ pairs of isodoublets take the form

$$\begin{aligned}
\mathcal{A}_3 &= 3 + \frac{N_f}{2}\left(3q_{10} + q_{\overline{5}} - 4\right) \\
\mathcal{A}_2 &= 2 + \frac{N_f}{2}\left(3q_{10} + q_{\overline{5}} - 4\right) + \frac{\tilde{N}}{2}(q_{h_u} + q_{h_d} - 2) \\
\mathcal{A}_1 &= \frac{N_f}{2}\left(3q_{10} + q_{\overline{5}} - 4\right) + \frac{3\tilde{N}}{10}(q_{h_u} + q_{h_d} - 2)
\end{aligned} \tag{4.38}$$

The corresponding conditions $\mathcal{A}_i = 0 \;\; mod(\eta)$ in the case $N_f = 3$, after we enforce the, common in flipped or $SU(5)$ assignments, mass relation $3q_{10} + q_{\overline{5}} - 4 = -(q_{h_u} + q_{h_d}) \;\; mod(N)$, amount to

$$\begin{aligned}
\mathcal{A}_3 \to & \quad 3(q_{h_u} + q_{h_d} - 2) = 0 \; mod(N) \\
\mathcal{A}_2 - \mathcal{A}_3 \to & \quad \tilde{N}(q_{h_u} + q_{h_d} - 2) = 2 \; mod(N) \\
\mathcal{A}_1 - \mathcal{A}_3 \to & \quad \frac{3\tilde{N}}{5}(q_{h_u} + q_{h_d} - 2) = 6 \; mod(N) \\
\to & \quad 24 = 0 \;\; mod(N)
\end{aligned} \tag{4.39}$$

From the first two equations we obtain in the familiar case $\tilde{N} = 1$ the standard non-trivial solutions $N = 3, 6$ with $q_{h_u} + q_{h_d} = 4 \; mod(N)$ discussed previously. In an analogous manner, for $\tilde{N} = 2$ we again obtain $N = 3, 6$ for the allowed symmetries but with a different relation for the Higgs charges, namely $q_{h_u} + q_{h_d} = 0 \ mod(N)$. In contrast, the $\tilde{N} = 3$ case has no solution for non-trivial symmetries ($N \geq 3$).

4.2.2 Discrete R-Symmetries with Anomaly Cancellation through the GS Mechanism

Another possibility for the realization of discrete R-symmetries is when gauge and gravitational anomalies are cancelled through the operation of the *Green-Schwarz mechanism.* The corresponding conditions on the anomaly coefficients (4.28) read

$$\begin{aligned} \mathcal{A}_3 = \mathcal{A}_2 = \mathcal{A}_1 = \rho \ mod(\eta) \\ \mathcal{A}_0 = 24\rho \ mod(\eta) \end{aligned} \tag{4.40}$$

From the explicit expressions of the coefficients (4.28) for the $MSSM$ content, enforcing the mass conditions, we obtain[5] for $\mathcal{Z}_N$ charges commuting with $SU(5)$ or flipped.

$$\begin{aligned} \mathcal{A}_3 &= \rho & &\rightarrow & \rho &= -3 \ mod(\eta) \\ \mathcal{A}_2 - \mathcal{A}_3 &= 0 & &\rightarrow & q_{h_u} + q_{h_d} &= 4 \ mod(N) \\ \mathcal{A}_1 - \mathcal{A}_3 &= 0 & &\rightarrow & 12 &= 0 \ mod(\eta) \\ \mathcal{A}_0 - 24\mathcal{A}_3 &= 0 & &\rightarrow & 18 + \sum_s (q_s - 1) &= 0 \ mod(\eta) \end{aligned} \tag{4.41}$$

Note that the first three of these conditions, for $\rho = 0$ (i.e. $N = 3, 6$), coincide with the conditions (4.30).

The last two conditions can be reconciled without the use of extra singlets in the case of $N = 3, 4, 6, 12$. Nevertheless, they can be compatible for all possible N if we extend $MSSM$ introducing extra singlets. A most straightforward possibility is that of a singlet S coupled to the Higgses as $S\, h_u\, h_d$. Then, its required charge would be $q_S = -2 \ mod(N)$. Two such singlets reconcile the last two conditions[6]. Thus, the allowed discrete R-symmetries would be those corresponding to $N = 3, 4, 6, 8, 12, 24$. Note that in the case $N = 3, 6$ only the quadratic term S^2 is allowed. In the $N = 4$

[5] We have included in $\mathcal{A}_0$ the contribution of the right-handed neutrino and that of the dilatino/axino. Enforcing the mass conditions on it we obtain $\mathcal{A}_0 = -14 - 10(q_{h_u} + q_{h_d}) + \sum_s (q_s - 1)$. The difference with $\mathcal{A}_3 = 3 - \frac{3}{2}(q_{h_u} + q_{h_d})$ is $\mathcal{A}_0 - 24\mathcal{A}_3 = 18 + 26(q_{h_u} + q_{h_d} - 4) + \sum_s (q_s - 1)$.

[6] Of course, such a "solution" is not unique. Six singlets of zero charge could do the job as well. Note also that, in the case of $N = 3, 6$, these conditions are met with an arbitrary number of such singlets, since $\sum_s (q_S - 1) = N_s(-3 \ mod(N))$, which is a subset of $0 \ mod(3)$.

N	q_Q	q_{d^c}	q_{N^c}	q_ℓ	q_{u^c}	q_{e^c}	q_{h_u}	q_{h_d}	$\tilde{q}_{h_u}$	$\tilde{q}_{h_d}$
4	1	1	1	1	1	1	0	0	4	8
4	3	3	3	3	3	3	0	0	4	8

Table 4.3: Allowed Flipped Charges for $\mathcal{Z}_4^{(R)}$

case, only a linear and a cubic term are allowed. Finally, in the $N = 8$ case, only a cubic term is allowed. Thus, in the $N = 8$ case the singlet extension coincided with $NMSSM$, in the $N = 4$ case it is a modified $NMSSM$ with an additional linear term. Finally, the cases $N = 12,\, 24$, where no singlet self-term is present, correspond to what has been termed $nMSSM$ [73].

Allowed $\mathcal{Z}_N$ R-symmetries commuting with $SU(5)$ or $SO(10)$ have been studied [78]. We shall go further in our analysis considering charge assignments that do not commute with these symmetries. More specifically, we shall consider *"flipped"* assignments

$$q_Q = q_{d^c} = q_{N^c} = q_{10},\;\; q_\ell = q_{u^c} = q_{\overline{5}},\;\; q_{e^c} = q_1\,, \tag{4.42}$$

which commute with the so-called *"flipped"* $SU(5) \times U(1)$. The anomaly coefficient conditions considered above are identical for all these cases, provided the mass relations are invoked. Nevertheless, the final charge assignments for matter correspond to distinct models. Turning now to the phenomenological features of the allowed models, we see that the condition on the Higgs charges (4.41) is sufficient to forbid the μ-term. In addition, dangerous $D = 5$ Baryon/Lepton Number violating operators are absent as well, due to (4.36) and (4.41). Indeed, we have

$$QQQ\ell,\; d^cd^cu^cN^c \;\rightarrow\; 3q_{10} + q_{\overline{5}} = 4 - q_{h_u} - q_{h_d} = 0 \neq 2 \;\; mod(N)$$

$$u^cu^cd^ce^c \;\rightarrow\; 2q_{\overline{5}} + q_{10} + q_1 = 4 - q_{h_u} - q_{h_d} = 0 \neq 2 \;\; mod(N)\,.$$

In order to conclude whether dangerous $D = 3,\, 4$ operators are present we need to proceed further with the matter and Higgs charge assignments for each particular value of N.

N = 3, 6. In these cases the first of the conditions (4.41) implies that $\rho = 0$. Thus, for these cases the analysis will coincide with the one carried on previously in anomaly-free case without the operation of the GS mechanism.

N = 4. The condition $q_{h_u} + q_{h_d} = 0 \;\; mod(4)$ can only be satisfied with even charges. These are the choices $q_{h_u} = q_{h_d} = 0$ and $q_{h_u} = q_{h_d} = 2$. The first choice leads to two $SO(10)$-invariant solutions with identical phenomenology. Note that these solutions

N	$\tilde{q}_Q$	$\tilde{q}_{d^c}$	$\tilde{q}_{N^c}$	$\tilde{q}_\ell$	$\tilde{q}_{u^c}$	$\tilde{q}_{e^c}$	$\tilde{q}_{h_u}$	$\tilde{q}_{h_d}$
8,12,24	-1	-1	-1	3	3	-5	24	28
8,12,24	1	1	1	-3	-3	5	28	24

Table 4.4: Allowed (Shifted) Flipped charges for $\mathcal{Z}_8^{(R)}$, $\mathcal{Z}_{12}^{(R)}$, $\mathcal{Z}_{\mathbf{24}}^{(\mathbf{R})}$

$N=5$	q_Q	q_ℓ	q_{d^c}	q_{u^c}	q_{e^c}	q_{N^c}	q_S	q_{h_u}	q_{h_d}
	4	4	1	1	1	1	3	2	2

Table 4.5: $\mathcal{Z}_5^{(R)}$ charges

allow for a right-handed neutrino Majorana mass. All constraints for the absence of $D = 3, 4$ operators are met. The second choice is discarded due to the presence of unwanted $D = 3, 4$ operators. The allowed cases are listed in Tab4.3.

N = 8, 12, 24. The condition $q_{h_u} + q_{h_d} = 4\ mod(N)$ allows again only even charges and can in principle be satisfied for various sets of Higgs charges. Among them two distinct general solutions, with $D = 3, 4$ operators absent, are listed in Tab.4.4, where, of course, these charges can always be modulo-shifted to lie in the first modulo of the given symmetry. The charge $\tilde{q}_{N^c} = 1$ of the second row allows always for a right-handed neutrino Majorana mass term.

A "non-unified" case ($\mathcal{Z}_\mathbf{5}^{(\mathbf{R})}$). The above found set of allowed $\mathcal{Z}_N$ does not exclude other *"non-unified"* possibilities. The anomaly coefficients for non-unified charges are

$$\begin{aligned}
\mathcal{A}_3 &= 3\left(q_Q + \frac{1}{2}q_{d^c} + \frac{1}{2}q_{u^c} - 2\right) + 3 \\
\mathcal{A}_2 &= 3\left(\frac{3}{2}q_Q + \frac{1}{2}q_\ell - 2\right) + \frac{\tilde{N}}{2}\left(q_{h_d} + q_{h_u} - 2\right) + 2 \\
\mathcal{A}_1 &= \frac{9}{5}\left(\frac{1}{6}q_Q + \frac{1}{3}q_{d^c} + \frac{4}{3}q_{u^c} + \frac{1}{2}q_\ell + q_{e^c} - \frac{10}{3}\right) + \frac{3\tilde{N}}{10}(q_{h_d} + q_{h_u} - 2)
\end{aligned} \tag{4.43}$$

where $\tilde{N}$ is the number of Higgs isodoublets. A particularly interesting case is that of a $\mathcal{Z}_5$ discrete R-symmetry, motivated by the solution to the μ-problem in terms of a minimal singlet extension of MSSM. The assigned charges [73] are given in Tab.4.5.

For these charges the anomaly coefficients are

$$\begin{aligned} \rho = \mathcal{A}_3 &= 2\ mod(5) \\ \mathcal{A}_2 &= \tilde{N}\ mod(5) \\ \mathcal{A}_1 &= \frac{24}{5} + \frac{3N_\sigma}{5}(q_\sigma + q_{\overline{\sigma}} - 2)\ mod(5) \end{aligned} \tag{4.44}$$

where we have introduced tentatively N_σ pairs of hypercharge ± 1 singlets. Finally, the condition on $\mathcal{A}_0 = 70 + N_\sigma(q_\sigma + q_{\overline{\sigma}} - 2) + \sum_s (q_s - 1)$ reads $2 + N_\sigma(q_\sigma + q_{\overline{\sigma}} - 2) + \sum_s (q_s - 1) = 0\ mod(5)$. The anomaly conditions corresponding to (4.44) can be met for $\tilde{N} = 2$, $N_\sigma = 1$ and the shifted charges[7] $q_\sigma = q_{\overline{\sigma}} = 7$. The gravitational anomaly condition can also be met if, apart from the charge-3 singlet S, we introduce also an additional neutral singlet. Note that for these charges all possible dangerous terms are disallowed ($\sigma\overline{\sigma}$, $e^c\sigma$, $\ell\overline{\sigma}h_d$, $h_d d_d \overline{\sigma}$, $h_u h_u \sigma$). Note also that a term $\sigma\,\overline{\sigma}\,S$ is allowed. Apart from being anomaly-free, this model allows for all standard terms, including neutrino Majorana masses, and forbids all unwanted $D \leq 5$ terms. The extra charged singlets introduced here for the sake of anomaly cancellation can obtain a mass through the vev of the singlet S.

4.2.3 Unification

Can these discrete symmetries be incorporated in a $4D$ grand unified theory? There are convincing arguments that, for a simple gauge group and $MSSM$ particle content at low energies, this is not possible [81]. We shall depart from both of these assumptions and allow on the one hand, additional matter at low energies, beyond $MSSM$, and on the other hand go beyond simple unifying groups. Specifically, we shall consider $SU(5) \times U(1)$ and use *"flipped"* $\mathcal{Z}_N$-charge assignments commuting with it. One of the motivations for this model is that it is accompanied by an elegant mechanism for *triplet-doublet splitting*. Note that this is one of the main problems that one has to face in promoting $MSSM$ to a GUT endowed with the discussed discrete R-symmetries.

Minimal Case. The standard matter content of $SU(5) \times U(1)$ comes in three copies of $F(10,\,1)$, $f^c(\overline{5},\,-3)$, $\ell^c(1,\,5)$ with corresponding $\mathcal{Z}_N$ charges q_{10}, $q_{\overline{5}}$, q_1. The standard Higgs content is $h(5,\,-2;\,q_h)$, $h^c(\overline{5},\,2;\,q_{h^c})$, $H(10,\,1;\,0)$, $\overline{H}(\overline{10},\,-1;\,0)$. The last pair, through its non-zero vev will achieve the breaking down to $MSSM$. Obviously, it has to be neutral under the discrete symmetry. The standard matter couplings

[7] Another working choice is $N_\sigma = 2$ and $q_\sigma = q_{\overline{\sigma}} = 4$.

are

$$F\,F\,h\,+\,F\,f^c\,h^c\,+\,f^c\,\ell^c\,h^c\,. \tag{4.45}$$

Nevertheless, in order to realize triplet-doublet splitting, the couplings $H\,H\,h + \overline{H}\,\overline{H}\,h^c$ are necessary. These couplings force the Higgs charges

$$q_h\,=\,q_{h^c}\,=\,2\;\;mod(N)\,. \tag{4.46}$$

Note that these charges do not allow a μ-term. Nevertheless, enforcing the mass conditions

$$2\,q_{10}\,+\,q_h\,=\,q_{10}\,+\,q_{\overline{5}}\,+\,q_{h^c}\,=\,q_{\overline{5}}\,+\,q_1\,+\,q_h\,=\,2\;mod(N)\,, \tag{4.47}$$

we are led to the $SO(10)$-invariant assignments

$$q_{10}\,=\,q_{\overline{5}}\,=\,q_1\,=\,\frac{N}{2}\;\;mod(N)\qquad(N\,>\,4)\,. \tag{4.48}$$

Note that, since the charges have to be integers, only even values of N are allowed. Note also that the case $N=4$, in which all matter fields have charge 2 $mod(4)$, is excluded since it allows the operator $F\overline{H}$. The anomaly coefficient conditions for general even $N\,>\,4$ are

$$\begin{aligned}\mathcal{A}_5 &=\,3N-3\,=\,\rho\;\;mod(N/2)\\ \mathcal{A}_X &=\,3N\,-3\,-\frac{5}{2}\,=\,\rho\;\;mod(N/2)\end{aligned} \tag{4.49}$$

and anomaly cancellation cannot be met without introducing extra matter. It can also be checked that anomaly cancellation cannot be saved by shifted charges.

Extra Fields. In what follows we shall introduce extra matter and at the same time avoid the $SO(10)$-symmetric assignment enforced by (4.46) in order to realize the split assignments found in the previous section. In order to avoid (4.46), we introduce an extra pair of Higgs fields $H'(10,1;\,q_{H'})$, $\overline{H}'(\overline{10},-1;\,q_{\overline{H}'})$. Assuming that the only non-zero vevs are those[8] of $\langle H\rangle=\langle\overline{H}\rangle$ and assuming that the couplings $H\,H'\,h$, $\overline{H}\,\overline{H}'\,h^c$ are present, after symmetry breaking, the pairs $d_h(3,\,1,\,-1/3;\,q_h)$, $d^c_{H'}(\overline{3},\,1,\,1/3;\,q_{H'})$ and $d^c_{h^c}(\overline{3},\,1,\,1/3;\,q_{h^c})$, $d_{\overline{H}'}(3,\,1,\,-1/3;\,q_{\overline{H}'})$ will obtain a large mass and will be removed from the spectrum. In addition, the fields Q_H, $Q^c_{\overline{H}}$, $N^c_H-N^c_{\overline{H}}$ will be *"higgsed away"*. The surviving fields, apart from the standard matter and a neutral Higgs singlet, will

[8] Thus, assuming a $\mathcal{Z}_N^{(R)}$-invariant vacuum.

be a *"hybrid"* pair $10' + \overline{10}'$ composed out of the $H'(\overline{H}')$ with their $d^c_{H'}(d_{\overline{H}'})$ replaced by $d^c_H(d_{\overline{H}})$. The conditions on the charges resulting from the existence of the standard couplings

$$F\,F\,h \,+\, F\,f^c\,h^c \,+\, f^c\,\ell^c\,h^c \,+\, H\,H'\,h \,+\, \overline{H}\,\overline{H}'\,h^c \tag{4.50}$$

are given by (4.47) and

$$q_{H'} \,+\, q_h \,=\, q_{\overline{H}'} \,+\, q_{h^c} \,=\, 2\ mod(N)\,. \tag{4.51}$$

Nevertheless, a number of unwanted terms are still allowed by gauge symmetry, namely, the terms

$$\begin{aligned} &h\,h^c \,+\, H'\,\overline{H} \,+\, \overline{H}'\,H \,+\, \overline{H}'\,H' \,+\, F\,\overline{H} \,+\, F\,\overline{H}' \\ &+\, H^2\,h \,+\, \overline{H}^2\,h^c \,+\, F\,H\,h \,+\, F\,H'\,h \,+\, H\,f^c\,h^c \,+\, H'\,f^c\,h^c\,, \end{aligned} \tag{4.52}$$

which are expected to be removed by the discrete symmetry.

Next let's consider the anomaly coefficients. The gauge anomaly coefficients are

$$\begin{aligned} \mathcal{A}_5 &= 5 + 3\left(\frac{3}{2}q_{10} + \frac{1}{2}q_{\overline{5}} - 2\right) \,+\, \frac{3}{2}\left(q_{H'} + q_{\overline{H}'} \,-4\right) \,+\, \frac{1}{2}\left(q_h + q_{h^c} \,-2\right) \\ \mathcal{A}_X &= \frac{3}{40}\left(10q_{10} + 45q_{\overline{5}} + 25q_1 - 80\right) + \frac{1}{4}\left(q_{H'} + q_{\overline{H}'} - 4\right) + \frac{1}{2}(q_h + q_{h^c} - 2) \end{aligned} \tag{4.53}$$

or, applying the conditions (4.47), (4.51),

$$\begin{aligned} \mathcal{A}_5 &= 4 \,-\, \frac{5}{2}(q_h + q_{h^c}) \,=\, \rho\ \ mod(\eta) \\ \mathcal{A}_X &= -1 \,-\, \frac{5}{4}(q_h + q_{h^c}) \,=\, \rho\ mod(\eta) \end{aligned} \tag{4.54}$$

Taking the familiar condition

$$q_h + q_{h^c} \,=\, 4\ \ mod(N)\,, \tag{4.55}$$

we obtain

$$\mathcal{A}_5 \,=\, \mathcal{A}_X \,=\, -6 \tag{4.56}$$

which satisfies the anomaly condition for any N.

The gravitational coefficient is[9] $\mathcal{A}_0 \;=\; -\,7 \,-\, 17(q_h + q_{h^c}) \,+\, \sum_s(q_s - 1)$ and the

[9]The contribution of a dilatino/axino is also considered.

corresponding condition becomes

$$\mathcal{A}_0 - 24\mathcal{A}_5 = 69 + \sum_s (q_s - 1) = 0 \ mod(\eta) \tag{4.57}$$

and can be satisfied, among other choices, for any N with a single neutral singlet of shifted charge $\tilde{q}_s = -68$. For all symmetries which are divisors of 138 this extra singlet is not required.

Note that the $D = 3,4$ Baryon/Lepton violating operators of the standard field content $\ell\, h_u$, $Q\ell d^c$, $u^c d^c d^c$, although not directly present due to the gauge symmetry, may appear through higher dimensional operators[10]. Thus, their absence will be eventually determined by their respective $\mathcal{Z}_N$ charges. On the other hand, the analogous $D = 5$ operators $FFFf^c$, $f^c f^c F \ell^c$ are always absent since they have charges $3q_{10} + q_{\overline{5}}$ and $2q_{\overline{5}} + q_{10} + q_1$, both equal to $4 - q_{h^c} - q_h = 0 \ mod(N)$, due to (4.55).

Taking (4.55) as our starting point, we proceed to consider phenomenologically allowed charge assignments. For even N, a general assignment that satisfies (4.55) and is compatible with (4.47), (4.51) of *shifted charges* is

$$\begin{aligned} &q_h = -2,\ q_{h^c} = 6,\ q_{H'} = 4,\ q_{\overline{H}'} = -4 \\ &q_{10} = 2 + N/2,\ q_{\overline{5}} = -6 + N/2,\ q_1 = 10 + N/2 \end{aligned} \tag{4.58}$$

The anomaly constraints are readily satisfied[11] $\mathcal{A}_5 = \mathcal{A}_X = 3N - 6$. Next, we demand that the dangerous terms (4.52) are absent. Their charges are

$$\begin{aligned} &hh^c \to 4,\ H'\overline{H} \to 4,\ \overline{H}'H \to -4,\ \overline{H}'H' \to 0 \\ &F\overline{H} \to 2 + N/2,\ F\overline{H}' \to -2 + N/2,\ H^2 h \to -2,\ \overline{H}^2 h^c \to 6 \\ &FHh \to N/2,\ FH'h \to 4 + N/2,\ Hf^c h^c \to N/2,\ H'f^c h^c \to 4 + N/2 \end{aligned} \tag{4.59}$$

and they should be $\neq 2 \ mod(N)$. The first two rows lead to the constraints $N \neq 4, 6, 8$, while the third does not give any additional constraint. Therefore, the allowed even values of N are

$$N = 2k \geq 10\,. \tag{4.60}$$

Note that, since F^2 has charge $4 + N$, no right-handed neutrino Majorana mass is

[10] $f^c h^c$, FFf^c are not gauge singlets but $Hf^c h^c$, $HFFf^c$ are.

[11] Note that the anomaly coefficients for the low energy spectrum with the new extra matter satisfy also $\mathcal{A}_3 = \mathcal{A}_2 = \mathcal{A}_1 = 3N - 6$.

N	q_{10}	$q_{\overline{5}}$	q_1	q_{h^c}	q_h	$q_{H'}$	$q_{\overline{H}'}$
10	7	9	5	6	8	4	6
12	8	0	4	6	10	4	8

Table 4.6: Allowed $\mathcal{Z}_{10}^{(R)}$, $\mathcal{Z}_{12}^{(R)}$ charges for $SU(5)\times U(1)_X$.

allowed. Furthermore the dangerous $D=3,\,4$ operators are absent since

$$\ell h_u \to N/2\,, \quad Q\ell d^c,\, d^c d^c u^c \to N/2-2+N \tag{4.61}$$

which are $\neq 2\ \ mod(N)$ for the allowed symmetries. The corresponding charges for the cases $N=10,\,12$ are shown in Tab4.6.

Next, starting again from (4.55), we proceed to investigate possible odd values of N and consider a general assignment of *shifted* charges compatible with (4.47), (4.51)

$$\begin{aligned} &q_h \;=\; 1,\, q_{h^c} \;=\; 3,\, q_{H'} \;=\; 1,\, q_{\overline{H}'} \;=\; -1 \\ &q_{10} \;=\; (N+1)/2,\, q_{\overline{5}} \;=\; (N-3)/2,\, q_1 \;=\; (N+5)/2 \end{aligned} \tag{4.62}$$

The anomaly constraints are readily satisfied[12] $\mathcal{A}_5 \;=\; \mathcal{A}_X \;=\; 3N\,-\,6\,.$ Again, we demand that the dangerous terms (4.52) are absent. Their charges are

$$\begin{aligned} &hh^c \to 4,\, H'\overline{H} \to 1,\, \overline{H}'H \to -1,\, \overline{H}'H' \to 0 \\ &F\overline{H} \to (N+1)/2,\, F\overline{H}' \to (N-1)/2,\, H^2h \to\, 1,\, \overline{H}^2h^c \to\, 3 \\ &FHh \to\, (N+3)/2,\, FH'h \to (N+5)/2,\, Hf^ch^c \to\, (N+3)/2, \\ &H'f^ch^c \to\, (N+5)/2 \end{aligned} \tag{4.63}$$

and they should be $\neq\, 2\ \ mod(N)$. The first two rows lead to the constraints $N\,\neq\, 3,\, 5$, while the rest do not supply us with any additional restriction. Thus, the allowed odd values of N are

$$N \;=\; 2k+1\,\geq\, 7\,. \tag{4.64}$$

Again, since F^2 has charge $N+1$, no right-handed neutrino Majorana mass is allowed. The $D=3,4$ operators are also absent. In Tab.4.7 we show the corresponding charges for the cases $N=7,\,9$.

It can also be checked that for any other Higgs charge assignments in the range

[12] Again, the anomaly coefficients for the low energy spectrum with the new extra matter satisfy also $\mathcal{A}_3 \;=\; \mathcal{A}_2 \;=\; \mathcal{A}_1 \;=\; 3N-6$.

N	q_{10}	$q_{\overline{5}}$	q_1	q_{h^c}	q_h	$q_{H'}$	$q_{\overline{H}'}$
7	4	2	6	3	1	1	6
9	5	3	7	3	1	1	8

Table 4.7: Allowed $\mathcal{Z}_7^{(R)}$, $\mathcal{Z}_9^{(R)}$ Charges for $SU(5) \times U(1)_X$.

$(0, 7)$ satisfying $q_h + q_{h^c} = 4 \ mod(N)$, the corresponding phenomenologically viable models also forbid the symmetries $\mathcal{Z}_3$, $\mathcal{Z}_4$, $\mathcal{Z}_5$, $\mathcal{Z}_6$, $\mathcal{Z}_8$. This fact is sufficient to forbid these symmetries for all possible models with $q_h + q_{h^c} = 4 \ mod(N)$ although departing from this relation may in principle allow some of them.

Before closing this section it is interesting to note that the above list of symmetries does not exhaust all possible symmetries for the given gauge group. As an example, consider the model of ref. [82] that is characterized by a radiative breaking of the $SU(5) \times U(1)$ symmetry. This model, having the same set of fields as the model considered above plus gauge singlets, possesses the discrete symmetry $\mathcal{Z}_3^{(R)} \times \mathcal{Z}_2$, which can readily promoted to be anomaly-free at the expense of introducing a massive pair of hypercharge ± 1 singlets of $\mathcal{Z}_3^{(R)}$- charge equal to 2.

Overview.

In our previous analysis we have reconsidered the issue of possible anomaly-free discrete R-symmetries $\mathcal{Z}_N^{(R)}$ that avoid the μ-problem and the dangerous $D \leq 5$ operators within MSSM and extensions of it. Freedom from anomalies was considered either through strictly vanishing anomaly coefficients for the $(\text{gauge})^2\mathcal{Z}_N$ anomalies or through the operations of the *Green-Schwarz mechanism* for the former as well as the $(\text{grav})^2\mathcal{Z}_N$ anomalies. We have extended known investigations by considering charge assignments, that do not commute with the standard $SU(5)$ or $SO(10)$ gauge groups but are, instead, compatible with a so-called flipped-$SU(5)$ symmetry. Staying within the framework of $MSSM$, we have found that, in the anomaly-free case, the symmetries $\mathcal{Z}_3^{(R)}$, $\mathcal{Z}_6^{(R)}$ with flipped assignments are possible. We have also investigated the possibility of multidoublet extensions of MSSM in this case. Phenomenologically acceptable charge assignments have been listed. In the same framework, in the case of GS-anomaly cancellation, we have arrived at phenomenologically allowed $\mathcal{Z}_N^{(R)}$ symmetries with flipped assignments for $N = 3, 4, 6, 8, 12, 24$. Phenomenologically acceptable charges for these cases have been listed. As an example of discrete symmetries non-commuting with any of the above unifying symmetries, we have also considered a $\mathcal{Z}_5^{(R)}$ symmetry with non-unified charges in the framework of a singlet extension of MSSM.

We have also considered the question of finding such symmetries for traditional $4D$

grand unified models. Having excluded simple gauge groups such as $SU(5)$ or $SO(10)$ or non-simple groups like $SU(5) \times U(1)_X$ with MSSM-low energy content, we arrived at an extended flipped $SU(5) \times U(1)_X$ model. For this model, $\mathcal{Z}_N^{(R)}$ symmetries were shown to be anomaly free (through GS) and phenomenologically viable for $N = 2k+7$ and $N = 2k + 10$.

Chapter 5

General overview

In our previous analysis we have focused on several open issues that necessarily arise within the Standard Model of particle physics and its standard extensions, namely supersymmetry and Grand Unified Theories. These issues, among others are crucial for our understanding on the fundamental laws of nature.

In what preceded we have followed a twofold approach. In the first two chapters we have reviewed the main frameworks of our study and illustrated certain aspects and selected topics therein. Specifically, in Chap.1 we have introduced the fundamental concepts of gauge symmetries and renormalizability. These concepts further allow to establish the general and concrete framework of Quantum Gauge Field Theories. Then, within this context, we are led to the Standard Model of particle physics. We describe its general structure as well as its phenomenological implications on the particle spectrum. We eventually conclude this chapter by presenting several inadequacies of this model, among which the "fermion masses and mixing puzzle", strongly suggesting in favour of a high energy completion. Thus, in Chap.2 we review the rather standard extensions of the SM, namely SUSY and SUSY-GUTS which seem to evade at least some of its technical problems while offering new possibilities for theoretical constructions. There, after introducing fundamental aspects and properties of global supersymmetry in its minimal version ($\mathcal{N} = 1$) we arrive at the Minimal Supersymmetric Standard Model. Among the virtues of this model, interesting also on its own, is its possible embedding within the framework of Grand Unified Theories. SUSY-GUTs, also discussed in Chap.2 offer many interesting and attractive theoretical realizations that may potentially answer some of the open questions the SM leaves behind.

In the second part of this thesis, and having established our general framework, we proceeded in Chap.3,4 to demonstrate the main aspects of our research. In Chap.3 we have revisited the puzzle of fermion masses and mixing in more detail. This arises within the context of the SM but also propagates to the MSSM due to the unconstrained Yukawa structure in both models. Then, in [48] we mainly focused on the issue of neutrino masses and mixing from the viewpoint of a non-minimal SUSY-$SU(5)$ model. As mentioned there a non-minimal approach is always required when one considers realizations of the $SU(5)$ group, even in its supersymmetric version, since current proton stability constraints have practically ruled out the minimal version. In this model, proton decay is evaded due to the extended field representation content which allows for unification at a higher scale, namely $M_G \sim 10^{17} GeV$. Admittedly, a certain amount of fine tuning is required to keep tree level masses below M_G and subsequently achieve successful unification. However, this situation is rather typical in various GUT models. Moreover, renormalization group analysis necessarily constrains one of the heavy neutrino masses to lie within $10^{13} \sim 10^{14} GeV$, a phenomenologically preferred region for acquiring light seesaw neutrinos at the sub-eV scale. Another interesting aspect of this realization is a prediction for a zero tree level mass for the lightest neutrino. This, as it turns out, is not a special property of this particular model but a rather more general feature of GUTs where two neutrinos belong to the same irreducible representation of the unified group. Current data cannot yet either support or exclude such a possibility for the neutrino mass spectrum. If a much lighter neutrino is eventually observed then this type of GUTs would offer a rather elegant explanation for such a much smaller neutrino mass.

Next, in the remaining sections of Chap.3 we have investigated the possibility of an alternative approach on the lopsided (asymmetric) idea for the Yukawa matrices of fermions [62]. In this approach we have imposed a lopsided structure in the neutrino sector to account for the large mixing-large hierarchy observed there. We have found that such a possibility, in practice orthogonal to the standard approach, shares not only the attractive properties of lopsided models but also displays many unique features which render this framework even more attractive for model building. It turns out that this lopsided structure has a natural embedding in the type-I (or -III) seesaw mechanism since this seesaw formula offers the possibility of a hidden lopsided structure underlying the symmetric light neutrino matrix. What is remarkable in this approach is that it is meaningful even outside the framework of GUTs. In fact what is only required is an analogous lopsided structure shared by all Yukawa couplings of fermions[1]. Then without any other assumption the observed pattern for fermion masses and mixing straightforwardly appears. Thus, in this way one obtains hierarchical fermion masses, small and hierarchical CKM mixing, and large PMNS angles if a standard seesaw

[1]Besides of course the right handed neutrino mass matrix which is by default symmetric.

mechanism is also present. Within this context one may fit all current lepton masses and mixing data. Large mixing angles may originate from both charged lepton and neutrino mass matrix or only the neutrino mass matrix with the latter option most appreciated by theoretical considerations. In this perspective, we considered an $SO(10)$ realization where the desired global lopsided Yukawa structure was obtained through an interplay of renormalizable and non-renormalizable operators. Within this model we were able to fit adequately all current fermion masses and mixing data. However as already mentioned such an alternative lopsided approach may apply beyond this GUT model and even beyond GUT considerations in general.

Finally, in Chap.4 we examined the phenomenological implications of a discrete R-symmetries, which appear only within the framework of SUSY. Such symmetries have been long considered as a possible way to avoid the phenomenological difficulties of the MSSM and various SUSY-GUTs in practice forbidding the presence of certain dangerous operators. Actually the famous *ad hoc* matter-parity of the MSSM also present in many of its extensions may be regarded as originating from such a symmetry. In this viewpoint we examined those discrete R-symmetries [67] that, besides satisfying obvious phenomenological constraints[2], they can also exhibit a dynamical origin. In other words we further required that these $\mathbb{Z}_N^R$ symmetries should be embeddable in abelian gauge groups. Therefore they should further satisfy non-trivial constraints for their possible charges, as these arise from the gauge anomaly cancellation conditions of their parent gauge symmetries. With our analysis we have extended previous investigations of anomaly free, discrete, R-symmetries to the case of R-charges commuting with the flipped-$SU(5)$ gauge group. We have also examined the possibilities of multidoublet and singlet extensions of the MSSM. Finally we have constructed a non-minimal flipped-$SU(5)$ with an extended but phenomenologically viable low energy spectrum.

[2]These are the existence of the MSSM renormalizable couplings, the absence of the μ-term and the absence of potentially dangerous baryon- and lepton- number violating operators

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