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Critical exponents from scaling with neglect of cutoffs

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The neglect of cutoff dependences enables the derivation of scaling relations by pure dimensional analysis. Scaling relations for systems in confined volumes are utilized to provide the Imry *et al.* interdimensional relations for static and dynamic critical exponents. Illustrations are given for ν, β, and z. For $n = 0$, the Flory value for ν is obtained, while for $n = 1$ the errors in the three-dimensional values of ν and β are 5 and 10%, respectively, from our simple algebraic recursion relation. Our approach is based upon asymptotic dimensional arguments for systems governed by Landau-Ginzburg-type free-energy functionals, and this formulation also enables the separation of relevant and irrelevant variables near the critical point. The utility of the scaling theory is illustrated by application to the problem of the description of the electronic structure of disordered materials where traditional renormalization-group methods yields runaway solutions. The present methods (neglecting cutoffs) yield nontrivial information concerning conductivity and density-of-states exponents near the mobility edge.

I. INTRODUCTION

The scaling hypothesis of critical phenomena[1,2] assumes that near the critical point there is only one relevant length scale. Thus, all the thermodynamic properties and correlation functions have their "critical behavior" dependent on this fundamental correlation length. This assumption gives rise to various scaling relations for thermodynamic functions and to interrelationships between critical exponents for different properties.[3]

The renormalization-group (RG) method[4,5] goes beyond scaling theories in two general aspects. Firstly, the RG theory proves that only certain parameters of the energy functional are relevant in the neighborhood of the critical point, thereby explaining the universal character of classes of critical phenomena. The RG method then drives scaling relations for experimental quantities in terms of this small set of relevant parameters. Scaling then emerges from the symmetries of the energy functional under RG transformations. The second general characteristic of the RG theory is its ability to produce computational algorithms for the evaluation of critical exponents. Scaling theories only provide interrelationships between different exponents.[3,5]

Imry *et al.*[6] have shown how approximate relationships may be obtained which described the dimensionality dependence of critical exponents. Their derivation invokes a host of known properties of critical phenomenon. Here we show how these interdimensional scaling relationships can be derived directly and extremely simply from "naive" scaling.[7] The derivation is of interest in itself because it explicitly focuses on the inherent approximations involved. In particular, it is shown that the interdimensional laws emerge by neglecting the cutoff dependences of the general scaling relations. If the cutoffs were absent, the naive scaling[7] would be exact, so the errors incurred are measures of the importance of the cutoffs.

While an analysis of the approximations inherent in the interdimensional laws and a simplified derivation are of considerable interest, of greater importance is the fact that the same methods can be utilized to generate scaling relations and interdimensional scaling laws for situations in which the traditional RG method fails because of the production of runaway solutions, etc. An example of the latter occurs for the case of the motion of electrons in random potentials[8] where the analog of a critical point is the mobility edge, the demarcation energy between conductive and nonconductive states. Our method differs from that of Imry *et al.* in the use of a single matching between d and d-1 dimensional "critical" regions, while theirs involves matching between four regions (the above two and their corresponding mean-field regions). We are able to utilize a single matching because the dimensionality dependences on the coupling strength are utilized in our approach. Here we show the new interdimensional scaling relations for electrical conductivity and density of states can be derived. Elsewhere we provide a discussion of the implications of these new results.[9]

In Sec. II scaling is obtained directly by the application of simple dimensional analysis to the Landau-Ginzburg functional representation[4,5] for the thermodynamic and dynamic properties of the system or from the analogous classical field theoretical

representation for the random electronic problem. In Sec. III we derive recursion relations between critical exponents for d and $d-1$. The particular examples given are ν, β, and z where the recursion relations are[6,10]

$$\frac{1}{\nu_d}=2+\frac{(d-4)}{(d-5)}\left[\frac{1}{\nu_{d-1}}-2\right], \tag{1.1}$$

$$\beta_d=\beta_{d-1}\left[\frac{2\nu_d}{d-5}+1\right]-\frac{\nu_d}{d-5}, \tag{1.2}$$

$$z_d=2+z_{d-1}\nu_{d-1}\left[\frac{2}{d-5}+\frac{1}{\nu_d}\right]-\frac{2(d-4)}{d-5}. \tag{1.3}$$

After submission of this manuscript we were made aware of derivations of Eqs. (1.1) and (1.2) by Imry *et al.*[8] Our method of derivation differs from theirs in a number of important respects. First, it demonstrates how the errors are associated with the neglect of cutoffs, and second, it enables the extension to treatments of problems where traditional critical phenomena theories are inadequate. Using the two dimensional ($d=2$) exact Ising ($n=1$) values in Eqs. (1.1) and (1.2), the $d=3$ ones are found to be too high by 5 and 10%, respectively, from the currently accepted "best" values. For the polymer problem ($n=0$), the use of the exact ν for $d=1$ generates almost the exact values for all higher dimensions. This supports the predictions for the electronically disordered problem (in Sec. IV) where previous information on many exponents are unavailable. (The explicitly quantum predictions for electrical conductivity differ markedly from those which are generated utilizing results from percolation theory.[11]) A consideration of scaling theories of polymer solutions[12] is presented elsewhere.[13]

II. SCALING RELATIONS

We consider the traditional Landau-Ginzburg expressions for the free-energy functional

$$F[M]=\int d\vec{R}\,M(\vec{R})[r-c\nabla^2]M(\vec{R})+u\int M^4(\vec{R})\,d\vec{R}, \tag{2.1}$$

where $M(\vec{R})$ is the scalar order parameter (or "magnetization") and integrations are over a d-dimensional volume $d\vec{R}$. The analysis is left, for simplicity, in R space, but care must be exercized in the analysis of four-point correlation functions where the use of Fourier space is more convenient (see Sec. IV). The parameter r is defined by $r\propto(T-T_c)/T_c$, where T is the temperature and T_c is the critical temperature. The most general $F[M]$ would contain ∇^4, etc., terms in the M^2 part, ∇^2, etc., terms in the M^4 part, and M^6, M^8, etc., contributions. Below we review how the condition $r\to 0$ suffices to indicate the irrelevancy of all of these variables above certain critical dimensions. The convergence of certain integrals requires the presence of a minimum length cutoff l. The derivation in this section provides results which are necessary in Secs. III and IV.

The correlation function is defined by

$$\chi(\vec{R}\,|r,c,u)=\frac{\int M(\vec{R})M(\vec{0})\exp(-F[M])\,\delta M}{\int\exp(-F[M])\,\delta M}\equiv\langle M(\vec{R})M(\vec{0})\rangle \tag{2.2}$$

and other properties are defined in the usual manner.[3-5] Consider the simple *change in variables* for the δM and $d\vec{R}$ integrations.

$$M'(\vec{R}')\equiv bM(\vec{R}), \tag{2.3a}$$

$$\vec{R}'=a\vec{R},\quad V'=Va^d, \tag{2.3b}$$

and V is the volume of the system. Equation (2.3) converts Eq. (2.1) to

$$F[M]\equiv a^{-d}b^{-2}\int_{V'}M'(\vec{R}')[r-a^2c(\nabla')^2]M'(R')\,d\vec{R}'+ua^{-d}b^{-4}\int_{V'}d\vec{R}'\,[M'(R')]^4. \tag{2.4}$$

Because Eq. (2.3) is just a change in variables of integration, a and b may be chosen totally arbitrarily, and the equality between Eqs. (2.1) and (2.4) remains a *pure identity*. The choice

$$a^2a^{-d}b^{-2}c\equiv 1 \tag{2.5}$$

converts the coefficient of $(\nabla')^2$ to unity. Then Eq. (2.4) yields the scaled form

$$F[M|r,u,c,V,l]=F[M'|rc^{-1}a^{-2},ua^{d-4}c^{-2},1,a^dV,al], \tag{2.6}$$

which is valid for *arbitrary* a. The convenient choice

$$a=(r/c)^{1/2} \tag{2.7}$$

converts Eq. (2.6) to

$$F[M]=F[M'|1,ur^{(d-4)/2}c^{-d/2},1,(r/c)^{d/2}V,(r/c)^{1/2}l]. \tag{2.8}$$

We define $u'=ur^{(d-4)/2}c^{-d/2}$ as the relevant dimensionless coupling constant. For $r\to 0$ and $d>4$, $u'\to 0$, the quartic part of Eq. (2.1) or Eq. (2.4) is ir-

relevant; mean-field theory ensures.

Were it not for the presence of the cutoff in Eq. (2.6) or Eq. (2.8), these relationships could be utilized to provide all the exact scaling laws. However, scaling relations must be established for two systems having the same cutoff, while Eqs. (2.6) and (2.8) contain different cutoffs on left- and right-hand sides. It is this difference which introduces the necessity for the decimation portion of the RG transformation as this decimation effectively reduces the transformed cutoff back to its original value so the resultant scaling relation compares systems with constant cutoffs. Here we invoke the sole approximation of ignoring the cutoff dependences of scaling relations like Eqs. (2.6) and (2.8). These may at first sight appear to be irrelevant in Eq. (2.8) for $r \to 0$, but otherwise divergent integrals make them relevant. Given this approximation, it is possible to proceed by pure dimensional analysis. However, the transformation (2.3) simply systematizes this process and is useful below when additional terms are appended to Eq. (2.4) or the electron problem is considered.

The transformation (2.3) and Eq. (2.5) converts Eq. (2.2) to

$$\chi(\vec{R}\,|r,u,c,V) = a^{d-2}c^{-1}\chi(a\vec{R}\,|rc^{-1}a^{-2}, ua^{d-4}c^{-2}, 1, a^dV) \ . \quad (2.9)$$

Because χ is an intensive variable the volume dependence in Eq. (2.9) may be dropped. Using Eq. (2.7) in Eq. (2.9) and dropping the arguments of unity gives

$$\chi(\vec{R}) = r^{(d-2)/2}c^{-d/2}\chi\left(\frac{\vec{R}\,(r/c)^{1/2}}{ur^{(d-4)/2}c^{-d/2}}\right) . \quad (2.10)$$

The correlation length is defined by

$$\xi_d^2 \equiv \frac{\int d\vec{R}\,\chi(\vec{R})\,\vec{R}^{\,2}}{\int d\vec{R}\,\chi(\vec{R})} \ . \quad (2.11)$$

Using the scaling form (2.1) the integration (2.11) implies

$$\xi_d^2 = (r/c)^{-1}f(ur^{(d-4)/2}c^{-d/2}) \ , \quad (2.12)$$

with f an unknown function. For $r \to 0$ and $d > 4$, u' becomes irrelevant, so the correlation length exponent ν_d becomes the mean-field value $\frac{1}{2}$ for $d > 4$. For $r \to 0$ and $d < 4$, u' is large, and the dominant behavior of f is assumed to be of the power-law variety giving

$$\xi_d^2 = c_d r^{-2\nu_d} u^{2(1-2\nu_d)/(d-4)} c^{-1+d(2\nu_d-1)/(d-4)} \ , \quad (2.13)$$

defining ν_d, where c_d is a pure number.

The average magnetization $\overline{M}$ is defined as $\int d\vec{R}\,\langle M\rangle$, so Eqs. (2.3) and (2.5) show it to scale as

$$\overline{M} = a^{-(d+2)/2}c^{-1/2}\overline{M}(rc^{-1}a^{-2}, ua^{d-4}c^{-2}, a^dV) \ . \quad (2.14)$$

Because $\overline{M}$ is extensive, an overall factor of a^dV must appear, whereupon use of Eq. (2.7) produces the scaling relation

$$\overline{M}_d = V|r|^{(d-2)/4}c^{-d/4}g(u|r|^{(d-4)/2}c^{-d/2}) \ . \quad (2.15)$$

Again $g(0) =$ constant yields the mean-field exponent β as $\frac{1}{4}(d-2)$. Large u' implies the power-law form

$$\overline{M}_d = Vc_d'|r|^{\beta_d}u^{[2\beta_d-(d-2)/2]/(d-4)}c^{[d/(d-4)][(1/2)-\beta_d]} \ . \quad (2.16)$$

The establishment of other scaling relations, like Eqs. (2.10) and (2.15), is straightforward, as the above is just dimensional analysis. The important feature of scaling is the "proof" that other length scales do not enter when the cutoff is neglected. This is readily established *in the limit* $r \to 0$ as follows: Terms in $F[M]$ of the form

$$\int d\vec{R}\,M(\vec{R})\nabla^{2n}M(\vec{R})u_{2n}$$

scale under Eq. (2.3) with a factor of $b^{-2}a^{-d+2n}$ which behaves as r^{n-1} when we have Eqs. (2.5) and (2.7). Thus, these terms are irrelevant for $n > 1$ as $r \to 0$. Likewise, the range of the spin-spin interaction adds terms like

$$u_2\int d\vec{R}\,M(\vec{R})^2\nabla^2M(\vec{R})^2$$

to $F[M]$. These scale under Eq. (2.3) with a new scaled coupling constant $u_2' = u_2a^{-d+2}b^{-4}$. u_2' is converted by the choices (2.5) and (2.7) to being *dimensionless* [c.f. Eq. (2.6)] and proportional to $\frac{1}{2}[r(d-2)]$. Hence, the range of the pair interaction is irrelevant for $d > 2$, and it is marginal for $d = 2$. Similarly, three spin interactions contribute to the free-energy functional in the leading form of $\int d\vec{R}\,M^6(\vec{R})$. From Eqs. (2.3), (2.5), and (2.7) this scales as r^{d-3}, being irrelevant for $d > 3$. Consequently, by neglecting the cutoff scaling emerges as a simple consequence of *asymptotic* dimensional analysis. The important feature is that, as $r \to 0$, we have a small parameter. Despite the fact that $u' \to \infty$ in this limit (for $d < 4$) and perturbation theory cannot be employed, the remaining other parameters in $F[M]$, which are absent in Eq. (2.1) are either irrelevant or marginal for $d = 3$.

The treatment of dynamical scaling proceeds similarly. In this case it is convenient to apply the scale

transformation directly to the time-dependent Landau-Ginzburg equation,

$$\frac{\partial M(\vec{R},t)}{\partial t} = \Gamma_0(-2c\nabla^2 + 2r)M(\vec{R},t) + 4\Gamma_0 u M^3(\vec{R},t) + \Gamma_0^{1/2} f(\vec{R},t) , \quad (2.17)$$

where Γ_0 is the bare transport coefficient and f is Gaussian random, $\langle f \rangle = 0$,

$$\langle f(\vec{R},t) f(\vec{R}_0 t_0) \rangle = \delta(R - R_0)\delta(t - t_0) .$$

Introducing the *change in variables*

$$M'(\vec{R}',t') = bM(\vec{R},t) , \quad \vec{R}' = a\vec{R} , \quad t' = \lambda t , \quad (2.18)$$

with the particular choices Eqs. (2.5) and (2.7) and $\lambda = r\Gamma_0$ yields the scaled equation

$$\frac{\partial M'(\vec{R}',t')}{\partial t'} = [-2(\vec{\nabla}')^2 + 2]M'(\vec{R}',t') + 4uc^{-d/2} r^{(d-4)/2}[M'(\vec{R},t')]^3 + f'(\vec{R}',t') , \quad (2.19)$$

where

$$\langle f'(\vec{R}',t') f'(\vec{R}_0',t_0') = \delta(\vec{R}' - \vec{r}_0')\delta(t' - t_0') .$$

The dynamic correlation function emerges as

$$\langle M(\vec{R},t) M(\vec{0},0) \rangle = c^{-d/2} r^{(d-2)/2} f_{\vec{R},t}[\vec{R}(r/c)^{1/2}, r\Gamma_0 t, uc^{-d/2} r^{(d-4)/2}] . \quad (2.20)$$

The static susceptibility (2.10) emerges from the $t = 0$ limit of Eq. (2.20), while the correlation frequency is defined by

$$\omega_c^{-1} \equiv \frac{\int_0^\infty dt\, t \langle M(\vec{0},t) M(\vec{0},0) \rangle}{\int_0^\infty dt \langle M(\vec{0},t) M(\vec{0},0) \rangle} = (r\Gamma_0)^{-1} f_\omega(uc^{-d/2} r^{(d-4)/2}) \equiv r^{-z\nu} u^{2(1-z\nu)/(d-4)} c^{-d(1-z\nu)/(d-4)} , \quad (2.21)$$

providing the definition of the dynamical exponent z.

Because $u' \to \infty$ where $r \to 0$ for $d < 4$, the evaluation of exponents like ν_d, β_d, and z_d in Eqs. (2.13), (2.16), and (2.21), respectively, is generally a difficult task. Renormalization-group methods provide asymptotic series expansions of these exponents in powers of $\epsilon = 4 - d$. Now we demonstrate how the Imry *et al.*[6] *nonclassical approximations* may be obtained directly from scaling theory, in a very elementary fashion. The results are almost exact in the limit of zero spin dimensionality ($n = 0$), but are only approximate for the Ising case ($n = 1$) of Eq. (2.1). Perhaps, methods can be developed for evaluating the corrections.

III. CRITICAL EXPONENTS FROM SCALING

Consider a system in d dimensions which is confined between a pair of hyperplanes which are separated by a spacing D. Under the transformation (2.3) the separation D scales to Da. Hence, pursuing the analysis from Eq. (2.12) to Eq. (2.13) gives the scaling relation,

$$\xi_{d,D}^2 = (r/c)^{-1} f_0(ur^{(d-4)/2} c^{-d/2}, D(r/c)^{1/2}) . \quad (3.1)$$

Physically, it is clear that the relevant dependence on D must arise in the combination[12,13] (D/ξ_d) because, when $(D/\xi_c) \to \infty$, the confinement is irrelevant and $\xi_{d,D}^2 \to \xi_d^2$. This behavior is summarized in the scaling form (homogeneity assumption)

$$\xi_{d,D}^2 = \xi_d^2 g_D(D/\xi_d) , \quad g_0(\infty) = 1 . \quad (3.2)$$

When $D/\xi_d \ll 1$, we enter the "critical" region where g_D is taken to have a power-law form

$$\xi_{d,D}^2 = \xi_d^2 (D/\xi_d)^y (\text{const}) , \quad D/\xi_d \ll 1 . \quad (3.3)$$

However, $D/\xi_d \to 0$ implies that the system is now a $(d-1)$ one-dimensional, so Eq. (3.3) gives

$$\xi_{d-1}^2 = (\text{const}) \xi_d^2 (D/\xi_d)^y . \quad (3.4)$$

Introducing Eq. (2.13) into Eq. (3.4) and comparing powers of ν and u on both sides of the equation yields ($D_{\min} \propto c^{-1}$), the simple algebraic recursion relation (1.1) between ν_d and ν_{d-1}. Note that this recursion relation is obtained independent of the number of spin components.

The polymer problem[12-14] corresponds to $n = 0$ where the chain length L is the inverse Laplace variable of r. The case of $d = 1$ trivially gives the exactly soluble rod limit of $\nu_1 = 1$. Use of this boundary condition in Eq. (1.1) gives

$$\nu_d = 3/(d+2) \quad 4 \geq d \geq 1 \quad (n = 0) , \quad = \tfrac{1}{2} \quad d \geq 4 \quad (n = 0) , \quad (3.5)$$

which is almost the exact result for $d < 4$.[15]

For the Ising ($n = 1$) model, the exact solution for $d = 2$ produces $\nu_2 = 1$. Substitution of this value in Eq. (1.1) yields $\nu_3 = \frac{2}{3}$ (for $n = 1$), 5% higher than the best renormalization-group[15] value of 0.63. For

$d \geq 4$ mean-field values ensue from Eq. (1.1) as is already obvious from the irrelevancy of u' for $d > 4$. At $d = 4$ Eq. (1.1) contains the 0/0-type ratio $(2\nu_d - 1)/(d-4)$. Hence in order to utilize calculations in $\epsilon = 4 - d$ dimensions, we require the limiting $\epsilon \to 0^+$ ratio of this quantity, or equivalently $2(\partial \nu_d / \partial d)_{d=4^-}$. The Ising renormalization-group result[4,5] for $\nu_{4-\epsilon} = \frac{1}{2} + \frac{1}{12}\epsilon$ produces the results (after setting $\epsilon \to 0$) from Eq. (1.1) of $\nu_3 = \frac{3}{5}$ and $\nu_2 = \frac{3}{4}(n=1)$. In the polymer case ($n = 0$) the expansion $\nu_{4-\epsilon} = \frac{1}{2} + \frac{1}{8}\epsilon$ generates $\nu_3 = \frac{2}{3}$, $\nu_2 = 1$, and $\nu_1 = 2$ which is clearly incorrect.

The calculation for β_d proceeds similarly. For the confined case scaling produces

$$M_{d,D} = V|r|^{(d-2)/4} c^{-d/4} \times g_D'(u|r|^{(d-4)/2} c^{-d/2}, D(r/c)^{1/2}) \ . \quad (3.6)$$

Again taking the D dependence to occur through D/ξ_d gives the result that

$$M_{d-1} = M_d (D/\xi_d)^{\gamma'} \ , \quad D/\xi_d << 1 \ , \quad (3.7)$$

whereupon a comparison of powers yields Eq. (1.2). Use of the exact ($n = 1$) Ising $B_2 = \frac{1}{8}$ and our calculated $\nu_3 = \frac{2}{3}$ gives $\beta_3 = \frac{9}{24} = 0.38$, about 10% higher than the best renormalization-group[15] value of 0.32. Use of $\nu_3 = 0.63$ would improve matters to $\beta_3 = 0.36$, while the ϵ-expansion renormalization-group value $\beta_{4-\epsilon} = \frac{1}{2} - \frac{1}{6}\epsilon$ yields $\beta_3 = \frac{3}{10}$.

As the exponents generated by our methods obviously obey "scaling," the scaling relations between exponents apply. Thus, $\nu_3 = \frac{2}{3}$ implies $\alpha_3 = 0$, and a direct calculation of α_3 from α_2 by recursion relations for α yields the same results. Likewise η and γ can be evaluated from β and ν.

The treatment of dynamical exponents proceeds just as in the static case. Equating the results of the confined problem, $\omega_{c,d,D}^{-1}$, in d dimensions $(D/\xi_d << 1)$ to that in $(d-1)$ dimensions yields

$$\omega_{c,d,D}^{-1} = \bar{c}_d r^{-z_d \nu_d} u^{2(-z_d\nu_d+1)/(d-4)} c^{(d-z_d\nu_d+1)/(d-4)} (D_{\min}/\xi_d)^{y_1} \ , \quad (D_{\min}/\xi_d) << 1 \ , \quad (3.8)$$

$$= \bar{c}_{d-1} r^{-z_{d-1}\nu_{d-1}} u^{2(-z_{d-1}\nu_{d-1}+1)/(d-5)} c^{(d-1)(-z_{d-1}\nu_{d-1}+1)/(d-5)} \ . \quad (3.9)$$

Equating coefficients of r and u yields Eq. (1.3). Values of z_2 of sufficient accuracy to test Eq. (1.3) do not appear to be available. Perhaps, the use of exact values for ν_d and ν_{d-1} in Eq. (1.3) can incorporate some of the important effects of the omitted anomalous dimensions.

Given the accuracy of present renormalization-group calculations, the accuracy of Eqs. (1.1) and (1.2) may not be impressive. The utter simplicity of their derivation is, however, a virtue. Also the fact that their errors are associated with neglect of cutoffs is an important one.

However, the scaling theory of Sec. II and the exponent recursion relation of this section can be applied in situations, e.g., electrons in disordered materials, where the traditional renormalization-group method[8] fails to give useful results.

IV. ELECTRONIC STRUCTURE IN RANDOM SYSTEMS

Consider the motion of an electron in a Gaussian random, white noise, potential,[16-18] $\phi(\vec{R})$. The one electron Green's function for a particular $\phi(\vec{R})$ is the solution to

$$\left[E + \frac{\hbar^2}{2m}\nabla_{\vec{R}}^2 - \phi(\vec{R})\right] G(\vec{R}, \vec{R}'; E|[\phi]) = \delta(\vec{R} - \vec{R}') \ , \quad (4.1)$$

while the ensemble averaged Green's function and "two-electron" Green's function are obtained from the averaging,

$$G(\vec{R} - \vec{R}'; E) = \langle G(\vec{R}, \vec{R}'; E|[\phi]) \rangle_\phi \ , \quad (4.2a)$$

$$G_2(\vec{R} - \vec{R}_0, \vec{R}' - \vec{R}_0, \vec{R}'' - \vec{R}_0; E, E') = \langle G(\vec{R}, \vec{R}_0; E|[\phi]) G(\vec{R}', \vec{R}''; E'|[\phi]) \rangle_\phi \ , \quad (4.2b)$$

respectively, over the random ϕ. An alternative formulation considers a path integral representation for G and G_2. The derivation of scaling relations within this representation is discussed elsewhere[9] along with the physical interpretation of the results.

We could proceed to apply asymptotic dimensional analysis directly to Eqs. (4.1) and (4.2).[9] Instead, we choose to recast the averages in Eq. (4.2) into the form of a zero-component field theory in order to illustrate how our methods can be applied to field theories for which the standard renormalization-group methods fail.[8]

$G([\phi])$ may be expressed in terms of a classical real field $X(\vec{R})$ by

$$G(\vec{R}-\vec{R}';E|[\phi]) = \alpha \int \delta X\, X(\vec{R})X(\vec{R}') \exp\{-F[X,\phi]\} / \int \delta X \exp\{-F[X,\phi]\} , \tag{4.3a}$$

$$F[X,\phi] = \alpha \int d\vec{R}\, X(\vec{R}) \left[E + \frac{\hbar^2}{2m}\nabla^2 - \phi(\vec{R})\right] X(\vec{R}) , \tag{4.3b}$$

with α any complex number such that the integrals in Eq. (4.3a) exist (see below). The averaging over ϕ is hindered by the $\int \delta X \exp\{-F[X,\phi]\}$ factor in the denominator, so Eqs. (4.3) and (4.2a) are rewritten as the limit[8,13]

$$G(\vec{R}-\vec{R}';E) = \alpha \lim_{n\to 0} \left\langle \int \delta X\, X(\vec{R})X(\vec{R}') \exp\{-F[X,\phi]\} \left[\int \delta X \exp\{-F[X,\phi]\}\right]^{n-1} \right\rangle_{\phi} . \tag{4.4}$$

Relabeling the dummy X-variables $X_1, X_2, \ldots, X_n$ in the individual functional integrals, the ϕ averaging proceeds by completion of the square[8,13] to yield

$$G(\vec{R}-\vec{R}';E) = \alpha \lim_{n\to\infty} \int \prod_{\beta=1}^{n} \delta X_\beta X_1(\vec{R})X_1(\vec{R}') \exp\left[-\alpha \int d\vec{R} \sum_{\beta=1}^{n} X_\beta(\vec{R}) \left[E + \frac{\hbar^2}{2m}\nabla^2\right] X_\beta(\vec{R}) + \alpha^2 \left(\frac{v}{\hbar^2}\right) \int d\vec{R} \left[\sum_{\beta=1}^{n} X_\beta^2(\vec{R})\right]^2\right] , \tag{4.5}$$

a standard looking zero-component field theory for any $\mathrm{Re}\,\alpha^2 < 0$. Introducing the Fourier representation (V the volume)

$$X_\beta(\vec{R}) = V^{-1} \sum_{\vec{K}} X_\beta(\vec{K}) \exp(i\vec{K}\cdot\vec{R}) , \tag{4.6}$$

converts the free-energy functional in Eq. (4.5) to

$$F_n[X,E] = \alpha V^{-1} \sum_{\beta,\vec{K}} |X_\beta(\vec{K})|^2 \left[E - \frac{\hbar^2}{2m}K^2\right] - \left(\frac{v}{\hbar^2}\right)\alpha^2 V^{-3} \sum_{\beta,\gamma} \sum_{\vec{K}_1,\vec{K}_2,\vec{K}_3} X_\beta(\vec{K}_1)X_\beta(\vec{K}_2)X_\gamma(\vec{K}_3)X_\gamma(-\vec{K}_1-\vec{K}_2-\vec{K}_3) , \tag{4.7}$$

$$G(\vec{R}-\vec{R}';E) = \alpha \lim_{\eta\to 0} V^{-2} \sum_{\vec{K}} \prod_{\beta,\vec{K}'} \int dX_\beta(\vec{K}') |X_1(\vec{K})|^2 \exp[i\vec{K}(\vec{R}-\vec{R}')] \exp[-F_n(X)] . \tag{4.8}$$

Standard renormalization-group treatments on Eqs. (4.7) and (4.8) or G_2, yield runaway solutions.[8]

Consider the scale transformation *(a change in variables)*

$$X_\beta'(\vec{K}') = BX_\beta(\vec{K}) , \quad \vec{K}' = A^{-1}\vec{K} , \quad V' = A^d V , \tag{4.9}$$

which converts Eq. (4.7) identically into

$$F_n[X,E] = \alpha(V')^{-1}A^d B^{-2} \sum_{\beta,\vec{K}'} |X_\beta'(\vec{K}')| \left[E - \frac{\hbar^2}{2m}(\vec{K}')^2 A^2\right] - \alpha^2 \left(\frac{v}{\hbar^2}\right)(V')^{-3}A^{3d}B^{-4} \sum_{\beta,\gamma} \sum_{\vec{K}_1',\vec{K}_2',\vec{K}_3'} X_\beta'(\vec{K}_1')X_\beta'(\vec{K}_2')X_\gamma'(\vec{K}_3')X_\gamma'(-\vec{K}_1'-\vec{K}_2'-\vec{K}_3') . \tag{4.10}$$

We consider here the choice[9] of

$$A = \left[\left(\frac{v}{\hbar^2}\right)\left(\frac{m}{\hbar^2}\right)^2\right]^{(4-d)^{-1}} ,$$

$$B = \left(\frac{v}{\hbar^2}\right)^{(d+2)/2(4-d)} \left(\frac{m}{\hbar^2}\right)^{3d/2(4-d)} , \quad d \neq 4 , \tag{4.11}$$

whereupon $G(\vec{R}-\vec{R}',E)$ is found to scale as

$$G(\vec{R}-\vec{R}';E|v,m) = \left(\frac{m}{\hbar^2}\right)^{d(4-d)^{-1}} \left(\frac{v}{\hbar^2}\right)^{(d-2)(4-d)^{-1}} \times f\left(A(\vec{R}-\vec{R}'), \frac{E}{E_c}\right) , \tag{4.12}$$

$$E_c = c\left[\left(\frac{v}{\hbar^2}\right)^2 \left(\frac{m}{\hbar^2}\right)^d\right]^{(4-d)^{-1}} , \tag{4.13}$$

where C is a constant and the Jacobian B^{Nn} disappears in the $n \to 0$ limit. (For the system in a volume V, N is finite, so the order of the limits $n \to 0$ then $V \to \infty$ is implied.) The infinitesimal part of E can be manipulated as in the unscaled case; consequently the density of states is obtained from $-1/\pi \operatorname{Im} G(\vec{0},E)$ as[9]

$$n(E) = (\text{const}) \left(\frac{m}{\hbar^2}\right)^{d(4-d)^{-1}} \left(\frac{v}{\hbar^2}\right)^{(d-2)(4-d)^{-1}} \times f_n\left(\frac{E-E_c}{E_c}\right) . \tag{4.14}$$

At the mobility edge, $E = E_c$, $f_n(0)$ is a constant, and Eq. (4.14) provides the m and v dependence of $n(E_c)$.

Another choice of transformation (4.9) is given by

$$B = A^{(d+2)/2} , \quad A = (E-E_c)^{1/2} , \tag{4.15}$$

which scales Eq. (4.7) into a problem with

$$m' = m , \quad v' = v(E-E_c)^{(d-4)/2} , \quad V' = V(E-E_c)^{d/2} . \tag{4.16}$$

The density of states then scales as

$$n(E) = (\text{const})(E-E_c)^{d/2-1} \times g_n\left(\frac{E}{E-E_c} , v(E-E_c)^{(d-4)/2}\right) . \tag{4.17}$$

For $E - E_c \approx 0$, Eq. (4.17) corresponds to a strong coupling problem, and indications of irrelevancy of other parameters, absent in Eq. (4.7), follow as in Sec. II. For $E >> E_c$, g_n may be treated by perturbation theory, and Eq. (4.17) then reduces to the well-known free-particle limit.

In order to evaluate G_2 it is necessary to introduce two sets of zero-component interacting fields,[8]

$$G_2(\vec{R}-\vec{R}_0, \vec{R}'-\vec{R}_0, \vec{R}''-\vec{R}_0; E, E') = \lim_{\substack{\eta\to 0 \\ m\to 0}} V^{-5}\alpha^2 \sum_{\vec{K}_1,\vec{K}_2,\vec{K}_3} \int \prod_{\beta,\vec{K}'} \int dX_\beta(\vec{K}')\, dY_\beta(\vec{K}') \times \exp\left(i \sum_{j=1}^{3} \vec{K}_j(\vec{R}_j - R_0)\right) X_1(\vec{K}_1) X_1(\vec{K}_2)\, Y_1(\vec{K}_3)\, Y_1(-\vec{K}_1-\vec{K}_2-\vec{K}_3) \times \exp\left\{-F_n[X,E] - F_n[Y,E'] + 2\alpha^2(v/\hbar^2) V^{-3} \sum_{\beta,\gamma} \sum_{\vec{K}_1,\vec{K}_2,\vec{K}_3} X_\beta(\vec{K}_1) X_\beta(\vec{K}_2)\, Y_\gamma(\vec{K}_3)\, Y_\gamma(-\vec{K}_1-\vec{K}_2-\vec{K}_3)\right\} . \tag{4.18}$$

The conductivity is obtained from dropping overall factors of $\hbar$,

$$e^2 m^{-2} \langle \operatorname{Tr}\{\vec{\nabla} \operatorname{Im} G(E,[\phi]) \cdot \vec{\nabla} \operatorname{Im} G(E,[\phi])\}\rangle_\phi , \tag{4.19}$$

where the trace is over spatial variables. Representing Eq. (4.19) in terms of (4.18) and then performing the transformation (4.9) and Eq. (4.11) provides the scaling relationship for $\sigma(E)$ as[9]

$$\sigma(E) \propto e^2 m^{2(d-2)(4-d)^{-1}} v^{(d-2)(4-d)^{-1}} g_\sigma\left(\frac{E-E_c}{E_c}\right) . \tag{4.20}$$

For $E - E_c > 0$ and small, a power-law form for g_σ is expected to apply [$g_\sigma(x) = 0$, $x < 0$] giving

$$\sigma(E) \propto e^2 m^{2(d-2-\sigma_d)(4-d)^{-1}} v^{(d-2-2\sigma)(4-d)^{-1}} (E-E_c)^\sigma \tag{4.21}$$

which provides a relationship between m, v, and $(E-E_c)$ exponents. The physical condition, that m and v have nonpositive values, provided nontrivial bounds[9] on σ. Equation (4.14) with $f_n(0) = \text{const}$ and Eq. (4.21) provide the first nonpercolation theory information about density of states and conductivity exponents for $E - E_c = 0^+$.

The evaluation of recursion relations for ν_d and σ_d (the value of σ in d dimensions) proceeds just as in Sec. III. ξ_d^2 may be defined from

$$\xi_d^2 = \frac{\int d\vec{R}\, \vec{R}^2 \langle |G(\vec{R};E[\phi])|^2\rangle_\phi}{\int d\vec{R}\, \langle |G(\vec{R};E[\phi])|^2\rangle_\phi} \tag{4.22}$$

and the use of $|G(\vec{R};E)|$ as a measure would produce the same scaling relation. The choice Eq. (4.11) yields

$$\xi_d^2 = A_d^{-2} g_\xi\left(\frac{E-E_c}{E_c}\right) \to A^{-2}\left(\frac{E-E_c}{E_c}\right)^{-2\nu_d} , \quad (E-E_c)/E_c \to 0 , \tag{4.23}$$

while the methods in Sec. III imply

$$\xi_{d-1}^2 = \xi_d^2 (D/\xi_d)^\lambda , \quad D/\xi_d << 1 . \tag{4.24}$$

Comparing powers of $E - E_c$ and v (or m) yields

$$\nu_{d-1} = \nu_d(4-d)/(5-d-2\nu_d) . \tag{4.25}$$

Likewise, the conductivity recursion relation is

$$\sigma_{d-1} = [2\nu_d - \sigma_d(5-d)]/[2\nu_d - (5-d)] \ . \quad (4.26)$$

Again Eqs. (4.25) and (4.26) are nontrivial results whose implications are discussed elsewhere.[9]

V. DISCUSSION

When cutoff dependences are ignored, scaling arises as a consequence of asymptotic dimensional analysis. The presence of a small parameter, $(T-T_c)/T_c$ or $(E-E_c)/E_c$, then enables the demonstration of the irrelevance of parameters in the free-energy functional. Applications are provided here to standard Landau-Ginzburg-type field theories for static and dynamic critical phenomena and to problems of electrons in random media. In the latter case the application of standard renormalization-group methods produces runaway solutions.[8] Nevertheless, scaling is directly established by the present methods, producing new information concerning conductivity and density-of-states exponents near the mobility edge. Applications to scaling theories of polymer solutions[12] are given elsewhere.[13] In the latter application there are n zero-component interacting quartic fields with the mean concentration n/V of these fields being finite.[12,13] (An equivalent path integral representation is convenient to fix the chain lengths.[13])

The cutoff free scaling relations are shown to enable the derivation of the Imry *et al.*[6] recursion relations between exponents for dimensionalities d and $d-1$. The results are illustrated for ν, β, and z. For the $n=0$ limit the boundary condition $\nu_1 = 1$ produces almost exact values of ν_d for $d > 1$. On the other hand, exact Ising values for ν_2 and β_2 yield nonclassical values of ν_3 and β_3 (to 5 and 10% accuracy, respectively).

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