Fractional Cointegration Analysis and Applications

by

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DECLARATION

I, Nakos Apostolos, do hereby assert that the research and analysis presented in this dissertation is the result of my own intellectual endeavors and original contribution. It is imperative to emphasize that in cases where information or ideas have been drawn from external sources, the necessary effort has been made to explicitly and comprehensively acknowledge and attribute the origins of such materials. The utmost care has been taken to ensure that no section of this dissertation includes content which has been previously submitted to the examiners of this esteemed university or any other institution, nor any material that has been previously utilized for any other form of academic evaluation. By adhering to these principles, the credibility and integrity of this dissertation are firmly upheld, thus affirming my commitment to scholarly excellence and ethical academic practice.

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ABSTRACT

The assemblage of seven nations, commonly referred to as the Group of Seven (G7), comprises countries acknowledged for their considerable level of industrialization, rather than being categorized as the wealthiest nations globally. Among this collection of countries, the United States of America distinguishes itself as a nation renowned for its extensive and firmly established conventional industries. Leveraging the fractional cointegrated vector autoregressive ($FCVAR$) model, we have performed a thorough examination of the enduring relationship that exists between various stock market indices over the timeframe encompassing the years 2000 to 2023. By incorporating the notion of fractional cointegration into our analysis, we have identified convincing evidence indicating the presence of a long-term correlation between these indicators when scrutinizing them in pairs. Throughout the aforementioned timeframe, the Italian stock market index demonstrates a noteworthy degree of price volatility, in stark contrast to the US stock market index, which showcases a comparatively minimal level of volatility. Additionally, alongside the insights gained from the b and d index, we have also derived valuable information concerning these indicators through the assessment of shareholder performance, ultimately enabling us to draw meaningful and substantiated conclusions.

Keywords: *FVCAR model*, *Financial*, *Econometrics*,*Log Returns*

1. CONTRIBUTIONS OF THIS THESIS

The contribution of this thesis in relation to previous research on the utilization of time series techniques in macro and financial variables is of paramount importance. To begin with, we employ a pioneering and highly sophisticated time series methodology known as the *FCVAR* model, which enables us to empirically examine various topics under the assumption of fractional cointegration in an unprecedented manner. In particular, we utilize this method to analyze the stock market indices of seven countries, focusing specifically on the daily closing price. This approach demonstrates remarkable flexibility and suitability for diverse environments, thereby empowering us to derive meaningful insights regarding both long-term relationships and short-term dynamics. Additionally, by simultaneously exploring the potential cointegrating relationships and the behavior of the error term, we have not only contributed to the existing literature but also expanded its horizons by introducing novel scenarios that could significantly impact decision-making processes for policy makers and other relevant stakeholders. Furthermore, the versatility of this methodology allows us to combine it with other techniques, thereby enhancing the accuracy and relevance of our findings and, consequently, making even more profound contributions to the field. The intricate details of these contributions are meticulously expounded upon in subsequent sections and chapters of this thesis.

2. INTRODUCTION

Fractional cointegration can be classified as a generalized category of cointegrated systems that offers the possibility to estimate not only the fractional memory orders but also the cofractional memory orders of time series, thus eliminating the restriction of fixing the memory parameters to integer values. Extensive empirical research has revealed that numerous macroeconomic and financial variables exhibit long memory characteristics in the long-run. However, the emphasis on the short-run adjustment dynamics of the fractional cointegrating relation has been relatively limited. The existence of nonlinear adjustments in the long-run equilibrium relation of cointegrating variables has been extensively documented in the $I(1)$=$I(0)$ cointegration literature. Despite the substantial evidence supporting the presence of long memory in macroeconomic and financial variables in the long-run, the investigation into the dynamics of short-run adjustment in the fractional cointegration relationship has received relatively less attention [Cheang, 2018]. The interrelationships between stock markets have been extensively explored from various perspectives. Numerous techniques such as the Efficient Market Hypothesis (EMH) proposed [Kim et al., 2009], the Capital Asset Pricing Model ($CAPM$) introduced by [Heimonen, 2002], and the Generalized Autoregressive Conditional Heteroskedasticity ($GARCH$) model employed by [Illueca and Lafuente, 2002] have been utilized to draw conclusions regarding the relationships, convergence, and co-movements among these markets. Additionally, a multitude of techniques, including the application of time series data as outlined in [Brooks et al., 2015], have been employed to investigate the integration and cointegration among different global economic regions, with a particular emphasis on the USA-EU relationship (as demonstrated by [Caporale et al., 2016]) as well as intraregional markets, such as the members of the European Monetary Union (EMU) as examined [Soares da Fonseca, 2013] and other scholars.

The primary objective of this paper is to conduct an extensive examination of financial integration across the seven major stock markets worldwide, namely Germany, France, United Kingdom, Japan, Italy, Canada, and the United States of America (U.S.A), during the time frame spanning from January 2000 to July 2023. This particular study adopts an econometric approach to shed light on the integration of stock markets, addressing a significant gap in the existing literature pertaining to the time series analysis of market cointegration. Consequently, our research offers a valuable contribution to the body of knowledge previously established in the field of stock market integration, specifically from the perspective of fractional cointegration vector autoregressive analysis. Despite the application of fractional cointegration in earlier investigations, the approach introduced by [Johansen and Nielsen, 2012] represents a novel contribution to the existing literature. This model, which is augmented to accommodate deterministic trends, possesses distinct advantages in the estimation of a system comprising fractional time series variables that potentially exhibit cointegration. Furthermore, the flexibility inherent in this model empowers researchers to ascertain the number of equilibrium relationships through rigorous statistical testing procedures, while simultaneously facilitating the joint estimation of adjustment coefficients and cointegrating relationships, thereby accounting for short-run dynamics.

The subsequent portion of this chapter has been meticulously arranged in the following manner in order to provide a comprehensive and coherent exposition of the subject matter at hand. To begin with, Section 3 undertakes an in-depth examination of the existing body of literature, with a particular emphasis placed on the various methodologies employed to investigate the intricate dynamics of stock markets. Subsequently, the focus of discussion shifts towards the application of integration and cointegration tests within diverse economic regions, thereby shedding light on their relevance and efficacy. Furthermore, this section incorporates pertinent references to prior research endeavors and their corresponding findings, thereby enriching the scholarly context of the present study. Moving forward, Section 4 expounds upon the methodology that has been adopted for the purpose of this research, elucidating the underlying principles and procedures that have been employed in a meticulous manner. In a similar vein, Section 5 meticulously presents the data and conducts a preliminary analysis, thereby setting the stage for a more comprehensive interpretation of the empirical findings. Moreover, Section 6 delves into a detailed discussion of the empirical results, drawing upon the established literature and the methodological framework to provide a comprehensive analysis and interpretation of the data. Finally, the concluding section, Section 7, encapsulates the key insights and implications of the research, thereby synthesizing the various threads of discussion and presenting a coherent set of conclusions that are grounded in empirical evidence.

3. LITERATURE REVIEW

The foundation for the long-term interaction between various volatility processes was created by fractional cointegrated systems. The *FCVAR* (Fractionally Cointegrated Vector Autoregressive) model represents an approach to evaluate the interdependence between the stock market indices. It is often used in studies in the field of economic analysis. This empirical model analyses data to find out how the stock market indices interact with each other in the long term. These interactions include how one index can affect other indices through the market interaction and basic economic variables. The *FCVAR model* can be used to estimate the degree of risk in the stock markets and to monitor the interconnection of different stock market indices in periods of market instability. The application of the fractional cointegration technique (*FCVAR model*), which is an extension of the CVAR approach [Johansen, 1995a], is sufficient from a theoretical perspective to provide more information about the cointegrating rank, the adjustments of the coefficients, and long-run relationships among different variables—in this case, financial markets [Gagnon et al., 2016].

The long run cointegrating relation of the fractionally cointegrated systems has received much attention through semiparametric approaches. The fractionally cointegrated vector autoregressive (FCVAR) model allows one to jointly estimate the memory parameters, the long-run cointegrating relations with the adjustment parameters, and the short-run lagged dynamics. It also allows one to count the number of equilibrium relations via the cointegrating rank test. The cointegrated systems' long-memory parameters may be flexible estimated using the FCVAR model, which simplifies to the typical cointegration scenario when the fractional and cofractional orders are equal to unity. [Johansen and Nielsen, 2012] established asymptotic theory for likelihood estimation and inference in the fractionally cointegrated VAR model. [Johansen and Nielsen, 2016] examined the function of observable and unobserved beginning values in conditional maximum likelihood estimators of the nonstationary fractional systems. In the early research on cointegration, the conventional $I(1)/I(0)$ kind of cointegration was primarily studied, in which the linear combinations of the $I(1)$ nonstationary processes are $I(0)$ stationary.

[Johansen, 1988, Johansen, 1991, Johansen, 1995a] created a fully parametric inference of cointegration in the error correction process. Cointegration is modeled using the vector autoregressive model for nonstationary variables in Johansen's framework. After choosing a cointegrating rank, it is possible to estimate the cointegrating vector, the rate of adjustment toward the long-run cointegrated relation, and the short-run dynamics. A statistical characteristic of linear combinations of two or more series having a fractional order lower than all the individual series considered in the analysis is known as the fractional cointegration relationship. Several semiparametric methods have been created for the fractionally cointegrated systems after these issues had been raised. [Johansen, 2008] created the well-known $I(1)/I(0)$ cointegrated vector autoregressive model of [Johansen, 1995b] by allowing for the modeling of long memory time series with non-integer order of integration. Empirical applications adopted Johansen's cointegration in the studies of prices parities by rational expectations theory in which the model based expectation restrictions provide testable information on cointegrating relations and short-run adjustments. The typical $I(1)/I(0)$ cointegration's underlying assumptions, according to which the time series variables are co-moved at order zero and integrated at order one, are quite constrictive. Many economic and financial time series have long-range dependency in the autocorrelation function, but they don't always display a unit root process, also known as the long memory process, according to a substantial body of literature. Examples include asset price volatility, long-term dependency of stock indices, forward premium, currency rates, interest rate term structure, and more. For a thorough analysis of long memory processes in econometrics, [Baillie, 1996]. If a time series has a long memory, the traditional unit root and stationarity hypothesis tests may produce contradictory results. The long memory approach in the time series process is supported by the semiparametric estimators of the fractional integrated order. However, if the parametric form, such as the number of lags, is incorrect, the completely parametric long memory estimator, such as the $ARFIMA$ model, is inconsistent. The fact that long memory processes entail distinct expectations of long run dynamics and consequences of shocks to the macroeconomic variables is one of the most significant reasons to separate long memory processes from $I(0)$ and $I(1)$ processes. The $I(1)/I(0)$ cointegration framework's arbitrary limitation of integrated orders d to integer values will lead to an incorrectly stated likelihood function and flawed statistical inference.

Despite the distinctiveness of the individual markets or the nation profiles, the financial markets, especially the stock markets, for emerging and established economies have lately become increasingly intricately intertwined. The relationship between national stock markets has been extensively researched since [Grubel, 1968] work on elaborating the advantages of worldwide portfolio diversification. This piques academics' curiosity for studying the relationships between international marketplaces. The US as a global participant has received a lot of attention in the literature on financial integration. [Ripley, 1973], [Lessard, 1976], and [Hilliard, 1979] and other earlier research usually discovered modest correlations between national stock markets, indicating the advantages of global diversity. Following the worldwide market meltdown in October 1987, interest in the connections between national stock markets has increased. People have come to know that different national equity markets are extremely intertwined since established economies like the US stock market have a significant impact on other markets, thanks to the crisis.

Since [Granger, 1981] and [Engle and Granger, 1987] made significant contributions, cointegration of nonstationary time series has been thoroughly studied during the past three decades. The representation theorem for a broad example of cointegration when the variables are fractionally integrated of a non-integer order d and cofractional of a smaller order *(d-b)*, for $0 \leq d\text{-}b < 1/2$, was presented by [Granger, 1986] and [Engle and Granger, 1987]. The level of cointegration between the variables is determined by the parameter b. Many approaches have been utilized in integration and cointegration research, including the unit root tests developed by [Dickey and Fuller, 1979] and [Dickey and Fuller, 1981] to determine the sequence of integration. Time series X_t is described by [Granger and Joyeux, 1980] and [Hosking, 1981] as a fractionally integrated process of order d for non-integer $d \in (0,1)$, denoted $X_t \in$ I(d), if it has an $I(0)$ stationary, invertible autoregressive moving average representation after fractional differencing, i.e. $\Delta^d X_t \in \mathrm{I}(0)$. Since a fractionally integrated error term will also exhibit mean-reverting behavior, the equilibrium error may be mean-reverting without being exactly $I(0)$ [Hosking, 1981]. [Granger and Joyeux, 1980], who modeled the US DGP and consumption time series, and [Hosking, 1981], who modeled the Nile River flows series, developed the first models proposed in the statistical literature with regard to these specific features in the same year. They both offered a fresh mathematical environment in which a time series is fractionally integrated rather than being integrated with integer values. Their model's originality is the fractional difference operator, which is a weighted sum of infinitely many observations with weight sizes determined by a single real integer, sometimes abbreviated d (the fractional order of a time series).

[Eun and Shim, 1989] show evidence of co-movements between the US stock market and other global equity markets by applying the vector autoregression models. The quantity of data accessible, for instance, for various time series financial assets, has significantly increased in the applied econometrics literature. The datasets include many observations, which are frequently sampled at high frequency, and the fractionally integrated model frequently does a good job of describing the autocorrelation characteristics of these series. The cross-country market efficiency hypothesis was then tested by looking at the cointegration of the variables using [Johansen, 1988, Johansen, 1991] multivariate cointegration test. The Efficient Market Hypothesis, which is one of the pillars of modern finance, was proposed by [Fama, 1970] in one of his most significant publications. Essentially, an efficient market is one in which the prices of the assets sold within reflect all information available to investors. The least constrictive type of market efficiency is called weak form efficiency. The empirical discussions of fractional cointegrated economic variables date back to the 1990s and include those of [Baillie and Bollerslev, 1994], [Cheung and Lai, 1993], [Dueker and Startz, 1998], and [Mohanty et al., 1998], among others. These authors argued that the macroeconomic variables have long memories and can be described as fractionally integrated processes, and that fitting the fractionally integrated variables into the standard case may lead to a false rejection of the economic hypothesis. The autoregressive fractionally integrated moving average ($ARFIMA$) model, which expanded the $ARMA$ model to consider the hyperbolic rate of fading autocorrelations, was used in early investigations of long memory data. The difficulty is that, when the cointegrating errors have long memory, they are correlated over long horizons, which makes the conventional least squares estimators inconsistent, as addressed by [Robinson, 1991, Robinson, 1994] and [Robinson and Hidalgo, 1997]. When there are more than two variables, the Johansen cointegration test provides more reliable findings than other cointegration tests for demonstrating common stochastic tendencies across stock markets [Gonzalo, 1994].

[Kasa, 1992] investigated the financial integration of five developed markets by using shared stochastic patterns in these series. In finance, further instances include [Baillie and Bollerslev, 1994] work, which sought to determine a dynamic link between nominal spot and future exchange rates in a fractional cointegration setup. Strong evidence for cointegration among many important European stock markets in the late 1970s and early 1980s was discovered by [Taylor and Tonks, 1989] and [Corhay et al., 1993]. [Dickinson, 2000] discovered that there is a cointegrating relationship between the major European stock markets, which may be partially driven by the long-run relationships of macroeconomic fundamentals among these countries, possibly through indirect channels of international interaction. When [Cheung and Ng, 1992] looked at the dynamic characteristics of stock returns in Tokyo and New York, they discovered that from January 1985 to December 1989, the US market was a significant worldwide driver. In their analysis of the aftermath of the October 1987 stock market crisis, [Lee and Kim, 1993] reported that there was a larger correlation between national stock markets when the US stock market was more volatile. [Jeon and Von Furstenberg, 1990] demonstrate using the VAR technique and the impulse response function analysis that the level of international co-movement in stock price indexes has dramatically grown since the 1987 crisis. However, [Van den Broeck et al., 1994] utilizes Bayesian techniques to get to the conclusion that there are no universal patterns in stock prices between nations. Markets are not always connected over the long term if cointegration does not exist, hence diversification may be advantageous. Testing for cointegration and any variations in its degree over time are crucial because of this. For instance, [Richards, 1995] showed that there is no cointegration across different stock markets, supporting the presence of investor gains from diversification. In addition, [Richards, 1995] and [Francis and Leachman, 1998] both looked into the possibility of cointegration links between the established European and American economies. The first indicated that markets are in a long-term state of equilibrium, but the second revealed that national return indexes are not cointegrated. On the other hand, a random walk was shown [Fernández Anaya et al., 2008] to show a change in long-term behavior. Additionally, several research employing univariate or multivariate cointegration models, such as those by [Yang et al., 2003], [Tian, 2007], and [Kenourgios et al., 2009], have found long-run co-movements across foreign stock markets.

In terms of daily price trends, [Gil-Alana et al., 2013] found that the US stock markets throughout the years 1971–2007 had very comparable patterns. Utilizing various methodologies, [Granger and Hyung, 2004] described the cointegration through structural fractures, demonstrating extended memory dependency. [Bessler and Yang, 2003] attempted to show the interdependence of nine major stock markets in a global context in 2003, concluding that they are not fully integrated. However, [Soares da Fonseca, 2013] demonstrated that throughout the first ten years of the EMU, the major stock markets in the Euro region were not completely integrated using a VAR model.

In conclusion, this method offers a means to illustrate various explanations of market integration in various circumstances. It is discovered that adding a level parameter to the FCVAR formulation has the benefit of lessening the bias of pre-sample observations. Furthermore, formulation of the deterministic term, offered by [Dolatabadi et al., 2016], permits deterministic linear temporal trends or drift in the variables. The fractionally cointegrated VAR system has been acknowledged as having empirical importance in the fields of political economy and financial markets. For the investigation of price discovery in commodity spot and futures for five non-ferrous metals (aluminum, copper, lead, nickel, and zinc), [Dolatabadi et al., 2015, Dolatabadi et al., 2016] applied the *FCVAR* model. Through the equilibrium connection, considerations of long-run contango or backwardation characteristic and disequilibrium mistakes are permitted. [Dolatabadi et al., 2015, Dolatabadi et al., 2016] showed higher evidence for price discovery in the spot compared to the outcome from the non-fractional scenario using the same data from [Figuerola-Ferretti and Gonzalo, 2010] who instead utilized the usual cointegrated VAR model. Although one of the most important studies in the field of unit root testing is presented in these publications, the test's strength is limited since it cannot account for extended memory processes [Caporale et al., 2015]. Cointegration has also been used to examine if investing in several stock markets offers diversification advantages, as demonstrated most recently by [Caporale et al., 2015]. The multivariate cointegration test of [Johansen, 1988, Johansen, 1991] was then used to analyze the cointegration of the variables and evaluate the cross-country market efficiency hypothesis. Other uses of empirical finance include the forecasting of stock prices based on a relationship between high and low prices [Caporin et al., 2013] and a no-arbitrage link between spot and futures [Rossi and De Magistris, 2013]. In addition, daily time series that only include the opening (or closing) asset price conceal intraday fluctuation, omitting crucial data ([Degiannakis et al., 2013]; [Haniff and Pok, 2010]. The authors show a possible profit performance of technical analysis tactics based on predictions of high and low prices by employing a long memory forecasting framework, a fractional vector autoregressive model with error correction ($FVECM$).

In [Carlini et al., 2011], a highly significant empirical application focuses on the potential for discovering a dynamic link between high-frequency pricing and cumulated order flows, or the cumulated total of the difference between transactions that were launched by buyers and sellers. In order to filter the series, they first used the fractional difference operator of type II. [Baruník and Dvořáková, 2015] analyzed the cointegration dynamics between daily high and low stock prices as well as the long memory characteristics of their linear combination, i.e. the range, of the key global stock market indexes over the 2003–2012 period. The results supported recent evidence that volatility may not be a stationary process by indicating that all of the indices' ranges exhibit long memory and are mostly in the non-stationary zone. [He and Wan, 2009] added that the prices at which an excess of demand alters its course are referred to as the highs and lows. [Alizadeh et al., 2002] demonstrate that the (log) range, commonly known as the difference between the greatest and lowest (log) prices of an asset during a specified sample period, is a very effective volatility measure. The range-based volatility estimator, which overcomes the shortcomings of conventional volatility models based on closing prices that fall to use the information contents inside the reference period of the prices, resulting in inaccurate forecasts, as noted by [Tietz et al., 2006], appears robust to microstructure noise such as bid-ask bounce. This approach is also used by [Gagnon et al., 2016] to examine the cointegration of risk-neutral moments across five significant European stock markets. They find that there is significant financial integration but that this integration is only partially complete when anticipations are considered. For example, despite the high correlation between market indexes, [Lucey and Muckley, 2011] have found evidence of valuable long-term diversification potential in European markets suggesting that other metrics of risk could be relevant. Volatility spillovers can be driven by contagion [King and Wadhwani, 1990] and during the Eurozone crisis of 2010-2011, there was strong evidence of spillovers [Kohonen, 2013]. Higher-order moments of return distributions are an issue for risk managers and regulators. It is well known in the literature that recovering distributional moments from historical data is difficult and that option data allows for more accurate estimation of distributional moments ([Bali et al., 2011, Birru and Figlewski, 2012, Conrad et al., 2013]). These techniques have demonstrated that risk-neutral moments have cross-sectional explanatory power and may anticipate realized returns ([Bali and Murray, 2013]). However, a thorough examination of global links at the level of expectations seems to be absent.

The literature has shown a strong interest in the relationships between global stock markets, and this interest has grown significantly as a result of the easing of financial regulations in both developed and developing markets, the development of trading and communications technology, and the introduction of new financial products that have increased the opportunities for global portfolio investments. The globalization, which provides additional fuel to the increased intertwining of global economies and financial markets, can also be credited with the interest. International fund managers have become interested in the new lucrative emerging stock markets as a chance to diversify their portfolios in recent years. Although the literature makes considerable use of this strategy, another area of the research concentrates on European stock markets. Essentially, the effect that the European stock market indices have with the indices of other countries such as America and Japan for example. In our study, we will investigate the long-term interdependence of the stock market indices related to the *G7* group. The *G7* is an international organization that gathers the top economies of the western hemisphere. The members of the *G7* are the United States, Canada, France, Germany, Italy, Japan, and the United Kingdom. The main issues discussed in *G7* meetings relate to the economy, trade, sustainability, climate change, security, and global health.

4. METHODOLOGY

The Benchmark Model: *Fractionally Cointegrated Vector Auto Regressive Model*

This bachelor's thesis applies the recently developed Fractionally Cointegrated Vector Auto Regressive (*FCVAR*) model by [Johansen, 2008], later extended by [Johansen and Nielsen, 2012], to the study of the short-run interdependence of the stock market indices in the *G7* countries from January 2000 to September 2023. The *FCVAR* model, which allows for processes to be integrated with fractional order *d* that cointegrate to fractional order *d-b*, is essentially a fractional generalization of the Cointegrated Vector Auto Regressive (*CVAR*) model in which the parameters *d* and *b* can take on both integer and fractional values. The *CVAR* model prevented this from happening.

Fractional processes play a crucial role in econometrics, as evidenced by the works of [Gil-Alana and Hualde, 2009] as well as [Henry and Zaffaroni, 2003]. These authors demonstrate the practical applications of fractional integration and long-range dependence within the domains of macroeconomics and finance. Specifically, [Henry and Zaffaroni, 2003] delve into the intricate relationship between fractional processes and these two fields. Furthermore, [Johansen, 2008] and [Johansen and Nielsen, 2012] proposal of the FC-VAR model represents a significant advancement in the field, building upon the foundation laid by [Johansen, 1995a] C-VAR model. This generalization of the FC-VAR model allows for the inclusion of fractional processes of order d, which cointegrate to order $d - b$. Consequently, this extended framework provides researchers with a more comprehensive understanding of the complexities at play within econometrics and its various sub-disciplines. Overall, the recognition and incorporation of fractional processes have greatly enriched the field of econometrics, leading to new insights and avenues of exploration. The error correction form represents the fractionally cointegrated VAR model for a dimension p time series, X_t. The form is:

$$\Delta^d X_t = \alpha\beta' L_b \Delta^{d-b} X_t + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b X_t + \epsilon_t \quad (1)$$

The distribution of ε_t is always p-dimensional independent, with mean **zero**, and the covariance matrix Ω. In the case, when $0 \leq \mathrm{r} \leq \mathrm{p}$, the parameters α and β are p $\times$ r matrices. The co-integration rank or co-fractional rank is known as r. The stationary combinations, or long-run equilibrium, are represented by $\beta' X_t$ in matrix β whereas the columns represent the co-integrating relationships. The parameters in α are the loading or adjustment coefficients, which describe the rate of adjustment for each variable towards equilibrium. In the auto-regressive augmentation, the parameters $\Gamma(=\Gamma_1,...,\Gamma_k)$ control the short-run dynamics of the variables. As a result,Γ_i reflects the short-run behavior of the variables, while $\alpha\beta'$ is the adjustment long-run.

To sum up, we extract critical data by estimating the FCVAR model. By using conditional maximum likelihood, the above model is estimated (for additional information, [ÃŸrregaard Nielsen and Ksawery Popiel, 2018]). Importantly, the model is able to incorporate empirically realistic $I(d)$ long-memory and their fractional co-integration while ensuring that the returns are $I(0)$ by independently parameterized the long-run and the short-run dynamics of the series ([Bollerslev et al., 2013]).

5. DATA AND PRELIMINARY ANALYSIS

Stock indices are tools utilized to represent the worth of shares of corporations that are traded on the stock exchange of a particular nation. It is worth mentioning that the *G7* group consists of the seven largest economies across the globe, namely the United States, Canada, France, Germany, Japan, the United Kingdom, and Italy. Investors attach great significance to stock indices due to their ability to facilitate the evaluation of markets and companies in relation to other countries and markets. Moreover, stock market indices play a crucial role in forecasting the economic growth of a given country. It is noteworthy to mention the specific indicators corresponding to the countries that comprise the G7 group, which include the *S&P 500* for the United States, *S&P TSX* Composite for Canada, *CAC 40* for France, *Nikkei 225* for Japan, *DAX* for Germany, *FTSE 100* for the United Kingdom, and *FTSE MIB* for Italy.

The *S&P 500* Index is a stock market index that tracks the performance of the 500 largest publicly traded companies in the United States. The total market capitalization of the companies included in the index corresponds to approximately 80% of the US capital market. The *S&P 500* is one of the most widely used stock market indices in the world and is often used as a stock market index to track the health and performance of the US economy. In addition, the index is divided into 11 sectors, as well as sub-sectors such as the technology, consumer staples and energy sectors, to provide further insight into the performance of the market as a whole. The *S&P 500* is often used by investors and businesses to track changes in the capital market.

The *S&P/TSX* Composite is Canada's main stock market index, and represents approximately 70% of Canada's total capital market. It evaluates the performance of approximately 230 companies traded on the Toronto Stock Exchange (*TSX*), Canada's largest stock exchange. The *S&P/TSX* Composite was established in 1977, and its structure is based on the market of some of the leading companies in each industry sector. The *S&P/TSX* Composite is important to investors because it represents a wide range of Canada's economy and allows investors to respond to market developments as a whole. It also is a widely used index for performance comparisons and for monitoring the Canadian capital market industry.

The *CAC 40* index is the main index of the Paris stock exchange (Euronext Paris). It includes the largest 40 companies listed on this stock exchange. The index is updated every minute and can be used as a measure of the health of the French economy, as it reflects the value of the leading companies in the country. The value of the index is calculated on the basis of the capitalization of the listed companies, i.e. their total shares, and reflects the performance of the stock market as a whole. The *CAC 40* index is often used as a measure to benchmark the performance of companies from different industries. It is an important indicator for investors who wish to invest in the French stock market, as it allows them to assess the accuracy of their results based on a specific market.

The *Nikkei 225* is Tokyo's main stock market index and represents the 225 largest companies by market capitalization on the Tokyo Stock Exchange. The *Nikkei 225* is often used to monitor the sentiment of the Japanese economy and makes investment decisions in the Japanese market. The index is a particular importance to companies that play a leading role in the robotics, telecommunications and automotive markets, as many of them are listed on the Tokyo Stock Exchange. The *Nikkei 225* also is used as a benchmark of the Japanese stock market against other markets in the world and provides the investors with information on the volatility of its companies' shares.

The *DAX* (Deutscher Aktienindex) is Germany's stock index, which tracks the performance of the 30 largest companies per capita traded on the Frankfurt Stock Exchange. The *DAX* index is the largest stock index in Europe and one of the most representative indices in the world. The value of the index reflects the development of the German economy and subsequently the European economy, as many of the companies included in the index have activities in several European countries. *DAX* is one of the most reliable indices in the stock market world.

The *FTSE 100* is the stock market index of the London Stock Exchange, comprising the 100 largest companies by market capitalization in the United Kingdom. The *FTSE 100* is often used as a measure of a country's economic growth and the level of business sector development. The performance of the index provides an insight into the state of the UK share market and is an indicator of the effective stewardship of its constituent companies. An increase in the *FTSE 100* mostly indicates a positive development of business activities in the UK and correspondingly a decrease in it indicates a negative trend in the economy.

The *FTSEMIB* index is the main stock index of the Milan Stock Exchange in Italy. This index consists of the largest and most valuable companies in Italy that are traded on the stock exchange. The *FTSEMIB* acts as a measure of the performance of the Italian stock market. This indicator gives investors a good picture of the country's economic situation and the health of the Milan stock market. Overall, the *FTSEMIB* index is an important tool for evaluating the stock market in Italy and monitoring the emerging trends of the country's economy.

For the purpose of conducting our empirical analysis, we employed the closing prices of the seven stock market indices belonging to the esteemed G7 group, which comprises the world's most economically influential nations. To obtain the necessary data, we meticulously collected the relevant information from the reputable website known as *Investing*, a platform widely recognized for its comprehensive coverage of financial markets. Moreover, in order to ensure the comprehensiveness and accuracy of our dataset, we sought to address any gaps in our records by extracting additional data from the esteemed W*all Street Journal (WSJ)* website. This endeavor was undertaken specifically to fill in any missing years pertaining to certain stock indices. Consequently, our meticulously curated series of data spans a daily timeframe, commencing from the month of January in the year 2000, and extending all the way through to the month of July in the year 2023. The sheer magnitude of our dataset is truly impressive, amounting to a staggering 4454 observations, thereby ensuring a robust foundation for our empirical analysis.

According to the *Table 5.1*, we offer the dynamics of our series along with the accompanying descriptive data. Standard deviation provides a quantified estimation of the uncertainty surrounding future returns. Skewness is a measurement of the distortion of a symmetrical distribution or the asymmetry within a given dataset. Skewness is illustrated on a bell curve when the data points are not symmetrically distributed on both sides of the median. Negative skew signifies a longer or fuller tail on the left side of the distribution, whereas positive skew implies a longer or fuller tail on the right side. Kurtosis is a statistical measure employed to depict a characteristic of a dataset. When normally distributed data is graphed, it typically assumes the shape of a bell. Distributions with a large kurtosis possess more tail data compared to normally distributed data, causing the tails to converge towards the mean. Conversely, distributions with low kurtosis exhibit a scarcity of tail data, which causes the tails of the bell curve to move away from the mean. For investors, a high kurtosis of the return distribution curve suggests that there have been numerous price fluctuations in the past, whether positive or negative, deviating from the average returns for the investment. Therefore, an investor may encounter significant price fluctuations when dealing with an investment characterized by high kurtosis. This phenomenon is commonly referred to as kurtosis risk. In our analysis, it is evident that the *SP&500* exhibits the lowest value among all the indices, while the *FTSEMIB* boasts the highest value. Consequently, there exists a substantial disparity between the two. The stock's skewness, namely *SPTSX* and *FTSE 100*, manifests as a negative value, indicating that the tail is more pronounced on the left side rather than the right. Therefore, the most extreme values are predominantly located to the left. Conversely, the remaining five indicators demonstrate a positive value in this regard. Notably, all indicators possess a kurtosis value below three, denoting their classification as Platykurtic. Platykurtic distributions have exhibited greater stability compared to other curves, as extreme price fluctuations have been infrequent in the past. As a result, a level of risk below moderate is observed.

Variable	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
CAC40	4411.06	914.8464	0.239443	-0.708473	2403.04	6856.76
FTSEMIB	26073.48	8862.7030	0.804572	-0.411982	10428	**50108.56**
S&P500	1539.282	547.5851	0.983891	-0.045781	**676.53**	3013.77
SPTSX	11844.91	2839.125	**-0.309681**	-1.018580	5695.33	16669.4
NIKKEI225	14049.79	4287.24	0.412852	-0.982922	7054.98	24270.62
DAX	7314.05	2804.447	0.477586	-0.793069	2202.96	13559.6
FTSE100	5846.25	1003.29	**-0.269555**	-0.692242	3287	7877.45

Table 5.1: **Descriptive statistics for the options data**

It is worth mentioning in this context that the two periods of crisis, as well as their durations, are determined by significant financial and economic events that have unfolded throughout history. To begin with, let us establish the first crisis period, known as the Great Financial Crisis (*GFC*), which spans from September 16, 2008, when Lehman Brothers infamously filed for bankruptcy, to March 31, 2009. This timeline has been officially recognized by esteemed institutions such as the Federal Reserve Board of St. Louis (2009) and the Bank for International Settlements (BIS, 2009). These institutions have meticulously documented the significant financial and economic events that transpired during this tumultuous period in their comprehensive timelines.

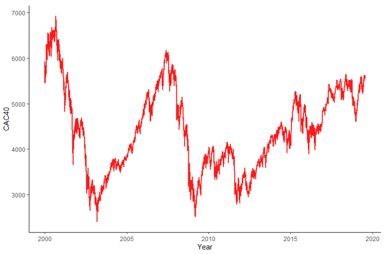
It is also noteworthy to mention that one of the pivotal moments in the aftermath of the *GFC* was the introduction of a monumental 1$ trillion package of reforms for the global financial industry, which was unveiled during the *G20* Summit on April 2, 2009. This crucial development played a vital role in restoring investor confidence and signifying the end of the global financial crisis. Such a significant event undoubtedly left a profound impact on the global economy and marked a turning point in the trajectory of financial recovery.

Moving forward, our analysis will focus on a comparative study of the American index, specifically the *S&P500*, in relation to the other six countries within the esteemed Group of Seven. By utilizing the *S&P500* as our benchmark, we aim to gain insights into the variations of indicators over time and discern the intricate dynamics at play within each index. This comparative approach will provide us with a comprehensive understanding of the movements and fluctuations within the time series, ultimately shedding light on the broader implications and ramifications for the global economy.

To facilitate our analysis, we have included a series of figures below, which effectively illustrate the dynamic nature of each index in relation to the benchmark index, namely the *S&P500*. These figures visually depict the intricate and nuanced patterns that emerge, enabling us to delve deeper into the underlying dynamics and intricacies of the respective indices. By meticulously examining these figures, we can glean valuable insights and discern the subtle trends that shape the trajectory and performance of each index.

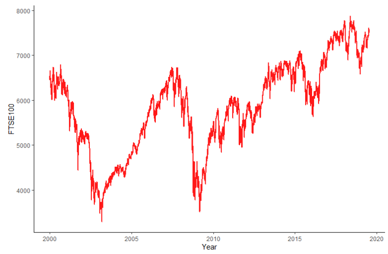
In summary, the periods of crisis that have unfolded throughout history are inherently connected to significant financial and economic events that have made a lasting impact on the global stage. The Great Financial Crisis, specifically, serves as a defining moment in economic history, with its official timeline extending from the consequential day of Lehman Brothers' declaration of bankruptcy to the subsequent actions taken to restore investor confidence. By conducting a comparative analysis of the American index and its counterparts within the Group of Seven, we can attain a comprehensive comprehension of the variations and dynamics that shape the performance of each index. *Figure 5.1* and *5.2* highlights the inclusion of visual representations, which further enhances our capability to decipher the intricate patterns and trends that underlie these indices, enabling us to draw meaningful conclusions and gather valuable insights into the global financial landscape.

Figure 5.2 deserves mention in that the indices for France and Italy are currently exhibiting a noticeable decline, while the indices for Canada, Germany, England, Japan, and the United States of America are notably displaying an upward trend. Additionally, it is of importance to highlight the significant disparity between the indices of the United States and Italy, whereas the distinction between the indices of the United States and France is comparatively smaller in magnitude. By directing our attention specifically to the index for the United States of America, it becomes evident that it consistently demonstrates a progressive inclining pattern with only minor fluctuations. Consequently, in order to offer a comprehensive visual representation of these trends, we present a graph that encompasses all seven of the aforementioned nations.



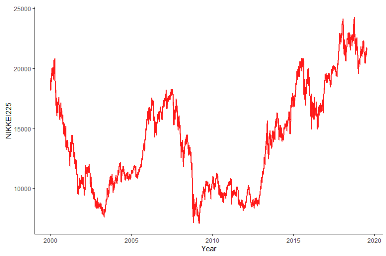
(a) CAC40

(b) DAX



(c) FTSE100

(d) FTSEMIB

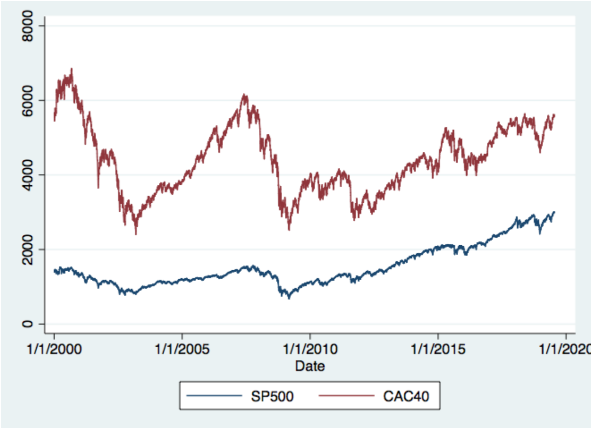


(e) NIKKEI225

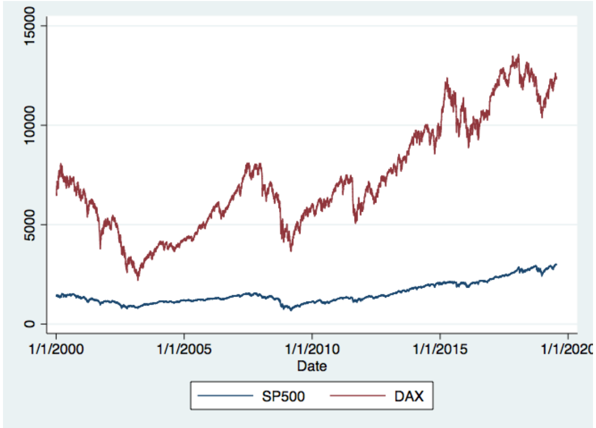
(f) SPTSX

(g) S&P500

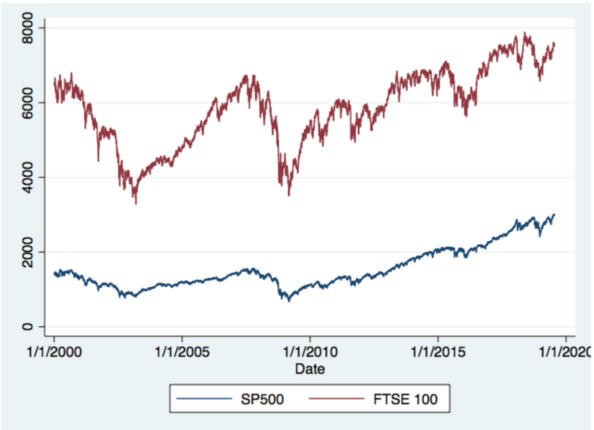
Figure 5.1: **Time Series Plot for each index.**



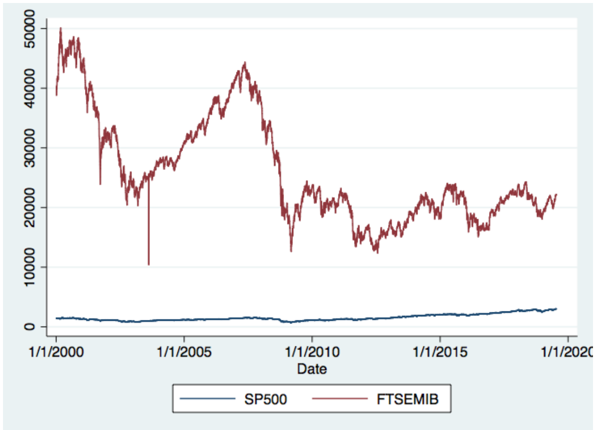
(a) S&P500-CAC40



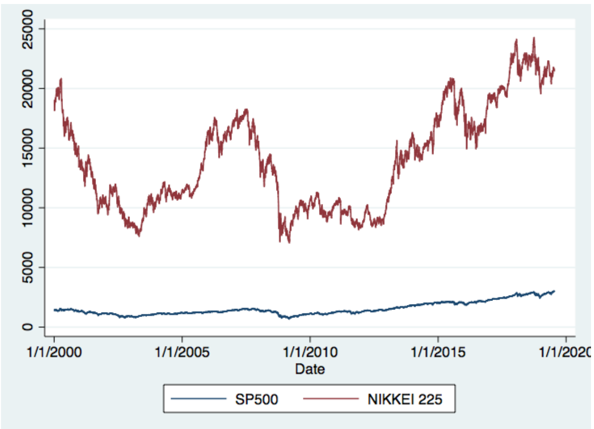
(b) S&P500-DAX



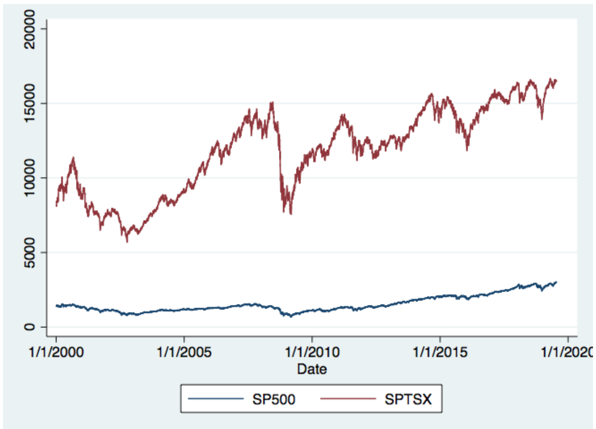
(c) S&P500-FTSE100



(d) S&P500-FTSEMIB



(e) S&P500-NIKKEI225



(f) S&P500-SPTSX

Figure 5.2: **Time Series Plot for each pair indicators.**

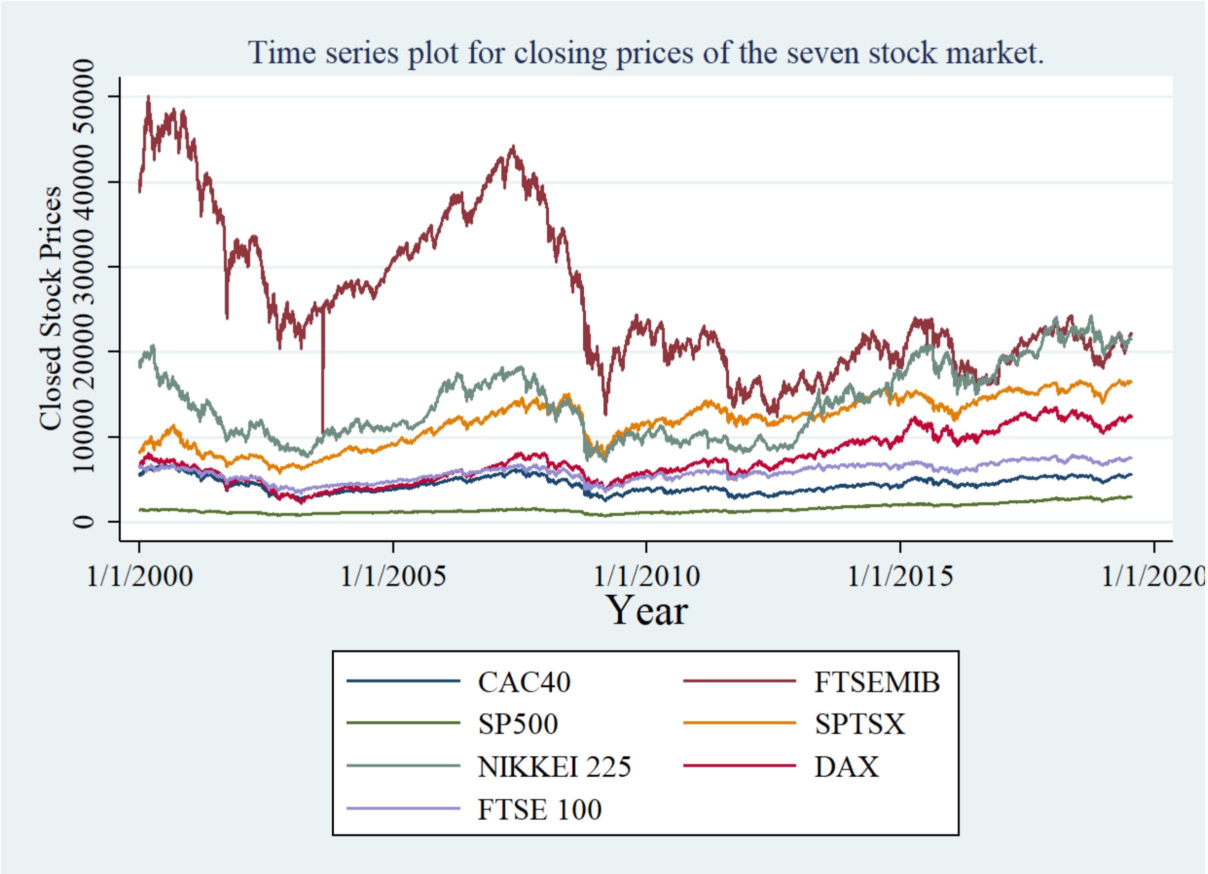


Figure 5.3: **Seven Stock Market**

5.1 Log Return Stock Price

In the realm of finance, the concept of "log return" is used to denote the percentage change in the natural logarithm of a stock price during a specified time interval. This particular metric serves as a means to quantify the rate of return on an investment in a stock, taking into consideration both the appreciation of capital and the dividends received, all the while accounting for the compound effects that occur over time. The utilization of log returns proves to be especially advantageous when examining stocks that exhibit highly variable price movements, as it provides a more steadfast and precise gauge of performance compared to the customary arithmetic returns. Log returns prove to be invaluable in the task of comparing returns on stocks that possess divergent initial prices. Due to the fact that log returns are denoted as a percentage alteration in the natural logarithm of the stock price, they remain unaffected by the stock's initial price. As such, it becomes easier to make accurate comparisons between returns on stocks that possess vastly different initial prices. Moreover, log returns serve as a powerful tool for analyzing the volatility of stock prices. By virtue of being calculated using the natural logarithm, log returns prove to be less reactive to significant price movements when compared to the traditional arithmetic returns. This endows them with the capacity to furnish a superior measure of volatility, especially in the case of stocks that exhibit highly variable price movements. Furthermore, log returns can also be employed to calculate the anticipated return on a stock, taking into account its historical volatility and the risk-free rate of return. Furthermore, log returns can also be harnessed to compute the beta coefficient, which is a metric that serves as an indicator of a stock's volatility in relation to the overall market. The beta coefficient is derived by conducting a regression analysis on the log returns of the stock against the log returns of a comprehensive market index, such as the *S&P500*. A beta coefficient exceeding 1 suggests that the stock in question is more volatile than the market, whereas a beta coefficient less than 1 indicates that the stock is less volatile than the market.

Table 5.2 provides descriptive statistics for log returns over a 23-year period. Each data series exhibits skewness, kurtosis, and standard deviation. Kurtosis is a statistical measure utilized to elucidate an inherent characteristic within a given dataset. By examining the kurtosis, one can discern the degree to which the data is concentrated within the tails. Notably, the *FTSEMIB* index demonstrates a higher degree of kurtosis in comparison to the other indices. Conversely, the remaining indicators display values above three, denoting greater stability than other curves due to the infrequent occurrence of extreme price movements in the past. Consequently, this translates to a level of risk that is less than moderate. Within the realm of probability theory and statistics, the notion of asymmetry arises as a method of quantifying the extent of asymmetry present in the probability distribution of a real-valued random variable relative to its mean. It is crucial to note that skewness itself can assume several states, including positive, zero, negative, or even undefined. When considering a unimodal distribution, a negative skewness value typically indicates that the tail of the distribution is situated on the left side, while a positive skewness value suggests that the tail is positioned on the right side. It is intriguing to observe that in this specific scenario, the *CAC40* and *FTSEMIB* indices exhibit positive values, resulting in a more prominent right-sided tail. As the distribution is positive, the assumption is that its skewness value is positive as well. Consequently, the majority of values fall short of the average value, implying that the most extreme values are situated on the right side. As an investor, one may experience small losses with a positive slope, but also have the potential to generate substantial profits, albeit to a lesser extent. Conversely, the remaining indicators exhibit a negative slope, signifying the opposite pattern. Standard deviation serves as a significant measure of investment risk. It is a fundamental mathematical concept that quantifies market volatility or the average deviation of individual data points from the mean. Higher standard deviation values indicate greater volatility, while lower values reflect a relatively calm market with minimal fluctuations. In the case of all indicators, prices are low, thus indicating the absence of significant price fluctuations.

Variable	Mean	Std.Dev	Skewness	Kurtosis	Min	Max
CAC40	-0.000004	0.014951	**0.036467**	5.998425	-0.094715	0.133048
FTSEMIB	-0.000135	0.024822	**0.200601**	**760.287066**	-0.892103	0.899269
S&P500	0.000170	0.012449	-0.262789	7.809632	-0.094695	0.104236
SPTSX	0.000157	0.011225	-0.590818	9.789625	-0.097880	0.093703
NIKKEI225	0.000027	0.015434	-0.410508	6.158925	-0.121110	0.132346
DAX	0.000141	0.015353	-0.006719	5.312869	-0.095756	0.134627
FTSE100	0.000032	0.012175	-0.053202	7.615678	-0.092646	0.111112

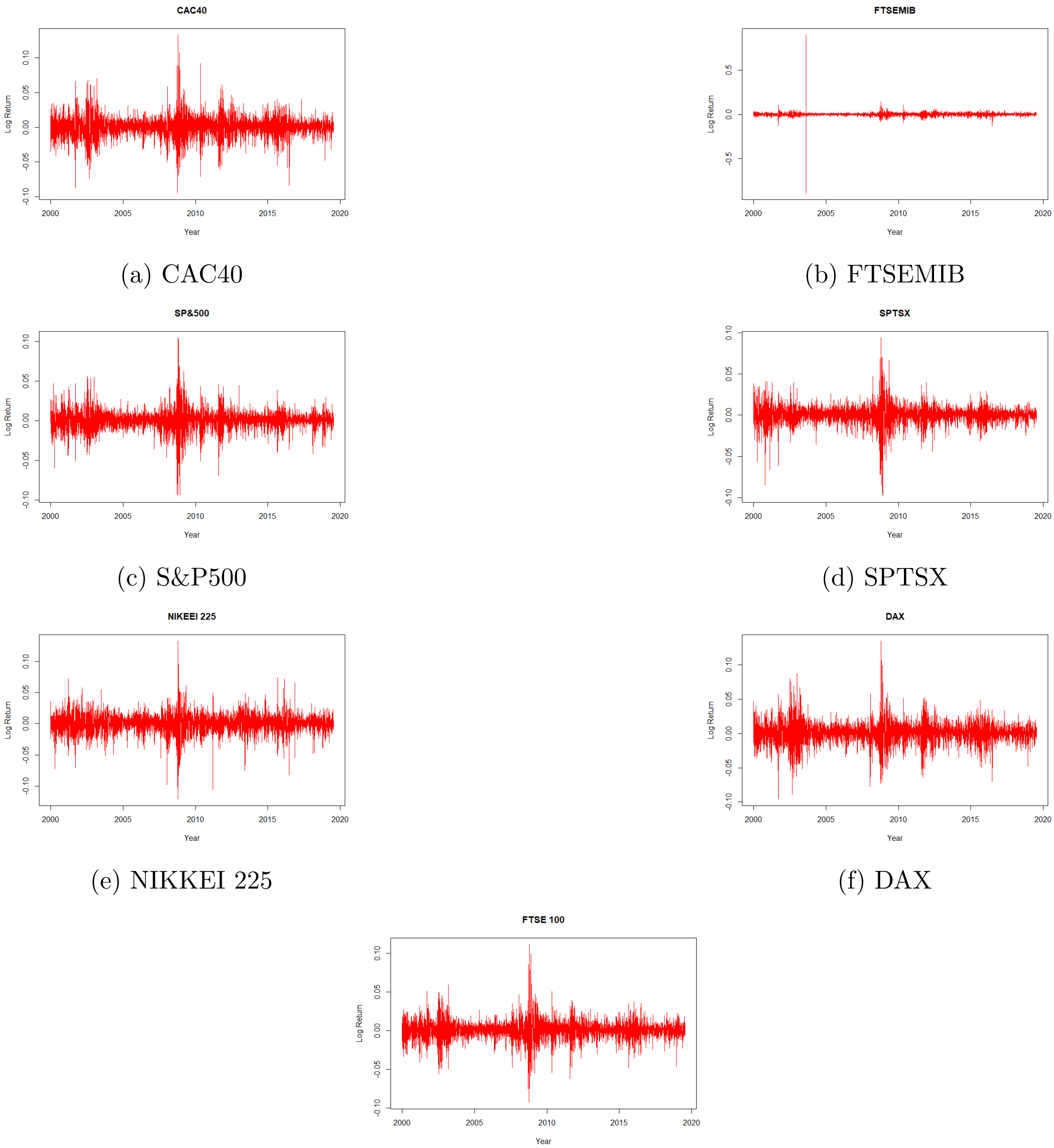
Table 5.2: **Descriptive data for 23-year log returns**

The global financial crisis of 2008, which is commonly referred to as the Great Recession, was an immensely consequential economic event that had profound implications for the worldwide economy. This crisis was precipitated by the collapse of the United States housing market, a cataclysmic event that instilled a pervasive sense of doubt in the financial system and subsequently resulted in a credit crunch of unprecedented proportions. Within the context of this crisis, the *G7*, a group comprised of the world's leading industrial nations, played a pivotal role in fashioning a response that aimed to mitigate the damaging effects of the crisis.

In the field of finance, the concept of return refers to the profit gained from an investment. The logarithmic return is a method used to determine the rate of return on an investment. This calculation takes into consideration the duration of the investment, allowing for the comparison of investments of varying lengths. However, when it comes to comparing our investment with others that offer a fixed interest rate, such as bank savings accounts, the logarithmic return is not as useful. This is due to the fact that fixed interest rates are typically quoted on an annual compounded basis, while the logarithmic return is continuously compounded. Log returns are commonly employed as a means of measuring investment performance, as they can be added together over time. It is important to note that a loss, rather than a profit, is indicated by a negative return, assuming that the initial investment amount is greater than zero. Return, in this context, measures the growth of an asset or liability or the decrease in value of a short position. A negative initial value is often associated with a liability or short position. If the initial value is negative and the final value becomes even more negative, a positive return will be observed. In such a scenario, the positive return represents a loss rather than a profit.

The fluctuations that have been observed in these indicators serve to bring attention to the intricate and multifaceted nature of the global financial crisis that occurred in 2008. Furthermore, it highlights the diverse array of impacts that this crisis had on different regions and sectors of the global economy. The presence of volatility in these indices necessitates a comprehensive and nuanced analysis in order to fully comprehend the intricacies and implications of this exceptional economic event. Thus, it is of utmost importance that both researchers and policymakers continue to delve deep into the complexities of this crisis, with the purpose of obtaining a more comprehensive understanding of its causes, consequences, and potential avenues for future prevention. It is crucial to acknowledge that all indicators during this time period were characterized by a significant degree of variation. In instances where the values of these indicators display negativity, it is important to take note of the losses that occur, provided that the initial investment amount surpasses zero.

The Figure 5.3 the logarithmic returns of the stock indices used in the statistical analysis. A pattern that emerges is that volatility increased during crisis.



(a) CAC40

(b) FTSEMIB

(c) S&P500

(d) SPTSX

(e) NIKKEI 225

(f) DAX

(g) FTSE 100

Figure 5.4: **Logarithmic returns of the stock indices**

6. FCVAR MODEL AND ESTIMATION

6.1 Empirical Analysis

In the present investigation, we proceed with the implementation of the *FCVAR* model proposed by [Johansen, 2008] and further developed by [Johansen and Nielsen, 2012] [Johansen and Nielsen, 2016] in order to analyze the indicators of seven distinct countries.

In the course of our empirical analysis, we aim to assess the impact of the correlation between different stock indices by considering a pair of indices at each instance. It is important to note that, during this process, we consistently correlate the US index while keeping it constant with the remaining six indices. The outcomes of all the pair bivariate *FC-VAR* models are presented in the tabular form, providing a comprehensive overview of the findings.

In the traditional VAR model, it becomes imperative to explicitly specify the augmentation of the lag structure of the system subsequent to undertaking an analysis of the presence of long memory in time series in an identical fashion. In this particular scenario, a lag value of $k = 3$ is employed and the generic to specific order proposed by [Jones et al., 2014] is adhered to. The determination of the subsequent lag level is predicated on an evaluation of the statistical significance of the coefficients in the highest order lag. The Table 6.1 provided herein showcases four estimations for each variable at the respective lag levels. The statistical significance of each level is established through the employment of the likelihood ratio (LR) test. In the event that the null hypothesis is not refuted, we proceed to the subsequent model by eliminating the highest order lag.

Table 6.1 showcases a comprehensive overview of the model specification in the first panel, which serves as a crucial foundation for further analysis. The second panel, on the other hand, presents an extensive list of lags (k) in the first column, thereby facilitating a more comprehensive understanding of the variables under consideration. The model itself is comprised of not one, but two variables (r), emphasizing the complexity and interplay between these elements. Additionally, one cannot overlook the significance of the fractional integration order (d) in this model, as it plays a pivotal role in capturing the intricacies and nuances of the data.

Moreover, the reported values of the cointegration order (b) are meticulously estimated for each lag, taking into account both the rank (r) and fractional integration order (d). This level of granularity allows for a more precise and nuanced understanding of the underlying dynamics at play. Moving forward, the fifth column in Table 6.1 is dedicated to showcasing the Log-likelihood ($LogL$) statistics for each lag, further augmenting the comprehensiveness and rigor of the analysis.

Furthermore, the likelihood ratio (*LR*) statistic and its corresponding p-values (*pv*) under the null hypothesis Γ_k=0 are diligently reported in the subsequent two columns. This presentation of statistical significance adds another layer of robustness and reliability to the analysis, ensuring that any conclusions drawn are firmly supported by empirical evidence. Additionally, the inclusion of the *AIC* and *BIC* information criteria in *Table 6.1* serves as a valuable benchmark for evaluating the model's performance and selecting the most appropriate specification. The Akaike Information Criterion (*AIC*), a statistical technique employed for the purpose of model selection, serves to facilitate the comparison of candidate models, ultimately leading to the identification of the most optimal option. The primary objective of the *AIC* is to determine the model capable of providing the most comprehensive explanation of the variability observed in the dependent variable, all while incorporating the fewest number of independent variables, also referred to as parameters. In essence, the *AIC* aims to favor the selection of simpler models, characterized by a smaller number of parameters, over more complex counterparts. In the field of statistics, the Bayesian information criterion (*BIC*) or Schwarz information criterion (alternatively referred to as *SIC*, *SBC*, or *SBIC*) assumes the role of a criterion for model selection within a finite set of models, with a notable preference being given to models exhibiting a lower BIC. The Bayesian information criterion (BIC) itself functions as a criterion for model selection within a limited set of models, drawing upon the likelihood function as a fundamental component and establishing a close relationship with the Akaike information criterion (*AIC*). Despite being derived from distinct perspectives, these two measures exhibit a close association. It is important to note that the sole divergence between the two lies in the incorporation of the number of observations within the formula for the BIC, a consideration that the *AIC* does not account for. Consequently, a lower *AIC* value corresponds to a superior model fit, with the absolute value of the *AIC* being of no particular consequence, as it may assume either a positive or negative value. Although the BIC consistently surpasses the *AIC* in magnitude, the superiority of a model is predicated upon the attainment of lower values for both of these measures.

Lastly, the remaining columns in *Table 6.1* provide p-values for white noise tests conducted on the residuals, further enhancing the overall reliability and validity of the analysis. Specifically, the first p-value corresponds to the multivariate Q-tests (*pmvQ*), which serve as a comprehensive assessment of the residuals' adherence to the white noise assumption.

To provide a more detailed assessment, the subsequent p-values, namely *pQ1* and *pLM1*, represent the univariate *Q-tests* and *LM* tests for the residuals in the first equation, respectively. Similarly, *pQ2* and *pLM2* signify the p-values for the same tests conducted on the residuals in the second equation. These additional tests ensure that the residuals conform to the white noise assumption, thereby reinforcing the credibility and accuracy of the model's findings.

Table. Lag Selection Result

	k	*r*	*d*	*b*	*log-likelihood*	*LR*	*p-value*	*AIC*	*BIC*	*pmvQ*	*pQ1*	*pLM1*	*pQ2*	*pLM2*
S&P500-CAC40	3	2	0.944	1.014	27068.75	3.08	0.545	-54097.51	-53969.49	0.00	0.00	0.64	0.00	0.20
	2	2	0.128	0,361	27067.21	106.22	0.000	-54102.43	-54000.01	0.00	0.00	0.72	0.00	0.21
	1	2	0.224	0.477	27014.11	-93.03	1,000	-54004.21	-53927.40	0.00	0.00	0.08	0.00	0.22
	0	2	0.692	0,766	27060.62	0.00	0.000	-54105.24*	-54054.03*	0.00	0.00	0.67	0.00	0.22
S&P500-FTSEMIB	3	2	0.218	0.405	24252.16	6.17	0.187	-48464.32	-48336.30	0.09	0.00	0.69	0.66	0.17
	2	2	0.966	1.030	24249.08	28.43	0.000	-48466.15*	-48363.73	0.02	0.01	0.74	0.28	0.05
	1	2	0.950	1.032	24234.86	-9.39	1.000	-48445.72	-48368.91	0.00	0.00	0.65	0.02	0.02
	0	2	0.077	0.010	24239.56	0.00	0.000	-48463.11	-48411.90*	0.00	0.00	0.73	0.00	0.01
S&P500-SPTSX	3	2	1.000	1.000	28913.73	15.20	0.004	-57787.46*	-57659.44	0.00	0.04	0.83	0.04	0.91
	2	2	0.983	1.015	36332.35	13.93	0.008	-72632.71	-72530.29	0.00	0.04	0.71	0.01	0.75
	1	2	0.979	1.017	36325.39	12.86	0.012	-72626.78	-72549.96	0.00	0.01	0.72	0.00	0.79
	0	2	0.968	1.030	36318.96	0.00	0.000	-72621.91	-72570.70*	0.00	0.00	0.72	0.00	0.80
S&P500-NIKKEI225	3	2	0.976	1.002	26206.27	16.80	0.002	-52372.54*	-52244.52	0.57	0.08	0.90	0.97	1.00
	2	2	0.971	1.011	26197.87	10.99	0.027	-52363.74	-52261.32	0.07	0.01	0.74	0.91	1.00
	1	2	0.967	1.021	26192.38	27.83	0.000	-52360.75	-52283.94	0.02	0.01	0.73	0.68	0.97
	0	2	0.962	1.027	26178.46	0.00	0.000	-52340.92	-52289.71*	0.00	0.00	0.69	0.41	0.95
S&P500-DAX	3	2	0.976	1.026	34355.02	10.12	0.038	-68670.03*	-68542.01	0.00	0.04	0.83	0.04	0.91
	2	2	0.983	1.015	28906.13	13.93	0.008	-57780.26	-57677.84	0.00	0.01	0.71	0.01	0.75
	1	2	0.979	1.017	28899.16	12.86	0.012	-57774.33	-57697.51	0.00	0.01	0.72	0.00	0.79
	0	2	0.968	1.030	28892.73	0.00	0.000	-57769.46	-57718.25*	0.00	0.00	0.72	0.00	0.80
S&P500-FTSE100	3	2	0.976	1.026	26928.79	10.11	0.039	-53817.57*	-53689.55	0.00	0.00	0.58	0.00	0.39
	2	2	0.980	1.017	26923.73	10.71	0.030	-53815.46	-53713.04	0.00	0.01	0.72	0.00	0.24
	1	2	0.977	1.018	26918.38	17.47	0.002	-53812.76	-53735.94	0.00	0.00	0.70	0.00	0.22
	0	2	0.962	1.032	26909.64	0.00	0.000	-53803.29	-53752.08*	0.00	0.00	0.66	0.00	0.15

Table 6.1: **Note. LR** = likelihood ratio; **AIC** = Akaike information criterion; **BIC** = Bayesian information criterion. *Indicate the best (that is, minimized) values of the respective information criteria.

In conclusion, the preference lies with lag 1 for the pair composed of the *S&P500* and *CAC40*. Lag 3 is the preferred choice for the pair consisting of the *SP&500* and *FTSEMIB*. Similarly, lag 3 is chosen for the pair made up of the *S&P500* and *SPTSX*, as well as the pair formed by the *S&P500* and *NIKKEI225*. The same preference for lag 3 is observed in the pair consisting of the *S&P500* and *FTSE100*. Lastly, lag 2 is preferred for the pair composed of the *S&P500* and *DAX*. Expanding our analysis further, we turn to the LR statistic and its corresponding p-value, which indicate that Γ_2 is statistically significant. This is evident as it rejects the null hypothesis of Γ_2=0 at a 10% level of significance.

The subsequent stage involves choosing the suitable rank, which is the total number of cointegrated vectors in the system, after determining the best fitting lag level for every model. We investigate the following hypothesis concerning the rank of systems: H_0: rank

= r and H_1: rank = p, where p is the total number of variables in the system and r = 0,1, 2,... The number of cointegrated vectors in the system is the first non-rejected value for the alternative ranks. The initial section of Table 6.2 provides the details of the model specification. Subsequently, the subsequent section presents the first column, denoting the roster of ranks. Each rank is characterized by a distinct fractional integration order, designated as d. Additionally, the cointegration order, represented by b, is estimated for each individual rank. The ensuing columns then proceed to showcase the log-likelihood and likelihood ratio (LR) tests corresponding to each rank.

Furthermore in *Table 6.2*, it is worth noting that if the fractional cointegration order b is greater than 1/2 and the p-values of the *Ljung-Box Q tests* are high, it means that the null hypothesis of white noise (no serial correlation) cannot to be rejected. Therefore, considering all these criteria collectively, we can confidently suggest that the appropriate model selection is based on the largest lag in all indicator pairs except the pair (*S*&5*P00-CAC40*). This integrated approach ensures that we have considered the various factors contributing to the analysis, resulting in a robust and reliable model selection.

Table LR Tests for Cointegrating Rank.

	Rank	*d*	*b*	Log-Likehihood	LR statitsic	*p value*
S&P500-CAC40	0	0.011	0.010	26914.529	199.153	—
	1	0.010	0.470	27055.229	-82.246	—
	2	0.224	0,477	27014.105	—	—
S&P500-FTSEMIB	0	0.067	0.010	24239.555	-9.390	—
	1	0.010	0.199	24232.851	4.018	—
	2	0.950	1.032	24234.860	—	—
S&P500-SPTSX	0	0.010	0.368	28890.067	18.194	—
	1	0.010	0.527	28889.803	18.722	—
	2	0.979	1.017	28899.164	—	—
S&P500-NIKKEI225	0	0.010	0.509	26196.211	-7.671	—
	1	0.010	0.742	26187.805	9.141	—
	2	0.967	1.021	26192.375	—	—
S&P500-DAX	0	0.010	0.388	34344.018	1.171	—
	1	0.010	0.711	34335.213	18.781	—
	2	0.977	1.018	34344.603	—	—
S&P500-FTSE100	0	0.010	0.129	26898.399	39.958	—
	1	0.010	0.388	26916.445	3.866	—
	2	0.977	1.018	26918.378	—	—

Table 6.2: **Note.** As it is pointed by Nielsen and Popiel (2014), p values for the cointegration rank tests are not provided by the main code. **LR** = likelihood ratio.

P-values not calculated for the rank test with rank=0 and rank=1. *P*-values are only calculated if: there are no deterministic terms, or there is only restricted constant and d = b, or there is only a level parameter and d = b.

6.2 Fractionally Cointegrated Var: Estimation Results

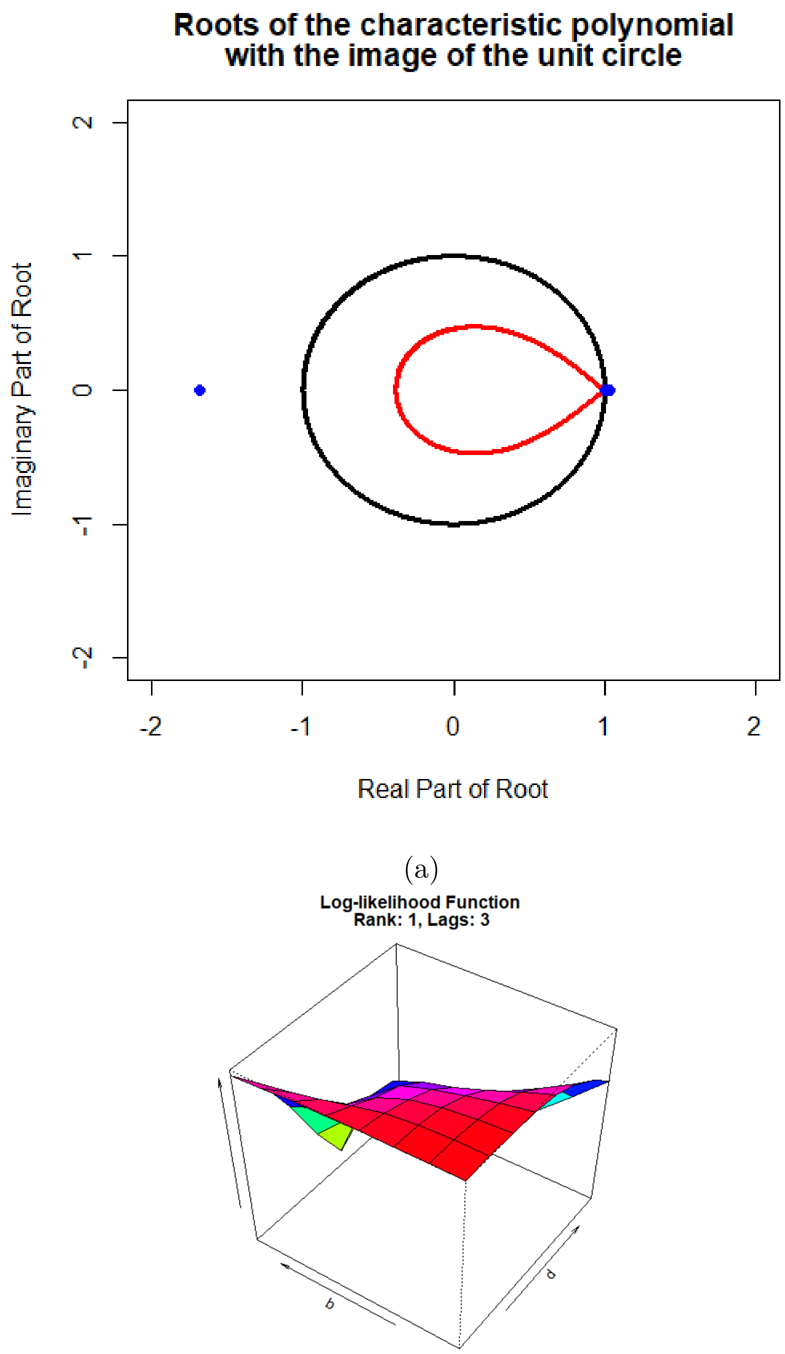
The outcomes of the bivariate FC-VAR models are shown in below tables. The application of the *FC-VAR* model is supported by the results of the fractional parameters of unrestricted model (*Panel A*), White noise tests (*Panel B*), and estimates of fractional parameters d and b (*Panel C*) of the aforementioned tables. These results are more appropriate for the examined pairs of indicators than the *C-VAR* model, which has exactly $d = b = 1$. The (mu) parameter in the level accommodates a nonzero initial value for the initial observation of the process.

FC-VAR results for the pair of S&P500 and CAC40		
Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	0.010	0.470
b (Stand. errors)	NaN	NaN
	S&P500	*CAC40*
CE(beta)	1.000	-1.266
Adj. matrix (alpha)	-0.018	0.033
Stand. errors	NaN	NaN
Level parameters (mu)	-0.000	-0.000
Stand. errors	NaN	NaN
Panel B: White noise tests [H_0: no autocorrelation against H_1: autocorrelation]		
Q	32.463	48.427
p-value	0.001	0.000
LM	9.181	18.592
p-value	0.687	0.099
Multivariate (p-value)	145.868	0.000
Panel C: Restriction tests		
H_0: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		27055.2290
Restricted log-likelihood		26357.855
Test Results (df=1)		
LR statistic		1394.748
P-value		0.000

Table 6.3: **S&P500 and CAC40**

After appropriately selecting the appropriate lag and rank for our systems, we move on to this crucial phase where we aim to evaluate and elucidate the characteristics of our models. In order to delve deeper into the long-term association between each pair of indicators, we meticulously analyze the findings that are presented in the table. The beta coefficient, which serves as an essential indicator, provides valuable insights into the volatility of a stock or portfolio in relation to market volatility. Put simply, it sheds light on the extent to which the price of a stock or the performance of a portfolio fluctuates when faced with movements in the stock market. When we mention the term "*market*", we are typically referring to the primary stock market index of a stock exchange. However, it is worth noting that other benchmarks, such as additional indices or even the prices of fundamental stocks, can also be utilized for analysis purposes. The value assumed by the coefficient plays a significant role in elucidating the correlation between the stock indices. If the beta coefficient surpasses unity, it indicates the presence of a positive correlation between the stock and the index, suggesting that the performance of the stock is more variable than that of the index. On the other hand, if the beta coefficient is lower than unity, it signifies that the share price demonstrates less "*aggressive*" movement in relation to the stock market index, with each change being smaller than that of the index. In the case where the beta coefficient takes on a negative value, it implies that the return on investment moves in the opposite direction to that of the market, indicating a negative correlation between the returns. In the exceptional scenario where the beta coefficient is equal to 0, it indicates that the correlation between the stock and the index is zero, and therefore, the prices move independently of each other. Furthermore, it is important to note that the estimation of beta in the long run is contingent upon the *LR* (Likelihood Ratio) index. The *LR* test, commonly known as the likelihood ratio test, is a comparative examination of two models, primarily focusing on the enhancement of the likelihood value. As additional predictor variables are incorporated into a linear model, the *R-square* value experiences a significant increase, which is also applicable to the probability value associated with the model. All *FC-VAR* models exhibit a cointegration association (r), which suggests the existence of a persistent relationship between the US financial index and the respective financial indices of the remaining six countries within the group. Therefore, based on the beta index (*Table 6.3*) it is positive indicating that this index is hedging during the sample period. Accordingly, based on the beta index in *Table 6.3*, which exhibits a negative value, our analysis leads us to the inference that the relationship between the *S&P500* and *CAC40* pair and market returns is characterized by a negative correlation.

The *Figure 6.1*, we present the roots of the characteristic polynomial of the unit circle as well as the Log likelihood function. It is of utmost importance to note that a system is deemed asymptotically stable if and only if all of its characteristic roots lie within the confines of the unit circle. Conversely, a system is classified as unstable if it satisfies one or both of the following conditions: *i)* at least one root is located outside the unit circle, or *ii)* there exist repeated roots on the unit circle. It is crucial to highlight that in the event that there are no roots present outside the circle, the system can be regarded as stable. In conclusion, the location of the characteristic roots with respect to the unit circle plays a pivotal role in determining the stability of a system. To ensure the dynamic stability of the model, the inverse roots of the AR polynomial were calculated. In the context of the *S&500-CAC40* pair, it is observed that there exists an out-of-cycle root, thereby confirming the dynamic stability of the model in this scenario.



(b)

Figure 6.1: **S&P500-CAC40,Roots,LogL**

FC-VAR results for the pair of S&P500 and FTSEMIB

Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	0.010	0.199
b (Stand. errors)	0.030	0.084
	S&P500	*FTSEMIB*
CE(beta)	1.000	0.235
Adj. matrix (alpha)	0.292	0.257
Stand. errors	0.392	0.332
Level parameters (mu)	-0.001	-0.001
Stand. errors	NaN	NaN
Panel B: White noise tests [H_0: no autocorrelation against H_1: autocorrelation]		
Q	32.495	30.432
p-value	0.001	0.002
LM	9.204	23.063
p-value	0.685	0.027
Multivariate (p-value)	100.587	0.000
Panel C: Restriction tests		
Ho: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		24232.851
Restricted log-likelihood		23478.374
Test Results (df=1)		
LR statistic		1508.954
P-value		0.000

Table 6.4: **S&P500 and FTSEMIB**

The beta estimate for the *FTSEMIB* index, as depicted in the aforementioned *Table 6.4*, showcases an affirmative and statistically noteworthy correlation, thus signifying its role as a protective gauge during the observed temporal realm. The subsequent beta estimates for this particular asset exhibit particularly pessimistic figures in the year 2004. However, as the passage of time ensues, relatively pessimistic valuations resurface once more during the period of the global financial crisis that spanned from 2008 to 2009. The *FTSEMIB* index, on the whole, experienced a substantial and enduring impact stemming from the aforementioned global financial crisis. The recovery of the index to its pre-crisis levels took several years, thereby indicating the persistent challenges faced by the Greek economy throughout this specific time period. Therefore, based on the beta index, it is positive but less than unity which means that this *S&P500* and *FTSEMIB* pair is less volatile than the whole market.

In Figure 6.2, denoted as the pair *S&P500-FTSEMIB*, it is evident that all the roots of both the characteristic polynomial, with the exception of one that resides outside, are located within the confines of the unit circle. This observation holds significant value as it indicates that the model is dynamic stable. Hence, the fact that all roots lie within the unit circle in this particular instance represents a favorable outcome, serving as an indicator of a robust system that possesses the ability to maintain stability over an extended period of time, excluding the point at which the root is observed and instability arises.

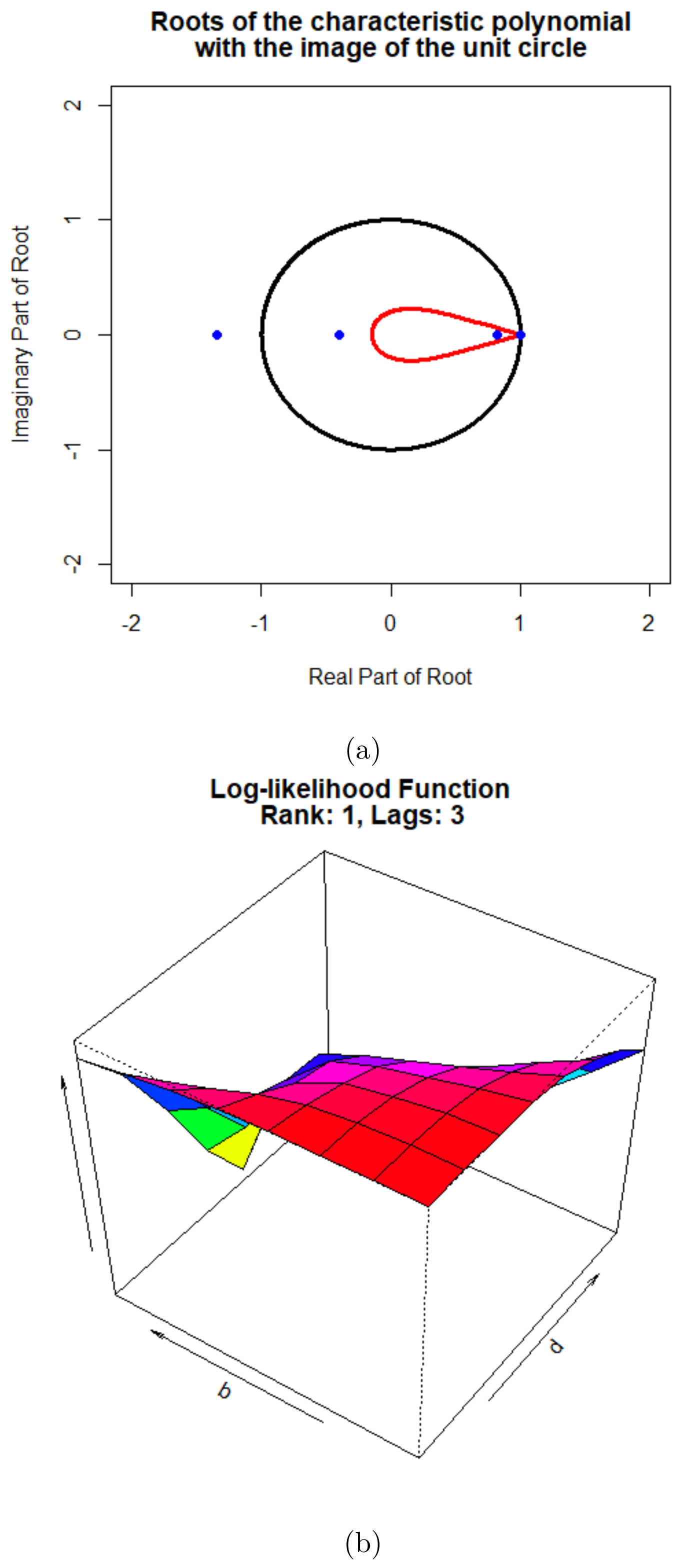


Figure 6.2: **S&P500-FTSEMIB,Roots,LogL**

According to the findings presented in *Table 6.5* pertaining to the *SPTSX* index, it becomes evident that the beta coefficient exhibits a negative sign, thereby indicating a statistically significant relationship. In other words, upon analyzing the beta index of the

S&P500 and *SPTSX* pair, it becomes apparent that it is inversely correlated with market returns. Conversely, when we examine the retrospective price behavior of the index, we are able to discern a discernible pattern characterized by substantial price fluctuations that occurred during the financial crisis.

FC-VAR results for the pair of S&P500 and SPTSX

Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	0.010	NaN
b (Stand. errors)	0.527	NaN
	S&P500	*SPTSX*
CE(beta)	1.000	-9.396
Adj. matrix (alpha)	-0.003	-0.001
Stand. errors	0.002	0.002
Level parameters (mu)	-0.001	-0.000
Stand. errors	0.001	0.000
Panel B: White noise tests [H_0: no autocorrelation against H_1: autocorrelation]		
Q	30.637	36.943
p-value	0.002	0.000
LM	8.883	10.251
p-value	0.713	0.594
Multivariate (p-value)	131.868	0.000
Panel C: Restriction tests		
H_0: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		28889.803
Restricted log-likelihood		28223.183
Test Results (df=1)		
LR statistic		1333.239
P-value		0.000

Table 6.5: **S&P500 and SPTSX**

Based on the findings presented in *Table 6.3*, it is evident that all the roots lie within the bounds of the circle. As a result, the model can be considered dynamically stable.

Turning our attention to the *NIKKEI 225* index, as demonstrated in *Table (6.6)*, we observe that the long-term beta estimate yields a negative result that is highly significant, thus suggesting an unstable dependence between the examined indices and the market. Furthermore, when looking at the retrospective beta estimates, we observe a wealth of variation within the index over the period under investigation, with particular emphasis on the period associated with the financial crisis. These variations are of utmost importance in our analysis.

The presentation of the Log-lihood values can be found in *Figure 6.4*, a comprehensive compilation that offers a detailed overview of the statistical measures obtained through rigorous analysis. As with the other cases observed, all the roots are within the bounds of the circle. This observation leads us to the conclusion that the system can be considered dynamically stable.

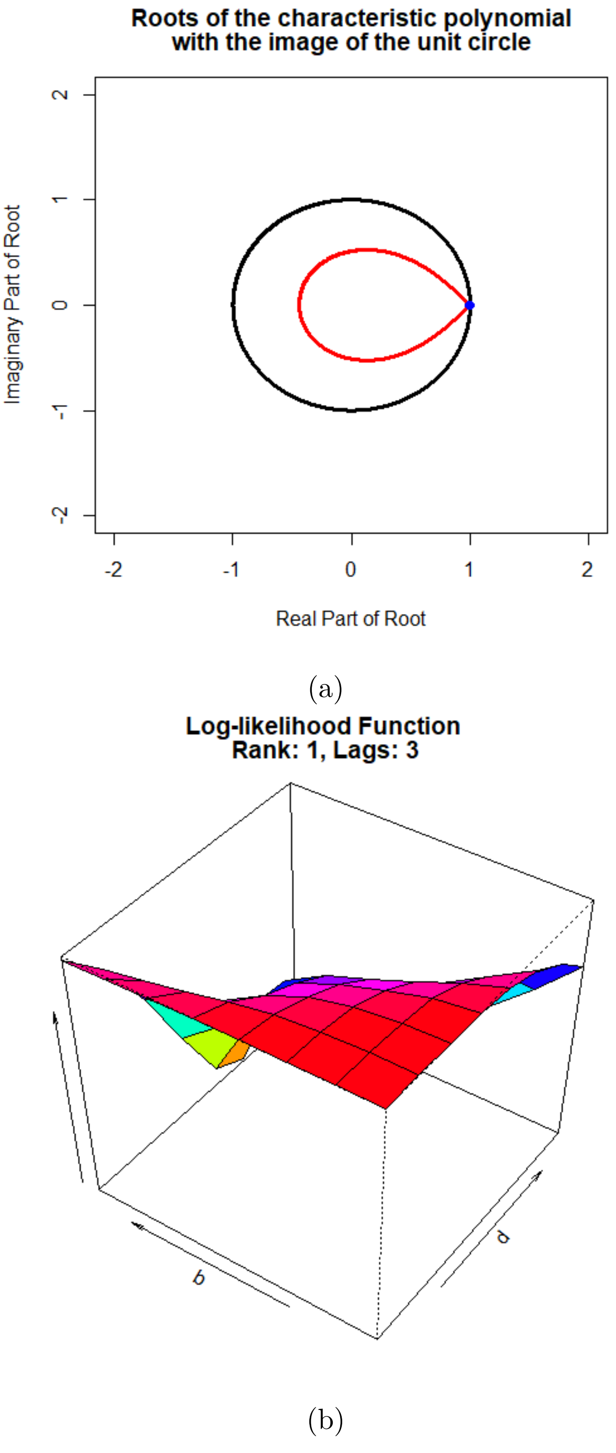
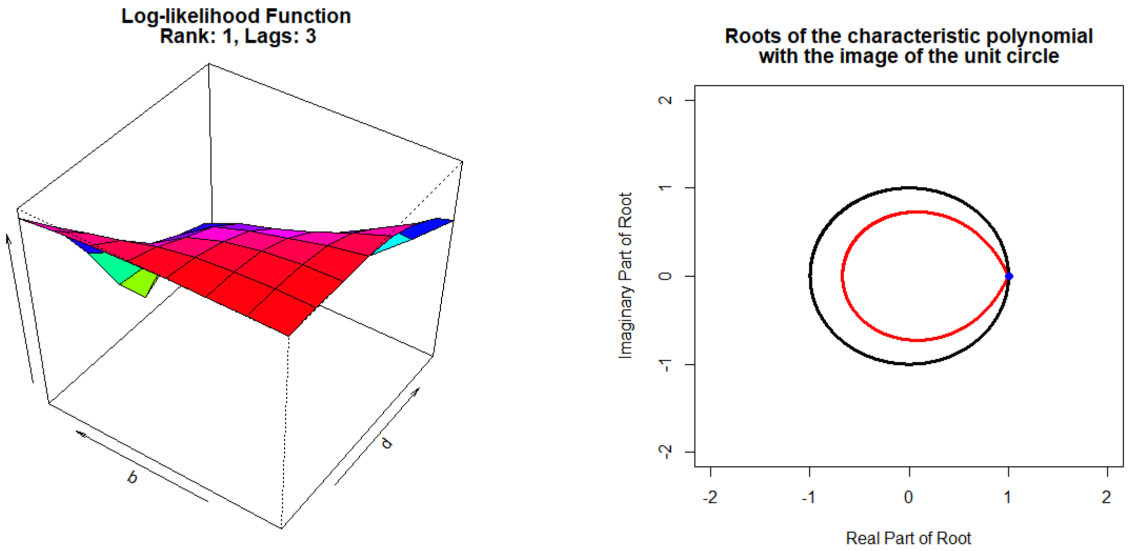


Figure 6.3: **S&P500-SPTSX,Roots,LogL**

The long-term beta estimate for the *DAX* index, as presented in *Table 6.7*, is negative, thus suggesting a weak correlation of the *S&P500* and *DAX* pair with the broader market under consideration. The retrospective beta ratings, in line with the various other indicators evaluated, show a similar pattern of behavior. It is worth noting that the economic crisis was marked by noticeable fluctuations in prices, reinforcing the importance of these

FC-VAR results for the pair of S&P500 and NIKKEI225		
Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	0.010	0.023
b (Stand. errors)	0.742	0.086
	S&P500	*NIKKEI225*
CE(beta)	1.000	-0.835
Adj. matrix (alpha)	0.009	0.018
Stand. errors	0.007	0.010
Level parameters (mu)	-0.000	-0.001
Stand. errors	0.000	0.000
Panel B: White noise tests [H_o: no autocorrelation against H_1: autocorrelation]		
Q	35.052	9.683
p-value	0.000	0.644
LM	9.635	4.076
p-value	0.648	0.982
Multivariate (p-value)	78.741	0.003
Panel C: Restriction tests		
H_0: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		26187.805
Restricted log-likelihood		25577.496
Test Results (df=1)		
LR statistic		1220.618
P-value		0.000

Table 6.6: **S&5P500 and NIKKEI225**



(a) (b)

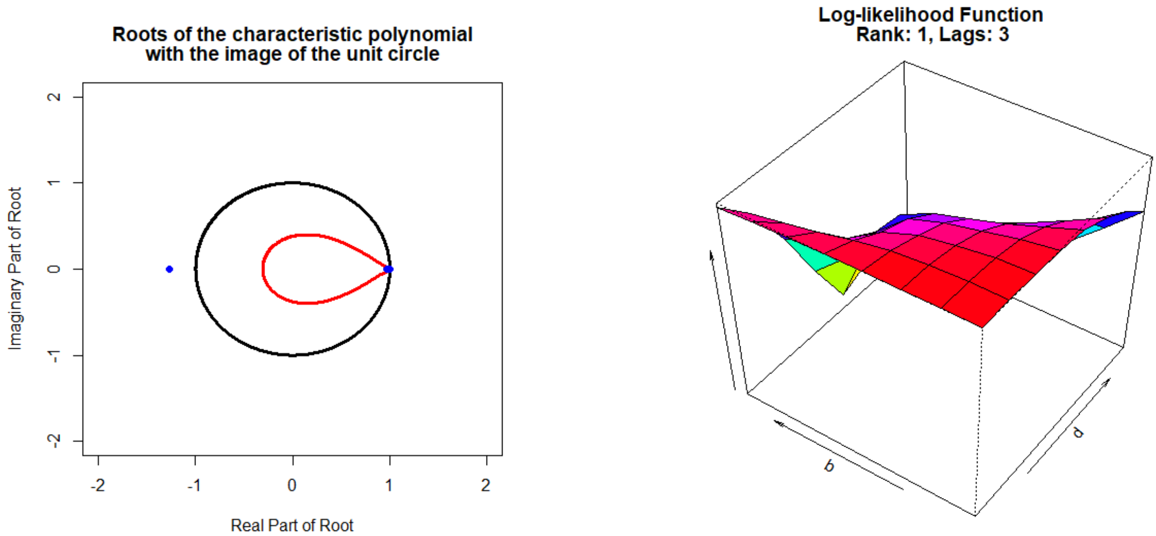
Figure 6.4: **S&P500-NIKKEI225,Roots,LogL**

findings.

Figure 6.6 depicting the Log-likelihood values and the roots of the characteristic polynomial of the unit circle on the pair *SP500-FTSE100*. Here we notice that all the roots are not inside the circle. So the model here is considered dynamically unstable.

FC-VAR results for the pair of S&P500 and DAX		
Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	0.010	0.029
b (Stand. errors)	0.388	0.173
	S&P500	*DAX*
CE(beta)	1.000	-0.708
Adj. matrix (alpha)	0.078	-0.003
Stand. errors	0.111	0.085
Level parameters (mu)	-0.000	-0.000
Stand. errors	0.000	0.000
Panel B: White noise tests [Ho: no autocorrelation against H_1: autocorrelation]		
Q	36.331	44.611
p-value	0.000	0.000
LM	9.808	17.153
p-value	0.633	0.144
Multivariate (p-value)	112.421	0.000
Panel C: Restriction tests		
H_0: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		26916.445
Restricted log-likelihood		26248.191
Test Results (df=1)		
LR statistic		1336.507
P-value		0.000

Table 6.7: **S&P500 and DAX**



(a) (b)

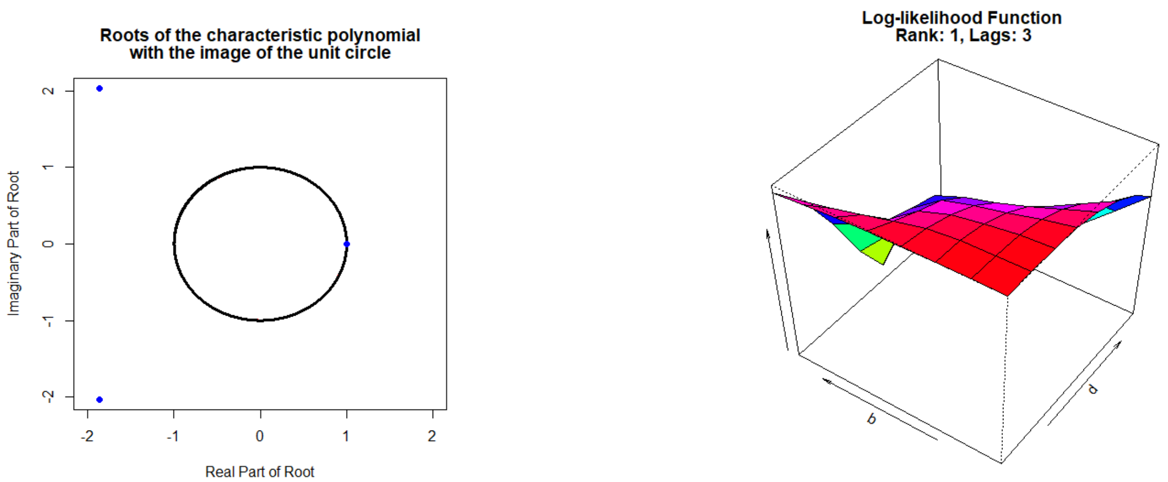
Figure 6.5: **S&P500-DAX,Roots,LogL**

The *FTSE 100* stock index *(Table 6.8)*, finally, shows a beta that is also negative, thus highlighting the negative relationship between the pair *S&P500* and *FTSE100* and the broader market. Furthermore, it is important to note that the overall trajectory of the index remains unaffected by external factors during this time period, even amid the significant price swings seen in 2008-2009. Consequently, it becomes evident that there is a remarkable and persistent correlation between the pair of indices, namely the *S&P500-FTSE100*, thus highlighting the importance of their long-term relationship.

FC-VAR results for the pair of S&P500 and FTSE100		
Panel A: Fractional parameters of unrestricted model		
d (Stand.errors)	1.000	1.000
b (Stand. errors)	0.028	0.052
	S&P500	*FTSE100*
CE(beta)	1.000	-0.987
Adj. matrix (alpha)	-0.616	0.927
Stand. errors	0.044	0.063
Level parameters (mu)	0.001	0.001
Stand. errors	0.012	0.012
Panel B: White noise tests [H_0: no autocorrelation against H_1: autocorrelation]		
Q	585.190	546.150
p-value	0.000	0.000
LM	209.163	209.281
p-value	0.000	0.000
Multivariate (p-value)	872.333	0.000
Panel C: Restriction tests		
Ho: C-VAR restricted (d=b=1) against H_1: FC-VAR unristricted model (d=b≠1)		
Unrestricted log-likelihood		26965.463
Restricted log-likelihood		26965.288
Test Results (df=1)		
LR statistic		0.350
P-value		0.554

Table 6.8: **S&P500 and FTSE100**

Figure 6.6 depicting the Log-likelihood values and the roots of the characteristic polynomial of the unit circle on the pair *S&P500-FTSE100*. Here we notice that all the roots are not inside the circle. In this case, the model here is dynamically unstable.



(a) (b)

Figure 6.6: **S&P500-FTSE100,Roots,LogL**

7. CONCLUSION

Briefly, we would like to draw attention to the conclusions drawn from the analysis carried out. Our study delves into the long-run relationship between various indices, focusing specifically on the stock market indices of the seven nations that make up the famous Group of Seven (*G7*). The time frame for our research spans from the year 2000 to 2023. Through the use of the *FCVAR* model, we were able to draw some conclusions from our data interpretation. It is important to emphasize that we also took into account the global financial crisis that unfolded between 2008 and 2009, examining its impact on the volatility of the indicators under consideration. In the regression process, we incorporated the index (*beta*), statistical index *LR* and other relevant indicators to thoroughly analyze our data under fractional cointegration. As part of our model testing, we focused on six different pairs of stock indices, fixing the US index as a benchmark. Our findings show that the FTSEMIB index exhibits significant price fluctuations, starting at extremely high prices and ending at reduced costs. In contrast, the *S&P500* index follows a more stable trajectory when prices are low. In addition, we found that the period that includes the global financial crisis shows the highest levels of volatility and price volatility, as evidenced by the "log-return" values of the indices. Furthermore, we found only the pair *S&P500-FTSEMIB* to have a positive value and less than unity. Based on this our beta indicator indicates that the return of this pair is less volatile to the various fluctuations of the whole market. In conclusion, in order to guarantee the dynamic stability of the model, the reciprocal solutions of the autoregressive polynomial were executed. When examining the pairs *S&P500-NIKKEI225* and *S&P500-SPTSX*, it is evident that the roots are situated within a circle and possess a magnitude smaller than one. This observation suggests that the model is characterized by dynamic stability.

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