UNIVERSITY OF IOANNINA

Department of Mathematics

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A contribution to Rigidity and Deformability theory of isometric immersions

Doctoral Thesis

Ioannina, 2023

Dedicated to my parents Vasilis and Fani
and to my brother Pantelis.

The present dissertation was carried out under the Ph.D. program of the Department of Mathematics of the University of Ioannina in order to obtain the degree of Doctor of Philosophy.

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Amalia-Sofia Tsouri

Acknowledgements

First of all, I would like to thank my supervisor Theodoros Vlachos for his endless support and encouragement. He was a mentor for me, not only scientifically but in life in general. It was a great honor to be his student.

Furthermore, I would like to thank the other six members of the evaluation commitee: Marcos Dajczer from IMPA, Luis J. Alias from the University of Murcia, Andreas Savas-Halilaj from the University of Ioannina, Stavros Papadakis from the University of Ioannina, Nikolaos Roidos from the University of Patras and Christos Saroglou from the University of Ioannina.

Moreover, I would like to thank my friends and colleagues: Tatiani Pappa, Vanesa Papastavrou, Konstantinos Zisis, Klio Papadopoulou, Athina-Eleni Kanellopoulou, Kleanthis Polumerakis, Christos Raent-Onti, Thodoris Kasioumis, Stelios Frestas, Fotini Tsiaklaganou, Ektoras Frestas, Giorgos Mitsos, Georgia Nefridou, Elina Mitsou, Lefteris Zois.

Last but not least, I would like to thank my parents Vasilis Tsouris and Tsikrika Fani, my brother Pantelis Tsouris and Alexandros Stavrou for their endless encouragement and help all these years.

Amalia-Sofia Tsouri

Ioannina, Greece, February 2023

ΠΕΡΙΛΗΨΗ

Στόχος της διατριβής είναι να ερευνήσουμε την ακαμψία και την παραμορφωσιμότητα ψευδοολόμορφων καμπυλών στην σχεδόν Kähler σφαίρα $\mathbb{S}^6$, μεταξύ των ελαχιστικών επιφανειών σε σφαίρες. Οι μη ολικά γεωδαισιακές ψευδοολόμορφες καμπύλες στην $\mathbb{S}^6$ είτε έχουν ουσιώδη συνδιάσταση σε μια ολικά γεωδαισιακή σφαίρα $\mathbb{S}^5 \subset \mathbb{S}^6$, είτε έχουν ουσιώδη συνδιάσταση στην $\mathbb{S}^6$. Στην τελευταία περίπτωση, η ψευδοολόμορφη καμπύλη είναι είτε μηδενικής στρέψης, είτε μη ισοτροπική. Οι μηδενικής στρέψης καμπύλες είναι ισοτροπικές. Η μελέτη του ανωτέρω προβλήματος ανάγεται στη μελέτη ξεχωριστά καθεμιάς από τις τρεις κλάσεις ψευδοολόμορφων καμπυλών.

Οι ψευδοολόμορφες καμπύλες που έχουν ουσιώδη συνδιάσταση στη σφαίρα $\mathbb{S}^5 \subset \mathbb{S}^6$ αποδεικνύεται ότι είναι παραμορφώσιμες. Μια ελαχιστική επιφάνεια σε σφαίρα είναι τοπικά ισομετρική με μια ψευδοολόμορφη καμπύλη στη σφαίρα $\mathbb{S}^5$ αν και μόνο αν ικανοποιείται η Ricci-like συνθήκη $\Delta \log(1-K) = 6K$, όπου K είναι η καμπυλότητα Gauss της επαγόμενης μετρικής και Δ είναι ο τελεστής Laplace της επαγόμενης μετρικής. Εκτός από ισόπεδες ελαχιστικές επιφάνειες σε σφαίρες, ευθέα αθροίσματα επιφανειών μελών της μονοπαραμετρικής οικογένειας ψευδοολόμορφων καμπυλών στην $\mathbb{S}^5$ ικανοποιούν τη Ricci-like συνθήκη. Και τα δύο αυτά είδη επιφανειών είναι exceptional επιφάνειες. Αυτές οι επιφάνειες είναι ελαχιστικές επιφάνειες των οποίων όλα τα διαφορικά Hopf είναι ολόμορφα, ή ισοδύναμα οι ελλείψεις καμπυλότητας, εκτός ίσως της τελευταίας, έχουν σταθερή εκκεντρότητα. Κάτω από διάφορες υποθέσεις, αποδεικνύουμε ότι ελαχιστικές επιφάνειες σε σφαίρες που ικανοποιούν τη Ricci-like συνθήκη είναι όντως exceptional. Επομένως, η ταξινόμηση αυτών των επιφανειών ανάγεται στην ταξινόμηση των exceptional επιφανειών οι οποίες είναι τοπικά ισομετρικές με μια ψευδοολόμορφη καμπύλη στην $\mathbb{S}^5$. Μάλιστα, αποδεικνύουμε μεταξύ άλλων ότι τέτοιες exceptional επιφάνειες σε περιττής διάστασης σφαίρες είναι είτε ισόπεδες είτε ευθέα αθροίσματα επιφανειών μελών της μονοπαραμετρικής οικογένειας μιας ψευδοολόμορφης καμπύλης στην $\mathbb{S}^5$.

Αποδεικνύουμε ότι οι συμπαγείς ισοτροπικές ψευδοολόμορφες καμπύλες της σχεδόν Kähler σφαίρας $\mathbb{S}^6$ είναι άκαμπτες μεταξύ των ελαχιστικών επιφανειών σε σφαίρες. Οι μη συμπαγείς ισοτροπικές ψευδοολόμορφες καμπύλες της σχεδόν Kähler σφαίρας $\mathbb{S}^6$ είναι άκαμπτες μεταξύ των exceptional επιφανειών σε σφαίρες.

Για μια μη ισοτροπική ψευδοολόμορφη καμπύλη g με ουσιώδη συνδιάσταση στη σχεδόν Kähler σφαίρα $\mathbb{S}^6$, δυνάμεθα να περιγράψουμε το χώρο των παραμορφώσεων όλων των μη

γεωμετρικά ισότιμων ελαχιστικών επιφανειών f οι οποίες είναι τοπικά ισομετρικές με την καμπύλη g, και έχουν ίσες κάθετες καμπυλότητες έως δεύτερης τάξης με την καμπύλη g. Επιπλέον, αποδεικνύουμε ένα θεώρημα τύπου Schur για ελαχιστικές επιφάνειες σε σφαίρες.

Abstract

The aim of the thesis is to investigate the rigidity and the deformability of pseudo-holomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$, among minimal surfaces in spheres. The nontotally geodesic pseudoholomorphic curves in $\mathbb{S}^6$ are either substantial in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$ or substantial in $\mathbb{S}^6$ (see [2]). In the latter case, the pseudo-holomorphic curve is either null torsion (studied by Bryant [3]) or non-isotropic. It turns out that null torsion curves are isotropic. In order to study the above problem we have to deal separately with these three classes of pseudoholomorphic curves.

Substantial pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$ turn out to be quite deformable. Being locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$ is equivalent to the Ricci-like condition $\Delta \log(1-K) = 6K$, where K is the Gaussian curvature of the induced metric and Δ is the Laplacian operator of the surface with respect to the induced metric. Besides flat minimal surfaces in spheres, direct sums of surfaces in the associated family of pseudoholomorphic curves in $\mathbb{S}^5$ do satisfy this Ricci-like condition. Surfaces in both classes are exceptional surfaces. These are minimal surfaces whose all Hopf differentials are holomorphic, or equivalently the curvature ellipses have constant eccentricity up to the last but one. Under appropriate global assumptions, we prove that minimal surfaces in spheres that satisfy this Ricci-like condition are indeed exceptional. Thus, the classification of these surfaces is reduced to the classification of exceptional surfaces that are locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$. In fact, we prove, among other results, that such exceptional surfaces in odd dimensional spheres are flat or direct sums of surfaces in the associated family of a pseudoholomorphic curve in $\mathbb{S}^5$.

We prove that compact substantial isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$ are rigid among minimal surfaces in spheres. Noncompact isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$ are rigid among exceptional surfaces in spheres.

For a substantial non-isotropic pseudoholomorphic curve g in the nearly Kähler sphere $\mathbb{S}^6$ we aim to describe the moduli space of all noncongruent minimal surfaces f in $\mathbb{S}^n$ that are locally isometric to the curve g, having the same normal curvatures up to order 2 with the curve g. Moreover, we prove a Schur type theorem (see [8, p. 36]) for minimal surfaces in spheres.

Contents

CHAPTER 1

Introduction

Rigidity and deformability problems of a given isometric immersion are fundamental problems of the theory of isometric immersions. Of particular interest is the classification of all noncongruent minimal surfaces in a space form, that are isometric to a given one. This problem was raised by Lawson in [28] and partial answers were provided by several authors. For instance, see [6, 22, 27, 28, 32, 33, 36, 37, 40, 42].

A classical result due to Ricci-Curbastro [35] asserts that the Gaussian curvature $K \leq 0$ of any minimal surface in $\mathbb{R}^3$ satisfies the so-called Ricci condition

$$\Delta \log(-K) = 4K,$$

away from totally geodesic points, where Δ is the Laplacian operator of the surface with respect to the induced metric ds^2. This condition is equivalent to the flatness of the metric $d\hat{s}^2 = (-K)^{1/2} ds^2$. Conversely (see [26]), a metric on a simply connected 2-dimensional Riemannian manifold with negative Gaussian curvature is realized on a minimal surface in $\mathbb{R}^3$, if the Ricci condition is satisfied. Hence, the Ricci condition is a necessary and sufficient condition for a 2-dimensional Riemannian manifold to be locally isometric to a minimal surface in $\mathbb{R}^3$.

Lawson [27] studied the above problem for minimal surfaces in a Euclidean space that are isometric to minimal surfaces in $\mathbb{R}^3$. Using the Ricci condition and the holomorphicity of the Gauss map, he classified all minimal surfaces in $\mathbb{R}^n$ that are isometric to a minimal surface in $\mathbb{R}^3$. Calabi [6] obtained a complete description of the moduli space of all noncongruent minimal surfaces in $\mathbb{R}^n$ which are isometric to a given holomorphic curve in the complex space $\mathbb{C}^n$.

The aforementioned problem has drown even more attention for minimal surfaces in spheres. That is mainly due to the difficulty that arises from the fact that the Gauss map is merely harmonic, in contrast to minimal surfaces in the Euclidean space where the Gauss map is holomorphic. The classification problem of minimal surfaces in spheres that are isometric to minimal surfaces in the sphere $\mathbb{S}^3$ was raised by Lawson in [27], where he stated a conjecture that is still open. This conjecture has been only confirmed for certain classes of minimal surfaces in spheres (see [33, 36, 37, 40, 42]).

It is worth noticing that a surface is locally isometric to a minimal surface in $\mathbb{S}^3$ if its Gaussian curvature K satisfies the spherical Ricci condition

$$\Delta \log(1-K) = 4K,$$

away from totally geodesic points, where Δ is the Laplacian operator of the surface with respect to its induced metric.

In this thesis, we turn our interest to a distinguished class of minimal surfaces in spheres, the so-called *pseudoholomorphic curves* in the nearly Kähler sphere $\mathbb{S}^6$. This class of surfaces was introduced by Bryant [3] and has been widely studied (cf. [2, 19, 18]). The pseudoholomorphic curves in $\mathbb{S}^6$ are nonconstant smooth maps from a Riemann surface into the nearly Kähler sphere $\mathbb{S}^6$, whose differential is complex linear with respect to the almost complex structure of $\mathbb{S}^6$ that is induced from the multiplication of the Cayley numbers.

In analogy with Calabi's aforementioned work [6], in the present thesis we focus on the following problem:

Classify noncongruent minimal surfaces in spheres that are isometric to a given pseudoholomorphic curve in the nearly Kähler sphere $\mathbb{S}^6$.

One of the aims in this thesis is to investigate the moduli space of all noncongruent substantial minimal surfaces $f\colon M \to \mathbb{S}^n$ that are isometric to a given pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$. By substantial, we mean that $f(M)$ is not contained in any totally geodesic submanifold of $\mathbb{S}^n$. It is known [3, 18] that any pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$ is 1-isotropic (for the notion of s-isotropic surface see Chapter 2). The nontotally geodesic pseudoholomorphic curves in $\mathbb{S}^6$ are either substantial in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$ or substantial in $\mathbb{S}^6$ (see [2]). In the latter case, the curve is either null torsion (studied by Bryant [3]) or non-isotropic. It turns out that null torsion curves are isotropic. In order to study the above problem we have to deal separately with these three classes of pseudoholomorphic curves.

The results of the present thesis are contained in [38, 39]. The present thesis is organised as follows:

In Chapter 2, we collect definitions and several facts about minimal surfaces in spheres.

In Chapter 3, we recall the nearly Kähler structure of the sphere $\mathbb{S}^6$ and we summarize well known properties of pseudoholomorphic curves in $\mathbb{S}^6$.

In Chapter 4, we deal with the case of pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$. A characterization of Riemannian metrics that arise as induced metrics on pseudoholomorphic curves in $\mathbb{S}^5$ was given in [19, 18]. In fact, the Gaussian curvature $K \leq 1$ of a pseudoholomorphic curve in $\mathbb{S}^5$ satisfies the condition

$$\Delta \log(1-K) = 6K, \tag{$*$}$$

away from totally geodesic points, where Δ is the Laplacian operator of the induced metric ds^2. This condition is equivalent to the flatness of the metric $d\hat{s}^2 = (1-K)^{1/3}ds^2$. Conversely, any two-dimensional Riemannian manifold (M, ds^2), with Gaussian curvature $K < 1$, that satisfies the Ricci-like condition $(*)$ can be locally isometrically immersed as a pseudoholomorphic curve in $\mathbb{S}^5$. Thus the classification of minimal surfaces in spheres that are locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5 \subset \mathbb{S}^6$ is equivalent to the classification of those surfaces whose induced metrics satisfy condition $(*)$.

Flat minimal surfaces in odd dimensional spheres (see [24, 4]) are obviously isometric to any flat pseudoholomorphic curve in $\mathbb{S}^5$. In the present thesis, we provide a method to produce nonflat minimal surfaces in odd dimensional spheres that are isometric to pseudoholomorphic curves in $\mathbb{S}^5$. More precisely, let $g_\theta, 0 \leq \theta < \pi$, be the associated family of a simply connected pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$. We consider the surface $\hat{g}\colon M \to \mathbb{S}^{6m-1}$ defined by

$$\hat{g} = a_1 g_{\theta_1} \oplus \cdots \oplus a_m g_{\theta_m}, \tag{1.1}$$

where $a_1, \dots, a_m$ are any real numbers with $\sum_{j=1}^m a_j^2 = 1$, $0 \leq \theta_1 < \cdots < \theta_m < \pi$, and $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of the Euclidean space $\mathbb{R}^{6m}$. It is easy to see that $\hat{g}$ is minimal and isometric to g.

Using strongly the fact that the curve g is pseudoholomorphic with respect to the almost complex structure of the nearly Kähler sphere $\mathbb{S}^6$, we prove that minimal surfaces given by (1.1) belong to the class of exceptional surfaces that was studied in [42, 43]. These are minimal surfaces whose all Hopf differentials are holomorphic, or equivalently all curvature ellipses of any order have constant eccentricity up to the last but one (see Section 2.3 for details). In fact, we prove that besides flat minimal surfaces in odd dimensional spheres, the only simply connected exceptional surfaces that are isometric to a pseudoholomorphic curve in $\mathbb{S}^5$ are of the type (1.1).

Passing to our main problem, at first we wish to describe the moduli space of noncongruent minimal surfaces in spheres that are isometric to a given nonflat pseudoholomorphic curve in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$. It turns out that this is a hard problem. However, under appropriate global assumptions, we prove (see Theorem 4.4.1) that minimal surfaces in spheres that are locally isometric to a pseudoholomorphic curve in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$, are exceptional. Therefore, it is quite natural to investigate this moduli space in the class of exceptional substantial surfaces in $\mathbb{S}^n$. We denote by $\mathcal{M}_n^{\mathrm{e}}(g)$ the moduli space of all noncongruent exceptional surfaces $f\colon M \to \mathbb{S}^n$ that are isometric to a given pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$. Given such a pseudoholomorphic curve g, we are able to show (see Theorem 4.0.1) that the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$ for odd n is empty unless $n \equiv 5 \bmod 6$, in which case the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$ splits as

$$\mathcal{M}_n^{\mathrm{e}}(g) = \mathbb{S}_*^{m-1} \times \Gamma_0,$$

where $m = (n+1)/6$,

$$\mathbb{S}_*^{m-1} = \Big\{(a_1, \ldots, a_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \colon \prod_{j=1}^{m} a_j \neq 0\Big\}$$

and Γ_0 is a subset of

$$\Gamma^m = \big\{(\theta_1, \ldots, \theta_m) \in \mathbb{R}^m \colon 0 \leq \theta_1 < \cdots < \theta_m < \pi\big\}.$$

Moreover, if Γ_0 is a proper subset of Γ^m then it is locally a disjoint finite union of d-dimensional real analytic subvarieties where $d = 0, \ldots, m-1$. We prove (see Theorem 4.2.1) that $\Gamma_0 = \Gamma^m$ in the case where the surface M is simply connected.

If M is compact and not homeomorphic to the torus, then it is shown that the set Γ_0 that shows up in the moduli space is a proper subset of Γ^m (see Theorem 4.4.3). As a result, we are able to prove the following theorem, which provides an answer to the aforementioned problem for minimal surfaces in spheres with low codimension.

Theorem. *Let $g\colon M \to \mathbb{S}^5$ be a compact pseudoholomorphic curve. If M is not homeomorphic to the torus, then the moduli space of all noncongruent substantial minimal surfaces in $\mathbb{S}^n$, $4 \leq n \leq 7$, that are isometric to g is empty, unless $n = 5$ in which case the moduli space is a finite set.*

The necessity of the assumption that the surface is not homeomorphic to the torus is justified by the class of flat tori in $\mathbb{S}^5$ (see Remark 4.4.2).

Moreover, we prove (see Theorem 4.4.2) that, under certain assumptions, there are no minimal surfaces in even dimensional spheres that satisfy the condition $(*)$.

It is worth noticing that a necessary and sufficient condition for a two dimensional Riemannian manifold to be locally isometric to a minimal Lagrangian (or totally real) surface in the complex projective plane $\mathbb{C}P^2$ (see [15, Theorem 3.8]) is that its induced metric satisfies condition $(*)$. Thus studying the minimal surfaces in spheres that are locally isometric to pseudoholomorpic surfaces in a totally geodesic $\mathbb{S}^5$ in the nearly Kähler $\mathbb{S}^6$ is equivalent to the study of those minimal surfaces in spheres that are locally isometric to minimal Lagrangian surfaces in $\mathbb{C}P^2$. Our results apply to minimal surfaces in spheres that are locally isometric to minimal Lagrangian surfaces in $\mathbb{C}P^2$. The study of Lagrangian submanifolds of a Kähler manifold was initiated in the early

1970's. A Lagrangian submanifold M of a Kähler manifold $\tilde{M}$ is a submanifold such that the almost complex structure of the ambient manifold $\tilde{M}$ carries each tangent space of M into the corresponding normal space of M. We notice that the interesting class of Lagrangian surfaces in $\mathbb{C}P^2$ has been widely investigated by many authors. In particular, methods of constructing minimal Lagrangian surfaces were provided in [7, 29, 30, 31].

In Chapter 5, we deal with isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$. It turns out that these surfaces are rigid, if they are compact. In fact, for compact minimal surfaces our result is stated as follows.

Theorem. *Let $f\colon M \to \mathbb{S}^n$ be a compact substantial minimal surface. If f is isometric to an isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$, then $n = 6$ and f is congruent to g.*

The same result holds if instead of the compactness of the surface we assume that the surface is exceptional.

Finally, in Chapter 6 we deal with the third class of pseudoholomorphic curves, the non-isotropic ones in $\mathbb{S}^6$. For a given pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$ that is non-isotropic, our aim is to describe the moduli space $\mathcal{M}_n^K(g)$ of all noncongruent minimal surfaces $f\colon M \to \mathbb{S}^n$ that are locally isometric to the curve g, having the same normal curvatures up to order 2 with the curve g (for the definition of the normal curvatures we refer the reader to Chapter 2). We are able to give the following description of the moduli space of a pseudoholomorphic curve in any of the three classes mentioned before.

Theorem. *Let $g\colon M \to \mathbb{S}^6$ be a pseudoholomorphic curve. The moduli space of all noncongruent minimal surfaces $f\colon M \to \mathbb{S}^6$ that are isometric to g and have the same normal curvatures with g, is either a circle or a finite set.*

For non-isotropic pseudoholomorphic curves substantial in $\mathbb{S}^6$, under an assumption on the Euler-Poincaré number of the second normal bundle (see Chapters 2 and 3 for details), we prove the following result that provides a partial answer to our problem.

Theorem. *Let $g\colon M \to \mathbb{S}^6$ be a compact substantial pseudoholomorphic curve that is non-isotropic. If the Euler-Poincaré number of the second normal bundle of g is nonzero, then there are at most finitely many minimal surfaces in $\mathbb{S}^6$ isometric to g having the same normal curvatures with g.*

The necessity of the assumption on the codimension and the global assumptions in the above theorem is justified by the fact that direct sums of the associated family of a simply connected non-isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$ yield minimal surfaces isometric to g (see Remark 6.2.1).

In addition, we prove the following theorem that may be viewed as analogous to the classical result of Schur (see [8, p. 36]) in the realm of minimal surfaces in spheres.

Theorem. *Let $g\colon M \to \mathbb{S}^6$ be a compact, non-isotropic and substantial pseudoholomorphic curve and $\hat{g}\colon M \to \mathbb{S}^n$ be a substantial minimal surface that is isometric to g. If $\hat{g}$ is not 2-isotropic and the second normal curvatures $K_2^{\perp}, \hat{K}_2^{\perp}$ of the surfaces g and $\hat{g}$ respectively satisfy the inequality $\hat{K}_2^{\perp} \leq K_2^{\perp}$, then $n = 6$. Moreover, the moduli space of all such noncongruent minimal surfaces $\hat{g}\colon M \to \mathbb{S}^6$ that are isometric to g, is either a circle or a finite set.*

CHAPTER 2

Preliminaries

In this chapter, we collect several facts and definitions about minimal surfaces in spheres. For more details we refer to [11] and [13].

2.1 Higher fundamental forms and higher normal subbundles

Let $f\colon M \to \mathbb{S}^n$ be an isometric immersion of a 2-dimensional Riemannian manifold. The k^{th}-*normal space* of f at $p \in M$ for $k \geq 1$ is defined as

$$N_k^f(p) = \operatorname{span}\left\{\alpha_{k+1}^f(X_1, \ldots, X_{k+1}) : X_1, \ldots, X_{k+1} \in T_pM\right\},$$

where the symmetric tensor

$$\alpha_s^f\colon TM \times \cdots \times TM \to N_fM, \quad s \geq 3,$$

given inductively by

$$\alpha_s^f(X_1, \ldots, X_s) = \left(\nabla_{X_s}^{\perp} \cdots \nabla_{X_3}^{\perp} \alpha^f(X_2, X_1)\right)^{\perp},$$

is called the s^{th}-*fundamental form* and $\alpha^f : TM \times TM \to N_fM$ stands for the standard second fundamental form of f with values in the normal bundle. Here, $\nabla^{\perp}$ denotes the induced connection in the normal bundle N_fM of f and $(\,\cdot\,)^{\perp}$ stands for the projection onto the orthogonal complement of $N_1^f \oplus \cdots \oplus N_{s-2}^f$ in N_fM. It is well known that if f is minimal, then $\dim N_k^f(p) \leq 2$ for all $k \geq 1$ and any $p \in M$ (cf. [11]).

A surface $f\colon M \to \mathbb{S}^n$ is called *regular* if for each k the subspaces N_k^f have constant dimension and thus form normal subbundles. Notice that regularity is always verified along connected components of an open dense subset of M.

Assume that an isometric immersion $f\colon M \to \mathbb{S}^n$ is minimal and substantial. By the latter, we mean that $f(M)$ is not contained in any totally geodesic submanifold of $\mathbb{S}^n$. In this case, the normal bundle of f splits along an open dense subset of M as

$$N_f M = N_1^f \oplus N_2^f \oplus \cdots \oplus N_m^f, \quad m = [(n-1)/2].$$

All higher normal bundles have rank two except possible the last one that has rank one if n is odd; see [9] or [11]. Moreover, if M is oriented, then an orientation is induced on each plane subbundle N_s^f given by the ordered basis

$$\alpha_{s+1}^f(X, \ldots, X), \quad \alpha_{s+1}^f(JX, \ldots, X),$$

where $0 \neq X \in TM$, and J is the complex structure of M determined by the orientation and the metric.

At any point $p \in M$ and for each N_r^f, $1 \leq r \leq m$, the r^{th}*-order curvature ellipse* $\mathcal{E}_r^f(p) \subset N_r^f(p)$ is defined by

$$\mathcal{E}_r^f(p) = \left\{ \alpha_{r+1}^f(Z^\varphi, \ldots, Z^\varphi)\colon\ Z^\varphi = \cos\varphi Z + \sin\varphi JZ \text{ and } \varphi \in [0, 2\pi) \right\},$$

where $Z \in T_x M$ is any vector of unit length.

A substantial regular surface $f\colon M \to \mathbb{S}^n$ is called *s-isotropic* if it is minimal and at any point $p \in M$ the curvature ellipses $\mathcal{E}_r^f(p)$ contained in all two-dimensional N_r^f's are circles for any $1 \leq r \leq s$. It is called *isotropic* if it is s-isotropic for any s.

The r-th *normal curvature* $K_r^\perp$ of f is defined by

$$K_r^\perp = \frac{2}{\pi}\mathrm{Area}(\mathcal{E}_r^f).$$

If $\kappa_r \geq \mu_r \geq 0$ denote the length of the semi-axes of the curvature ellipse $\mathcal{E}_r^f$, then

$$K_r^\perp = 2\kappa_r \mu_r. \tag{2.1}$$

Clearly, the curvature ellipse $\mathcal{E}_r^f(p)$ at a point $p \in M$ is a circle if and only if $\kappa_r(p) = \mu_r(p)$.

The *eccentricity* ε_r of the curvature ellipse $\mathcal{E}_r^f$ is given by

$$\varepsilon_r = \frac{\left(\kappa_r^2 - \mu_r^2\right)^{1/2}}{\kappa_r},$$

where $\left(\kappa_r^2 - \mu_r^2\right)^{1/2}$ is the distance from the center to a focus, and can be thought of as a measure of how far $\mathcal{E}_r^f$ deviates from being a circle.

The *a-invariants* (see [43]) are the functions

$$a_r^{\pm} = \kappa_r \pm \mu_r = \left(2^{-r}\|\alpha_{r+1}^f\|^2 \pm K_r^{\perp}\right)^{1/2}.$$

These functions determine the geometry of the r-th curvature ellipse.

Denote by τ_f^o the index of the last plane bundle, in the orthogonal decomposition of the normal bundle. Let $\{e_1, e_2\}$ be a local tangent orthonormal frame and $\{e_\alpha\}$ be a local orthonormal frame of the normal bundle such that $\{e_{2r+1}, e_{2r+2}\}$ span N_r^f for any $1 \leq r \leq \tau_f^o$ and e_{2m+1} spans the line bundle N_{m+1}^f if $n = 2m+1$. For any $\alpha = 2r+1$ or $\alpha = 2r+2$, we set

$$h_1^\alpha = \langle \alpha_{r+1}^f(e_1, \ldots, e_1), e_\alpha \rangle, \ h_2^\alpha = \langle \alpha_{r+1}^f(e_1, \ldots, e_1, e_2), e_\alpha \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard metric of $\mathbb{S}^n$. Introducing the complex valued functions

$$H_\alpha = h_1^\alpha + i h_2^\alpha \ \text{ for any } \ \alpha = 2r+1 \ \text{ or } \ \alpha = 2r+2,$$

it is not hard to verify that the r-th normal curvature is given by

$$K_r^{\perp} = i\left(H_{2r+1}\overline{H}_{2r+2} - \overline{H}_{2r+1}H_{2r+2}\right). \tag{2.2}$$

The length of the $(r+1)$-th fundamental form α_{r+1}^f is given by

$$\|\alpha_{r+1}^f\|^2 = 2^r\left(|H_{2r+1}|^2 + |H_{2r+2}|^2\right), \tag{2.3}$$

or equivalently (cf. [1])

$$\|\alpha_{r+1}^f\|^2 = 2^r(\kappa_r^2 + \mu_r^2). \tag{2.4}$$

In particular, it follows from the Gauss equation that

$$\|\alpha_2^f\|^2 = 2(1-K). \tag{2.5}$$

Each plane subbundle N_r^f inherits a Riemannian connection from that of the normal bundle. Its *intrinsic curvature* K_r^* is given by the following proposition (cf. [1]).

Proposition 2.1.1. *The intrinsic curvature K_r^* of each plane subbundle N_r^f of a minimal surface $f\colon M \to \mathbb{S}^n$ is given by*

$$K_1^* = K_1^{\perp} - \frac{\|\alpha_3^f\|^2}{2K_1^{\perp}} \ \textit{and} \ K_r^* = \frac{K_r^{\perp}}{(K_{r-1}^{\perp})^2}\frac{\|\alpha_r^f\|^2}{2^{r-2}} - \frac{\|\alpha_{r+2}^f\|^2}{2^r K_r^{\perp}} \ \textit{for} \ 2 \leq r \leq \tau_f^o.$$

2.2 The associated family

Let $f\colon M \to \mathbb{S}^n$ be a minimal isometric immersion. If M is simply connected, there exists a one-parameter *associated family* of minimal isometric immersions $f_\theta\colon M \to \mathbb{S}^n$, where $\theta \in \mathbb{S}^1 = [0,\pi) = \mathbb{R}/\pi\mathbb{Z}$. To see this, for each $\theta \in \mathbb{S}^1$ consider the orthogonal parallel tensor field

$$J_\theta = \cos\theta I + \sin\theta J,$$

where I is the identity endomorphism of the tangent bundle and J is the complex structure of M induced by the metric and the orientation. Then, the symmetric section $\alpha^f(J_\theta\cdot,\cdot)$ of the bundle $\mathrm{Hom}(TM\times TM, N_fM)$ satisfies the Gauss, Codazzi and Ricci equations, with respect to the same normal connection; see [12] for details. Therefore, there exists a minimal isometric immersion $f_\theta\colon M \to \mathbb{S}^n$ whose second fundamental form is given by

$$\alpha^{f_\theta}(X,Y) = T_\theta\alpha^f(J_\theta X, Y), \tag{2.6}$$

where $T_\theta\colon N_fM \to N_{f_\theta}M$ is a parallel vector bundle isometry that identifies the normal subspaces N_s^f with $N_s^{f_\theta}$, $s \geq 1$.

2.3 Hopf differentials and Exceptional surfaces

Let $f\colon M \to \mathbb{S}^n$ be a minimal surface. The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the eigenspaces $T'M$ and $T''M$ of the complex structure J, corresponding to the eigenvalues i and $-i$. The $(r+1)$-th fundamental form α^f_{r+1}, which takes values in the normal subbundle N_r^f, can be complex linearly extended to $TM\otimes\mathbb{C}$ with values in the complexified vector bundle $N_r^f\otimes\mathbb{C}$ and then decomposed into its (p,q)-components, $p+q=r+1$, which are tensor products of p differential 1-forms vanishing on $T''M$ and q differential 1-forms vanishing on $T'M$. The minimality of f is equivalent to the vanishing of the $(1,1)$-part of the second fundamental form. Hence, the (p,q)-components of α^f_{r+1} vanish unless $p=r+1$ or $p=0$, and consequently for a local complex coordinate $z=x+iy$ on M, we have the following decomposition

$$\alpha^f_{r+1} = \alpha^{(r+1,0)}_{r+1}dz^{r+1} + \alpha^{(0,r+1)}_{r+1}d\bar{z}^{r+1},$$

where

$$\alpha^{(r+1,0)}_{r+1} = \alpha^f_{r+1}(\partial,\dots,\partial),\ \ \alpha^{(0,r+1)}_{r+1} = \overline{\alpha^{(r+1,0)}_{r+1}} \quad \text{and} \quad \partial = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big).$$

The *Hopf differentials* are the differential forms (see [41])

$$\Phi_r = \langle \alpha^{(r+1,0)}_{r+1}, \alpha^{(r+1,0)}_{r+1}\rangle dz^{2r+2}$$

of type $(2r+2,0), r=1,\dots,[(n-1)/2]$, where $\langle\cdot,\cdot\rangle$ denotes the extension of the usual Riemannian metric of $\mathbb{S}^n$ to a complex bilinear form. These forms are defined on the open subset where the minimal surface is regular and are independent of the choice of coordinates, while Φ_1 is globally well defined.

Let $\{e_1, e_2\}$ be a local orthonormal frame in the tangent bundle. It will be convenient to use complex vectors, and we put

$$E = e_1 - ie_2 \text{ and } \phi = \omega_1 + i\omega_2,$$

where $\{\omega_1, \omega_2\}$ is the dual frame. We choose a local complex coordinate $z = x + iy$ such that $\phi = Fdz$.

From the definition of Hopf differentials, we easily obtain

$$\Phi_r = \frac{1}{4}\left(\overline{H}^2_{2r+1} + \overline{H}^2_{2r+2}\right)\phi^{2r+2}.$$

Moreover, using (2.2) and (2.3), we find that

$$\left|\langle\alpha^{(r+1,0)}_{r+1}, \alpha^{(r+1,0)}_{r+1}\rangle\right|^2 = \frac{F^{2r+2}}{2^{2r+4}}\left(\|\alpha^f_{r+1}\|^4 - 4^r(K^\perp_r)^2\right). \tag{2.7}$$

Thus, the zeros of Φ_r are precisely the points where the r-th curvature ellipse $\mathcal{E}^f_r$ is a circle. From (2.1) and (2.4) we obtain the following:

Lemma 2.3.1. *Let $f\colon M \to \mathbb{S}^n$ be a minimal surface. Then the following assertions are equivalent:*

(i) The surface f is s-isotropic.

(ii) The Hopf differentials satisfy $\Phi_r = 0$ for any $1 \le r \le s$.

(iii) The length of the $(r+1)$-th fundamental form α^f_{r+1} and the r-th normal curvature $K^\perp_r$ satisfy

$$\|\alpha^f_{r+1}\|^2 = 2^r K^\perp_r,$$

for any $1 \le r \le s$. In particular, the surface f is 1-isotropic if and only if the first normal curvature $K^\perp_1$ satisfies

$$K^\perp_1 = 1 - K.$$

The Codazzi equation implies that Φ_1 is always holomorphic (cf. [9, 10]). Besides Φ_1, the rest Hopf differentials are not always holomorphic. The following characterization of the holomorphicity of Hopf differentials was given in [42], in terms of the eccentricity of curvature ellipses of higher order.

Theorem 2.3.1. *Let $f\colon M \to \mathbb{S}^n$ be a minimal surface. Its Hopf differentials $\Phi_2, \dots, \Phi_{r+1}$ are holomorphic if and only if the higher curvature ellipses have constant eccentricity up to order r.*

A minimal surface in $\mathbb{S}^n$ is called *r-exceptional* if all Hopf differentials up to order $r+1$ are holomorphic, or equivalently if all higher curvature ellipses up to order r have constant eccentricity. A minimal surface in $\mathbb{S}^n$ is called *exceptional* if it is r-exceptional for $r = [(n-1)/2 - 1]$. This class of minimal surfaces may be viewed as the next simplest to superconformal ones. In fact, superconformal minimal surfaces are indeed exceptional, characterized by the fact that all Hopf differentials vanish up to the last but one, which is equivalent to the fact that all higher curvature ellipses are circles up to the last but one. As a matter of fact, there is an abundance of exceptional surfaces.

We recall some results for exceptional surfaces proved in [42], that will be used in the proofs of our main results. A regular point is a point $p \in M$ where the normal spaces have the maximum possible dimension.

Proposition 2.3.1. *Let $f\colon M \to \mathbb{S}^n$ be an $(r-1)$-exceptional surface. At regular points the following hold:*

(i) For any $1 \le s \le r-1$, we have

$$\Delta \log \|\alpha_{s+1}\|^2 = 2\big((s+1)K - K_s^*\big),$$

where Δ is the Laplacian operator with respect to the induced metric ds^2.

(ii) If $\Phi_r \neq 0$, then

$$\Delta \log \left(\|\alpha_{r+1}\|^2 + 2^r K_r^{\perp}\right) = 2\big((r+1)K - K_r^*\big)$$

and

$$\Delta \log \left(\|\alpha_{r+1}\|^2 - 2^r K_r^{\perp}\right) = 2\big((r+1)K + K_r^*\big).$$

(iii) If $\Phi_r = 0$, then

$$\Delta \log \|\alpha_{r+1}\|^2 = 2\big((r+1)K - K_r^*\big).$$

(iv) The intrinsic curvature of the s-th normal bundle N_s^f is $K_s^* = 0$ if $1 \le s \le r-1$ and $\Phi_s \neq 0$.

A remarkable property of exceptional surfaces is that singularities of the higher normal bundles are of holomorphic type and can be smoothly extended to vector bundles. This fact was proved in [42, Proposition 4].

Proposition 2.3.2. *Let $f\colon M \to \mathbb{S}^n$ be an r-exceptional surface. Then the set L_0, where f fails to be regular, consists of isolated points and all N_s^f's and the Hopf differentials Φ_s's extend smoothly to L_0 for any $1 \leq s \leq r$.*

2.4 Absolute value type functions

For the proof of our results, we use the notion of absolute value type functions introduced in [15, 16]. A smooth complex valued function p defined on a Riemann surface is called of *holomorphic type* if locally $p = p_0 p_1$, where p_0 is holomorphic and p_1 is smooth without zeros. A function $u\colon M \to [0, +\infty)$ defined on a Riemann surface M is called of *absolute value type* if there is a function p of holomorphic type on M such that $u = |p|$.

The zero set of such a function on a connected compact oriented surface M is either isolated or the whole of M, and outside its zeros the function is smooth. If u is a nonzero absolute value type function, i.e., locally $u = |t_0|u_1$, with t_0 holomorphic, the order $k \geq 1$ of any point $p \in M$ with $u(p) = 0$ is the order of t_0 at p. Let $N(u)$ be the sum of the orders for all zeros of u. Then $\Delta \log u$ is bounded on $M \smallsetminus \{u = 0\}$ and its integral is computed in the following lemma that was proved in [15, 16].

Lemma 2.4.1. *Let (M, ds^2) be a compact oriented two-dimensional Riemannian manifold with area element dA.*

(i) If u is an absolute value type function on M, then

$$\int_M \Delta \log u dA = -2\pi N(u).$$

(ii) If Φ is a holomorphic symmetric $(r, 0)$-form on M, then either $\Phi = 0$ or $N(\Phi) = -r\chi(M)$, where $\chi(M)$ is the Euler-Poincaré characteristic of M.

The following lemma, that was proved in [33], provides a sufficient condition for a function to be of absolute value type.

Lemma 2.4.2. *Let D be a plane domain containing the origin with coordinate z and u be a real analytic nonnegative function on D such that $u(0) = 0$. If u is not identically zero and $\log u$ is harmonic away from the points where $u = 0$, then u is of absolute value type and the order of the zero of u at the origin is even.*

We will also need the following result [43], concerning the Euler-Poincaré number of the plane normal subbundles of exceptional surfaces.

Lemma 2.4.3. *Let $f\colon M \to \mathbb{S}^n$ be a compact exceptional surface. The Euler-Poincaré number $\chi(N_r^f M)$ of the r-th plane normal bundle and the Euler-Poincaré characteristic $\chi(M)$ of M satisfy the following:*

(i) If $\Phi_r \neq 0$ for some $1 \leq r < m$, where $m = [(n-1)/2]$, then

$$\chi(N_r^f M) = 0 \quad \textit{and} \quad (r+1)\chi(M) = -N(a_r^+) = -N(a_r^-).$$

(ii) If $\Phi_r = 0$ for some $1 \leq r \leq m$, then

$$(r+1)\chi(M) - \chi(N_r^f M) = -N(a_r^+).$$

(iii)If $\Phi_m \neq 0$, then

$$(m+1)\chi(M) \mp \chi(N_m^f M) = -N(a_m^\pm).$$

CHAPTER 3

Pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$

In this chapter, we recall the nearly Kähler structure of the sphere $\mathbb{S}^6$ and we summarize some well known facts about pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$.

3.1 The nearly Kähler sphere $\mathbb{S}^6$

A finite dimensional algebra $\mathbb{A}$ over $\mathbb{R}$ with euclidean inner product is called *normed* if $\|ab\| = \|a\|\|b\|$ for any $a, b \in \mathbb{A}$. We have an orthogonal decomposition $\mathbb{A} = \mathbb{R}\cdot 1 \oplus \mathbb{A}'$ where $\mathbb{A}'$ is called the space of *imaginary* elements of $\mathbb{A}$. Every nonzero $a \in \mathbb{A}$ has an inverse $a^{-1} = \overline{a}/\|a\|^2$ where $\overline{a} = a_0 - a'$ for $a = a_0 + a'$ with $a_0 \in \mathbb{R}$ and $a' \in \mathbb{A}'$. There are only four normed algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (real and complex numbers, quaternions and octonions), and the octonions $\mathbb{O} \cong \mathbb{R}^8$ contain all the others. Octonions are not associative, but still computations are easy if one observes the following three rules which follow almost immediately from the equation $\|ab\| = \|a\|\|b\|$:

(1) Any unit vector $a \in \mathbb{O}'$ generates a subalgebra isomorphic to $\mathbb{C}$ where a plays the role of i.

(2) Any two orthonormal $a, b \in \mathbb{O}'$ generate a subalgebra isomorphic to $\mathbb{H}$ where a, b, ab play the roles of i, j, k; they are associative and anticommutative, $ab = -ba$.

(3) Any three orthonormal $a, b, c \in \mathbb{O}'$ with $c \perp ab$ (normed Cayley triples) generate the algebra $\mathbb{O}$; they are antiassociative, $a(bc) = -(ab)c$.

Let $1, i, j, k, l, il, jl, kl$ be the standard basis of $\mathbb{O} = \mathbb{H} + \mathbb{H}l$. Then (i, j, l) is a normed Cayley triple, and so is its image $(\alpha i, \alpha j, \alpha l)$ under any automorphism α of $\mathbb{O}$; note that α is orthogonal. Vice versa, given any normed Cayley triple (a, b, c), there is precisely one automorphism α of $\mathbb{O}$ with $a = \alpha i, b = \alpha j, c = \alpha l$. Thus the space of normed Cayley triples is a manifold of dimension $6 + 5 + 3 = 14$ on which the exceptional group $G_2 = Aut(\mathbb{O}) \subset SO_7$ acts simply transitively. In particular, G_2 acts transitively on $\mathbb{S}^6$.

For $a \in \mathbb{O}'$ and $b \in \mathbb{O}$ we have (using rule (2))

$$a(ab) = a^2 b = -\langle a, a\rangle b,$$

and this remains true for $a \in \mathbb{O}'_c, b \in \mathbb{O}$ where $\langle , \rangle$ is the complexified inner product. In particular $a(ab) = 0$ when $\langle a, a\rangle = 0$. Other useful formulas which extend for all $a, b, c \in \mathbb{O}_c$, are

$$\langle ab, ac\rangle = \langle a, a\rangle\langle b, c\rangle$$

and the antisymmetry of $\langle ab, c\rangle$ in all three variables.

The sphere $\mathbb{S}^6$ plays a similar role for the octonions $\mathbb{O}$ as the sphere $\mathbb{S}^2$ for the quaternions $\mathbb{H}$: they are unit spheres in $\mathbb{A}'$, the imaginary part of the division algebra $\mathbb{A} = \mathbb{O}, \mathbb{H}$, respectively. Each $p \in \mathbb{S}$ satisfies $(L_p)^2 = -I$ where $L_p : x \mapsto px$ denotes the left multiplication with p. Hence L_p is a complex structure preserving the plane $\text{Span}\{1, p\}$ and its orthogonal complement, the tangent space $T_p\mathbb{S}$. Thus $J_p := L_p|T_p\mathbb{S}$ is a complex structure on $T_p\mathbb{S}$ and defines an almost complex structure J on $\mathbb{S}$. It is convenient to use the *cross product* $a \times b$ which is the imaginary ($\mathbb{A}'-$) part of the product ab for any $a, b \in \mathbb{A}'$:

$$a \times b = (ab)' = \begin{cases} ab & \text{if } a \perp b, \\ 0 & \text{if } a, b \text{ are linearly dependent.} \end{cases}$$

Then each J_p extends to a linear map on $\mathbb{A}'$,

$$J_p(v) = p \times v, \tag{3.1}$$

and the derivative of the matrix-valued linear map $J : \mathbb{A}' \to \text{End}(\mathbb{A}') : p \mapsto J_p$ is $(\partial_v J)w = v \times w$. Denoting by $\nabla = \partial^T$ the Levi-Civitá derivative on $\mathbb{S}^6$, we have

$$(\nabla_v J)w = (v \times w)_{p^\perp} = v \times w - \langle v \times w, p\rangle p,$$

where $p \in \mathbb{S}^6$ is the position vector and $v, w \in T_p\mathbb{S} = p^\perp$. In particular $(\partial_v J)v = v \times v = 0$ and therefore

$$(\nabla_v J)v = 0.$$

A Riemannian manifold with an almost complex structure J with this property is called *nearly Kähler*.

An orthogonal linear map g on $\mathbb{O}'$ which preserves the almost complex structure J satisfies $gJ_p(v) = J_{gp}(gv)$ for any $p, v \in \mathbb{O}'$ with $v \perp p$. By (3.1) this is equivalent to $g(pv) = (gp)(gv)$ which holds if and only if $g \in G_2 = \text{Aut}(\mathbb{O}) \subset SO_7$. Thus G_2 is precisely the group of isometries g on $\mathbb{S}^6$ which are pseudoholomorphic.

3.2 Pseudoholomorphic curves and their properties

A *pseudoholomorphic curve* is a nonconstant smooth map $g\colon M \to \mathbb{S}^6$ from a Riemann surface M into the nearly Kähler sphere $\mathbb{S}^6$, whose differential is complex linear. This concept was introduced by Bryant [3].

It is known [3, 18] that any pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$ is 1-isotropic. The nontotally geodesic pseudoholomorphic curves in $\mathbb{S}^6$ are either substantial in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$ or substantial in $\mathbb{S}^6$ (see [2]). In the latter case, the curve is either null torsion (studied by Bryant [3]) or non-isotropic. It turns out that null torsion curves are isotropic.

The following theorem (see [18]) provides a characterization of Riemannian metrics that arise as induced metrics on pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of the nearly Kähler sphere $\mathbb{S}^6$.

Theorem 3.2.1. *Let (M, ds^2) be a simply connected Riemann surface, with Gaussian curvature $K \leq 1$ and Laplacian operator Δ. Suppose that the function $1 - K$ is of absolute value type. Then there exists an isometric pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$ if and only if*

$$\Delta \log(1 - K) = 6K. \tag{$*$}$$

In fact, up to translations with elements of G_2, that is the set $Aut(\mathbb{O}) \subset SO(7)$, there is precisely one associated family of such maps.

The above result shows that a minimal surface in a sphere is locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$ if its Gaussian curvature satisfies the condition $(*)$ at points where $K < 1$, or equivalently if the metric $d\hat{s}^2 = (1 - K)^{1/3} ds^2$ is flat.

Let $g\colon M \to \mathbb{S}^5$ be a pseudoholomorphic curve and let $\xi \in \Gamma(N_f M)$ be a smooth unit vector field that spans the extended line bundle N_2^g over the isolated set of points where f fails to be regular (see Proposition 2.3.2). The surface $g^*\colon M \to \mathbb{S}^5$ defined by $g^* = \xi$ is called the *polar surface* of g. It has been proved in Corollary 3 in [43] that the surfaces g and g^* are congruent.

The following lemma is crucial for our proofs.

Lemma 3.2.1. *Let $f\colon (M, ds^2) \to \mathbb{S}^n$ be a nontotally geodesic minimal surface. If (M, ds^2) satisfies the Ricci-like condition $(*)$, at points with Gauss curvature $K < 1$, then the function $1 - K$ is of absolute value type with isolated zeros of even order. Moreover, if M is compact and $p_j, j = 1, \ldots, m$, are the isolated zeros of $1 - K$ with corresponding order $\mathrm{ord}_{p_j}(1 - K) = 2k_j$, then we have*

$$\sum_{j=1}^{m} k_j = -3\chi(M), \tag{3.2}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. In particular, M cannot be homeomorphic to the sphere $\mathbb{S}^2$.

Proof. Let M_0 be the set of points where $K = 1$. The open subset $M \smallsetminus M_0$ is dense on M, since minimal surfaces in spheres are real analytic. Around each point $p_0 \in M_0$, we choose a local complex coordinate z such that p_0 corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$. The Gaussian curvature K is given by

$$K = -\frac{2}{F}\partial\bar{\partial}\log F.$$

Moreover, condition $(*)$ is equivalent to

$$4\partial\bar{\partial}\log(1-K) = 6KF.$$

Thus we have

$$\partial\bar{\partial}\log\left((1-K)F^3\right) = 0.$$

According to Lemma 2.4.2, the function $1-K$ is of absolute value type with isolated zeros $p_j, j = 1, \ldots, m$, and corresponding order $\mathrm{ord}_{p_j}(1-K) = 2k_j$. Then, (3.2) follows from Lemma 2.4.1(i) and condition $(*)$. ☐

We recall the following theorem (see [18]), which provides a characterization of Riemannian metrics that arise as induced metrics on isotropic substantial pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$.

Theorem 3.2.2. *Let (M, ds^2) be a simply connected Riemann surface, with Gaussian curvature $K \leq 1$ and Laplacian operator Δ. Suppose that the function $1-K$ is of absolute value type. Then there exists an isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$, unique up to translations with elements of G_2, with induced metric ds^2 if and only if*

$$\Delta\log(1-K) = 6K - 1. \qquad (**)$$

The following theorem [43] provides a characterization of Riemannian metrics that arise as induced metrics on non-isotropic substantial pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$.

Theorem 3.2.3. *Let (M, ds^2) be a simply connected Riemann surface, with Gaussian curvature $K \leq 1$ and Laplacian operator Δ. Suppose that the function $1-K$ is of absolute value type. Then there exists a non-isotropic substantial pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$, unique up to translations with elements of G_2, with induced metric ds^2 if and only if*

$$\Delta\log\left((1-K)^2\left(1-6K+\Delta\log\left(1-K\right)\right)\right) = 12K.$$

Moreover the following holds:

$$6K - 1 < \Delta \log(1 - K) < 6K. \tag{3.3}$$

CHAPTER 4

Isometric deformations of pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of $\mathbb{S}^6$

In this chapter, we study the nontrivial isometric deformations of pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of the nearly Kähler sphere $\mathbb{S}^6$. Given a pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$, one wishes to describe the moduli space of all noncongruent substantial minimal surfaces $f\colon M \to \mathbb{S}^n$ that are locally isometric to the curve g. It turns out that this is a hard problem. However, under appropriate global assumptions, we prove (see Theorem 4.4.1) that minimal surfaces in spheres that are locally isometric to a pseudoholomorphic curve in a totally geodesic $\mathbb{S}^5 \subset \mathbb{S}^6$, are exceptional. Therefore, it is quite natural to investigate this moduli space in the class of exceptional substantial surfaces in $\mathbb{S}^n$. We denote by $\mathcal{M}_n^e(g)$ the moduli space of all noncongruent substantial exceptional surfaces $f\colon M \to \mathbb{S}^n$ that are isometric to the curve g.

One of the main results of this chapter is Theorem 4.2.1, which gives a partial answer to our problem of the chapter. Theorem 4.2.1 below implies that if M is nonflat simply connected and n is odd, then $n \equiv 5 \bmod 6$,

$$\mathcal{M}_n^{\mathrm{e}}(g) = \mathbb{S}_*^{m-1} \times \Gamma^m,$$

where $m = (n+1)/6$,

$$\mathbb{S}_*^{m-1} = \Big\{\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \colon \prod_{j=1}^{m} a_j \neq 0\Big\}$$

and

$$\Gamma^m = \big\{\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in [0,\pi) \times \cdots \times [0,\pi) \colon 0 \leq \theta_1 < \cdots < \theta_m < \pi\big\}.$$

For not necessarily simply connected surfaces we prove the following theorem which provides properties of the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$ of exceptional surfaces that are locally isometric to a pseudoholomorphic curve in a totally geodesic $\mathbb{S}^5$ in the nearly Kähler sphere $\mathbb{S}^6$.

Theorem 4.0.1. *If g is a nonflat pseudoholomorphic curve in $\mathbb{S}^5$, and n is odd, then the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$ splits as $\mathbb{S}_*^{m-1} \times \Gamma_0$, where Γ_0 is a subset of Γ^m. If Γ_0 is a proper subset of Γ^m, then it is locally a disjoint finite union of d-dimensional real analytic subvarieties where $d = 0, \dots, m-1$. Moreover, the subset Γ_0 has the property that for each point $\boldsymbol{\theta} \in \Gamma_0$, every straight line through $\boldsymbol{\theta}$ that is parallel to every coordinate axis of $\mathbb{R}^m$ either intersects Γ_0 at finitely many points, or at a line segment.*

Under appropriate global assumptions, we prove some global results related to the problem mentioned before. Among them, the following theorem provides an answer to this problem for minimal surfaces in spheres with low codimension.

Theorem 4.0.2. *Let $g\colon M \to \mathbb{S}^5$ be a compact pseudoholomorphic curve. If M is not homeomorphic to the torus, then the moduli space of all noncongruent substantial minimal surfaces in $\mathbb{S}^n$, $4 \leq n \leq 7$, that are isometric to g is empty, unless $n = 5$ in which case the moduli space is a finite set.*

4.1 A class of minimal surfaces that are locally isometric to pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of $\mathbb{S}^6$

The aim of this section is to study a class of minimal surfaces that are exceptional, nonflat and locally isometric to a pseudoholomorphic curve in a totally geodesic $\mathbb{S}^5$ in the nearly Kähler sphere $\mathbb{S}^6$.

This class of surfaces is constructed as follows. Let $g\colon M \to \mathbb{S}^5$ be a simply connected pseudoholomorphic curve with Gaussian curvature $K < 1$, with respect to the induced metric $\langle\cdot,\cdot\rangle = ds^2$, and let $g_\theta, \theta \in \mathbb{S}^1$, be its associated family.

Take

$$\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m \text{ with } \prod_{j=1}^m a_j \neq 0$$

and

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1, \text{ where } 0 \leq \theta_1 < \cdots < \theta_m < \pi.$$

We consider the map $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}\colon M \to \mathbb{S}^{6m-1} \subset \mathbb{R}^{6m}$ defined by

$$\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}} = a_1 g_{\theta_1} \oplus \cdots \oplus a_m g_{\theta_m}, \tag{4.1}$$

where $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of $\mathbb{R}^{6m}$. Its differential is given by

$$d\hat{g} = a_1 dg_{\theta_1} \oplus \cdots \oplus a_m dg_{\theta_m}.$$

It is obvious that $\hat{g}$ is an isometric immersion. We can easily see that the second fundamental form of the surface $\hat{g}$ is given by

$$\hat{\alpha}_2(X,Y) = \sum_{j=1}^{m} a_j \alpha^{g_{\theta_j}}(X,Y),\ X,Y \in TM,$$

which implies that $\hat{g}$ is minimal.

The following proposition provides several properties for the above class of minimal surfaces. More important is that these surfaces turn out to be exceptional. The proof of the latter strongly uses the fact that the surface g is pseudoholomorphic with respect to the almost complex structure of the nearly Kähler sphere $\mathbb{S}^6$. In fact we are able to compute all Hopf differentials of $\hat{g}$ using Lemma 5 in [43], where it is shown how the pseudoholomorphicity provides information on the third fundamental form of the surface g, besides being 1-isotropic.

Proposition 4.1.1. *For any simply connected pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$, the minimal surface $\hat{g}\colon M \to \mathbb{S}^{6m-1}$ given by (4.1) is substantial and isometric to g. Moreover, it is an exceptional surface and the following hold:*

(i) The length of its $(s+1)$-th fundamental form is given by

$$\|\hat{\alpha}_{s+1}\|^2 = \begin{cases} \hat{b}_s(1-K)^{s/3} & \text{if } s \equiv 0 \bmod 3, \\ \hat{b}_s(1-K)^{(s+2)/3} & \text{if } s \equiv 1 \bmod 3, \\ \hat{b}_s(1-K)^{(s+1)/3} & \text{if } s \equiv 2 \bmod 3, \end{cases} \tag{4.2}$$

for any $1 \leq s \leq 3m-1$, where $\hat{b}_s$ are positive numbers.

(ii) Its s-th normal curvature is given by

$$\hat{K}_s^{\perp} = \begin{cases} \hat{c}_s\,(1-K)^{s/3} & \text{if } s \equiv 0 \bmod 3, \\ \hat{c}_s\,(1-K)^{(s+2)/3} & \text{if } s \equiv 1 \bmod 3, \\ \hat{c}_s\,(1-K)^{(s+1)/3} & \text{if } s \equiv 2 \bmod 3, \end{cases} \tag{4.3}$$

for any $1 \leq s < 3m-1$, where $\hat{c}_s$ are positive numbers.

(iii) Its s-th Hopf differential is given by

$$\hat{\Phi}_s = \begin{cases} \hat{d}_s \Phi^{(s+1)/3} & \text{if } s \equiv 2 \bmod 3, \\ 0 & \text{otherwise}, \end{cases}$$

for any $1 \leq s \leq 3m-1$, *where* $\hat{d}_s \in \mathbb{C}$ *and* Φ *is the second Hopf differential of* g.

Proof. We consider a local orthonormal frame $\{e_1, e_2\}$ in the tangent bundle away from totally geodesic points of g. Moreover, we choose a local orthonormal frame field $\{\xi_1, \xi_2, \xi_3\}$ in the normal bundle of g such that

$$\xi_1 = \frac{\alpha(e_1, e_1)}{\|\alpha(e_1, e_1)\|}, \;\; \xi_2 = \frac{\alpha(e_1, e_2)}{\|\alpha(e_1, e_2)\|}.$$

From Lemma 5 in [43] it follows that $h_1^3 = \kappa$, $h_2^3 = 0$, $h_1^4 = 0$ and $h_2^4 = \kappa$, where κ is the radius of the first circular curvature ellipse. Hence $H_3 = \kappa$ and $H_4 = i\kappa$. Moreover, we have that $h_2^5 = 0$ and $h_1^5 = \kappa$. Therefore, it follows that

$$\langle \nabla^{\perp}_{e_1} \xi_1, \xi_3 \rangle = 1, \;\; \langle \nabla^{\perp}_{e_1} \xi_2, \xi_3 \rangle = 0,$$

$$\langle \nabla^{\perp}_{e_2} \xi_1, \xi_3 \rangle = 0, \;\; \langle \nabla^{\perp}_{e_2} \xi_2, \xi_3 \rangle = -1,$$

or equivalently

$$\langle \nabla^{\perp}_{\overline{E}} \xi_3, \xi_1 - i\xi_2 \rangle = 0, \;\; \langle \nabla^{\perp}_{\overline{E}} \xi_3, \xi_1 + i\xi_2 \rangle = -2, \tag{4.4}$$

where $E = e_1 - ie_2$.

In order to show that the minimal surface $\hat{g}$ is substantial, it is sufficient to prove that

$$\sum_{j=1}^{m} a_j \langle g_{\theta_j}, w_j \rangle = 0 \tag{4.5}$$

for $(w_1, \ldots, w_m) \in \mathbb{R}^{6m} = \mathbb{R}^6 \oplus \cdots \oplus \mathbb{R}^6$ implies that $w_j = 0$ for any $j = 1, \ldots, m$.

Assume to the contrary that $w_j \neq 0$ for all $j = 1, \ldots, m$. Differentiating (4.5) we obtain

$$\sum_{j=1}^{m} a_j \langle dg_{\theta_j}, w_j \rangle = 0, \tag{4.6}$$

and

$$\sum_{j=1}^{m} a_j \left\langle \alpha^{g_{\theta_j}}, w_j \right\rangle = 0.$$

Using (2.6), we have that

$$\sum_{j=1}^{m} a_j \left\langle T_{\theta_j} \alpha^{g} (J_{\theta_j} \overline{E}, \overline{E}), w_j \right\rangle = 0,$$

where $T_{\theta_j}\colon N_gM \to N_{g_{\theta_j}}M$ is a parallel vector bundle isometry. Since $J_\theta \overline{E} = e^{-i\theta}\overline{E}$, it follows that

$$\sum_{j=1}^{m} a_j e^{-i\theta_j} \left\langle T_{\theta_j}(\xi_1 + i\xi_2), w_j \right\rangle = 0. \tag{4.7}$$

Differentiating with respect to $\overline{E}$, and using the Weingarten formula, we obtain

$$\sum_{j=1}^{m} a_j e^{-i\theta_j} \left\langle \nabla^{\perp}_{\overline{E}} T_{\theta_j}(\xi_1 + i\xi_2), w_j \right\rangle = \sum_{j=1}^{m} a_j e^{-i\theta_j} \left\langle dg_{\theta_j} \circ A_{T_{\theta_j}(\xi_1+i\xi_2)}(\overline{E}), w_j \right\rangle,$$

where $A_{T_{\theta_j}\eta}$ is the shape operator of g_{θ_j} with respect to its normal direction $T_{\theta_j}\eta$. It follows from (2.6) that

$$A_{T_{\theta_j}(\xi_1+i\xi_2)} = e^{i\theta_j} A_{\xi_1+i\xi_2}.$$

This and (4.6) yield

$$\sum_{j=1}^{m} a_j e^{-i\theta_j} \left\langle T_{\theta_j} \left(\nabla^{\perp}_{\overline{E}} (\xi_1 + i\xi_2) \right), w_j \right\rangle = 0.$$

Using (4.4) and (4.7), the above is written as

$$\sum_{j=1}^{m} a_j e^{-i\theta_j} \left\langle T_{\theta_j}\xi_3, w_j \right\rangle = 0,$$

or equivalently

$$\sum_{j=1}^{m} a_j e^{-i\theta_j} \langle g^*_{\theta_j}, w_j \rangle = 0,$$

where $g^*_{\theta_j} = T_{\theta_j}\xi_3$ is the polar surface of g_{θ_j}. This is equivalent to

$$\sum_{j=1}^{m} a_j \cos\theta_j \langle g^*_{\theta_j}, w_j \rangle = 0 \ \text{ and } \ \sum_{j=1}^{m} a_j \sin\theta_j \langle g^*_{\theta_j}, w_j \rangle = 0.$$

Eliminating $\langle g^*_{\theta_m}, w_m \rangle$, we can easily see that

$$\sum_{j=1}^{m-1} a_j \langle g^*_{\theta_j}, w_j^{(m)} \rangle = 0, \tag{4.8}$$

where $w_j^{(m)} = \sin(\theta_m - \theta_j) w_j \neq 0$. Using the fact that the polar surface of g_{θ_j} is congruent to g_{θ_j} (cf. [14, Lemma 11], or [43, Corollary 3]) and arguing as for (4.8), we have that

$$\sum_{j=1}^{m-2} a_j \langle g_{\theta_j}, w_j^{(m-1)} \rangle = 0,$$

where $w_j^{(m-1)} = \sin(\theta_{m-1} - \theta_j) w_j^{(m)}$. By an induction argumnet, we obtain

$$a_1 \langle g_{\theta_1}, w \rangle = 0 \ \text{ or } \ a_1 \langle g^*_{\theta_1}, w \rangle = 0$$

for a vector $w \in \mathbb{R}^6 \smallsetminus \{0\}$, which is a contradiction. Hence $\hat{g}$ is substantial.

Claim. We now claim that the higher fundamental forms of $\hat{g}$ are given by

$$\hat{\alpha}_s(\overline{E}, \dots, \overline{E}) = \begin{cases} \kappa^{s/3} \sum_{j=1}^m c_j^s g_{\theta_j} & \text{if } \ s \equiv 0 \bmod 6, \\ \kappa^{(s-1)/3} \sum_{j=1}^m c_j^s dg_{\theta_j}(\overline{E}) & \text{if } \ s \equiv 1 \bmod 6, \\ \kappa^{(s+1)/3} \sum_{j=1}^m c_j^s T_{\theta_j}(\xi_1 + i\xi_2) & \text{if } \ s \equiv 2 \bmod 6, \\ \kappa^{s/3} \sum_{j=1}^m c_j^s T_{\theta_j} \xi_3 & \text{if } \ s \equiv 3 \bmod 6, \\ \kappa^{(s-1)/3} \sum_{j=1}^m c_j^s T_{\theta_j}(\xi_1 - i\xi_2) & \text{if } \ s \equiv 4 \bmod 6, \\ \kappa^{(s+1)/3} \sum_{j=1}^m c_j^s dg_{\theta_j}(E) & \text{if } \ s \equiv 5 \bmod 6, \end{cases} \tag{4.9}$$

where the complex vectors $\mathbf{C}_s = (c_1^s, \dots, c_m^s) \in \mathbb{C}^m \smallsetminus \{0\}$, $2 \le s \le 3m$ satisfy the following orthogonality conditions, with respect to the standard Hermitian product $(\cdot, \cdot)$ on $\mathbb{C}^m$:

$$(\mathbf{C}_t, \overline{\mathbf{C}}_{t'}) = 0 \ \text{ if } \ t \equiv 1 \bmod 6 \text{ and } t' \equiv 5 \bmod 6, \ \text{ or } t \equiv 2 \bmod 6 \text{ and } t' \equiv 4 \bmod 6, \tag{4.10}$$

$$(\mathbf{C}_t, \overline{\mathbf{C}}_{t'}) = 0 = (\mathbf{C}_t, \mathbf{C}_{t'}) \text{ if } t \neq t' \text{ and } t, t' \equiv 0 \bmod 6, \ \text{ or } t, t' \equiv 3 \bmod 6, \tag{4.11}$$

$$(\mathbf{C}_t, \mathbf{C}_{t'}) = 0 = (\overline{\mathbf{C}}_t, \overline{\mathbf{C}}_{t'}) \text{ if } \ t \neq t' \text{ and } t, t' \equiv 1 \bmod 6, \ \text{ or } t, t' \equiv 2 \bmod 6, \ \text{ or } t, t' \equiv 4 \bmod 6, \ \text{ or } t, t' \equiv 5 \bmod 6 \tag{4.12}$$

and

$$(\mathbf{C}_t, \mathbf{a}) = 0 \text{ if } \ t \equiv 0, 1, 5 \bmod 6. \tag{4.13}$$

In particular, these complex vectors are defined inductively by

$$\mathbf{C}_{s+1} = \begin{cases} \mathbf{C}_s - \sum\limits_{t \equiv 1 \bmod 6}^{s} \frac{(\mathbf{C}_s, \mathbf{C}_t)}{\|\mathbf{C}_t\|^2} \mathbf{C}_t - \sum\limits_{t \equiv 5 \bmod 6}^{s} \frac{(\mathbf{C}_s, \overline{\mathbf{C}}_t)}{\|\mathbf{C}_t\|^2} \overline{\mathbf{C}}_t & \text{if } s \equiv 0 \bmod 6, \\ 2T_{\boldsymbol{\theta}} \mathbf{C}_s & \text{if } s \equiv 1 \bmod 6, \\ 2\mathbf{C}_s & \text{if } s \equiv 2 \bmod 6, \\ -\mathbf{C}_s + \sum\limits_{t \equiv 2 \bmod 6}^{s} \frac{(\mathbf{C}_s, \overline{\mathbf{C}}_t)}{\|\mathbf{C}_t\|^2} \overline{\mathbf{C}}_t + \sum\limits_{t \equiv 4 \bmod 6}^{s} \frac{(\mathbf{C}_s, \mathbf{C}_t)}{\|\mathbf{C}_t\|^2} \mathbf{C}_t & \text{if } s \equiv 3 \bmod 6, \\ -2T_{\boldsymbol{\theta}} \mathbf{C}_s & \text{if } s \equiv 4 \bmod 6, \\ -2\mathbf{C}_s & \text{if } s \equiv 5 \bmod 6, \end{cases} \tag{4.14}$$

where $\mathbf{C}_1 = \mathbf{a}$ and $T_{\boldsymbol{\sigma}} \colon \mathbb{C}^m \to \mathbb{C}^m$ denotes the unitary transformation given by

$$T_{\boldsymbol{\sigma}}\mathbf{u} = (u_1 e^{-i\sigma_1}, \dots, u_m e^{-i\sigma_m}), \quad \mathbf{u} = (u_1, \dots, u_m) \in \mathbb{C}^m$$

for any $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_m) \in \mathbb{R}^m$. It is worth noticing that (4.9) implies that $\mathbf{C}_s \neq 0$ for every $2 \leq s \leq 3m$, since the surface $\hat{g}$ is substantial.

To prove the claim, we proceed by induction on s. Using that

$$d\hat{g}(\overline{E}) = \sum_{j=1}^{m} a_j dg_{\theta_j}(\overline{E}),$$

the Gauss formula for g and g_{θ_j}, $j = 1, \dots, m$, yields

$$\hat{\alpha}_2(\overline{E}, \overline{E}) = \kappa \sum_{j=1}^{m} c_j^2 (\xi_1^{\theta_j} + i\xi_2^{\theta_j}),$$

where $c_j^2 = 2a_j e^{-i\theta_j}$. Hence, $\mathbf{C}_2 = 2T_{\boldsymbol{\theta}}\mathbf{C}_1 = 2T_{\boldsymbol{\theta}}\mathbf{a}$ and this proves (4.9) for $s = 2$.

Let us assume that (4.9)-(4.14) hold for any $t \leq s$. We shall prove that it is also true for $t = s + 1$. From the definition of the higher fundamental forms, we have that

$$\begin{aligned} \hat{\alpha}_{s+1}(\overline{E}, \dots, \overline{E}) &= \left(\overline{\nabla}_{\overline{E}} \hat{\alpha}_s(\overline{E}, \dots, \overline{E})\right)^{N_s^{\hat{g}}} \\ &= \kappa^{\lambda_s} \left(\tilde{\nabla}_{\overline{E}} \left(\frac{1}{\kappa^{\lambda_s}} \hat{\alpha}_s(\overline{E}, \dots, \overline{E})\right)\right)^{N_s^{\hat{g}}}, \end{aligned} \tag{4.15}$$

where $\overline{\nabla}$ is the induced connection of $g^*(T\mathbb{S}^{6m-1})$, $\tilde{\nabla}$ is the induced connection of the induced bundle $(i \circ g)^*(T\mathbb{R}^{6m})$, $i \colon \mathbb{S}^{6m-1} \to \mathbb{R}^{6m}$ is the standard inclusion, λ_s is the exponent of the function κ in (4.9) and $(\,\cdot\,)^{N_s^{\hat{g}}}$ stands for the projection onto the s-th normal bundle of $\hat{g}$. Taking (4.4) into account, we obtain

$$\tilde{\nabla}_{\overline{E}} \left(\frac{1}{\kappa^{\lambda_s}} \hat{\alpha}_s(\overline{E}, \dots, \overline{E})\right) = \sum_{j=1}^{m} c_j^s dg_{\theta_j}(\overline{E}) \ \text{ if } s \equiv 0 \bmod 6, \tag{4.16}$$

$$\begin{aligned} \tilde{\nabla}_{\overline{E}} \left(\frac{1}{\kappa^{\lambda_s}} \hat{\alpha}_s(\overline{E}, \dots, \overline{E})\right) &= \frac{1}{2} \sum_{j=1}^{m} c_j^s \Big(\langle \nabla_{\overline{E}} \overline{E}, E \rangle dg_{\theta_j}(\overline{E}) \\ &\quad + 4\kappa e^{-i\theta_j} T_{\theta_j}(\xi_1 + i\xi_2) \Big) \ \text{ if } s \equiv 1 \bmod 6, \end{aligned} \tag{4.17}$$

$$\tilde{\nabla}_{\overline{E}} \left(\frac{1}{\kappa^{\lambda_s}} \hat{\alpha}_s(\overline{E}, \dots, \overline{E})\right) = \sum_{j=1}^{m} c_j^s \left(-i\langle \nabla^{\perp}_{\overline{E}} \xi_1, \xi_2 \rangle T_{\theta_j}(\xi_1 + i\xi_2) + 2T_{\theta_j}\xi_3 \right) \ \text{ if } s \equiv 2 \bmod 6, \tag{4.18}$$

$$\tilde{\nabla}_{\overline{E}}\left(\frac{1}{\kappa^{\lambda_s}}\hat{\alpha}_s(\overline{E},\ldots,\overline{E})\right) = -\sum_{j=1}^{m} c_j^s T_{\theta_j}(\xi_1 - i\xi_2) \text{ if } s \equiv 3 \bmod 6, \tag{4.19}$$

$$\begin{aligned}\tilde{\nabla}_{\overline{E}}\left(\frac{1}{\kappa^{\lambda_s}}\hat{\alpha}_s(\overline{E},\ldots,\overline{E})\right) &= \sum_{j=1}^{m} c_j^s\Big(-2\kappa e^{-i\theta_j} dg_{\theta_j}(E) \\ &\quad + i\langle \nabla^{\perp}_{\overline{E}}\xi_1, \xi_2\rangle T_{\theta_j}(\xi_1 - i\xi_2)\Big) \text{ if } s \equiv 4 \bmod 6\end{aligned} \tag{4.20}$$

and

$$\tilde{\nabla}_{\overline{E}}\left(\frac{1}{\kappa^{\lambda_s}}\hat{\alpha}_s(\overline{E},\ldots,\overline{E})\right) = \frac{1}{2}\sum_{j=1}^{m} c_j^s \left(\langle \nabla_{\overline{E}}E, \overline{E}\rangle dg_{\theta_j}(E) - 4g_{\theta_j}\right) \text{ if } s \equiv 5 \bmod 6, \tag{4.21}$$

where ∇ is the Levi-Civitá connection on M.

Using (4.15) and (4.16)-(4.21), after some tedious computations, we derive that (4.9) holds for $t = s+1$. Taking into account (4.9) for $t = s+1$, the orthogonality of the higher normal bundles and (4.10)-(4.13) for $t \leq s$, we obtain that (4.10)-(4.13) are also true for $t = s+1$, and this completes the proof of the claim.

From (2.4) and since the length of the semi-axes κ_s and μ_s of the s-th curvature ellipse satisfy

$$\kappa_s^2 + \mu_s^2 = 2^{-2s}\left\|\hat{\alpha}_{s+1}(\overline{E},\ldots,\overline{E})\right\|^2,$$

we have

$$\|\hat{\alpha}_{s+1}\|^2 = 2^{-s}\left\|\hat{\alpha}_{s+1}(\overline{E},\ldots,\overline{E})\right\|^2.$$

Clearly (4.2) follows from (4.9) with

$$\hat{b}_s = \begin{cases} 2^{(3-4s)/3}\|\mathbf{C}_{s+1}\|^2 & \text{if } s \equiv 0 \bmod 3, \\ 2^{(1-4s)/3}\|\mathbf{C}_{s+1}\|^2 & \text{if } s \equiv 1 \bmod 3, \\ 2^{-(1+4s)/3}\|\mathbf{C}_{s+1}\|^2 & \text{if } s \equiv 2 \bmod 3. \end{cases}$$

Furthermore, the s-th normal curvature is given by

$$\hat{K}_s^{\perp} = 2^{-2s}\left(\left\|\hat{\alpha}_{s+1}(\overline{E},\ldots,\overline{E})\right\|^4 - |\langle \hat{\alpha}_{s+1}(E,\ldots,E), \hat{\alpha}_{s+1}(E,\ldots,E)\rangle|^2\right)^{1/2}.$$

This, combined with (4.9) yields (4.3), where

$$\hat{c}_s = \begin{cases} 2^{(3-7s)/3}\|\mathbf{C}_{s+1}\|^2 & \text{if } s \equiv 0 \bmod 3, \\ 2^{(1-7s)/3}\|\mathbf{C}_{s+1}\|^2 & \text{if } s \equiv 1 \bmod 3, \\ 2^{-(1+7s)/3}\left(\|\mathbf{C}_{s+1}\|^4 - |(\mathbf{C}_{s+1}, \overline{\mathbf{C}}_{s+1})|^2\right)^{1/2} & \text{if } s \equiv 2 \bmod 3. \end{cases}$$

Using (4.9) and the fact that the s-th Hopf differential of $\hat{g}$ is written as

$$\hat{\Phi}_s = 4^{-(s+1)} \langle \hat{\alpha}_{s+1}(E,\ldots,E), \hat{\alpha}_{s+1}(E,\ldots,E) \rangle \, \phi^{2s+2},$$

we obtain

$$\hat{\Phi}_s = 4^{-(s+1)} \kappa^{2\frac{s+1}{3}} \sum_{j=1}^{m} (\bar{c}_j^{s+1})^2 \phi^{2s+2} \ \text{ if } \ s \equiv 2 \bmod 3$$

and $\hat{\Phi}_s = 0$ otherwise.

The fact that the second Hopf differential Φ of g is given by $\Phi = 2^{-2}\kappa^2\phi^6$ completes the proof of part (iii), where $\hat{d}_s = 2^{-4(s+1)/3}(\overline{\mathbf{C}}_{s+1}, \mathbf{C}_{s+1})$. Obviously, all Hopf differentials are holomorphic and consequently $\hat{g}$ is exceptional according to Theorem 2.3.1. □

In the subsequent lemma, we determine the associated family of any surface $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}$ given by (4.1).

Lemma 4.1.1. *The associated family $\hat{g}_\varphi$ of any minimal surface $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}$ is given by $\hat{g}_\varphi = g_{\mathbf{a},\boldsymbol{\varphi}}$, where $\boldsymbol{\varphi} = (\theta_1 + \varphi, \ldots, \theta_m + \varphi)$.*

Proof. Let $f \colon M \to \mathbb{S}^{6m-1}$ be the minimal surface given by $f = g_{\mathbf{a},\boldsymbol{\varphi}}$. From (4.14) we can easily see that the complex vectors $\mathbf{C}_s^f$, $\mathbf{C}_s \in \mathbb{C}^m \smallsetminus \{0\}$ associated to f and $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}$, respectively, satisfy

$$\mathbf{C}_s^f = e^{-i\varphi}\mathbf{C}_s \ \text{ for any } \ 2 \le s \le 3m.$$

Moreover, Proposition 4.1.1(iii) implies that the s-th Hopf differential of f is given by

$$\Phi_s^f = \begin{cases} d_s^f \Phi^{(s+1)/3} & \text{if } \ s \equiv 2 \bmod 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $d_s^f = 2^{-4(s+1)/3}(\overline{\mathbf{C}}_{s+1}^f, \mathbf{C}_{s+1}^f)$. Equivalently, we have

$$\Phi_s^f = \begin{cases} 2^{-4(s+1)/3} e^{2i\varphi} (\overline{\mathbf{C}}_{s+1}, \mathbf{C}_{s+1}) \Phi^{(s+1)/3} & \text{if } \ s \equiv 2 \bmod 3, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the Hopf differentials of f and $\hat{g}$ satisfy

$$\Phi_s^f = e^{2i\varphi}\hat{\Phi}_s \ \text{ for any } \ 1 \le s \le 3m-1.$$

According to Theorem 5.2 in [41], the associated family of the surface $\hat{g}$ is $g_{\mathbf{a},\boldsymbol{\varphi}}$ and this completes the proof. □

4.2 Isometric deformations of simply connected pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of $\mathbb{S}^6$

In this section, we study exceptional surfaces that satisfy the Ricci-like condition $(*)$ or equivalently are locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$. To prove our main results, we need the following proposition.

Proposition 4.2.1. *Let $f\colon M \to \mathbb{S}^n$ be a nonflat r-exceptional surface which satisfies the Ricci-like condition $(*)$. Then the following hold:*

For any $1 \leq s \leq r+1$, we have:

$$\|\alpha_{s+1}\|^2 = \begin{cases} b_s(1-K)^{s/3} & \text{if } s \equiv 0 \bmod 3, \\ b_s(1-K)^{(s+2)/3} & \text{if } s \equiv 1 \bmod 3, \\ b_s(1-K)^{(s+1)/3} & \text{if } s \equiv 2 \bmod 3, \end{cases} \tag{4.22}$$

where $\{b_s\}, \{\rho_s\}$ are sequences of positive numbers such that $b_1 = 2, b_{s+1} = \rho_s^2 b_s, \rho_s \leq 1$ and $\rho_s = 1$ if $s \equiv 0, 1 \bmod 3$.

Moreover, for any $1 \leq s \leq r$, the following hold:

$$\Phi_s = 0 \quad \text{if } s \equiv 0, 1 \bmod 3, \tag{4.23}$$

$$K_s^* = \begin{cases} K & \text{if } s \equiv 0 \bmod 3, \\ -K & \text{if } s \equiv 1 \bmod 3, \\ 0 & \text{if } s \equiv 2 \bmod 3, \end{cases} \tag{4.24}$$

and

$$K_s^\perp = \begin{cases} c_s(1-K)^{s/3} & \text{if } s \equiv 0 \bmod 3, \\ c_s(1-K)^{(s+2)/3} & \text{if } s \equiv 1 \bmod 3, \\ c_s(1-K)^{(s+1)/3} & \text{if } s \equiv 2 \bmod 3, \end{cases} \tag{4.25}$$

where $c_s = 2^{-s}\rho_s b_s$.

Proof. We set $\rho_s = 2^s K_s^\perp / \|\alpha_{s+1}\|^2$. Since f is r-exceptional, the function ρ_s is constant for any $1 \leq s \leq r$. We proceed by induction on r.

Assume that f is 1-exceptional. The Gauss equation implies $\|\alpha_2\|^2 = 2(1-K)$. Then from Proposition 2.3.1(i) for $s = 1$ and the Ricci-like condition $(*)$, we find $K_1^* = -K$. Moreover, we have $K_1^\perp = \rho_1(1-K)$. We claim that $\rho_1 = 1$. Assume to the contrary that $\rho_1 \neq 1$. Then $\Phi_1 \neq 0$ and Proposition 2.3.1(ii) combined with condition $(*)$ yield

$K_1^* = K$, which is a contradiction. Hence $\rho_1 = 1$ and Proposition 2.1.1 yields (4.22) for $s = 2$ with $b_2 = 2$. This settles the case $r = 1$.

Suppose now that (4.23)-(4.25) hold if f is r-exceptional. We shall prove that (4.23)-(4.25) also hold assuming that f is $(r+1)$-exceptional. By Theorem 2.3.1, the Hopf differential Φ_{r+1} is holomorphic, hence either it is identically zero or its zeros are isolated.

At first we assume that $r \equiv 0 \bmod 3$. From the inductive assumption, we have

$$\|\alpha_{r+2}\|^2 = b_{r+1}(1-K)^{(r+3)/3}.$$

We claim that $\rho_{r+1} = 1$. Arguing indirectly, we assume that $\Phi_{r+1} \neq 0$. Then Proposition 2.3.1(iv) yields $K_{r+1}^* = 0$. Taking into account condition $(*)$, Proposition 2.3.1(ii) implies that M is flat and this is a contradiction. Thus Φ_{r+1} is identically zero, or equivalently $\rho_{r+1} = 1$. From Proposition 2.3.1(iii) and condition $(*)$, we obtain $K_{r+1}^* = -K$. Furthermore, we have that

$$K_{r+1}^{\perp} = 2^{-(r+1)} \rho_{r+1} \|\alpha_{r+2}\|^2,$$

or equivalently

$$K_{r+1}^{\perp} = c_{r+1}(1-K)^{(r+3)/3},$$

with $c_{r+1} = 2^{-(r+1)} b_{r+1}$. Then using Proposition 2.1.1, we obtain

$$\|\alpha_{r+3}\|^2 = b_{r+2}(1-K)^{(r+3)/3},$$

with $b_{r+2} = b_{r+1}$.

Assume now that $r \equiv 1 \bmod 3$. From the inductive assumption, we have that

$$\|\alpha_{r+2}\|^2 = b_{r+1}(1-K)^{(r+2)/3}.$$

If $\Phi_{r+1} \neq 0$, then Proposition 2.3.1(iv) yields $K_{r+1}^* = 0$. If Φ_{r+1} is identically zero, or equivalently $\rho_{r+1} = 1$, then Proposition 2.3.1(iii) and condition $(*)$ imply that $K_{r+1}^* = 0$. Furthermore, we have that

$$K_{r+1}^{\perp} = 2^{-(r+1)} \rho_{r+1} \|\alpha_{r+2}\|^2,$$

or equivalently

$$K_{r+1}^{\perp} = c_{r+1}(1-K)^{(r+2)/3},$$

with $c_{r+1} = 2^{-(r+1)} \rho_{r+1} b_{r+1}$. From Proposition 2.1.1, we obtain

$$\|\alpha_{r+3}\|^2 = b_{r+2}(1-K)^{(r+2)/3},$$

with $b_{r+2} = \rho_{r+1}^2 b_{r+1}$.

Finally, we suppose that $r \equiv 2 \bmod 3$. From the inductive assumption, we have that

$$\|\alpha_{r+2}\|^2 = b_{r+1}(1-K)^{(r+1)/3}.$$

We claim that $\rho_{r+1} = 1$. Assume to the contrary that $\rho_{r+1} \neq 1$ or equivalently $\Phi_{r+1} \neq 0$. Then Proposition 2.3.1(iv) yields $K^*_{r+1} = 0$. Taking into account the Ricci-like condition $(*)$, Proposition 2.3.1(ii) implies that M is flat, which is a contradiction. Hence Φ_{r+1} is identically zero, or equivalently $\rho_{r+1} = 1$. From Proposition 2.3.1(iii) and condition $(*)$, we obtain $K^*_{r+1} = K$. Furthermore, we have

$$K^{\perp}_{r+1} = 2^{-(r+1)}\rho_{r+1}\|\alpha_{r+2}\|^2,$$

or equivalently

$$K^{\perp}_{r+1} = c_{r+1}(1-K)^{(r+1)/3},$$

with $c_{r+1} = 2^{-(r+1)}\rho_{r+1}b_{r+1}$. Using Proposition 2.1.1, it follows that

$$\|\alpha_{r+3}\|^2 = b_{r+2}(1-K)^{(r+4)/3}$$

with $b_{r+2} = b_{r+1}$ and this completes the proof. □

We are now ready to prove the main result of this section. The following theorem provides an answer to the problem stated in the introduction of this chapter, in the case of simply connected exceptional surfaces.

Theorem 4.2.1. *Let $f\colon M \to \mathbb{S}^n$ be a nonflat simply connected exceptional surface with substantial odd codimension. If f satisfies the Ricci-like condition $(*)$ away from the isolated points with Gaussian curvature $K = 1$, then $n = 6m-1$ and there exists $\mathbf{a} = (a_1,\ldots,a_m) \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$ with $\Pi_{j=1}^m a_j \neq 0$ and $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_m) \in \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, where $0 \leq \theta_1 < \cdots < \theta_m < \pi$ such that $f = g_{\mathbf{a},\boldsymbol{\theta}}$.*

Proof. We claim that $n \equiv 5 \bmod 6$. Arguing indirectly, we suppose at first that $n = 6l+1$. Since f is $(3l-1)$-exceptional, (4.22) yields $\|\alpha_{3l+1}\|^2 = b_{3l}(1-K)^l$. Moreover, viewing f as a minimal surface in $\mathbb{S}^{6l+2}$, we obviously have $K^{\perp}_{3l} = K^*_{3l} = 0$. Then from Proposition 2.3.1(ii), we obtain

$$\Delta \log \|\alpha_{3l+1}\|^2 = 2(3l+1)K.$$

Thus $K = 0$, which is a contradiction.

Suppose that $n = 6l+3$. Since f is $3l$-exceptional, (4.22) yields that

$$\|\alpha_{3l+2}\|^2 = b_{3l+1}(1-K)^{l+1}.$$

Moreover, viewing f as a minimal surface in $\mathbb{S}^{6l+4}$, we obviously have $K^{\perp}_{3l+1} = K^{*}_{3l+1} = 0$. Then from Proposition 2.3.1(ii), it follows that

$$\Delta \log \|\alpha_{3l+2}\|^2 = 2(3l+2)K.$$

Thus $K = 0$, which is a contradiction.

Hence $n \equiv 5 \bmod 6$ and we may set $n = 6m - 1$. According to (4.23), we have that $\Phi_r = 0$ if $r \equiv 0, 1 \bmod 3$. Let

$$r_0 = \min \{r : 2 \leq r \leq 3m-1 \ \text{ with } \ \Phi_r \neq 0\}.$$

Obviously $r_0 \equiv 2 \bmod 3$. Let z be a local complex coordinate such that the induced metric is given by $ds^2 = F|dz|^2$. From the definition of Hopf differentials we know that $\Phi_r = f_r dz^{2r+2}$, where $f_r = \langle \alpha^{(r+1,0)}_{r+1}, \alpha^{(r+1,0)}_{r+1} \rangle$.

For any $r \equiv 2 \bmod 3$ and $r \geq r_0$ such that $\Phi_r \neq 0$, we may write $\Phi_r = |f_r| e^{i\sigma_r} dz^{2r+2}$. Using (4.22), (4.25) and (2.7), we obtain

$$\Phi_r = 2^{-(r+2)} b_r F^{r+1} e^{i\sigma_r} (1-K)^{(r+1)/3} \left(1-\rho_r^2\right)^{1/2} dz^{2r+2}.$$

We pick a branch h of $f_{r_0}^{3/(r_0+1)}$ and define the form $\Phi = c_0 h dz^6$, where c_0 is given by

$$c_0 = \begin{cases} \left(\frac{2}{b_{r_0}\left(1-\rho_{r_0}^2\right)^{1/2}}\right)^{3/(r_0+1)} & \text{if } r_0 < 3m-1, \\ \left(\frac{2}{b_{r_0}}\right)^{3/(r_0+1)} & \text{if } r_0 = 3m-1. \end{cases}$$

It is obvious that Φ is well defined and holomorphic. It follows that

$$\Phi_r = \frac{1}{2} b_r \left(1-\rho_r^2\right)^{1/2} e^{i\left(\sigma_r - \frac{r+1}{r_0+1}\sigma_{r_0}\right)} \Phi^{(r+1)/3}.$$

From the holomorphicity of Φ_r and Φ, we deduce that $\sigma_r - \frac{r+1}{r_0+1}\sigma_{r_0}$ is constant. Moreover, we easily see that

$$|c_0 h|^2 = \left(\frac{F}{2}\right)^6 (1-K)^2.$$

Using Theorem 11.1 in [18], we infer that there exists a pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$ whose second Hopf differential is Φ.

We consider a surface $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}$ which according to Proposition 4.1.1 is exceptional for any $\mathbf{a} \in \mathbb{S}^{m-1}$ and $\boldsymbol{\theta}$. Setting $\hat{\rho}_s = 2^s \hat{K}^{\perp}_s / \|\hat{\alpha}_{s+1}\|^2$, it follows from Proposition 4.1.1 that

$$\hat{\rho}_s = \begin{cases} \left(1 - \frac{\left|\left(\mathbf{C}_{s+1}, \overline{\mathbf{C}}_{s+1}\right)\right|^2}{\|\mathbf{C}_{s+1}\|^4}\right)^{1/2} & \text{if } \ s \equiv 2 \bmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

We now claim that we can choose $\mathbf{a} \in \mathbb{S}^{m-1}$ and $\boldsymbol{\theta}$ such that

$$b_r = \hat{b}_r, \ c_r = \hat{c}_r \ \text{ and } \ \rho_r = \hat{\rho}_r \ \text{ for every } \ 1 \leq r \leq 6m-3,$$

where $\hat{b}_r, \hat{c}_r, b_r, c_r$ and ρ_r are the sequences in Proposition 4.1.1 and Proposition 4.2.1, respectively. Obviously, Proposition 4.2.1 gives that

$$b_r = \hat{b}_r = 2 \ \text{ for } \ r = 1, 2, \ \ c_1 = \hat{c}_1 = 1 \ \text{ and } \ \rho_1 = \hat{\rho}_1 = 1.$$

We choose $\mathbf{a}$ and $\boldsymbol{\theta}$ such that the unitary transformation $T_{2\boldsymbol{\theta}}$ satisfies

$$|(T_{2\boldsymbol{\theta}}\mathbf{a}, \mathbf{a})|^2 = 1 - \rho_2^2.$$

According to (4.14), the above is equivalent to $\rho_2 = \hat{\rho}_2$. Then using Proposition 4.2.1, we obtain that

$$b_{r+1} = \hat{b}_{r+1}, \, c_r = \hat{c}_r \ \text{ and } \ \rho_r = \hat{\rho}_r \ \text{ for } \ 1 \leq r \leq 4.$$

Similarly, we may choose $\mathbf{a}$ and $\boldsymbol{\theta}$ such that

$$\frac{|(T_{2\boldsymbol{\theta}}\mathbf{C}_4, \mathbf{C}_4)|^2}{\|\mathbf{C}_4\|^4} = 1 - \rho_5^2,$$

or equivalently $\rho_5 = \hat{\rho}_5$, according to (4.14). Repeating this argument, and choosing $\mathbf{a}$ and $\boldsymbol{\theta}$ such that $\rho_r = \hat{\rho}_r$ for any $r \equiv 2 \bmod 3$, the claim follows inductively.

Thus, Proposition 4.2.1 implies that the a-invariants of the minimal surface f coincide with those of $\hat{g} = g_{\mathbf{a},\boldsymbol{\theta}}$ for appropriate $\mathbf{a}$ and $\boldsymbol{\theta}$.

It follows from Theorem 5.2 in [41] that f is a member of the associate family of $\hat{g}$, which in view of Lemma 4.1.1 completes the proof. ☐

For the proof of Theorem 4.2.2 below, we recall the following well known fact [15].

Lemma 4.2.1. *Let M be a two-dimensional Riemannian manifold and let $f\colon M \to \mathbb{R}$ be a smooth function such that $\Delta f = P(f)$ and $\|\nabla f\|^2 = Q(f)$ for smooth functions $P, Q\colon \mathbb{R} \to \mathbb{R}$, where ∇f denotes the gradient of f. Then the Gaussian curvature K satisfies*

$$2KQ + (2P - Q')(P - Q') + Q(2P' - Q'') = 0,$$

on $\{p \in M : \nabla f(p) \neq 0\}$.

For minimal surfaces in substantial even codimension, we prove the following result.

Theorem 4.2.2. *(i) Substantial exceptional surfaces in $\mathbb{S}^{6m}$ cannot satisfy the Ricci-like condition $(*)$.*

(ii) Substantial $\left[\frac{n-1}{2}\right]$-exceptional surfaces in an even dimensional sphere $\mathbb{S}^n$ cannot satisfy the Ricci-like condition $(*)$.

Proof. (i) Assume to the contrary that $f\colon M \to \mathbb{S}^{6m}$ is a substantial exceptional surface that satisfies condition $(*)$. Since f is $(3m-2)$-exceptional, Proposition 4.2.1 yields $\|\alpha_{3m}\|^2 = b_{3m-1}(1-K)^m$ and $K^*_{3m-2} = -K$. Moreover, combining Proposition 2.1.1 with Proposition 4.2.1, we find that

$$K^*_{3m-1} = \frac{2^{3m-1}K^{\perp}_{3m-1}}{b_{3m-1}(1-K)^m}.$$

By Theorem 2.3.1, Φ_{3m-1} is holomorphic. Hence either it is identically zero or its zeros are isolated. If Φ_{3m-1} is identically zero, then f is $(3m-1)$-exceptional, and (4.24) yields $K^*_{3m-1} = 0$. Then the above equation implies $K^{\perp}_{3m-1} = 0$. This means that f lies in a totally geodesic $\mathbb{S}^{6m-1}$ of $\mathbb{S}^{6m}$ (cf. [34, p. 96]), which is a contradiction. Suppose now that $\Phi_{3m-1} \neq 0$. By virtue of Proposition 2.3.1(ii), we derive that

$$\begin{aligned}
\Delta \log\left(\|\alpha_{3m}\|^2 + 2^{3m-1}K^{\perp}_{3m-1}\right) &= 2\big(3mK - K^*_{3m-1}\big),\\
\Delta \log\left(\|\alpha_{3m}\|^2 - 2^{3m-1}K^{\perp}_{3m-1}\right) &= 2\big(3mK + K^*_{3m-1}\big).
\end{aligned}$$

Using condition $(*)$ and setting $\rho = 2^{3m-1}K^{\perp}_{3m-1}/\|\alpha_{3m}\|^2$, the above equations are equivalent to

$$\Delta \log(1+\rho) = -2K^*_{3m-1} \quad\text{and}\quad \Delta \log(1-\rho) = 2K^*_{3m-1}. \tag{4.26}$$

Since $\rho = 2^{3m-1}K^{\perp}_{3m-1}/\|\alpha_{3m}\|^2$, we obtain $K^*_{3m-1} = \rho$.

Then equations (4.26) are written equivalently

$$\Delta\rho = -2\rho(1+\rho^2) \quad\text{and}\quad \|\nabla\rho\|^2 = 2\rho^2(1-\rho^2).$$

If the function ρ is constant, then $\rho = 0$ and consequently $K^{\perp}_{3m-1} = 0$, which contradicts the fact that f is substantial. If ρ is not constant, then Lemma 4.2.1 yields $K = -8$, which contradicts the Ricci-like condition $(*)$.

(ii) Assume that $f\colon M \to \mathbb{S}^n$ is a substantial $[(n-1)/2]$-exceptional surface which satisfies condition $(*)$, where n is even. It suffices to consider the case $n = 6m+2$ and $n = 6m+4$, since the case $n = 6m$ was settled in (i).

At first let us suppose that $n = 6m+2$. Since f is $3m$-exceptional, (4.23) and (4.24) yield $\Phi_{3m} = 0$ and $K^*_{3m} = K$. By virtue of Proposition 2.1.1, we obtain

$$K^*_{3m} = \frac{K^{\perp}_{3m} \|\alpha_{3m}\|^2}{2^{3m-2}\left(K^{\perp}_{3m-1}\right)^2}.$$

Then, using (4.22) and (4.25), we have that $K^*_{3m} = 1$, which is a contradiction.

We suppose now that $n = 6m+4$. Since f is $(3m+1)$-exceptional, (4.23) and (4.24) yield $\Phi_{3m+1} = 0$ and $K^*_{3m+1} = -K$. From Proposition 2.1.1 it follows that

$$K^*_{3m+1} = \frac{K^{\perp}_{3m+1} \|\alpha_{3m+1}\|^2}{2^{3m-1}\left(K^{\perp}_{3m}\right)^2}.$$

Using (4.22) and (4.25), we find that $K^*_{3m+1} = 1 - K$, which is a contradiction, and this completes the proof. □

4.3 Isometric deformations of nonsimply connected pseudoholomorphic curves in a totally geodesic $\mathbb{S}^5$ of $\mathbb{S}^6$

Now we focus on the study of the moduli space of noncongruent isometric deformations of a nonsimply connected pseudoholomorphic curve $g\colon M \to \mathbb{S}^5$. We consider the covering map $\Pi\colon \tilde{M} \to M$, $\tilde{M}$ being the universal cover of M with the metric and orientation that make Π an orientation preserving local isometry. Corresponding objects on $\tilde{M}$ are denoted with tilde. Then the map $\tilde{g}\colon \tilde{M} \to \mathbb{S}^5$ with $\tilde{g} = g \circ \Pi$ is a pseudoholomorphic curve. Obviously, since $\tilde{g}$ is simply connected, we know from Theorem 4.2.1 that

$$\mathcal{M}^{\mathrm{e}}_n(\tilde{g}) = \mathbb{S}^{m-1}_* \times \Gamma^m.$$

For any $(\mathbf{a}, \boldsymbol{\theta}) \in \mathbb{S}^{m-1}_* \times \overline{\Gamma}^m$, where $\overline{\Gamma}^m$ is the closure of Γ^m, we consider the minimal surface $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}\colon \tilde{M} \to \mathbb{S}^{6m-1} \subset \mathbb{R}^{6m}$ defined by

$$\tilde{g}_{\mathbf{a},\boldsymbol{\theta}} = a_1 \tilde{g}_{\theta_1} \oplus \cdots \oplus a_m \tilde{g}_{\theta_m},$$

where $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of $\mathbb{R}^{6m}$. Each surface $\tilde{g}_{\theta_j}\colon \tilde{M} \to \mathbb{S}^5$, $j = 1, \ldots, m$, is a member of the associated family of $\tilde{g}$.

Clearly, given an exceptional surface $f\colon M \to \mathbb{S}^n$ in the moduli space of the curve g, the minimal surface $\tilde{f}\colon \tilde{M} \to \mathbb{S}^n$ with $\tilde{f} = f \circ \Pi$ belongs to the moduli space $\mathcal{M}^{\mathrm{e}}_n(\tilde{g})$ of the curve $\tilde{g}$. Therefore, the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$ can be described as the subset of all $(\mathbf{a},\boldsymbol{\theta})$ in $\mathcal{M}_n^{\mathrm{e}}(\tilde{g})$ such that $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ factors as $F\circ\Pi$ for some exceptional surface $F\colon M\to\mathbb{S}^n$. We follow this notation throughout this chapter.

The group $\mathcal{D}$ of *deck transformations* of the covering map $\Pi\colon \tilde{M}\to M$ consists of all diffeomorphisms $\sigma\colon\tilde{M}\to\tilde{M}$ such that $\Pi\circ\sigma=\Pi$.

We need the following lemmas.

Lemma 4.3.1. *For each $\sigma\in\mathcal{D}$ the surfaces $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ and $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}\circ\sigma$ are congruent for every $(\mathbf{a},\boldsymbol{\theta})\in\mathbb{S}_*^{m-1}\times\overline{\Gamma}^m$, that is there exists $\Phi_{\boldsymbol{\theta}}(\sigma)\in\mathrm{O}(n+1)$ such that*

$$\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}\circ\sigma=\Phi_{\boldsymbol{\theta}}(\sigma)\circ\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}.$$

Proof. It follows from Proposition 9 in [14] that the surfaces $\tilde{g}_\theta$ and $\tilde{g}_\theta\circ\sigma$ are congruent for all $\theta\in[0,\pi)$. Therefore, there exists $\Psi_\theta(\sigma)\in\mathrm{O}(7)$ such that

$$\tilde{g}_\theta\circ\sigma=\Psi_\theta(\sigma)\circ\tilde{g}_\theta \tag{4.27}$$

for every $\theta\in[0,\pi)$.

We define the isometry $\Phi_{\boldsymbol{\theta}}(\sigma)\in\mathrm{O}(n+1)$ given by

$$\Phi_{\boldsymbol{\theta}}(\sigma)=\Psi_{\theta_1}(\sigma)\oplus\cdots\oplus\Psi_{\theta_m}(\sigma),$$

with respect to an orthogonal decomposition $\mathbb{R}^{6m}=\mathbb{R}^6\oplus\cdots\oplus\mathbb{R}^6$. That

$$\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}\circ\sigma=\Phi_{\boldsymbol{\theta}}(\sigma)\circ\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$$

holds, follows directly from (4.27). □

Remark 4.3.1. *The isometry $\Phi_{\boldsymbol{\theta}}(\sigma)$ is real analytic with respect to $\boldsymbol{\theta}$ (cf. [17]).*

Lemma 4.3.2. *If $(\mathbf{a},\boldsymbol{\theta})$ belongs to $\mathcal{M}_n^{\mathrm{e}}(\tilde{g})$, then $(\mathbf{a},\boldsymbol{\theta})$ belongs to $\mathcal{M}_n^{\mathrm{e}}(g)$ if and only if*

$$\Phi_{\boldsymbol{\theta}}(\mathcal{D})=\{\mathrm{Id}\}. \tag{4.28}$$

Proof. Let $(\mathbf{a},\boldsymbol{\theta})\in\mathcal{M}_n^{\mathrm{e}}(g)$. There exists an exceptional surface $F\colon M\to\mathbb{S}^n$ such that

$$F\circ\pi=\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}.$$

Composing with an arbitrary $\sigma\in\mathcal{D}$ and using Lemma 4.3.1, we obtain

$$\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}=\Phi_{\boldsymbol{\theta}}(\sigma)\circ\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}.$$

The fact that $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ has substantial codimension yields (4.28).

Conversely, assume that (4.28) holds. We will prove that $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ factors as $F \circ \Pi$ where $F\colon M \to \mathbb{S}^n$ is an exceptional surface. At first we claim that $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ remains constant on each fiber of the covering map Π. Indeed, let $\tilde{p}_1, \tilde{p}_2$ belong to the fiber $\Pi^{-1}(p)$ for some $p \in M$. Then there exists a deck transformation σ such that $\sigma(\tilde{p}_1) = \tilde{p}_2$. Using Lemma 4.3.1 and (4.28), we obtain

$$\begin{aligned} \tilde{g}_{\mathbf{a},\boldsymbol{\theta}}(\tilde{p}_2) &= \tilde{g}_{\mathbf{a},\boldsymbol{\theta}} \circ \sigma(\tilde{p}_1) \\ &= \Phi_{\boldsymbol{\theta}}(\sigma) \circ \tilde{g}_{\mathbf{a},\boldsymbol{\theta}}(\tilde{p}_1) \\ &= \tilde{g}_{\mathbf{a},\boldsymbol{\theta}}(\tilde{p}_1). \end{aligned}$$

Then $\tilde{g}_{\mathbf{a},\boldsymbol{\theta}}$ factors as $F \circ \Pi$, where $F\colon M \to \mathbb{S}^n$ is a minimal surface. It remains to prove that $F \in \mathcal{M}_n^{\mathrm{e}}(g)$. Since Π is an orientation preserving local isometry, it is obvious that F is an exceptional surface. □

Proof of Theorem 4.0.1. Lemma 4.3.2 implies that $\mathbb{S}_*^{m-1} \times \{\boldsymbol{\theta}\}$ is contained in $\mathcal{M}_n^{\mathrm{e}}(g)$ for each $(\mathbf{a}, \boldsymbol{\theta}) \in \mathcal{M}_n^{\mathrm{e}}(g)$. Therefore, the moduli space splits as

$$\mathcal{M}_n^{\mathrm{e}}(g) = \mathbb{S}_*^{m-1} \times \Gamma_0,$$

where Γ_0 is a subset of Γ^m. Additionally, Lemma 4.3.2 implies that $\boldsymbol{\theta} \in \Gamma_0$ if and only if $\Phi_{\boldsymbol{\theta}}(\mathcal{D}) = \{\mathrm{Id}\}$. Fix $\sigma \in \mathcal{D}$. Then $\Phi_{\boldsymbol{\theta}}(\sigma) = \mathrm{Id}$ and Γ_0 is a real analytic set (see Remark 4.3.1). If Γ_0 is a proper subset of Γ^m, according to Lojasiewicz's structure theorem [25, Theorem 6.3.3]) the set Γ_0 locally decomposes as

$$\Gamma_0 = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \cdots \cup \mathcal{V}^{m-1},$$

where each $\mathcal{V}^d$, $0 \le d \le m-1$, is either empty or a disjoint finite union of d-dimensional real analytic subvarieties.

Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_l, \ldots, \theta_m) \in \Gamma_0$. Suppose that the straight line through $\boldsymbol{\theta}$ that is parallel to the l-th coordinate axis of $\mathbb{R}^m$ is not a finite set. Thus, this line contains a sequence $\boldsymbol{\theta}^{(i)} = (\theta_1, \ldots, \theta_l^{(i)}, \ldots, \theta_m), i \in \mathbb{N}$. By passing if necessary to a subsequence, we may assume that this sequence converges to $\boldsymbol{\theta}^\infty = (\theta_1, \ldots, \theta_l^\infty, \ldots, \theta_m)$, where $\theta_l^\infty = \lim \theta_l^{(i)}$. Clearly $\theta_{l-1} \le \theta_l^\infty \le \theta_{l+1}$. At first we suppose that $\theta_{l-1} < \theta_l^\infty < \theta_{l+1}$, that is $\boldsymbol{\theta}^\infty \in \Gamma_0$. Fix $\sigma \in \mathcal{D}$. Lemma 4.3.2 implies that $\Phi_{\boldsymbol{\theta}^{(i)}}(\sigma) = \mathrm{Id}$ and consequently $\Phi_{\boldsymbol{\theta}^\infty}(\sigma) = \mathrm{Id}$. We define the function

$$h(\theta) = \left(\Phi_{(\theta_1,\ldots,\theta_{l-1},\theta,\theta_{l+1},\ldots\theta_m)}(\sigma)\right)_{ij},\ \theta \in [\theta_{l-1}, \theta_{l+1}),$$

where $\left(\Phi_{\boldsymbol{\theta}}(\sigma)\right)_{ij}$ denotes the (i,j)-element of the matrix of $\Phi_{\boldsymbol{\theta}}(\sigma)$ with respect to the standard basis of $\mathbb{R}^{n+1}$. From the mean value theorem we have that there exists $\xi_1^{(i)}$ between $\theta_l^{(i)}$ and θ_l^∞ such that $(dh/d\theta)(\xi_1^{(i)}) = 0$ and hence $(dh/d\theta)(\theta_l^\infty) = 0$. Applying again the mean value theorem, we obtain that there exists $\xi_2^{(i)}$ between $\xi_1^{(i)}$ and θ_l^∞ such that $(d^2h/d\theta^2)(\xi_2^{(i)}) = 0$. Inductively, we have that the k-th derivative satisfies $(d^kh/d\theta^k)(\theta_l^\infty) = 0$ for any k. The analyticity of h (see Remark 4.3.1) yields that $h = \delta_{ij}$ on $[\theta_{l-1}, \theta_{l+1})$, where δ_{ij} is the Krönecker delta. Hence, the proof follows from Lemma 4.3.2.

Now without loss of generality, assume that $\theta_{l-1} = \theta_l^\infty < \theta_{l+1}$. Clearly $\boldsymbol{\theta}^\infty \notin \Gamma_0$. We fix $\sigma \in \mathcal{D}$ and extend $\Phi_{\boldsymbol{\theta}}$ in the obvious way. Then $\Phi_{\boldsymbol{\theta}^{(i)}}(\sigma) = \mathrm{Id}$ and consequently $\Phi_{\boldsymbol{\theta}^\infty}(\sigma) = \mathrm{Id}$ and the claim follows as before. □

4.4 Global results

In this section, we prove results for compact minimal surfaces that satisfy the condition $(*)$ and are not homeomorphic to the torus. We recall from Lemma 3.2.1 that such surfaces cannot be homeomorphic to the sphere $\mathbb{S}^2$.

Theorem 4.4.1. *Let $f\colon M \to \mathbb{S}^n$ be a compact substantial minimal surface with genus $g \geq 2$ which satisfies the Ricci-like condition $(*)$ away from isolated points where the Gaussian curvature satisfies $K = 1$. If the eccentricity ε_r of the higher curvature ellipses of order $r \equiv 0 \bmod 3$ for any $1 \leq r \leq s$ satisfies condition*

$$\int_M \frac{\varepsilon_r}{(1-K)^\gamma} dA < \infty$$

for some constant $\gamma \geq 4/3$, then f is s-exceptional.

Proof. According to Lemma 3.2.1, the function $1 - K$ is of absolute value type with nonempty zero set $M_0 = \{p_1, \ldots, p_m\}$ and corresponding order $\mathrm{ord}_{p_j}(1-K) = 2k_j$. For each point $p_j, j = 1, \ldots, m$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$. Around p_j, we have that

$$1 - K = |z|^{2k_j} u_0, \tag{4.29}$$

where u_0 is a smooth positive function.

We shall prove that f is s-exceptional by induction. At first we show that f is 1-exceptional. In fact, we can prove that f is 1-isotropic. We know that the first Hopf differential $\Phi_1 = f_1 dz^4$ is holomorphic. Hence either Φ_1 is identically zero, or its zeros are isolated. Supose to the contrary that Φ_1 is not identically zero. The Gauss equation (2.5) yields that each p_j is a totally geodesic point. From the definition of the first Hopf differential we have that Φ_1 vanishes at each p_j. Thus we may write

$f_1 = z^{l_1(p_j)}\psi_1$ around p_j, where $l_1(p_j)$ is the order of Φ_1 at p_j, and ψ_1 is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\frac{1}{4}\|\alpha_2\|^4 - (K_1^{\perp})^2 = (2F^{-1})^4|\psi_1|^2|z|^{2l_1(p_j)} \tag{4.30}$$

around p_j. We now consider the function $u_1\colon M \setminus M_0 \to \mathbb{R}$ defined by

$$u_1 = \frac{\left(\frac{1}{4}\|\alpha_2\|^4 - (K_1^{\perp})^2\right)^3}{(1-K)^4}.$$

In view of (4.29) and (4.30) we find that the function u_1 around p_j, is written as

$$u_1 = (2F^{-1})^{12}u_0^{-4}|\psi_1|^6|z|^{6l_1(p_j)-8k_j}. \tag{4.31}$$

Using (2.5) we obtain $u_1 \leq (1-K)^2$. Thus, from (4.29) and (4.31) we deduce that $l_1(p_j) \geq 2k_j$ and we can extend u_1 to a smooth function on M. It follows from Proposition 2.3.1(ii) for $r = 1$, that

$$\Delta \log\left(\|\alpha_2\|^2 + 2K_1^{\perp}\right) = 2\big(2K - K_1^*\big)$$

and

$$\Delta \log\left(\|\alpha_2\|^2 - 2K_1^{\perp}\right) = 2\big(2K + K_1^*\big).$$

Summing up, we obtain

$$\Delta \log\left(\|\alpha_2\|^4 - 4(K_1^{\perp})^2\right) = 8K.$$

Combining the last equation with condition $(*)$, we have

$$\Delta \log\left(\|\alpha_2\|^4 - 4(K_1^{\perp})^2\right)^3 = \Delta \log(1-K)^4,$$

which implies that $\log u_1$ is harmonic away from the isolated zeros of u_1. By continuity, the function u_1 is subharmonic everywhere on M. Using the maximum principle, we deduce that u_1 is a positive constant. This contradicts the fact that $K = 1$ on M_0.

Suppose now that f is $(r-1)$-exceptional for $r \geq 2$. We note that M cannot be flat due to our assumption on the genus. We shall prove that f is also r -exceptional. From Proposition 4 in [42], we know that $\Phi_r = f_r dz^{2r+2}$ is globally defined and holomorphic. Hence either $\Phi_r = 0$ or its zeros are isolated. In the former case, f is r-exceptional.

Supose to the contrary that Φ_r is not identically zero. Obviously, Φ_r vanishes at p_j. Hence we may write $f_r = z^{l_r(p_j)}\psi_r$ around p_j, where $l_r(p_j)$ is the order of Φ_r at p_j, and ψ_r is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\|\alpha_{r+1}\|^4 - 4^r(K_r^{\perp})^2 = 4^{r+2}F^{-2(r+1)}|\psi_r|^2|z|^{2l_r(p_j)} \tag{4.32}$$

around p_j. We now consider the smooth function $u_r\colon M \smallsetminus M_0 \to \mathbb{R}$ (see Proposition 2.3.2) given by

$$u_r = \frac{\left(\|\alpha_{r+1}\|^4 - 4^r (K_r^{\perp})^2\right)^3}{(1-K)^{2(r+1)}}.$$

In view of (4.29) and (4.32), we find that

$$u_r = 4^{3(r+2)} F^{-6(r+1)} u_0^{-2(r+1)} |\psi_r|^6 |z|^{6l_r(p_j)-4k_j(r+1)}, \tag{4.33}$$

We claim that $r \equiv 2 \bmod 3$. Arguing indirectly, we at first assume that $r \equiv 0 \bmod 3$. Since $\varepsilon_r^2/(2-\varepsilon_r^2) \leq \varepsilon_r$, our assumption implies

$$\int_M \frac{\varepsilon_r^2}{(2-\varepsilon_r^2)\,(1-K)^{\gamma}} dA < \infty,$$

or equivalently, bearing in mind (2.1) and (2.4),

$$\int_M \frac{\left(\|\alpha_{r+1}\|^4 - 4^r (K_r^{\perp})^2\right)^{1/2}}{(1-K)^{\gamma}\,\|\alpha_{r+1}\|^2} dA < \infty.$$

Taking into account (4.22), the above becomes

$$\int_M \frac{\left(\|\alpha_{r+1}\|^4 - 4^r (K_r^{\perp})^2\right)^{1/2}}{(1-K)^{\gamma+\frac{r}{3}}} dA < \infty.$$

We consider the subset

$$U_\delta(p_j) = \{p \in M : |z(p)| < \delta\}, j = 1, \ldots, m.$$

Using (4.29) and (4.32), the above inequality implies that

$$\int_{U_{\delta_0}(p_j)\smallsetminus U_\delta(p_j)} |z|^{l_r(p_j)-2k_j(\gamma+\frac{r}{3})} dA < c$$

for any $\delta < \delta_0$, where c is a positive constant and δ_0 is small enough. We set $z = \rho e^{i\theta}$. Since $dA = F\rho d\rho \wedge d\theta$, we deduce that

$$\int_0^{\delta_0} \rho^{l_r(p_j)-2k_j(\gamma+\frac{r}{3})+1} d\rho < \infty.$$

This implies that

$$l_r(p_j) > 2k_j(\gamma + \frac{r}{3}) - 2.$$

Summing up, we obtain

$$N(\Phi_r) + 2m > 2(\gamma + \frac{r}{3})\sum_{j=1}^{m} k_j.$$

Using Lemma 2.4.1(ii) and (3.2) in Lemma 3.2.1, it follows that

$$\chi(M)(3\gamma - 1) + m > 0.$$

On the other hand, (3.2) implies that $m \leq -3\chi(M)$, which contradicts the above and the hypothesis that $\chi(M) < 0$.

Now assume that $r \equiv 1 \bmod 3$. Bearing in mind (4.22), we deduce that $u_r \leq b_r^6(1-K)^2$. Using (4.29) and (4.33), we obtain $3l_r(p_j) \geq 2k_j(r+2)$. Then from Lemma 3.2.1, we conclude that

$$N(\Phi_r) \geq -2(r+2)\chi(M).$$

Due to Lemma 2.4.1(ii), the above contradicts our hypothesis on the genus.

Therefore, we conclude that $r \equiv 2 \bmod 3$. By virtue of (4.22), we obtain $u_r \leq b_r^6$. Then (4.33) implies $3l_r(p_j) \geq 2k_j(r+1)$, and we can extend u_r to a smooth function on M. It follows from Proposition 2.3.1(ii) and the Ricci-like condition $(*)$ that $\log u_r$ is harmonic away from the zeros which are isolated, and consequently by continuity u_r is subharmonic everywhere on M. By the maximum principle, we deduce that the function u_r is a positive constant. This shows that the r-th curvature ellipse has constant eccentricity, i.e., the surface f is r-exceptional. This completes the proof. □

For compact minimal submanifolds in spheres with low codimension, we prove the following result.

Corollary 4.4.1. *Let $f\colon M \to \mathbb{S}^n$ be a substantial minimal surface with $4 \leq n \leq 7$. If M is compact and not homeomorphic to the torus, then it cannot be locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$, unless $n = 5$.*

Proof. From Lemma 3.2.1, we have that the genus of M satisfies $g \geq 2$. We assume that $n \neq 5$. For $n = 4$ and $n = 6$, the result follows immediately from Theorem 4.4.1 and Theorem 4.2.2(ii). In the case where $n = 7$, Theorem 4.4.1 implies that the surface is exceptional and the result follows from Theorem 4.2.1. □

Remark 4.4.1. *The assumption in Theorem 4.4.1 on the eccentricity of curvature ellipses of order $r \equiv 0 \bmod 3$ could be replaced by the condition*

$$\varepsilon_r \leq (1-K)^{\beta}$$

for positive constants c and $\beta > 1/3$. Both conditions claim that the curvature ellipses of order $r \equiv 0 \bmod 3$ tend to be circles close to totally geodesic points. We don't know whether Theorem 4.4.1 holds without this assumption in any codimension.

The following global result is complementary to Theorem 4.2.2.

Theorem 4.4.2. *Let $f\colon M \to \mathbb{S}^{6m+4}, m \geq 1$, be a substantial exceptional surface. If M is compact with genus $g \geq 2$, then it cannot be locally isometric to a pseudoholomorphic curve in $\mathbb{S}^5$.*

Proof. We assume to the contrary that the surface satisfies the Ricci-like condition $(*)$. Since f is $3m$-exceptional, from Proposition 2.3.2 we know that the Hopf differential $\Phi_{3m+1} = f_{3m+1}dz^{6m+4}$ is globally defined and holomorphic. Hence either $\Phi_{3m+1} = 0$ or its zeros are isolated.

Theorem 4.2.2(ii) implies that the Hopf differential Φ_{3m+1} cannot vanish identically.

According to Lemma 3.2.1, the function $1 - K$ is of absolute value type with nonempty zero set $M_0 = \{p_1, \ldots, p_m\}$ and corresponding order $\mathrm{ord}_{p_j}(1 - K) = 2k_j$. For each point $p_j, j = 1, \ldots, m$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$. Around p_j, we have that

$$1 - K = |z|^{2k_j} u_0, \tag{4.34}$$

where u_0 is a smooth positive function.

Obviously, Φ_{3m+1} vanishes at p_j. Hence we may write $f_{3m+1} = z^{l(p_j)}\psi$ around p_j, where $l(p_j)$ is the order of Φ_{3m+1} at p_j, and ψ is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\|\alpha_{3m+2}\|^4 - 4^{3m+1}(K^{\perp}_{3m+1})^2 = 2^{6(m+1)} F^{-2(3m+2)} |\psi|^2 |z|^{2l(p_j)} \tag{4.35}$$

around p_j. We consider the smooth function $u\colon M \smallsetminus M_0 \to \mathbb{R}$ (see Proposition 2.3.2) given by

$$u = \frac{\left(\|\alpha_{3m+2}\|^4 - 4^{3m+1}(K^{\perp}_{3m+1})^2\right)^3}{(1 - K)^{2(3m+2)}}.$$

In view of (4.34) and (4.35), we find that

$$u = 2^{18(m+1)} F^{-6(3m+2)} u_0^{-2(3m+2)} |\psi|^6 |z|^{6l(p_j) - 4k_j(3m+2)}. \tag{4.36}$$

Using (4.22), it follows that $u \leq b^6_{3m+1}(1 - K)^2$. Then (4.34) and (4.36) imply that $l(p_j) \geq 2k_j(m + 1)$. By Lemma 3.2.1, we deduce that

$$N(\Phi_{3m+1}) \geq -6(m + 1)\chi(M).$$

It follows from Lemma 2.4.1(ii) that the above contradicts our hypothesis on the genus and the theorem is proved. □

The following result provides properties of the structure of the moduli space of a compact pseudoholomorphic curve in $\mathbb{S}^5$.

Theorem 4.4.3. *If g is a compact pseudoholomorphic curve in $\mathbb{S}^5$ that is not homeomorphic to the torus, then the moduli space $\mathcal{M}_n^{\mathrm{e}}(g)$, with n odd, is given by $\mathcal{M}_n^{\mathrm{e}}(g) = \mathbb{S}_*^{m-1} \times \Gamma_0$, where Γ_0 is a proper subset of Γ^m that is locally a disjoint finite union of d-dimensional real analytic subvarieties where $d = 0, \ldots, m-1$. Moreover, every straight line through each point $\boldsymbol{\theta} \in \Gamma_0$ that is parallel to every coordinate axis of $\mathbb{R}^m$ intersects Γ_0 at finitely many points.*

Proof. Suppose to the contrary that the intersection of Γ_0 with the straight line through $\boldsymbol{\theta}$ that is parallel to the first coordinate axis is an infinite set. For a fixed $\mathbf{a} \in \mathbb{S}_*^{m-1}$, we choose $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N \in \Gamma_0$ that belong to this straight line. Hence $(\mathbf{a}, \boldsymbol{\theta}_j) \in \mathcal{M}_n^{\mathrm{e}}(g)$ for all $\boldsymbol{\theta}_j = (\theta_{j1}, \ldots, \theta_{jm}),\ j = 1, \ldots, N$. Consequently there exist exceptional surfaces $F_j \colon M \to \mathbb{S}^n$ such that $F_j \circ \pi = \tilde{g}_{\mathbf{a}, \boldsymbol{\theta}_j}$.

We claim that the set of all coordinate functions of all surfaces F_j's associated to vectors $\mathbf{v} = (v_1, 0, \ldots, 0)$ in $\mathbb{R}^{6m}$ are linearly independent. It is sufficient to prove that if

$$\sum_{j=1}^{N} \langle F_j, \mathbf{v} \rangle = 0, \tag{4.37}$$

then $\mathbf{v} = 0$. From (4.37) we obtain

$$\sum_{j=1}^{N} \langle F_j \circ \pi, \mathbf{v} \rangle = 0,$$

or equivalently

$$a_1 \sum_{j=1}^{N} \langle \tilde{g}_{\theta_{j1}}, v_1 \rangle = 0.$$

In analogy with the argument in the proof of Theorem 2 in [14], we finally conclude that $v_1 = 0$ and the claim is proved.

The contradiction follows easily since the coordinate functions of the surfaces F_j's are eigenfunctions of the Laplacian operator with corresponding eigenvalue 2 and the vector space of the eigenfunctions has finite dimension. Hence $\Gamma_0 \neq \Gamma^m$ and the proof follows from Theorem 4.0.1. $\square$

Remark 4.4.2. *The assumption in Theorem 4.4.3 that the pseudoholomorphic curve g is not homeomorphic to the torus is essential and can not be dropped. According to results due to Kenmotsu [23, 24] the moduli space of all minimal surfaces in odd dimensional spheres that are isometric to a flat pseudoholomorphic torus in $\mathbb{S}^5$ is not a finite set.*

Proof of Theorem 4.0.2. It follows from Theorem 4.4.1 and Corollary 4.4.1 that any minimal surface $f\colon M \to \mathbb{S}^n$ that is isometric to g is exceptional and $n = 5$. The proof follows from Theorem 4.4.3. □

Open problem. *It remains an open problem to describe the moduli space of all minimal surfaces in spheres that are locally isometric to a pseudoholomorphic curve in* $\mathbb{S}^5$*, without the assumption of being exceptional.*

CHAPTER 5

Rigidity of isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$

In this chapter, we investigate the moduli space of all noncongruent substantial minimal surfaces $f\colon M \to \mathbb{S}^n$ that are isometric to a given isotropic pseudoholomorphic curve g in $\mathbb{S}^6$. It turns out that these surfaces are rigid among minimal surfaces in spheres.

5.1 A global result

We prove the following theorem.

Theorem 5.1.1. *Let $f\colon M \to \mathbb{S}^n$ be a compact substantial minimal surface. If f is isometric to an isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$, then $n = 6$ and f is congruent to g.*

Proof. According to Theorem 2 in [43], the function $1 - K$ is of absolute value type. If the zero set of the function $1 - K$ is empty, then from condition $(**)$ it follows that f is homeomorphic to the sphere. From [5] we have that f is isotropic and from [41] it follows that $n = 6$ and f is congruent to g. Now suppose that the zero set of the function $1 - K$ is the set $M_0 = \{p_1, \ldots, p_m\}$ with corresponding order $\text{ord}_{p_j}(1 - K) = 2k_j$. For each point $p_j, j = 1, \ldots, m$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$. On a neighbourhood of p_j, we have that

$$1 - K = |z|^{2k_j} u_0, \tag{5.1}$$

where u_0 is a smooth positive function.

We claim that f is 1-isotropic. The first Hopf differential $\Phi_1 = f_1 dz^4$ is globally defined and holomorphic. Hence either Φ_1 is identically zero, or its zeros are isolated. Suppose to the contrary that Φ_1 is not identically zero. The Gauss equation (2.5) yields that each p_j is a totally geodesic point. It follows from the definition of the first Hopf differential that Φ_1 vanishes at each p_j. Hence we may write $f_1 = z^{l_{1j}}\psi_1$ around p_j, where l_{1j} is the order of Φ_1 at p_j, and ψ_1 is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\frac{1}{4}\,\|\alpha_2\|^4 - (K_1^{\perp})^2 = (2F^{-1})^4|\psi_1|^2|z|^{2l_{1j}} \tag{5.2}$$

around p_j. We now consider the function $u_1\colon M \smallsetminus M_0 \to \mathbb{R}$ defined by

$$u_1 = \frac{\left(\frac{1}{4}\,\|\alpha_2\|^4 - (K_1^{\perp})^2\right)^3}{(1-K)^4}.$$

From (5.1) and (5.2) it follows that the function u_1 around p_j, is written as

$$u_1 = (2F^{-1})^{12}u_0^{-4}|\psi_1|^6|z|^{6l_{1j}-8k_j}. \tag{5.3}$$

Using (2.5) we obtain $u_1 \leq (1-K)^2$. Thus, from (5.1) and (5.3) we deduce that $l_{1j} \geq 2k_j$ and we can extend u_1 to a smooth function on M. From Proposition 2.3.1(ii) for $r = 1$, it follows that

$$\Delta \log \left(\|\alpha_2\|^2 + 2K_1^{\perp}\right) = 2\big(2K - K_1^*\big)$$

and

$$\Delta \log \left(\|\alpha_2\|^2 - 2K_1^{\perp}\right) = 2\big(2K + K_1^*\big).$$

Summing up, we obtain

$$\Delta \log \left(\|\alpha_2\|^4 - 4(K_1^{\perp})^2\right) = 8K.$$

Combining the last equation with condition $(**)$, we have

$$\Delta \log \left(\|\alpha_2\|^4 - 4(K_1^{\perp})^2\right)^3 = \Delta \log(1-K)^4 + 4,$$

or equivalently $\Delta \log u_1 = 4$ away from the isolated zeros of u_1. Thus, by continuity $\Delta u_1 \geq 4u_1 \geq 0$, and from the maximum principle we have that this holds only if $u_1 \equiv 0$, or equivalently only if $\|\alpha_2\|^4 = 4(K_1^{\perp})^2$. Lemma 2.3.1 implies that $\Phi_1 = 0$ and this contradicts our assumption that Φ_1 is not identically zero. Hence, Φ_1 is identically zero and from Lemma 2.3.1 yields that f is 1-isotropic. Proposition 2.3.1(i) for $s = 1$ implies that

$$\Delta \log(1-K) = 2\big(2K - K_1^*\big),$$

which using condition $(**)$ yields

$$K_1^* = \frac{1}{2} - K. \tag{5.4}$$

Since f is 1-isotropic, we know from Lemma 2.3.1 that $K_1^{\perp} = 1 - K$. Proposition 2.1.1 in Section 2.1 and (5.4) yield that

$$\|\alpha_3\|^2 = 1 - K. \tag{5.5}$$

We now claim that f is also 2-isotropic. From Proposition 2.3.2 we know that $\Phi_2 = f_2 dz^6$ is globally defined. Theorem 2.3.1 implies that it is also holomorphic. Hence either Φ_2 is identically zero or its zeros are isolated. In the former case, from Lemma 2.3.1 we have that f is 2-isotropic. Assume now to the contrary that Φ_2 is not identically zero. Obviously, we have that α_3 vanishes at each p_j and consequently from the definition of the second Hopf differential, also Φ_2 vanishes at each p_j. Hence we may write $f_2 = z^{l_{2j}}\psi_2$ around p_j, where l_{2j} is the order of Φ_2 at p_j, and ψ_2 is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\|\alpha_3\|^4 - 16(K_2^{\perp})^2 = 2^8 F^{-6} |\psi_2|^2 |z|^{2l_{2j}} \tag{5.6}$$

around p_j. We now consider the function $u_2 \colon M \smallsetminus M_0 \to \mathbb{R}$ defined by

$$u_2 = \frac{\|\alpha_3\|^4 - 16(K_2^{\perp})^2}{(1-K)^2}.$$

In view of (5.1) and (5.6), it follows that the function u_2 around p_j is written as

$$u_2 = 2^8 F^{-6} u_0^{-2} |\psi_2|^2 |z|^{2l_{2j} - 4k_j}. \tag{5.7}$$

Using (5.5) we derive that $u_2 \leq 1$. From (5.1) and (5.7) we deduce that $l_{2j} \geq 2k_j$ and we can extend u_2 to a smooth function on M. Proposition 2.3.1(ii) for $r = 2$ implies that

$$\Delta \log \left(\|\alpha_3\|^2 + 4K_2^{\perp} \right) = 2\big(3K - K_2^*\big)$$

and

$$\Delta \log \left(\|\alpha_3\|^2 - 4K_2^{\perp} \right) = 2\big(3K + K_2^*\big).$$

Summing up, we obtain

$$\Delta \log \left(\|\alpha_3\|^4 - 16(K_2^{\perp})^2 \right) = 12K.$$

Combining the last equation with condition $(**)$, we have that $\Delta \log u_2 = 2$ away from the isolated zeros of u_2. Thus, by continuity $\Delta u_2 \geq 2u_2 \geq 0$, and from the maximum principle we have that this holds only for $u_2 \equiv 0$, or equivalently only if $\|\alpha_3\|^4 = 16(K_2^{\perp})^2$. Lemma 2.3.1 implies that $\Phi_2 = 0$ and this contradicts our assumption that Φ_2 is not identically zero. Hence, Φ_2 is identically zero and Lemma 2.3.1 implies that f is 2-isotropic. Now Proposition 2.3.1(i) for $s = 2$ yields

$$\Delta \log \|\alpha_3\|^2 = 2\big(3K - K_2^*\big),$$

and combining this with condition $(**)$ we obtain $K_2^* = 1/2$.

Since f is 2-isotropic, from Lemma 2.3.1 and (5.5) we have $K_2^{\perp} = (1-K)/4$. Using that $K_2^* = 1/2$, Proposition 2.1.1 for $r = 2$ implies that $\alpha_4 = 0$. Therefore $n = 6$ and the surface f is congruent to g (cf. [41, Theorem A]). □

5.2 A local result

The rigidity result in Theorem 5.1.1 still holds if instead of the compactness of the surface we assume that the surface is exceptional. In fact, we now prove the following local result for exceptional surfaces.

Theorem 5.2.1. *Let $f\colon M \to \mathbb{S}^n$ be a substantial exceptional surface that is isometric to an isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$. Then $n = 6$ and f is congruent to g.*

Proof. We set $\rho_s := 2^s K_s^{\perp} / \|\alpha_{s+1}\|^2$, for any $1 \leq s \leq r$, where $r = [(n-1)/2 - 1]$. Using (2.1) and (2.4) it follows that $\rho_s = 2\kappa_s\mu_s/(\kappa_s^2 + \mu_s^2)$. Since f is exceptional, by the definition we have that the s-th ellipse has constant eccentricity or equivalently the ratio of the semiaxes κ_s, μ_s is constant. Then it is clear that the function ρ_s is constant.

Using equation (2.5), Proposition 2.3.1(i) for $s = 1$ and condition $(**)$, we find

$$K_1^* = \frac{1}{2} - K. \tag{5.8}$$

Moreover, from the definition of ρ_1 we have that $K_1^{\perp} = \rho_1(1-K)$. We claim that f is 1-isotropic, which is equivalent to $\rho_1 = 1$ due to Lemma 2.3.1. Assume to the contrary that $\rho_1 \neq 1$. Then from Lemma 2.3.1 we have that $\Phi_1 \neq 0$. Consequently Proposition 2.3.1(ii) for $r = 1$ yields

$$\Delta \log \left(\|\alpha_2\|^2 - 2K_1^{\perp}\right) = 2\big(2K + K_1^*\big).$$

Using (2.5) and Lemma 2.3.1(iii) we obtain

$$\Delta \log(1-K) = 4K + 2K_1^*.$$

From (5.8) it follows that

$$\Delta \log(1-K) = 2K+1.$$

Combining this with condition (∗∗) we have that $K = 1/2$, which is a contradiction. Hence $\rho_1 = 1$ and consequently f is 1-isotropic. From Proposition 2.1.1, equation (5.8) and Lemma 2.3.1 it follows that

$$\|\alpha_3\|^2 = 1-K. \tag{5.9}$$

From Proposition 2.3.2 we know that $\Phi_2 = f_2 dz^6$ is globally defined. Theorem 2.3.1 implies that it is also holomorphic. Hence either Φ_2 is identically zero or its zeros are isolated. Moreover, we have that $K_2^{\perp} = 2^{-2}\rho_2 \|\alpha_3\|^2$. Similarly, we claim that $\rho_2 = 1$. Assume to the contrary that $\rho_2 \neq 1$. Then from Lemma 2.3.1, the Hopf differential $\Phi_2 \neq 0$. Proposition 2.3.1(ii) for $r = 2$ yields that

$$\Delta \log\left(\|\alpha_3\|^2 + 4K_2^{\perp}\right) = 2\big(3K - K_2^*\big)$$

and

$$\Delta \log\left(\|\alpha_3\|^2 - 4K_2^{\perp}\right) = 2\big(3K + K_2^*\big),$$

which due to (5.9) implies that

$$\Delta \log(1-K) = 6K.$$

This contradicts (∗∗), hence $\rho_2 = 1$ and consequently Φ_2 is identically zero. From Proposition 2.3.1(iii) for $r = 2$ and condition (∗∗), we obtain $K_2^* = 1/2$. Proposition 2.1.1 for $r = 2$ yields $\alpha_4 = 0$, which completes our proof. □

Open problem. *It should be interesting to know whether Theorem 5.1.1 still holds under the weaker assumption of completeness instead of compactness.*

CHAPTER 6

Isometric deformations of non-isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$

In this chapter, we mostly deal with noncongruent isometric deformations of pseudoholomorphic substantial curves in $\mathbb{S}^6$ that are always 1-isotropic (see Chapter 3) but in general not 2-isotropic. For a given non-isotropic substantial pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$, our aim is to describe the moduli space $\mathcal{M}_n^K(g)$ of all noncongruent minimal surfaces $f\colon M \to \mathbb{S}^n$ that are locally isometric to the curve g, having the same normal curvatures up to order 2 with the curve g.

At first, we prove the following theorem.

Theorem 6.0.1. *Let $g\colon M \to \mathbb{S}^6$ be a pseudoholomorphic curve. The moduli space of all noncongruent minimal surfaces $f\colon M \to \mathbb{S}^6$ that are isometric to g and have the same normal curvatures with g, is either a circle or a finite set.*

Note that this is a result concerning pseudoholomorphic curves of all types. From Corollary 5.4(ii) in [41], we know that two locally isometric 1-isotropic surfaces in $\mathbb{S}^6$ with the same normal curvatures, belong locally to the same associated family. This, in particular, implies that if g is simply connected then $\mathcal{M}_6^K(g) = \mathbb{S}^1$.

For compact non-isotropic pseudoholomorphic curves, we prove the following theorem under a topological assumption.

Theorem 6.0.2. *Let $g\colon M \to \mathbb{S}^6$ be a compact substantial pseudoholomorphic curve that is non-isotropic. If the Euler-Poincaré number of the second normal bundle of g is nonzero, then there are at most finitely many minimal surfaces in $\mathbb{S}^6$ isometric to g having the same normal curvatures with g.*

To conclude, we prove the following theorem that may be viewed as analogous to the classical result of Schur (see [8, p. 36]) in the realm of minimal surfaces in spheres.

Theorem 6.0.3. *Let $g\colon M \to \mathbb{S}^6$ be a compact, non-isotropic and substantial pseudoholomorphic curve and $\hat{g}\colon M \to \mathbb{S}^n$ be a substantial minimal surface that is isometric to g. If $\hat{g}$ is not 2-isotropic and the second normal curvatures $K_2^{\perp}, \hat{K}_2^{\perp}$ of the surfaces g and $\hat{g}$ respectively satisfy the inequality $\hat{K}_2^{\perp} \leq K_2^{\perp}$, then $n = 6$. Moreover, the moduli space of all such noncongruent minimal surfaces $\hat{g}\colon M \to \mathbb{S}^6$ that are isometric to g, is either a circle or a finite set.*

6.1 A local result

Hereafter we deal with substantial pseudoholomorphic curves not necessarily simply connected. We consider the covering map $\Pi\colon \tilde{M} \to M$, $\tilde{M}$ being the universal cover of M equipped with the metric and orientation that make Π an orientation preserving local isometry. Corresponding objects on $\tilde{M}$ are denoted with tilde. Then the map $\tilde{g}\colon \tilde{M} \to \mathbb{S}^6$ with $\tilde{g} = g \circ \Pi$ is up to congruence a pseudoholomorphic curve. Hence, the moduli space $\mathcal{M}_6^K(g)$ of the curve g can be described as the set of all $\theta \in \mathcal{M}_6^K(\tilde{g}) = \mathbb{S}^1$ such that $\tilde{g}_\theta$ factors as $\tilde{g}_\theta = g_\theta \circ \Pi$ for a minimal surface $g_\theta\colon M \to \mathbb{S}^6$ and $\tilde{g}_\theta$ is a member in the associated family of $\tilde{g}$. We follow this notation throughout this chapter.

Lemma 6.1.1. *(i) For each $\sigma \in \mathcal{D}$, the surfaces $\tilde{g}_\theta$ and $\tilde{g}_\theta \circ \sigma$ are congruent for every $\theta \in [0, \pi]$, that is there exists a unique $\Psi_\theta(\sigma) \in \mathrm{O}(7)$ such that*

$$\tilde{g}_\theta \circ \sigma = \Psi_\theta(\sigma) \circ \tilde{g}_\theta. \tag{6.1}$$

(ii) If θ belongs to $\mathcal{M}_6^K(\tilde{g})$, then θ belongs to $\mathcal{M}_6^K(g)$ if and only if

$$\Psi_\theta(\mathcal{D}) = \{\mathrm{Id}\}, \tag{6.2}$$

where $\Psi_\theta \in \mathrm{O}(7)$.

Proof. (i) From Proposition 9 in [14] we have that for any σ in the group $\mathcal{D}$, the surfaces $\tilde{g}_\theta\colon \tilde{M} \to \mathbb{S}^6$ and $\tilde{g}_\theta \circ \sigma\colon \tilde{M} \to \mathbb{S}^6$ are congruent for any $\theta \in \mathcal{M}_6^K(g)$. Therefore, there exists $\Psi_\theta(\sigma) \in \mathrm{O}(7)$ such that (6.1) holds for every $\theta \in \mathcal{M}_6^K(g)$.

(ii) Take $\theta \in \mathcal{M}_6^K(g)$. Then, there exists a minimal surface $g_\theta\colon M \to \mathbb{S}^6$ such that $g_\theta \circ \pi = \tilde{g}_\theta$. Composing with an arbitrary $\sigma \in \mathcal{D}$ and using (6.1), we obtain

$$\tilde{g}_\theta = \Psi_\theta(\sigma) \circ \tilde{g}_\theta.$$

Since $\tilde{g}_\theta$ has substantial codimension (6.2) yields.

Conversely, assume that (6.2) holds. We will prove that $\tilde{g}_\theta$ factors as $\tilde{g}_\theta = g_\theta \circ \Pi$, where $g_\theta \colon M \to \mathbb{S}^6$ is a minimal surface. At first we claim that $\tilde{g}_\theta$ remains constant on each fiber of the covering map Π. Indeed, let $\tilde{p}_1, \tilde{p}_2$ belong to $\Pi^{-1}(p)$ for some $p \in M$. Then there exists a deck transformation σ such that $\sigma(\tilde{p}_1) = \tilde{p}_2$. Using (6.1), we obtain

$$\begin{aligned} \tilde{g}_\theta(\tilde{p}_2) &= \tilde{g}_\theta \circ \sigma(\tilde{p}_1) \\ &= \Psi_\theta(\sigma) \circ \tilde{g}_\theta(\tilde{p}_1) \\ &= \tilde{g}_\theta(\tilde{p}_1). \end{aligned}$$

Then $\tilde{g}_\theta$ factors as $\tilde{g}_\theta = g_\theta \circ \Pi$, where $F \colon M \to \mathbb{S}^n$ is a minimal surface. It remains to prove that $g_\theta \in \mathcal{M}_6^K(g)$. Since Π is an orientation preserving local isometry, it is obvious that F is a minimal surface. □

Proof of Theorem 6.0.1. If g is substantial in a totally geodesic $\mathbb{S}^5$, then from Theorem 1 in [14], we know that the moduli space of g is either a circle or a finite set.

If g is isotropic and substantial in $\mathbb{S}^6$, then Theorem 5.2.1 implies that the moduli space of g consists of a single point.

Suppose now that g is substantial in $\mathbb{S}^6$ and non-isotropic. Assume that the moduli space $\mathcal{M}_6^K(g)$ is not finite. Thus, there exists a sequence $\theta^{(i)}, i \in \mathbb{N}$, that belongs to $\mathcal{M}_6^K(g)$. By passing if necessary to a subsequence, we assume that this sequence converges to $\theta^\infty \in [0, \pi]$. From Lemma 6.1.1(ii), we derive that $\Psi_{\theta^{(i)}}(\mathcal{D}) = \{\mathrm{Id}\}$ for every $i \in \mathbb{N}$ and $\Psi_{\theta^\infty}(\mathcal{D}) = \{\mathrm{Id}\}$. Fix a $\sigma \in \mathcal{D}$. We define the function

$$h(\theta) = \left(\Psi_\theta(\sigma)\right)_{ij}, \theta \in [0, \pi],$$

where $\left(\Psi_\theta(\sigma)\right)_{ij}$ denotes the (i,j)-element of the matrix of $\Psi_\theta(\sigma)$ with respect to the standard basis of $\mathbb{R}^7$. By the mean value theorem, there exists $\xi_1^{(i)}$ between $\theta^{(i)}$ and θ^∞ such that $(dh/d\theta)(\xi_1^{(i)}) = 0$ and hence $(dh/d\theta)(\theta^\infty) = 0$. Applying repeatedly the mean value theorem, we obtain inductively that the k-th derrivative of h satisfies $(d^k h/d\theta^k)(\theta^\infty) = 0$ for any k. The analyticity of h (cf. Remark 4.3.1) implies that $h = \delta_{ij}$, where δ_{ij} is the Krönecker delta. Hence, the proof follows from Lemma 6.1.1(ii). □

6.2 Isometric deformations of compact non-isotropic pseudoholomorphic curves in the nearly Kähler sphere $\mathbb{S}^6$ with substantial codimension

We now turn our attention to the study of isometric deformations of compact non-isotropic pseudoholomorphic curves in $\mathbb{S}^6$. We will need the following lemmas:

Lemma 6.2.1. *Let $g\colon M \to \mathbb{S}^6$ be a substantial non-isotropic pseudoholomorphic curve. For each $g_\theta, \theta \in \mathcal{M}_6^K(g)$, there exists a parallel and orthogonal bundle isomorphism $T_\theta\colon N_gM \to N_{g_\theta}M$ such that the second fundamental forms of g and g_θ are related by*

$$\alpha^{g_\theta}(X,Y) = T_\theta \circ \alpha^g(J_\theta X, Y), \quad X, Y \in TM.$$

Proof. Since g and g_θ have the same normal curvatures, it follows from Corollary 5.4(ii) in [41] that for any simply connected subset U of M there exists a parallel and orthogonal bundle isomorphism $T_\theta^U : N_gU \to N_{g_\theta}U$ such that the second fundamental forms of the surfaces $g|_U$ and $g_\theta|_U$ are related by

$$\alpha^{g_\theta|_U}(X,Y) = T_\theta^U \circ \alpha^{g|_U}(J_\theta X, Y), \quad X, Y \in TM.$$

Let U, V be simply connected subsets of M with $U \cap V \neq \varnothing$. Then on $U \cap V$ we have

$$T_\theta^U \circ \alpha^{g|_U}(J_\theta X, Y) = T_\theta^V \circ \alpha^{g|_V}(J_\theta X, Y),$$

for every $X, Y \in TM$. Equivalently we obtain

$$\left(T_\theta^U - T_\theta^V\right) \circ \alpha^{g|_{U\cap V}}(X,Y) = 0$$

and obviously $\left(T_\theta^U - T_\theta^V\right)(N_1^{g|_{U\cap V}}) = 0$.

Differentiating we obtain $\left(T_\theta^U - T_\theta^V\right)(N_2^{g|_{U\cap V}}) = 0$, which yields that $T_\theta^U = T_\theta^V$ on $U \cap V$. Thus, T_θ^U is globally well defined. □

The following lemma concerns the connection forms of an appropriately chosen orthonormal frame of substantial non-isotropic pseudoholomorphic curves.

Lemma 6.2.2. *Let $g\colon M \to \mathbb{S}^6$ be a substantial non-isotropic pseudoholomorphic curve and let M_1 be the zero set of the second Hopf differential Φ_2. Around each point of $M \smallsetminus M_1$, there exist a local complex coordinate (U, z), $U \subset M \smallsetminus M_1$ and orthonormal frames $\{e_1, e_2\}$ in TU, $\{e_3, e_4\}$ in N_1^gU and $\{e_5, e_6\}$ in N_2^gU which agree with the given orientations such that:*

(i) The vector fields e_5 and e_6 give respectively the directions of the major and the minor axes of the second curvature ellipse, and

(ii) We have that $H_5 = \kappa_2$, $H_6 = i\mu_2$ and κ_2 and μ_2 are smooth real functions. Moreover, the connection and the normal connection forms, with respect to this frame, are given respectively, by

$$\omega_{12} = -\frac{1}{6} * d\log(\kappa_2^2 - \mu_2^2),\ \omega_{34} = 2\omega_{12} + {*d}\log\kappa_1,\ \omega_{56} = \frac{\kappa_2\mu_2}{\kappa_2^2 - \mu_2^2} * d\log\frac{\mu_2}{\kappa_2}, \quad (6.3)$$

where $*$ stands for the Hodge operator.

Proof. (i) Take an arbitrary orthonormal frame $\{E_1, E_2\}$ in TU. Arguing pointwise in U we have that

$$\max_{\theta\in[0,2\pi)} \|\alpha_3^g(X_\theta, X_\theta, X_\theta)\| = \kappa_2 \quad \text{and} \quad \min_{\theta\in[0,2\pi)} \|\alpha_3^g(X_\theta, X_\theta, X_\theta)\| = \mu_2,$$

where $X_\theta = \cos\theta E_1 + \sin\theta E_2$.

Assume that the function $f(\theta) = \|\alpha_3^g(X_\theta, X_\theta, X_\theta)\|^2$ attains its maximum at $\theta_0 \in [0, 2\pi)$. Since $f'(\theta_0) = 0$, we find that

$$\langle \alpha_3^g(X_{\theta_0}, X_{\theta_0}, X_{\theta_0}), \alpha_3^g(X_{\theta_0}, X_{\theta_0}, X_{\theta_0})\rangle = 0,$$

or equivalently

$$2\langle \alpha_3^g(E_1, E_1, E_1), \alpha_3^g(E_1, E_1, E_2)\rangle = \big(\|\alpha_3^g(E_1, E_1, E_1)\|^2 - \|\alpha_3^g(E_1, E_1, E_2)\|^2\big)\tan 6\theta.$$

Since the second curvature ellipse is not a circle, we choose a smooth function σ such that

$$\tan\sigma = \frac{2\langle \alpha_3^g(E_1, E_1, E_1), \alpha_3^g(E_1, E_1, E_2)\rangle}{\|\alpha_3^g(E_1, E_1, E_1)\|^2 - \|\alpha_3^g(E_1, E_1, E_2)\|^2},$$

or

$$\cot\sigma = \frac{\|\alpha_3^g(E_1, E_1, E_1)\|^2 - \|\alpha_3^g(E_1, E_1, E_2)\|^2}{2\langle \alpha_3^g(E_1, E_1, E_1), \alpha_3^g(E_1, E_1, E_2)\rangle}.$$

We now consider the orthonormal frame $\{e_1, e_2\}$ in TU with

$$e_1 = \cos\sigma E_1 + \sin\sigma E_2 \text{ and } e_2 = -\sin\sigma E_1 + \cos\sigma E_2.$$

We may also consider the orthonormal frame $\{e_3, e_4\}$ in $N_1^g U$ given by

$$e_3 = \frac{1}{\kappa_1}\alpha^g(e_1, e_1) \text{ and } e_4 = \frac{1}{\kappa_1}\alpha^g(e_1, e_2)$$

and the orthonormal frame $\{e_5, e_6\}$ in $N_2^g U$ such that

$$e_5 = \frac{1}{\kappa_2}\alpha_3^g(e_1, e_1, e_1) \text{ and } e_6 = \frac{1}{\mu_2}\alpha_3^g(e_1, e_1, e_2).$$

Let $\{\tilde{e}_5, \tilde{e}_6\}$ be an orthonormal frame in $N_2^g U$ chosen as in Lemma 5 in [43]. Then the complex valued functions $\tilde{H}_5, \tilde{H}_6$ associated to the frame $\{\tilde{e}_5, \tilde{e}_6\}$ satisfy

$$\tilde{H}_6 = i(\kappa_1 - \tilde{H}_5). \tag{6.4}$$

We easily find that

$$\tilde{H}_5 = \cos\varphi H_5 + \sin\varphi H_6 \tag{6.5}$$

and

$$\tilde{H}_6 = -\sin\varphi H_5 + \cos\varphi H_6, \tag{6.6}$$

where φ is the angle between e_5 and $\tilde{e}_5$. Since $H_5 = \kappa_2$ and $H_6 = i\mu_2$, equations (6.4), (6.5) and (6.6) yield $\varphi = 0$ and consequently the orthonormal frames $\{e_5, e_6\}$ and $\{\tilde{e}_5, \tilde{e}_6\}$ coincide.

(ii) It follows directly from Lemma 6 in [42] that the connection forms ω_{34} and ω_{56} are given by (6.3).

From $\alpha_3(e_1, e_1, e_1) = \kappa_2 e_5$, we obtain

$$\omega_{35}(e_1) = -\omega_{45}(e_2) = \frac{\kappa_2}{\kappa_1} \text{ and } \omega_{36}(e_1) = \omega_{46}(e_2) = 0.$$

Similarly, $\alpha_3(e_1, e_1, e_2) = \mu_2 e_6$ implies that

$$\omega_{46}(e_1) = \omega_{36}(e_2) = \frac{\mu_2}{\kappa_1} \text{ and } \omega_{45}(e_1) = \omega_{35}(e_2) = 0.$$

Therefore,

$$\omega_{35} = \frac{\kappa_2}{\kappa_1}\omega_1, \ \omega_{45} = -\frac{\kappa_2}{\kappa_1}\omega_2, \ \omega_{36} = \frac{\mu_2}{\kappa_1}\omega_2 \ \text{ and } \omega_{46} = \frac{\mu_2}{\kappa_1}\omega_1.$$

Using the above, the Ricci equations

$$\langle R^{\perp}(e_1, e_2)e_3, e_5\rangle = 0 \text{ and } \langle R^{\perp}(e_1, e_2)e_4, e_6\rangle = 0,$$

where $R^{\perp}$ is the curvature tensor of the normal bundle, are written equivalently as

$$3\omega_{12}(e_1) = \frac{\mu_2}{\kappa_2}\omega_{56}(e_1) + e_2\big(\log\frac{\kappa_2}{\kappa_1}\big) - *d\log\kappa_1(e_1)$$

and

$$3\omega_{12}(e_1) = \frac{\kappa_2}{\mu_2}\omega_{56}(e_1) + e_2\big(\log\frac{\mu_2}{\kappa_1}\big) - *d\log\kappa_1(e_1)$$

respectively. From these and from the fact that the normal connection form ω_{56} is given by (6.3), one can easily obtain

$$\omega_{12}(e_1) = -\frac{1}{6} * d\log(\kappa_2^2 - \mu_2^2)(e_1). \tag{6.7}$$

Arguing similarly by using now the Ricci equations

$$\langle R^{\perp}(e_1, e_2)e_3, e_6\rangle = 0 \text{ and } \langle R^{\perp}(e_1, e_2)e_4, e_5\rangle = 0,$$

we find that

$$\omega_{12}(e_2) = -\frac{1}{6} * d\log(\kappa_2^2 - \mu_2^2)(e_2),$$

which combined with (6.7) yields the connection form ω_{12} of (6.3). □

Let $g\colon M \to \mathbb{S}^6$ be a substantial pseudoholomorphic curve. Assume hereafter that g is non-isotropic. For each point $p \in M \smallsetminus M_1$, we consider an orthonormal frame $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ on a neighborhood $U \subset M \smallsetminus M_1$ of p as in Lemma 6.2.2. We note that the connection form ω_{56} cannot vanish on any open subset of $M \smallsetminus M_1$. Suppose to the contrary that $\omega_{56} = 0$. Then (6.3) implies that $\mu_2 = \lambda\kappa_2$ for some $\lambda \in \mathbb{R}^+$ and from Theorem 5(iii) in [43] we obtain

$$\kappa_2 = \frac{\kappa_1}{\lambda + 1} \quad \text{and} \quad \mu_2 = \frac{\lambda\kappa_1}{\lambda + 1}.$$

From (6.3) it follows that the connection form is given by

$$\omega_{12} = -\frac{1}{3} * d\log\kappa_1,$$

which implies

$$6K = \Delta\log\kappa_1^2 = \Delta\log(1 - K).$$

According to Theorem 3.2.1, this would imply a reduction of codimension, which is a contradiction.

For any $\theta \in \mathcal{M}_6^K(g)$, let $\{e_1, e_2, T_\theta e_3, T_\theta e_4, T_\theta e_5, T_\theta e_6\}$ be an orthonormal frame along g_θ, where T_θ is the bundle isomorphism of Lemma 6.2.1. The complex valued functions H_3, H_4, H_5, H_6 of g, associated to the orthonormal frame $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the corresponding functions $H_3^\theta, H_4^\theta, H_5^\theta, H_6^\theta$ of g_θ, associated to the orthonormal frame $\{e_1, e_2, T_\theta e_3, T_\theta e_4, T_\theta e_5, T_\theta e_6\}$ satisfy

$$H_3^\theta = e^{-i\theta}H_3,\ H_4^\theta = e^{-i\theta}H_4,\ H_5^\theta = e^{-i\theta}H_5 \ \text{and}\ H_6^\theta = e^{-i\theta}H_6. \tag{6.8}$$

Using (6.8) and the Weingarten formula for g_θ, we obtain

$$\tilde{\nabla}_E T_\theta e_3 = \omega_{34}(E)T_\theta e_4 + \frac{\kappa_2}{\kappa_1}T_\theta e_5 - \frac{i\mu_2}{\kappa_1}T_\theta e_6 - \kappa_1 e^{i\theta}dg_\theta(\overline{E}), \tag{6.9}$$

$$\tilde{\nabla}_E T_\theta e_4 = -\omega_{34}(E)T_\theta e_3 + \frac{i\kappa_2}{\kappa_1}T_\theta e_5 + \frac{\mu_2}{\kappa_1}T_\theta e_6 + i\kappa_1 e^{i\theta}dg_\theta(\overline{E}), \tag{6.10}$$

$$\tilde{\nabla}_E T_\theta e_5 = \omega_{56}(E)T_\theta e_6 - \frac{\kappa_2}{\kappa_1}\left(T_\theta e_3 + iT_\theta e_4\right), \tag{6.11}$$

$$\tilde{\nabla}_E T_\theta e_6 = -\omega_{56}(E)T_\theta e_5 + \frac{i\mu_2}{\kappa_1}\left(T_\theta e_3 + iT_\theta e_4\right), \tag{6.12}$$

where $E = e_1 - ie_2$ and $\tilde{\nabla}$ stands for the usual connection in the induced bundle $(i_1 \circ f)^*(T\mathbb{R}^7)$, with $i_1\colon \mathbb{S}^6 \to \mathbb{R}^7$ being the inclusion map.

Lemma 6.2.3. *Suppose that for $\theta_j \in \mathcal{M}_6^K(g), j = 1, \ldots, m$, there exist vectors $v_j \in \mathbb{R}^7$, such that*

$$\sum_{j=1}^{m} \langle g_{\theta_j}, v_j \rangle = 0 \quad on \quad U.$$

Then the following hold:

$$\sum_{j=1}^{m} e^{i\theta_j} \left(\kappa_2 \langle T_{\theta_j} e_5, v_j \rangle - i\mu_2 \langle T_{\theta_j} e_6, v_j \rangle \right) = 0, \tag{6.13}$$

away from the zeros of ω_{56}, and

$$\overline{E}\Big(\sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_6, v_j \rangle \Big) = -\omega_{56}(\overline{E}) \sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_5, v_j \rangle. \tag{6.14}$$

Proof. Our assumption implies that

$$\sum_{j=1}^{m} \langle dg_{\theta_j}, v_j \rangle = 0.$$

Differentiating and using the Gauss formula we obtain

$$\sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_3 - iT_{\theta_j} e_4, v_j \rangle = 0. \tag{6.15}$$

Differentiating (6.15) with respect to E and using (6.8), (6.9) and (6.10), it follows that

$$\sum_{j=1}^{m} e^{i\theta_j} \left(\overline{H}_5 \langle T_{\theta_j} e_5, v_j \rangle + \overline{H}_6 \langle T_{\theta_j} e_6, v_j \rangle \right) = 0.$$

Using that $H_5 = \kappa_2$ and $H_6 = i\mu_2$ (see Lemma 6.2.2(ii)), the above yields (6.13).

From (6.12), we compute that

$$\overline{E}\Big(\sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_6, v_j \rangle \Big) = -\omega_{56}(\overline{E}) \sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_5, v_j \rangle - \frac{i\mu_2}{\kappa_1} \sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_3 - iT_{\theta} e_4, v_j \rangle,$$

which in view of (6.15) yields (6.14). □

Proof of Theorem 6.0.2. According to Theorem 6.0.1, the space $\mathcal{M}_6^K(g)$ of the isometric deformations that are isometric to g is either $\mathbb{S}^1$ or a finite set. Suppose to the contrary that $\mathcal{M}_6^K(g) = \mathbb{S}^1$. We claim that the coordinate functions of the minimal surfaces $g_\theta, \theta \in \mathbb{S}^1$, are linearly independent. Since these functions are eigenfunctions of the Laplace operator of M with corresponding eigenvalue 2, this contradicts the fact that the eigenspaces of the Laplace operator are finite dimensional. To prove that the coordinate functions are linearly independent, it is enough to prove that if

$$\sum_{j=1}^{m} \langle g_{\theta_j}, v_j \rangle = 0, \tag{6.16}$$

for $0 < \theta_1 < \cdots < \theta_m < \pi$, then $v_j = 0$ for all $1 \leq j \leq m$.

Assume to the contrary that $v_j \neq 0$ for all $1 \leq j \leq m$. Let $M_1 = \{p_1, \ldots, p_k\}$ be the zero set of Φ_2. Around each point $p \in M \smallsetminus M_1$, we choose local complex coordinate (U, z) and an orthonormal frame $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ on $U \subset M \smallsetminus M_1$ as in Lemma 6.2.2. We consider the complex valued function

$$\psi := \Big(\sum_{j=1}^{m} e^{i\theta_j} \langle T_{\theta_j} e_6, v_j \rangle \Big)^2,$$

where $T_{\theta_j} \colon N_g M \to N_{g_{\theta_j}} M$ is the bundle isomorphism of Lemma 6.2.1. Obviously ψ is well defined on $M \smallsetminus M_1$. Equations (6.3) imply that

$$E(\kappa_2) = i\mu_2 \omega_{56}(E) - 3i\kappa_2 \omega_{12}(E),$$

and

$$E(\mu_2) = i\kappa_2 \omega_{56}(E) - 3i\mu_2 \omega_{12}(E).$$

These yield

$$\omega_{56}(\overline{E}) = \frac{i}{\kappa_2^2 - \mu_2^2} \left(\kappa_2 \overline{E}(\mu_2) - \mu_2 \overline{E}(\kappa_2) \right).$$

Then (6.13) and (6.14) imply that $\overline{E}\big(\psi \left(1 - \mu_2^2/\kappa_2^2\right)\big) = 0$, and hence the function $\Psi := \psi \left(1 - \mu_2^2/\kappa_2^2\right) : M \smallsetminus M_1 \to \mathbb{C}$ is holomorphic. Since Ψ is bounded, its isolated singularities are removable and consequently there exists a constant c such that

$$\psi(\kappa_2^2 - \mu_2^2) = c\kappa_2^2 \;\text{ on } M \smallsetminus M_1. \tag{6.17}$$

We claim that $c = 0$. Indeed, if $\kappa_2(p_l) = \mu_2(p_l) > 0$ for some $1 \leq l \leq k$, then taking the limit in (6.17) along a sequence of points in $M \smallsetminus M_1$ that converges to p_l, we deduce that $c = 0$.

Suppose now that $\kappa_2(p_l) = \mu_2(p_l) = 0$ for all $1 \leq l \leq k$. Let (V, z) be a local complex coordinate around p_l with $z(p_l) = 0$. From the proof of Proposition 4 in [42] for $s = 2$ we obtain

$$d\overline{H}_5 - 3i\overline{H}_5 \omega_{12} - \overline{H}_6 \omega_{56} \equiv 0 \bmod \phi,$$

and

$$d\overline{H}_6 - 3i\overline{H}_6\omega_{12} + \overline{H}_5\omega_{56} \equiv 0 \bmod \phi.$$

Writing $\phi = Fdz$, we deduce that

$$\frac{\partial \overline{H}_5}{\partial \bar{z}} = 3i\overline{H}_5\omega_{12}(\overline{\partial}) + \overline{H}_6\omega_{56}(\overline{\partial})$$

and

$$\frac{\partial \overline{H}_6}{\partial \bar{z}} = 3i\overline{H}_6\omega_{12}(\overline{\partial}) - \overline{H}_5\omega_{56}(\overline{\partial}).$$

Due to Chern [9, p. 32], we may write

$$\overline{H}_5 = z^{m_l}H_5^* \quad \text{and} \quad \overline{H}_6 = z^{m_l}H_6^*,$$

where m_l is a positive integer and H_5^*, H_6^* are nonzero smooth complex functions. Since

$$\alpha_3(E, E, E) = 4(\overline{H}_5 e_5 + \overline{H}_6 e_6),$$

we obtain

$$\alpha_3^{(3,0)} = z^{m_l}\alpha_3^{*(3,0)} \quad \text{on } V, \tag{6.18}$$

where $\alpha_3^{*(3,0)}$ is a tensor field of type $(3,0)$ with $\alpha_3^{*(3,0)}|_{p_l} \neq 0$. We now define the N_2^g-valued tensor field $\alpha_3^* := \alpha_3^{*(3,0)} + \overline{\alpha_3^{*(3,0)}}$. It is clear that α_3^* maps the unit circle on each tangent plane into an ellipse, whose length of the semi-axes are denoted by $\kappa_2^* \geq \mu_2^* \geq 0$. We furthermore consider the differential form of type $(6,0)$

$$\Phi_2^* := \langle \alpha_3^{*(3,0)}, \alpha_3^{*(3,0)} \rangle dz^6,$$

which in view of (6.18), is related to the Hopf differential of g by $\Phi_2 = z^{2m_l}\Phi_2^*$. We split Φ_2 and Φ_2^*, with respect to an arbitrary orthonormal frame $\{\xi_1, \ldots \xi_6\}$, where $\{\xi_1, \xi_2\}$ and $\{\xi_5, \xi_6\}$ are arbitrary orthonormal frames of TV and N_2^gV respectively as

$$\Phi_2 = \frac{1}{4}\big(\overline{H}_5^2 + \overline{H}_6^2\big)\phi^6 = \frac{1}{4}k_2^+ k_2^- \phi^6,$$

$$\Phi_2^* = \frac{1}{4}\big(\overline{H}_5^{*2} + \overline{H}_6^{*2}\big)\phi^6 = \frac{1}{4}k_2^{*+} k_2^{*-} \phi^6,$$

where $k_2^{\pm} = \overline{H}_5 \pm i\overline{H}_6, k_2^{*\pm} = \overline{H}_5^* \pm i\overline{H}_6^*$,

$$H_5^* = \langle \alpha_3^*(e_1, e_1, e_1), e_5 \rangle + i\langle \alpha_3^*(e_1, e_1, e_2), e_5 \rangle$$

and

$$H_6^* = \langle \alpha_3^*(e_1, e_1, e_1), e_6 \rangle + i\langle \alpha_3^*(e_1, e_1, e_2), e_6 \rangle.$$

From (6.18), we obtain $\overline{H}_a = z^{m_l}\overline{H}^*_a$ for $a = 5, 6$, or equivalently, $k_2^{\pm} = z^{m_l}k_2^{*\pm}$. Observe that $a_2^{\pm} = |k_2^{\pm}|$. Hence

$$\kappa_2 = |z|^{m_l}\kappa_2^*,\ \ \mu_2 = |z|^{m_l}\mu_2^*. \tag{6.19}$$

Now (6.17) yields

$$\psi(\kappa_2^{*2} - \mu_2^{*2}) = c\kappa_2^{*2} \ \text{ on } V \smallsetminus \{p_l\}. \tag{6.20}$$

If $\kappa_2^*(p_l) > \mu_2^*(p_l)$ for all $1 \le l \le k$, then (6.19) implies that

$$N(a_2^+) = \sum_{l=1}^{k} m_l = N(a_2^-).$$

Hence, Lemma 2.4.3 yields $\chi(N_2^f) = 0$, which contradicts our assumption. Thus, $\kappa_2^*(p_l) = \mu_2^*(p_l)$ for some $1 \le l \le k$. Taking the limit in (6.20), along a sequence of points in $V \smallsetminus \{p_l\}$ which converges to p_l, we obtain $c\kappa_2^{*2}(p_l) = 0$. Since $\alpha_3^*|_{p_l} \neq 0$, we derive that $c = 0$.

In view of (6.17), we conclude that $\psi = 0$ on $M \smallsetminus M_1$. This implies that

$$\sum_{j=1}^{m} e^{i\theta_j}\langle T_{\theta_j}e_6, v_j\rangle = 0,$$

which due to (6.13) gives that

$$\sum_{j=1}^{m} e^{i\theta_j}\langle T_{\theta_j}e_5, v_j\rangle = 0.$$

Differentiating this with respect to E, and using (6.11) and the above, we obtain

$$\sum_{j=1}^{m} e^{i\theta_j}\langle T_{\theta_j}e_3 + iT_{\theta_j}e_4, v_j\rangle = 0$$

which combined with (6.15) yields

$$\sum_{j=1}^{m} e^{i\theta_j}\langle T_{\theta_j}e_3, v_j\rangle = 0 = \sum_{j=1}^{m} e^{i\theta_j}\langle T_{\theta_j}e_4, v_j\rangle. \tag{6.21}$$

Differentiating (6.21) with respect to E we find that

$$\sum_{j=1}^{m} e^{2i\theta_j}\langle dg_{\theta_j}(\overline{E}), v_j\rangle = 0.$$

Differentiating once more with respect to E and using the minimality of each g_{θ_j} we obtain

$$\sum_{j=1}^{m} e^{2i\theta_j}\langle g_{\theta_j}, v_j\rangle = 0.$$

Combining this with (6.16), we obtain

$$\sum_{j=2}^{m}\langle g_{\theta_j}, w_j\rangle = 0,$$

where $w_j := \lambda_j v_j \neq 0, j = 2,\dots,m$ and $\lambda_j = \cos 2\theta_m - \cos 2\theta_1$ or $\lambda_j = \sin 2\theta_m - \sin 2\theta_1$. By induction, we finally conclude that $\langle g_{\theta_m}, w\rangle = 0$, for some nonzero vector w. Therefore, g_{θ_m} lies in a totally geodesic $\mathbb{S}^5$, which is a contradiction and the theorem is proved. ☐

Remark 6.2.1. *The global assumptions and the assumption on the codimension in Theorem 6.0.2 are essential and can not be dropped. In fact, locally we can produce minimal surfaces in spheres that are isometric to a non-isotropic pseudoholomorphic curve g in $\mathbb{S}^6$. More precisely, let $g_\theta, 0 \leq \theta < \pi$, be the associated family of a simply connected non-isotropic pseudoholomorphic curve $g\colon M \to \mathbb{S}^6$. We consider the surface $G\colon M \to \mathbb{S}^{7m-1}$ defined by*

$$G = a_1 g_{\theta_1} \oplus \cdots \oplus a_m g_{\theta_m},$$

where $a_1,\dots,a_m$ are any real numbers with $\sum_{j=1}^{m} a_j^2 = 1$, $0 \leq \theta_1 < \cdots < \theta_m < \pi$, and $\oplus$ denotes the orthogonal sum with respect to an orthogonal decomposition of the Euclidean space $\mathbb{R}^{7m}$. Arguing as in Section 4.1, it is easy to see that the surface G is minimal and isometric to g.

Proposition 6.2.1. *Let $g\colon M \to \mathbb{S}^6$ be a compact non-isotropic and substantial pseudoholomorphic curve. If $\hat{g}\colon M \to \mathbb{S}^n$ is a minimal surface that is isometric to g, then $\hat{g}$ is 1-isotropic.*

Proof. According to Theorem 2 in [43], the function $1-K$ is of absolute value type. If the zero set of the function $1-K$ is empty, then from condition (3.3) it follows that M is homeomorphic to the sphere. From [5] we have that $\hat{g}$ is full isotropic and from [41] it follows that $n = 6$ and $\hat{g}$ is congruent to g. Now suppose that the zero set of the function $1-K$ is the nonempty set $M_0 = \{p_1,\dots,p_m\}$ with corresponding order $\mathrm{ord}_{p_j}(1-K) = 2k_j$. For each point $p_j, j = 1,\dots,m$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$. Around p_j, we have that

$$1 - K = |z|^{2k_j} u_0, \tag{6.22}$$

where u_0 is a smooth positive function.

We know that the first Hopf differential $\hat{\Phi}_1 = \hat{f}_1 dz^4$ of $\hat{g}$ is globally defined and holomorphic. We claim that $\hat{\Phi}_1$ is identically zero. We assume to the contrary that it is not identically zero. Hence its zeros are isolated. Each p_j is totally geodesic, according to (2.5), and obviously, $\hat{\Phi}_1$ vanishes at each p_j. Thus we may write $\hat{f}_1 = z^{l_{1j}}\psi_1$ around p_j, where l_{1j} is the order of $\hat{\Phi}_1$ at p_j, and ψ_1 is a nonzero holomorphic function. Bearing in mind (2.7), we obtain

$$\frac{1}{4}\|\hat{\alpha}_2\|^4 - (\hat{K_1}^{\perp})^2 = 2^4 F^{-4}|\psi_1|^2|z|^{2l_{1j}} \tag{6.23}$$

around p_j, where $\hat{\alpha}_2$ and $\hat{K_1}^{\perp}$ are respectedly the second fundmental form and the first normal curvature of $\hat{g}$. We now consider the function $u_1 \colon M \smallsetminus M_0 \to \mathbb{R}$ defined by

$$u_1 = \frac{\frac{1}{4}\|\hat{\alpha}_2\|^4 - (\hat{K_1}^{\perp})^2}{(1-K)^2}.$$

In view of (6.22) and (6.23) we have that

$$u_1 = 2^4 F^{-4} u_0^{-2} |\psi_1|^2 |z|^{2(l_{1j}-2k_j)}.$$

Using (2.5) we find that $u_1 \leq 1$, thus from the above and (6.22) we deduce that $l_{1j} \geq 2k_j$. Hence we can extend u_1 to a smooth function on M. Applying Proposition 2.3.1(i) for $s = 1$ for g and Proposition 2.3.1(ii) for $r = 1$ for $\hat{g}$ we have that

$$\Delta \log \|\alpha_2\|^2 = 2\big(2K - K_1^*\big),$$

$$\Delta \log \left(\|\hat{\alpha}_2\|^2 + 2\hat{K}_1^{\perp}\right) = 2\big(2K - \hat{K}_1^*\big)$$

and

$$\Delta \log \left(\|\hat{\alpha}_2\|^2 - 2\hat{K}_1^{\perp}\right) = 2\big(2K + \hat{K}_1^*\big).$$

Combining these equation we obtain

$$\Delta \log u_1 = 4K_1^*, \tag{6.24}$$

away from the isolated zeros of u_1, where K_1^* is the intrinsic curvature of the first normal bundle N_1^g. Moreover, Proposition 2.3.1(iii) for $r = 1$, in combination with (3.3) provides

$$K - \frac{1}{2} < -K_1^* < K,$$

or more specific

$$K_1^* + K > 0.$$

Hence, (6.24) yields that $\Delta \log u_1 + 4K > 0$ and consequently using Lemma 2.4.1 and the Gauss-Bonnet theorem we have that

$$N(u_1) \leq 4\chi(M) \leq 0,$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. This implies that $N(u_1) = 0$, which contradicts our assumption that $\Phi_1 = 0$. □

6.3 A Schur type theorem

In view of Proposition 6.2.1, the surface $\hat{g}$ in Theorem 6.0.3 is 1-isotropic and consequently the Hopf differential $\hat{\Phi}_2$ of $\hat{g}$ is not identically zero. The following lemma will be used in the proof of Theorem 6.0.3.

Lemma 6.3.1. *Under the assumptions of Theorem 6.0.3, the following assertions hold:*

(i) The a-invariants of g and $\hat{g}$ satisfy the inequality

$$a_2^- \hat{a}_2^+ \leq a_2^+ \hat{a}_2^- .$$

(ii) The eccentricities $\varepsilon_2, \hat{\varepsilon}_2$ of the second curvature ellipses of g and $\hat{g}$ respectively satisfy the inequality $\varepsilon_2 \leq \hat{\varepsilon}_2$.

(iii) There exists a constant $c \geq 1$ such that the lengths κ_2, μ_2 and $\hat{\kappa}_2, \hat{\mu}_2$ of the semi-axes of the second curvature ellipses of the surfaces g and $\hat{g}$ respectively satisfy

$$\kappa_2^2 - \mu_2^2 = c(\hat{\kappa}_2^2 - \hat{\mu}_2^2). \tag{6.25}$$

(iv) At a point $p \in M$, we have that $a_2^+(p) = 0$ if and only if $\hat{a}_2^+(p) = 0$.

(v) If $\hat{a}_2^+(p) > 0$ at a point $p \in M$, then $\hat{a}_2^-(p) = 0$ if and only if $a_2^-(p) = 0$.

Proof. (i) It follows from Proposition 6.2.1, Propositions 2.1.1 and 2.3.1 and the Gauss equation that $\|\hat{\alpha}_3\| = \|\alpha_3\|$. This means that

$$\hat{\kappa}_2^2 + \hat{\mu}_2^2 = \kappa_2^2 + \mu_2^2. \tag{6.26}$$

Combining the above with our assumption $\hat{\kappa}_2\hat{\mu}_2 \leq \kappa_2\mu_2$, we have that

$$\hat{\kappa}_2 + \hat{\mu}_2 \leq \kappa_2 + \mu_2 \text{ and } \kappa_2 - \mu_2 \leq \hat{\kappa}_2 - \hat{\mu}_2.$$

The proof of part (i) follows easily.

(ii) Since $\hat{K}_2^{\perp} \leq K_2^{\perp}$, equation (6.26) implies that

$$\frac{\hat{\kappa}_2 \hat{\mu}_2}{\hat{\kappa}_2^2 + \hat{\mu}_2^2} \leq \frac{\kappa_2 \mu_2}{\kappa_2^2 + \mu_2^2}.$$

We set $\hat{t}_2 := \hat{\mu}_2 / \hat{\kappa}_2$ and $t_2 := \mu_2 / \kappa_2$. Obviously, $0 \leq \hat{t}_2, t_2 \leq 1$ and

$$\frac{\hat{t}_2}{1 + \hat{t}_2^2} \leq \frac{t_2}{1 + t_2^2}.$$

This immediately implies that $\varepsilon_2 \leq \hat{\varepsilon}_2$.

(iii) From Proposition 2.3.1(ii) we have that

$$\Delta \log(\kappa_2 + \mu_2) = 3K - K_2^*, \quad \Delta \log(\kappa_2 - \mu_2) = 3K + K_2^*, \tag{6.27}$$

and

$$\Delta \log(\hat{\kappa}_2 + \hat{\mu}_2) = 3K - \hat{K}_2^*, \quad \Delta \log(\hat{\kappa}_2 - \hat{\mu}_2) = 3K + \hat{K}_2^*, \tag{6.28}$$

where $\hat{K}_2^*$ denotes the second intrinsic curvature of $\hat{g}$. Equations (6.27) and (6.28) imply that

$$\Delta \log\left(\|\alpha_3\|^4 - 4^2 (K_2^{\perp})^2\right) = 12K \text{ and } \Delta \log\left(\|\hat{\alpha}_3\|^4 - 4^2 (\hat{K}_2^{\perp})^2\right) = 12K.$$

Inequality $\hat{K}_2^{\perp} \leq K_2^{\perp}$ yields

$$|f_2|^2 \leq |\hat{f}_2|^2, \tag{6.29}$$

where $\Phi_2 = f_2 dz^6$ and $\hat{\Phi}_2 = \hat{f}_2 dz^6$. For each point $p_j \in M_0 = \{p_1, \ldots, p_m\}, j = 1, \ldots, m$, where M_0 is the union of the zero sets of Φ_2 and $\hat{\Phi}_2$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$.

Suppose that $\hat{\Phi}_2(p_j) = 0$ for some $j = 1, \ldots, m$. Then Lemma 6.3.1(ii) implies that $\Phi_2(p_j) = 0$. Thus we may write $f_2 = z^{m(p_j)} u$ and $\hat{f}_2 = z^{\hat{m}(p_j)} \hat{u}$ around p_j, where $m(p_j)$ and $\hat{m}(p_j)$ are the orders of Φ_2 and $\hat{\Phi}_2$ respectively at p_j and u and $\hat{u}$ are nonzero holomorphic functions. From (6.29) we have that $\hat{m} \leq m$, and therefore the function $u_2 = |f_2|^2 / |\hat{f}_2|^2 \colon M \smallsetminus M_0 \to \mathbb{R}$ can be extended to a smooth function on M.

Suppose now that $\hat{\Phi}_2(p_j) \neq 0$ for some $j = 1, \ldots, m$. We have that the function $u_2 = |z|^{2m(p_j)} u$, with u a positive smooth function, can be extended to a smooth function on M.

In both cases, we have that the function u_2 is subharmonic and the maximum principle yields (6.25). Obviously (6.25) gives that the zeros of the second Hopf differential

Φ_2 of the curve g coincide with the zeros of the second Hopf differential $\hat{\Phi}_2$ of the surface $\hat{g}$.

(iv) If $a_2^+(p) = 0$ at a point $p \in M$, we obtain $\kappa_2(p) = \mu_2(p) = 0$. It follows from (6.26) that $\hat{\kappa}_2(p) = \hat{\mu}_2(p) = 0$, which is $\hat{a}_2^+(p) = 0$.

(v) Part (v) follows immediately from (i) and (6.25) which is equivalently written as $a_2^+ a_2^- = c\hat{a}_2^+ \hat{a}_2^-$. □

Proof of Theorem 6.0.3. Equations (6.27) and (6.28) yield

$$\Delta \log \frac{a_2^- \hat{a}_2^+}{a_2^+ \hat{a}_2^-} = 2(K_2^* - \hat{K}_2^*), \tag{6.30}$$

on $M \setminus M_0$, where $M_0 = \{p_1, \ldots, p_m\}$ is the union of the zero sets of Φ_2 and $\hat{\Phi}_2$. For each point $p_j \in M_0 = \{p_1, \ldots, p_m\}, j = 1, \ldots, m$, we choose a local complex coordinate z such that p_j corresponds to $z = 0$ and the induced metric is written as $ds^2 = F|dz|^2$.

We now claim that the function $u = (a_2^- \hat{a}_2^+)/(a_2^+ \hat{a}_2^-) \colon M \setminus M_0 \to \mathbb{R}$ can be extended to a smooth function on M. To this aim, we distinguish the following cases:

Case I: Suppose that $\hat{a}_2^+(p_j) = 0$ for some $j = 1, \ldots, m$. Then Lemma 6.3.1(iv) implies that $a_2^+(p_j) = 0$. Hence $\hat{a}_2^-(p_j) = a_2^-(p_j) = 0$. The a-invariants are absolute value type functions, thus we may write $a_2^+ = |z|^{2m_+} u_+$, $a_2^- = |z|^{2m_-} u_-$, $\hat{a}_2^+ = |z|^{2\hat{m}_+} \hat{u}_+$ and $\hat{a}_2^- = |z|^{2\hat{m}_-} \hat{u}_-$ around p_j, where $m_+, m_-, \hat{m}_+$ and $\hat{m}_-$ are the orders of a_2^+, a_2^-, $\hat{a}_2^+$ and $\hat{a}_2^-$ respectively at p_j and $u_+, u_-, \hat{u}_+$ and $\hat{u}_-$ are nonvanishing smooth functions. From Lemma 6.3.1(i) it follows that

$$m_-(p_j) + \hat{m}_+(p_j) \geq m_+(p_j) + \hat{m}_-(p_j).$$

Therefore the function $u = (a_2^- \hat{a}_2^+)/(a_2^+ \hat{a}_2^-)$ can be extended to a smooth function around p_j.

Case II: Suppose that $\hat{a}_2^+(p_j) > 0$ for some $j = 1, \ldots, m$. Lemma 6.3.1(v) implies that either $\hat{a}_2^-(p_j) a_2^-(p_j) > 0$ or $\hat{a}_2^-(p_j) = a_2^-(p_j) = 0$. In the former case, by Lemma 6.3.1(i) we have that $a_2^+(p_j) > 0$. Thus u is well defined at p_j.

Now assume that $\hat{a}_2^-(p_j) = a_2^-(p_j) = 0$. Clearly (6.26) implies that $a_2^+(p_j) > 0$. Since the a-invariants are absolute value type functions, we may write $a_2^- = |z|^{2m_-} u_-$ and $\hat{a}_2^- = |z|^{2\hat{m}_-} \hat{u}_-$ around p_j, where m_- and $\hat{m}_-$ are the orders of a_2^-, and $\hat{a}_2^-$ respectively at p_j and u_- and $\hat{u}_-$ are nonvanishing smooth functions. Lemma 6.3.1(i) yields

$$m_-(p_j) \geq \hat{m}_-(p_j),$$

therefore the function $u = (a_2^- \hat{a}_2^+)/(a_2^+ \hat{a}_2^-) \colon M \setminus M_0 \to \mathbb{R}$ can be extended to a smooth function around p_j.

It follows from Proposition 2.1.1 and (6.30) that

$$\Delta \log u = \frac{2\,\|\alpha_2\|^2}{(K_1^{\perp})^2}(K_2^{\perp} - \hat{K}_2^{\perp}) + \frac{2\,\|\hat{\alpha}_4\|^2}{4\hat{K}_2^{\perp}} \tag{6.31}$$

away from the isolated zeros of u. Hence $\Delta \log u \geq 0$ on $M \smallsetminus M_0$. By continuity, the function u is subharmonic on M and from the maximum principle we have that u is constant. Then from (6.31) it follows that $\hat{K}_2^{\perp} = K_2^{\perp}$, and $\hat{\alpha}_4 = 0$. Hence $f(M)$ is contained in a totally geodesic sphere $\mathbb{S}^6$ in $\mathbb{S}^n$.

The fact that the set of all noncongruent minimal surfaces $\hat{g}$, as in the statement of the theorem, that are isometric to g is either a circle or a finite set, follows directly from Theorem 6.0.1. □

Corollary 6.3.1. *Let $g\colon M \to \mathbb{S}^6$ be a compact non-isotropic and substantial pseudo-holomorphic curve with second normal curvature $K_2^{\perp}$. Any substantial minimal surface $\hat{g}$ in $\mathbb{S}^n, n > 6$, whose second normal curvature $\hat{K}_2^{\perp}$ satisfies the inequality $\hat{K}_2^{\perp} \leq K_2^{\perp}$, cannot be isometric to g.*

Proof. Assume that $\hat{g}$ is isometric to g. Proposition 6.2.1 implies that $\hat{g}$ is 1-isotropic. Suppose that $n > 6$. Then Theorem 6.0.3 implies that $\hat{g}$ is 2-isotropic. Hence $\hat{\kappa}_2 = \hat{\mu}_2$. The inequality $\hat{K}_2^{\perp} \leq K_2^{\perp}$, in combination with (6.26) implies that $\kappa_2 = \mu_2$, which is a contradiction. □

Open problem. *No example of a surface as in Theorem 6.0.1 is known where the moduli space is a finite set. If there exist such examples, then it would make sense to estimate the number of the noncongruent classes in terms of the topology or the geometry of the surface.*

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