MASTER THESIS

$U(1)$ extensions to the Standard Model and Phenomenological Applications

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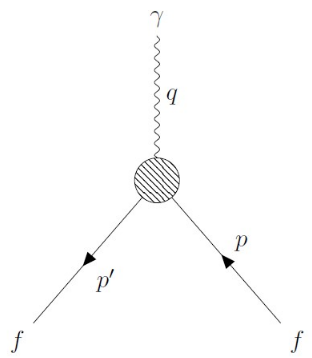

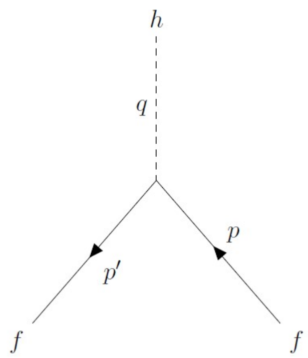

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Dedicated to the Memory of my Father

Abstract

In this thesis, I study a particular extension to the Standard Model of Particle Physics, namely the Beyond Standard Model Scenario of an additional $U(1)'$ symmetry to the SM symmetry group, which through Spontaneous Symmetry Breaking gives rise to a massive Z' gauge boson. The phenomenology that is analyzed and studied is around the muon's anomalous magnetic moment and the origins of the Yukawa couplings. The models presented here, and which attempt to explain these phenomena, require a clear understanding of one-loop radiative contributions, the seesaw mechanism, and effective field theory, which are presented analytically and in great detail. Both models introduce a vector-like lepton. In the first model, the existence of a vector-like lepton explains the small muon mass by a seesaw mechanism, based on lepton-specific two Higgs doublet models with a local $U(1)'$ symmetry. The physical muon mass is generated due to the mixing between the vector-like lepton and the muon after the leptophilic Higgs doublet and the dark Higgs acquire VEVs. The non-decoupling effects of the vector-like lepton give rise to leading contributions to the muon $g-2$, thanks to the light Z' and the light dark Higgs boson. In the second model, the ideas of inducing flavourful Z' couplings via mixing with a vector-like fourth family, which carries gauged $U(1)'$ charges, and furthermore generate fermion mass hierarchies and mixing patterns without introducing any family symmetry, are fused to provide a connection between $R_{K^{(*)}}$ and the origin of Yukawa couplings in the quark sector. In the last chapter, I suggest a different approach to the Origins of the Yukawa couplings, by introducing a modified form of the symmetry breaking mechanism, namely modifying the covariant derivative to include spinor fields. This mechanism generates in a natural way the Yukawa coupling terms to the theory, with the cost of introducing a scalar source and a tree-level Higgs-fermion scattering term. Despite this accomplishment, this mechanism needs to be modified in order to respect the observable measurements and the phenomenology of the SM.

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Contents

1 Introduction and the Standard Model

Introduction

The Standard Model of particle physics describes a wide range of precise experimental measurements, and that alone is a remarkable achievement. It is a model constructed from a number of beautiful and profound theoretical ideas put together in a somewhat ad hoc fashion in order to reproduce the experimental data. It consists of 25 parameters: 14 associated with the *Higgs field* (Yukawa couplings, VEV, Higgs mass), 8 associated with the *flavour sector* (parametrized PMNS and CKM mixing angles) and 3 associated with the *gauge interactions* (gauge coupling strengths). In principle, there is one further parameter in the Standard Model; the Lagrangian of QCD can contain a phase that would lead to CP violation in the strong interaction. Experimentally, this strong CP phase is known to be extremely small, and is usually taken to be zero. However, this theoretical framework alone can not answer all the unanswered questions about the phenomena that describe Nature [16]. For instance, it does not explain *dark matter* and *dark energy* even though our observations show that they make up the most of the energy content in the universe. It does not explain the matter-antimatter asymmetry and also the masses of the neutrinos. Most importantly, it fails to include the description of gravity, which at very high energy scales plays an important role in the structure of the universe. These are the reasons that we suspect that there has to be Physics *Beyond the Standard Model* that can answer these open questions. In this thesis, we present a rather simple extension to the SM, the addition of a simple $U(1)$ to the SM symmetry group, and study some phenomenological applications regarding the muon $g-2$ and the origin of the Yukawa couplings.

This thesis is structured as follows. We review briefly the contents of the SM in Chapter 1. In Chapter 2 we present the Z' models, discussing about their couplings, the new mass spectrum of the theory and their kinetic mixing with the SM Z boson. In Chapter 3, we analyze some of the SM contributions to the muon $g-2$, that are important in the description of the model of the next chapter. In Chapter 4 we present and analyze a model that attempts to explain a part of the discrepancy in the anomalous magnetic moment of the muon between the experiment and the theory, utilizing the seesaw mechanism from heavy vector-like leptons, in a Z' theoretical framework. In Chapter 5 we present a model that explains the origin of Yukawa couplings by utilizing effective field theory in a Z' theoretical framework, and is applied to the problem of $R_{K^{(*)}}$. In Chapter 6 a new mechanism that generates the Yukawa couplings is suggested. In Chapter 7 we present the conclusions.

The Standard Model

The first steps towards the SM was Sheldon Glashow's discovery in 1961 of a way to combine the electromagnetic and weak interactions. In 1967 Steven Weinberg and Abdus Salam incorporated the Higgs Mechanism into Glashow's electroweak interaction, giving it its modern form. As a *Quantum Field Theory*, it treats fundamental particles as excitations of fields that permeate spacetime. The whole dynamics and propagation of these fields are encoded in the *Lagrangian densities* (*Lagrangians* for short), which one can obtain the equations of motion and construct mathematical expressions for observables.

The Standard Model of Elementary particles consists of the following ingredients [10]:

1. **A symmetry.**
   Its proposers guessed that the smallest possible symmetry could correspond to a simple group but it must contain 3 group factors, it is of the form

$$G_{SM} = SU(3)_C \otimes SU(2)_I \otimes U(1)_Y, \tag{1.1}$$

   each of them being a special unitary group in dimensions 3, 2 and 1 respectively. The first refers to the colour group (C), the second the weak isospin (I), and the last behaves just like a number whose value is indicated by Y, called *hypercharge*. After writing down the symmetry of the theory, we also need the particle content that is described under these symmetries.

2. **The gauge bosons.**
   These are spin-1 particles, they must transform like the adjoined representation of the group, and their structure and interactions are fixed by the symmetry. The number of gauge bosons is the same as the number of generators of the group, in this case $(3^2 - 1) + (2^2 - 1) + 1 = 12$. They are indicated as follows:

$$\begin{aligned} &G^i_\mu, \quad \text{for } SU(3)_C, \quad (i = 1, \ldots, 8), \\ &W^j_\mu, \quad \text{for } SU(2)_I, \quad (j = 1, 2, 3), \\ &B_\mu, \quad \text{for } U(1)_Y. \end{aligned}$$

3. **Fermions (Matter particles)**
   The fermions are put in chiral multiplets of some representation of the above symmetry. The chiral components of a fermion field f are given as follows:

$$f_L = \frac{1}{2}(1 - \gamma_5) f, \quad f_R = \frac{1}{2}(1 + \gamma_5) f, \quad \text{such that} \quad \gamma_5 f_L = -f_L, \gamma_5 f_R = f_R, \tag{1.2}$$

where the *left-handed* doublets in $SU(2)_I$ have weak isospin value $I = 1/2$, whereas the right-handed singlets in $SU(2)_I$ have $I = 0$. The quantum number Y called *hypercharge* is defined so that

$$Q = I_3 + \frac{Y}{2}, \quad Q = \text{the electric charge in units of e.} \tag{1.3}$$

The fermions are split in two categories: the *leptons* and the *quarks*, which appear both in three flavors.
For the *leptons*, we have the following generations:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \quad e_R^-, \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \quad \mu_R^-, \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \quad \tau_R^-, \tag{1.4}$$

where $l_L \Leftrightarrow (I, Y) = (1/2, -1), l_R \Leftrightarrow (I, Y) = (0, -2)$, with l the lepton family.
For the *quarks*:

$$\begin{pmatrix} u^a \\ d^a \end{pmatrix}_L, \quad u_R^a, \quad d_R^a, \quad \begin{pmatrix} c^a \\ s^a \end{pmatrix}_L, \quad c_R^a, \quad s_R^a, \quad \begin{pmatrix} t^a \\ b^a \end{pmatrix}_L, \quad t_R^a, \quad b_R^a, \tag{1.5}$$

with $a = r, g, b$ colour indices. The quarks correspond to the following charges:

$$\begin{aligned} q_L^a &\Leftrightarrow (I, Y) = (1/2, 1/3), \\ (u^a, c^a, t^a)_R &\Leftrightarrow (I, Y) = (0, 4/3), \\ (d^a, s^a, b^a)_R &\Leftrightarrow (I, Y) = (0, -2/3). \end{aligned} \tag{1.6}$$

The weak interaction acts vertically between members of the isodoublet, while the strong interaction acts horizontally between quarks of the same flavor with different color indices. Under the group $SU(3)_c$, the *quarks* are color triplets, indicated as $(r, g, b) \Leftrightarrow$ (red, green, blue).

4. **A scalar boson(Higgs)**
A complex scalar particle (spin zero) with zero colour and also represented as an isodoublet

$$\begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}, (I, Y) = (1/2, -1), \quad \text{(Higgs doublet)} \tag{1.7}$$

which was found at LHC, CERN in 2012. It was very crucial for the success of the SM to cause the spontaneous symmetry breaking.

1.1 Quantum Electrodynamics

In order to construct the *Lagrangian of QED*, we first need to formulate the *Dirac Lagrangian* for a free fermion

$$\mathcal{L} = \overline{\psi}(x)\big(i\partial\!\!\!/ - m\big)\psi(x) \tag{1.8}$$

in the natural units framework ($\hbar = c = 1$), where $\psi(x)$ a fermion field and m the mass of the corresponding particle. QED needs to be invariant under $U(1)$ gauge transformations. These transformations correspond to a multiplication of the fermion field with a complex number of absolute value 1. If we consider *global* $U(1)$ gauge transformations,

$$\begin{aligned}\psi(x) \to \psi'(x) = S\psi(x) = e^{ia}\psi(x)\\ \overline{\psi}(x) \to \overline{\psi}'(x) = S^{\dagger}\overline{\psi}(x) = e^{-ia}\overline{\psi}(x)\end{aligned} \tag{1.9}$$

the Lagrangian of Eq(1.8) is invariant under this transformation. However, by imposing a local gauge transformation of the form

$$\begin{aligned}\psi(x) \to \psi'(x) = S(x)\psi(x) = e^{ia(x)}\psi(x)\\ \overline{\psi}(x) \to \overline{\psi}'(x) = S^{\dagger}(x)\overline{\psi}(x) = e^{-ia(x)}\overline{\psi}(x)\end{aligned} \tag{1.10}$$

we notice that the derivative of the fermion field does not transform the same way:

$$\partial_\mu\psi'(x) = \partial_\mu\big(S(x)\psi(x)\big) = S(x)\partial_\mu\psi(x) + \partial_\mu\big(S(x)\big)\psi(x) \neq S(x)\partial_\mu\psi(x), \tag{1.11}$$

and similarly for $\partial_\mu\overline{\psi}'(x)$. In order to make our theory gauge invariant under *local gauge transformations*, we generalize the notion of the derivative $\partial_\mu \to D_\mu$:

$$D_\mu \equiv \partial_\mu - igA_\mu(x) \tag{1.12}$$

where D_μ is the *covariant derivative.* The *gauge field* plays the role of the *photon field*, and $g = -e$ the QED coupling. We demand the covariant derivative to transform like ψ itself, $D'_\mu\psi' = e^{ig\theta(x)}D_\mu\psi$. Then we have,

$$\begin{aligned}D'_\mu\psi' &= \big(\partial_\mu - igA'_\mu\big)S(x)\psi(x)\\ &= S(x)\big(\partial_\mu - igA'_\mu\big)\psi(x) + \partial_\mu S(x)\psi(x)\\ &= S(x)\big(\partial_\mu - igA'_\mu\big)\psi(x) + ig\partial_\mu\theta(x)S(x)\psi(x)\\ &= S(x)\big[(\partial_\mu - igA_\mu) + ig(A_\mu - A'_\mu) + ig\partial_\mu\theta(x)\big]\psi(x)\\ D'_\mu\psi' &= S(x)D_\mu\psi(x) + ig\big[A_\mu - A'_\mu + \partial_\mu\theta(x)\big]S(x)\psi(x),\end{aligned} \tag{1.13}$$

and for the second term to be vanished, the gauge field needs to transform as

$$A'_\mu = A_\mu + \partial_\mu\theta(x). \tag{1.14}$$

Now, we may write the gauge invariant QED Lagrangian,

$$\begin{aligned}\mathcal{L}_{QED} &= \overline{\psi}(x)\big(i\not{D} - m\big)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \overline{\psi}(x)\big(i\not{\partial} - m\big)\psi(x) - eA_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \mathcal{L}_{Dirac} + \mathcal{L}_{int} + \mathcal{L}_{\gamma,kinetic},\end{aligned} \tag{1.15}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the *electromagnetic field tensor*. Summarizing, the Lagrangian in Eq(1.15) contains the kinematic and interacting terms of a massive fermion, while also the a spin-1 gauge boson, the photon. This Lagrangian is invariant under the local $U(1) = U(1)_{em}$ gauge transformations, and describes electromagnetism at the quantum level, while also at the classical level as well, since the $\mathcal{L}_{\gamma,kinetic}$ term leads to the Maxwell equations.

1.2 Quantum Chromodynamics

The strong interaction is governed by the $SU(3)_c$ group of the SM, with a set of traceless generators, the *Gell-Mann* matrices λ^α. Associated with this symmetry, we have a set of 8 gauge bosons G^a_μ, with μ the Lorentz index and $a = 1, 2, \ldots, 8$, the *gluons*. The gluons remain massless even after the spontaneous symmetry breaking. Thus, this symmetry remains unbroken. The merging theory is called *Quantum Chromodynamics*, and is described by the Lagrangian:

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^i_{\mu\nu}F^{i,\mu\nu} + \sum_r \overline{q}_{r\alpha}\not{D}_{\alpha\beta}q_{\beta r}, \tag{1.16}$$

where r is the quark flavour index, α and β are color indices, and D^μ is the covariant derivative

$$D_\mu = \partial_\mu - ig_sG^i_\mu\frac{\lambda^i}{2}, \tag{1.17}$$

and

$$F^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g_s f_{ijk}G^j_\mu G^k_\nu \tag{1.18}$$

where g_s is the strong gauge coupling constant, and f_{ijk} are the structure constants of $SU(3)$ given by:

$$[\lambda^i, \lambda^j] = 2if^{ijk}\lambda_k. \tag{1.19}$$

Quarks and gluons are the only particles in the SM that carry color charge. Combinations of quarks and virtual gluons can create composite structures, the *hadrons*. *Mesons* are hadrons consisting of a quark-antiquark pair, while *baryons* consist of three quarks. All composite structures that have been observed in nature do not carry free colour charge, since it is confined in the hadrons. This phenomenon is called *confinement*: colour-charged particles can not be isolated at low energies, they can only exist in the form of colorless combinations.

1.3 Weak interactions and the Electroweak theory

The charged-current weak interaction differs in almost all respects from the $\overline{\psi}(p')\gamma^\mu\psi(p)$ form of QED and QCD. It is mediated by massive charged $W^\pm$ bosons and consequently couples together fermions differing by one unit of electric charge. Furthermore, the parity violating nature of the interaction can be directly related to the form of the interaction vertex. From experiment, it is known that the weak charged current due to the exchange of $W^\pm$ bosons is a vector minus axial vector $(V - A)$ interaction of the form $\gamma^\mu - \gamma^\mu\gamma^5$, with a vertex factor of

$$-\frac{ig_w}{2\sqrt{2}}\gamma^\mu(1-\gamma^5) \tag{1.20}$$

where g_w is the weak coupling constant, and the corresponding four-vector current is given by

$$J^\mu_w = \frac{g_w}{2\sqrt{2}}\overline{\psi}(p')\gamma^\mu(1-\gamma^5)\psi(p). \tag{1.21}$$

The charged-current weak interaction is associated with invariance under $SU(2)_L$ local phase transformations,

$$\psi(x) \to \psi'(x) = e^{ig_w\vec{\mathbf{a}}(x)\cdot\vec{\mathbf{T}}}\psi(x) \tag{1.22}$$

where $\vec{\mathbf{T}} = \frac{1}{2}\vec{\sigma}$ are the three generators of the $SU(2)$ group that written in terms of the Pauli spin matrices, and $\vec{\mathbf{a}}$ are three functions which specify the local phase at each point in spacetime. Because the generators of the $SU(2)$ gauge transformation are the 2×2 Pauli spin-matrices, the wavefunction must be written in terms of two components, a *weak isospin doublet.* The weak isospin doublets contain flavours differing by one unit of electric charge, for instance

$$\psi_e(x) = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}, \tag{1.23}$$

with the third component of weak isospin being $I_3(\nu_e) = +\frac{1}{2}$, $I_3(e^-) = -\frac{1}{2}$. Since the observed form of the weak charged-current interaction couples only to left-handed chiral particle states and right-handed chiral antiparticle states, we place LH particle and RH antiparticle states into weak isospin doublets, while the RH particle and LH antiparticle chiral states are placed in *weak isospin singlets*, with $I_3 = 0$, and are therefore unaffected by the $SU(2)_L$ local gauge transformation. After the spontaneous symmetry breaking of $SU(2)_L$, the fields related to the "charged" the generators will generate the physical $W^{\pm}$ bosons

$$W_\mu^{\pm} = \frac{1}{\sqrt{2}}\big(W_\mu^{(1)} \mp iW_\mu^{(2)}\big). \tag{1.24}$$

As we shall see in the next subsection, the third "neutral" generator of $SU(2)_L$ combined with the generator of the *hypercharge* abelian group $U(1)_Y$, will provide the theory with the physical Z and photon fields, in terms of the weak mixing angle, θ_w. In comparison with the vertex factor of the W boson, the Z's boson vertex factor does not couple in a universal $V-A$ way to fermions as the W boson:

$$\frac{-ig_Z}{2}\gamma^\mu(c_V^f - c_A^f\gamma^5). \tag{1.25}$$

The vector and axial couplings to fermions are expressed by the coefficients c_V^f and c_A^f respectively, which depend on the particular fermion involved.

1.4 The Higgs Mechanism

The success of the Standard Model in describing the experimental data, including the high-precision electroweak measurements, places the local gauge principle on a solid experimental basis. However, the required local gauge invariance of the Standard Model is broken by the terms in the Lagrangian corresponding to particle masses. For example, if the photon were massive, the Lagrangian of QED would contain an additional term $\frac{1}{2}m_\gamma^2 A^\mu A_\mu$,

$$\mathcal{L}_{QED} \to \overline{\psi}(i\partial\!\!\!/ - m)\psi + e\overline{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m_\gamma^2 A_\mu A^\mu. \tag{1.26}$$

The photon field under the $U(1)$ symmetry transforms as

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\theta \tag{1.27}$$

and the new mass term becomes

$$\frac{1}{2}m_\gamma^2 A_\mu A^\mu = \frac{1}{2}m_\gamma^2\big(A_\mu + \partial_\mu\theta\big)\big(A^\mu + \partial^\mu\theta\big) \neq \frac{1}{2}m_\gamma^2 A_\mu A^\mu \tag{1.28}$$

from which it is clear that the photon mass term is not gauge invariant. Hence the required $U(1)$ local gauge symmetry can only be satisfied if the gauge boson of an interaction is massless. This restriction is not limited to the $U(1)$ local gauge symmetry of QED, it also applies to the $SU(2)_L$ and $SU(3)_c$ gauge symmetries of the weak interaction and QCD. Whilst the local gauge principle provides an elegant route to describing the nature of the observed interactions, it works only for massless gauge bosons. This is not a problem for QED and QCD where the gauge bosons are massless, but it is in apparent contradiction with the observation of the large masses of W and Z bosons.

The problem with particle masses is not restricted to the gauge bosons. For a fermion field ψ, the mass term in QED Lagrangian can be written in terms of the chiral particle states as

$$\begin{aligned} -m\overline{\psi}\psi &= -m\overline{\psi}\left[\frac{1}{2}(1-\gamma^5)+\frac{1}{2}(1-\gamma^5)\right]\psi \\ &= -m\left[\overline{\psi}_R\psi_L+\overline{\psi}_L\psi_R\right]. \end{aligned} \tag{1.29}$$

In the $SU(2)_L$ gauge transformation of the weak interaction, left-handed particles transform as weak isospin doublets and right-handed particles as singlets, and therefore the mass term of Eq(1.29) breaks the required gauge invariance. The Higgs mechanism provides the solution for the theory to obtain the masses of the fermions and the bosons. In the Salam–Weinberg model, the Higgs mechanism is embedded in the $SU(2)_L \times U(1)_Y$ local gauge symmetry of the electroweak sector of the Standard Model. Three *Goldstone bosons* will be required to provide the longitudinal degrees of freedom of the W^+, W^- and Z bosons. In addition, after symmetry breaking, there will be (at least) one massive scalar particle corresponding to the field excitations in the direction picked out by the choice of the physical vacuum. The simplest Higgs model, which has the necessary four degrees of freedom, consists of two complex scalar fields.

The minimal Higgs model consists of two complex scalar fields, placed in a weak isospin doublet [16]

$$\phi = \begin{pmatrix}\phi^+ \\ \phi^0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} \tag{1.30}$$

where one of the scalar fields must be neutral, written as ϕ^0, and the other must be charged such that ϕ^+ and $(\phi^+)^* = \phi^-$ give the longitudinal degrees of freedom of the W^+ and W^-. The Lagrangian for this doublet of scalar fields is

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) \tag{1.31}$$

with the Higgs potential

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{1.32}$$

The covariant derivative D_μ in $SU(2)_L \times U(1)_Y$ is written as

$$D_\mu = \partial_\mu + ig_w \mathbf{T} \cdot \mathbf{W}_\mu + ig' \frac{Y}{2} B_\mu \tag{1.33}$$

where $\mathbf{T} = \frac{1}{2}\boldsymbol{\tau}$ are the three generators of the $SU(2)_L$ symmetry group, and Y the generator of the $U(1)_Y$ hypercharge's symmetry group. The potential $V(\phi)$ has a manifold of minima, satisfying

$$\phi^\dagger \phi = \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = \frac{\mu^2}{2\lambda} = \frac{\upsilon^2}{2}. \tag{1.34}$$

After symmetry breaking, the photon is required to remain massless, and the minimum of the potential must correspond to a non-zero vacuum expectation value of the neutral scalar field,

$$\langle \phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon \end{pmatrix}, \tag{1.35}$$

while in the unitary gauge, the Higgs doublet is written,

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon + H(x) \end{pmatrix}. \tag{1.36}$$

Inserting the above doublet into the kinetic term we obtain the mass terms of the $\mathbf{W}_\mu$ and B_μ, and also the interactions between them and the Higgs boson. The mass terms are the following:

$$\frac{1}{8}\upsilon^2 g_w^2 \left(W_\mu^{(1)} W^{(1)\mu} + W_\mu^{(2)} W^{(2)\mu} \right) + \frac{1}{8}\upsilon^2 \left(g_w W_\mu^{(3)} - g' B_\mu \right) \left(g_w W^{(3)\mu} - g' B^\mu \right), \tag{1.37}$$

and in order to go to the physical mass eigenstates, we use the following linear combinations of the fields

$$\begin{aligned} A_\mu &= \cos\theta_w B_\mu + \sin\theta_w W_\mu^{(3)}, \\ Z_\mu &= -\sin\theta_w B_\mu + \cos\theta_w W_\mu^{(3)}, \\ W_\mu^{\pm} &= \frac{1}{\sqrt{2}} \big(W_\mu^{(1)} \mp W_\mu^{(2)} \big) \end{aligned} \tag{1.38}$$

where θ_w is the *weak mixing angle*, and

$$\cos\theta_w = \frac{g_w}{\sqrt{g_w^2 + g'^2}}. \tag{1.39}$$

Then, in the physical mass eigenbasis, the mass of the W boson is

$$m_w = \frac{1}{2} g_w \upsilon \tag{1.40}$$

and for the Z and the photon:

$$\frac{\upsilon^2}{8}\begin{pmatrix} W_\mu^{(3)} & B_\mu \end{pmatrix}\begin{pmatrix} g_w^2 & -g_w g' \\ -g_w g' & g'^2 \end{pmatrix}\begin{pmatrix} W^{(3)\mu} \\ B^\mu \end{pmatrix} = \frac{\upsilon^2}{8}\begin{pmatrix} A_\mu & Z_\mu \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & g_w^2+g'^2 \end{pmatrix}\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix}, \tag{1.41}$$

we obtain

$$m_\gamma = 0, \quad m_z = \frac{1}{2}\upsilon\sqrt{g_w^2+g'^2}. \tag{1.42}$$

The Higgs Mechanism has given mass to the gauge fields $W^\pm$ and Z, which absorbed the degrees of freedom from the corresponding Goldstone bosons, whereas the photon has remained massless. The remaining degree of freedom has given mass to a scalar field H, the Higgs field, with mass

$$m_H = \sqrt{2\lambda}\upsilon \tag{1.43}$$

and from the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g_w^2}{8m_w^2} = \frac{1}{2\upsilon^2} \tag{1.44}$$

where G_F is the Fermi constant of the weak interaction, we obtain the vacuum expectation value

$$\upsilon \approx 246\text{GeV}. \tag{1.45}$$

For the fermion masses, one needs to add them by hand in the Lagrangian. As it was mentioned in Eq(1.29), the theory needs gauge invariant terms for the fermion masses. For the leptons, this can be done by adding to the Lagrangian terms of the following form:

$$\mathcal{L}_l = -y_l\left[\begin{pmatrix} \overline{\nu}_l & \bar{l} \end{pmatrix}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} l_R + \bar{l}_R \begin{pmatrix} \phi^{+*} & \phi^{0*} \end{pmatrix}\begin{pmatrix} \nu_l \\ l \end{pmatrix}_L\right] \tag{1.46}$$

where y_l is a constant known as the Yukawa coupling of the lepton to the Higgs field. After the spontaneous symmetry breaking, the lepton mass terms become

$$\mathcal{L}_l = -\frac{y_l\upsilon}{\sqrt{2}}\left(\bar{l}_R l_L + \bar{l}_L l_R\right) - \frac{y_l}{\sqrt{2}}H\left(\bar{l}_R l_L + \bar{l}_L l_R\right). \tag{1.47}$$

The Yukawa coupling y_l is not predicted by the Higgs mechanism, but can be chosen to be consistent with the observed lepton mass,

$$y_l = \sqrt{2}\frac{m_l}{\upsilon} \tag{1.48}$$

meaning that

$$\mathcal{L}_l = -m_l\bar{l}l - \frac{m_l}{\upsilon}\bar{l}Hl. \tag{1.49}$$

The quarks however need a different kind of treatment. The reason behind this is that the combination of fields $\overline{L}\phi R + \overline{R}\phi^\dagger L$ can only generate the masses for the fermion in the lower component of an $SU(2)_L$ doublet. In particular, we need a mechanism that generates masses for the up-type quarks. This can be achieved by introducing the following conjugate doublet ϕ^c:

$$\phi^c = -i\sigma_2\phi^* = \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi_3 + i\phi_4 \\ \phi_1 - i\phi_2 \end{pmatrix}. \tag{1.50}$$

In order to have gauge invariant mass terms for the up-type quarks, we construct the following terms

$$\mathcal{L}_u = y_u \begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix}_L \begin{pmatrix} -\phi^{0*} \\ \phi^- \end{pmatrix} u_R + \text{h.c.} \tag{1.51}$$

which after the symmetry breaking becomes

$$\mathcal{L}_u = -\frac{y_u v}{\sqrt{2}}\big(\overline{u}_L u_R + \overline{u}_R u_L\big) - \frac{y_u}{\sqrt{2}} H\big(\overline{u}_L u_R + \overline{u}_R u_L\big). \tag{1.52}$$

The down-type quarks obtain their masses in the same way as the leptons. As it was previously mentioned, the Yukawa coupling terms are put manually to the SM theory. A mechanism should generate them naturally, coming from a more fundamental theory. In the last chapters we present models that can, up to a point, achieve that goal.

2 Z' models

One of the simplest and well-motivated extensions of the Standard Model is the addition of an extra $U(1)$ gauge factor to its $SU(3)_c \times SU(2)_L \times U(1)_Y$ structure ([11], [12]). Such extensions were motivated by grand unified theories of rank higher than that of the SM ($SO(10)$, E_6 or larger), or by geometric compactifications of heterotic string models which possessed a low-energy spectrum sharing many features with E_6 GUT's. Furthermore, in both string theories and in supersymmetric versions of grand unification with extra $U(1)'$s below the string or GUT scale, both the $U(1)'$ and the $SU(2)_L \times U(1)_Y$ breaking scales are generally tied to the soft supersymmetry breaking scale. Therefore, if supersymmetry is observed at the LHC there is a strong motivation that a string or GUT induced Z' would also have a mass at an observable scale.

The experimental discovery of a new Z' would be exciting, but the implications would be much greater than just the existence of a new vector boson. Breaking the $U(1)'$ symmetry would require an extended Higgs sector, with significant consequences for collider physics and cosmology (direct searches, the μ problem, dark matter, electroweak baryogenesis). Anomaly cancellation usually requires the existence of new exotic particles that are vectorlike with respect to the standard model but chiral under $U(1)'$, with several possibilities for their decay characteristics. The expanded Higgs and exotic sectors can modify or maintain the approximate gauge coupling unification of the minimal supersymmetric standard model (MSSM). In some constructions (especially string derived) the $U(1)'$ charges are family nonuniversal, which can lead to flavor changing neutral current (FCNC) effects, e.g., in rare B decays. Finally, the decays of a heavy Z' may be a useful production mechanism for exotics and superpartners. The constraints from the $U(1)'$ symmetry can significantly alter the theoretical possibilities for neutrino mass. $U(1)'$ interactions can couple to a hidden sector, possibly playing a role in supersymmetry breaking or mediation. In this chapter, we briefly mention how the Z' couplings may rise in a theory after the symmetry breaking of a $U(1)'$ symmetry group, and the shift on the mass of the SM Z boson that results from the kinetic mixing of the field tensors.

2.1 The Z' couplings

In the SM the neutral current interactions of the fermions are described by the Lagrangian

$$-\mathcal{L}_{NC}^{SM} = gJ_3^\mu W_{3\mu} + g' J_Y^\mu B_\mu = eJ_{em}^\mu A_\mu + g_1 J_1^\mu Z_{1\mu}^0 \tag{2.1}$$

where g and g' are the $SU(2)_L$ and $U(1)_Y$ gauge couplings, $W_{3\mu}$ is the (weak eigenstate) gauge boson associated with the third (diagonal) component of $SU(2)_L$, and

B_μ is the $U(1)_Y$ gauge boson. In the first form the currents are

$$\begin{aligned} J_3^\mu &= \sum_i \overline{f}_i \gamma^\mu \big[t_{3i_L} P_L + t_{3i_R} P_R\big] f_i \\ J_Y^\mu &= \sum_i \overline{f}_i \gamma^\mu \big[y_{i_L} P_L + y_{i_R} P_R\big] f_i \end{aligned} \tag{2.2}$$

where f_i is the field of the i^{th} fermion and $P_{L,R} = (1 \mp \gamma^5)/2$ are the left and right chiral projections. The currents in the new basis are

$$\begin{aligned} J_{em}^\mu &= \sum_i q_i \overline{f}_i \gamma^\mu f_i \\ J_1^\mu &= \sum_i \overline{f}_i \gamma^\mu \big[\epsilon_{iL}^1 P_L + \epsilon_{iR}^1 P_R\big] f_i \end{aligned} \tag{2.3}$$

with the chiral couplings

$$\epsilon_{iL}^1 = t_{3i_L} - \sin^2\theta_w q_i, \quad \epsilon_{iR}^1 = t_{3i_R} - \sin^2\theta_w q_i. \tag{2.4}$$

In the extension $SU(2)_L \times U(1)_Y \times U(1)^n$, $n \geq 1$, $\mathcal{L}_{NC}$ becomes [11]:

$$-\mathcal{L}_{NC} = e J_{em}^\mu A_\mu + \sum_{\alpha=1}^{n+1} g_\alpha J_\alpha^\mu Z_{\alpha\mu}^0 \tag{2.5}$$

where g_1, $Z_{1\mu}^0$, and J_1^μ are respectively the gauge coupling, boson, and current of the Standard Model Z. Similarly, g_α and $Z_{\alpha\mu}^0$, $\alpha = 2, \ldots, n+1$, are the gauge couplings and bosons for the additional $U(1)'$s [11]. The additional currents are

$$\begin{aligned} J_\alpha^\mu &= \sum_i \overline{f}_i \gamma^\mu \big[\epsilon_{iL}^\alpha P_L + \epsilon_{iR}^\alpha P_R\big] f_i \\ &= \frac{1}{2} \sum_i \overline{f}_i \gamma^\mu \big[g_{iV}^\alpha - g_{iA}^\alpha \gamma^5\big] f_i. \end{aligned} \tag{2.6}$$

The chiral couplings $\epsilon_{iL,R}^\alpha$, which may be unequal for a chiral gauge symmetry, are respectively the $U(1)_\alpha$ charges of the left and right handed components of fermion f_i, and $g_{iV,A}^\alpha = \epsilon_{iL}^\alpha \pm \epsilon_{iR}^\alpha$ are the corresponding vector and axial couplings. If we simply relate the $U(1)_\alpha$ charges with the chiral couplings, $\epsilon_L^\alpha(f) = Q_f^\alpha$, $\epsilon_R^\alpha(f) = Q_{f^c}^\alpha$ the diagonal (neutral current) part of the gauge covariant derivative of an individual field ϕ_i is

$$D_\mu \phi_i = \left(\partial_\mu + i e q_i A_\mu + i \sum_\alpha^{n+1} g_\alpha Q_{i\alpha} Z_{\alpha\mu}^0\right) \phi_i \tag{2.7}$$

where q_i and Q_i^α are respectively the electric and $U(1)_\alpha$ charges of ϕ_i.

2.2 Masses of the Bosons

Assuming that the neutral scalar fields ϕ_i acquire VEVs (and leaving the photon massless), the $Z^0_{\alpha\mu}$ fields acquire masses, and are described by the following term in the Lagrangian:

$$\mathcal{L}^Z_{mass} = \frac{1}{2} M^2_{\alpha\beta} Z^0_{\alpha\mu} Z^{0\mu}_\beta, \tag{2.8}$$

where

$$M^2_{\alpha\beta} = 2 g_\alpha g_\beta \sum_i Q_{\alpha i} Q_{\beta i} |\langle \phi_i \rangle|^2. \tag{2.9}$$

If we ignore the mixings between the Z^0's, then the matrix element $M^2_{11} = M^2_{Z^0}$ is the tree level Z boson's mass of the SM. In order to obtain the massive eigenstates, we diagonalize $M^2_{\alpha\beta}$ [11], to get to the new basis of fields

$$Z_{\alpha\mu} = \sum_\beta^{n+1} O_{\alpha\beta} Z^0_{\alpha\mu} \tag{2.10}$$

where O is an orthogonal mixing matrix. Now, studying the case $n = 1$, we have the following mixing matrix

$$\begin{aligned} M^2_{Z-Z'} &= \begin{pmatrix} 2g_1^2 \sum_i t_{3i}^2 |\langle \phi_i \rangle|^2 & 2g_1 g_2 \sum_i t_{3i} Q_i |\langle \phi_i \rangle|^2 \\ 2g_1 g_2 \sum_i t_{3i} Q_i |\langle \phi_i \rangle|^2 & 2g_2^2 \sum_i Q_i^2 |\langle \phi_i \rangle|^2 \end{pmatrix} \\ &\equiv \begin{pmatrix} M^2_{Z^0} & \Delta^2 \\ \Delta^2 & M^2_{Z'} \end{pmatrix}. \end{aligned} \tag{2.11}$$

Depending on the model we are studying, we can determine the mass parameters (matrix elements) in the above equation. Considering the scenario that our models contains an SU(2) singlet scalar field s, and two doublets ϕ_u, ϕ_d, we get

$$\begin{aligned} M^2_{Z^0} &= \frac{1}{4} g_1^2 \big(|v_u|^2 + |v_d|^2 \big), \\ \Delta^2 &= \frac{1}{2} g_1 g_2 \big(Q_u |v_u|^2 - Q_d |v_d|^2 \big), \\ M^2_{Z'} &= g_2^2 \big(Q_u^2 |v_u|^2 + Q_d^2 |v_d|^2 + Q_s^2 |v_s|^2 \big). \end{aligned} \tag{2.12}$$

For a general $M^2_{Z-Z'}$, the eigenvalues are [11]:

$$M^2_{1,2} = \frac{1}{2} \left[M^2_{Z^0} + M^2_{Z'} \mp \sqrt{(M^2_{Z^0} - M^2_{Z'})^2 + 4\Delta^4} \right], \tag{2.13}$$

where we use the rotation matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.14}$$

with

$$\theta = \frac{1}{2}\arctan\left(\frac{2\Delta^2}{M_{Z^0}^2 - M_{Z'}^2}\right), \tag{2.15}$$

and θ is related to the masses by

$$\tan^2\theta = \frac{M_{Z^0}^2 - M_1^2}{M_2^2 - M_{Z'}^2}. \tag{2.16}$$

2.3 Kinetic mixing

The phenomenon of kinetic mixing can significantly shift the predicted couplings of the Z' to SM states away from their canonical values, as well as changing the relationship between other SM observables. Furthermore, kinetic mixing is in general generated by renormalization group running down from the high (i.e., GUT) scale to the weak scale ([11],[12]). In $U(1)_\alpha \times U(1)_\beta$, the most general kinetic energy term for two gauge bosons $Z^0_{\alpha\mu}$ and $Z^0_{\beta\mu}$ is

$$\mathcal{L}_{kin} = -\frac{c_\alpha}{4}F_\alpha^{0\mu\nu}F^0_{\alpha\mu\nu} - \frac{c_\beta}{4}F_\beta^{0\mu\nu}F^0_{\beta\mu\nu} - \frac{c_{\alpha\beta}}{2}F_\alpha^{0\mu\nu}F^0_{\beta\mu\nu} \tag{2.17}$$

where $F^0_{\alpha\mu\nu} = \partial_\mu Z^0_{\alpha\nu} - \partial_\nu Z^0_{\beta\mu}$. A particular choice would be $c_\alpha = c_\beta = 1$, by rescaling the fields, and $c_{\alpha\beta} = \sin\chi$. In order to discuss the physical implications of the new gauge boson, it is necessary to work in the physical eigenbasis for the $Z - Z'$ system. Going to the physical eigenbasis requires both diagonalizing the field strength terms and the mass terms. This can be seen as a two-step process in which we first diagonalize the field strengths via a $GL(2, R)$ transformation [12]:

$$\begin{pmatrix} Z^0_{1\mu} \\ Z^0_{2\mu} \end{pmatrix} = \begin{pmatrix} 1 & -\tan\chi \\ 0 & 1/\cos\chi \end{pmatrix} \begin{pmatrix} \hat{Z}^0_{1\mu} \\ \hat{Z}^0_{2\mu} \end{pmatrix} = V \begin{pmatrix} \hat{Z}^0_{1\mu} \\ \hat{Z}^0_{2\mu} \end{pmatrix} \tag{2.18}$$

where V is non-unitary. In the new Z' basis, the mass matrix becomes $V^T M^2_{Z-Z'} V$, which can be diagonalized by an orthogonal matrix U . The interaction term then becomes

$$\begin{aligned}
\begin{pmatrix} g_1 J_1^{\mu} & g_2 J_2^{\mu} \end{pmatrix} \begin{pmatrix} Z_{1\mu}^0 \\ Z_{2\mu}^0 \end{pmatrix} &\equiv \mathcal{J}^T \begin{pmatrix} Z_{1\mu}^0 \\ Z_{2\mu}^0 \end{pmatrix} \\
&= \mathcal{J}^T V \begin{pmatrix} \hat{Z}_{1\mu}^0 \\ \hat{Z}_{2\mu}^0 \end{pmatrix} \\
&= \mathcal{J}^T V U^T \begin{pmatrix} Z_{1\mu} \\ Z_{2\mu} \end{pmatrix},
\end{aligned} \tag{2.19}$$

where $Z_{1,2}$ are the mass eigenstates. Then, depending on the approximations of the model, one may find the new eigenvalues of the diagonalized $M^2_{Z-Z'}$ matrix.

3 The Anomalous Magnetic Moment

One of the greatest successes of the Dirac equation was its prediction that the magnetic dipole moment $\vec{\mu}$, of a spin $|\vec{s}| = 1/2$ particle such as the electron (or muon) is given by

$$\vec{\mu}_l = g_l \frac{e}{2m_l} \vec{s}, \qquad l = e, \mu, \cdots \tag{3.1}$$

with gyromagnetic ratio $g_l = 2$, a value already implied by early atomic spectroscopy. Later, it was realized that a relativistic quantum field theory such as quantum electrodynamics (QED) can give rise via quantum fluctuations to a shift in g_l. These quantum corrections induce a deviation from the Dirac moment that is traditionally expressed as the magnetic moment anomaly,

$$a_l \equiv \frac{g_l - 2}{2}. \tag{3.2}$$

In a now classic QED calculation, Julian Schwinger showed that the one-loop $\big(O(e^2)\big)$ quantum correction to the electron's magnetic moment contributes [1]

$$a_l = \frac{\alpha}{2\pi} \approx 0.001162 \tag{3.3}$$

where the numerical value reflects the value of the fine structure constant, $\alpha = 1/137$. This contribution is due to quantum fluctuations via virtual lepton photon interactions and in QED is universal for all leptons. The predicted value of a_l can be confronted by experiment, which is easier to measure in the case of electron and muon, rather than the tau lepton, due to its very short lifetime $\tau_\tau = (290.3 \pm 0.5) \times 10^{-15} s$. Over the last decades, experiments have shown that the muon anomalous magnetic moment deviates from the predicted SM value. This observation was achieved by the even more precise experiments that were conducted, and also by the state-of-the-art evaluations of the contributions from quantum electrodynamics (QED) to tenth order, hadronic vacuum polarization, hadronic light-by-light, and electroweak processes. The theoretical prediction for $a_\mu(SM)$ is generally divided into three contributions,

$$a_\mu(SM) = a_\mu^{QED} + a_\mu^{EW} + a_\mu^{hadronic}. \tag{3.4}$$

The first results from FNAL [4] show the difference $a_\mu(Exp) - a_\mu(SM) = (251 \pm 59) \times 10^{-11}$, which has a significance of 4.2σ, and also confirm the BNL experiment result. This result will further motivate the development of SM extensions, including those having new couplings to leptons. In this chapter, we present the one-loop contributions with a virtual photon, SM Higgs boson, and SM Z boson to the anomalous magnetic moment of a fermion.

3.1 Construction of the Form Factors

Our first task, before calculating the radiative corrections to the anomalous magnetic moment of a lepton, is to present the derivation of the *form factors* in the *vertex function*. In the non-relativistic limit, the Dirac equation with electromagnetic radiation takes the form [14]:

$$i\frac{\partial}{\partial t}\psi = \left[\vec{\boldsymbol{\alpha}} \cdot \left(\vec{\mathbf{P}} - \frac{e}{c}\vec{\mathbf{A}}\right) + e\phi + mc^2\beta\right]\psi \tag{3.5}$$

where the Dirac matrices $\beta = (\gamma^0)^{-1}$, and $\alpha^i = \beta\gamma^i$ in the Pauli-Dirac representation are

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{3.6}$$

with the two component wavefunction

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imc^2 t}. \tag{3.7}$$

Substituting the two component wavefunction of Eq(3.7) into Eq(3.5), and through some simple manipulations in the calculations in the non-relativistic limit, we arrive at

$$i\frac{\partial}{\partial t}\psi = \left[\frac{1}{2m}\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^2 - \frac{e}{m}\frac{\vec{\boldsymbol{\sigma}}}{2} \cdot \vec{\mathbf{B}} + e\phi\right]\psi, \tag{3.8}$$

where $\vec{\boldsymbol{\sigma}}/2$ is the spin $\vec{\mathbf{S}}$ of the particle. Therefore, the Hamiltonian in the non-relativistic approximation is

$$H = \frac{1}{2m}\left(\vec{\mathbf{P}} - e\vec{\mathbf{A}}\right)^2 - \frac{e}{m}\vec{\mathbf{S}} \cdot \vec{\mathbf{B}} + e\phi. \tag{3.9}$$

Comparing this result with the magnetic moment $\vec{\mathbf{M}}$ of a particle

$$\vec{\mathbf{M}} = \frac{e}{m}\vec{\mathbf{S}} = g\frac{e}{2m}\vec{\mathbf{S}} \tag{3.10}$$

the *Lande g factor* is equal to $g = 2$. Moving on at the tree level of QED, the electromagnetic interaction of a charged particle takes the form

$$\mathcal{H}_I = -eA_\mu(x)\overline{\psi}(x)\gamma^\mu\psi(x) \tag{3.11}$$

where $A_\mu(x)$ is the four-vector potential of the electromagnetic field, e is the magnitude of the electric charge of the charged lepton and γ^μ is the Dirac algebra representation matrix given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{3.12}$$

with σ^μ being the Pauli matrices. Eventually, at the tree level $g = 2$ just like in the non-relativistic limit. Going beyond the tree level the value of g acquires additional contributions from loop effects. It is important now to study the most general vertex function Γ^μ in the following amplitude:

$$i\mathcal{M} = -ie\tilde{A}_\mu(p' - p)\overline{u}(p')\Gamma^\mu u(p), \tag{3.13}$$

where $\tilde{A}_\mu(q)$ is the Fourier transform of $A_\mu(x)$, and Γ^μ is the effective vertex function, involving all Lorentz scalars. The general form of Γ^μ includes γ^μ, $q^\mu = p'^\mu - p^\mu$, and $P^\mu = p'^\mu + p^\mu$, some contractions with the anti-symmetric tensor element $\epsilon^{\mu\nu\alpha\beta}$ with the momentums, and constants such as m, e and pure numbers. Through some manipulations, utilizing the following relations

$$\begin{aligned} \epsilon^{\mu\nu\alpha\beta} &= i\gamma^{[\mu}\gamma^\nu\gamma^\alpha\gamma^{\beta]}\gamma^5, \\ \not{p}u(p) &= mu(p), \\ \overline{u}(p')\not{p}' &= \overline{u}(p')m, \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}, \end{aligned} \tag{3.14}$$

the effective vertex function takes the form

$$\Gamma^\mu = \gamma^\mu F_1 + \frac{P^\mu}{2m}F_2 + i\frac{q^\mu}{2m}F_3 + \gamma^\mu\gamma^5 F_4 + \frac{q^\mu}{2m}\gamma^5 F_5 + i\frac{P^\mu}{2m}\gamma^5 F_6 \tag{3.15}$$

where the factor $1/2m$ ensures the corresponding F's to be *dimensionless*, and the factor i ensures the corresponding F's to be real so that $\epsilon_\mu(q)\overline{u}(p')\Gamma^\mu u(p)$ is *hermitian*, where ϵ_μ is the *polarization vector* of the electromagnetic field. Utilizing the following relations:

$$\begin{aligned} &q_\mu P^\mu = 0, \\ &q_\mu \overline{u}(p')\gamma^\mu u(p) = 0, \\ &q_\mu \overline{u}(p')\gamma^\mu\gamma^5 u(p) = 2m\overline{u}(p')\gamma^5 u(p), \end{aligned} \tag{3.16}$$

since $\overline{u}(p')\Gamma^\mu u(p)$ is a conserved current, then $q_\mu\overline{u}(p')\Gamma^\mu u(p) = 0$, and we have

$$q_\mu\overline{u}(p')\Gamma^\mu u(p) = \overline{u}(p')\left(i\frac{q_\mu q^\mu}{2m}F_3 + 2m\gamma^5 F_4 + \frac{q_\mu q^\mu}{2m}\gamma^5 F_5\right)u(p) \tag{3.17}$$

which leads to $F_3 = 0$ and $F_5 = -F_4 4m^2/q^2$. Then Γ^μ takes the form

$$\Gamma^\mu = \gamma^\mu F_1 + \frac{P^\mu}{2m} F_2 + \left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right) F_4 + i\frac{P^\mu}{2m}\gamma^5 F_6, \tag{3.18}$$

and by further using the Gordon identities (Appendix C), which allow us to swap the P^μ term for one involving $\sigma^{\mu\nu}q_\nu$,

$$\begin{aligned} \overline{u}(p')\frac{P^\mu}{2m}u(p) &= \overline{u}(p')\left(\gamma^\mu - i\frac{\sigma^{\mu\nu}q_\nu}{2m}\right)u(p), \\ \overline{u}(p')\frac{P^\mu}{2m}\gamma^5 u(p) &= \overline{u}(p')\left(- i\frac{\sigma^{\mu\nu}q_\nu}{2m}\gamma^5\right)u(p), \end{aligned} \tag{3.19}$$

we have

$$\Gamma^\mu = \gamma^\mu F_E(q^2) + \left(\gamma^\mu - \frac{2mq^\mu}{q^2}\right)\gamma^5 F_A(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_M(q^2) + \frac{\sigma^{\mu\nu}q_\nu}{2m}\gamma^5 F_D(q^2) \tag{3.20}$$

where the renamed F coefficients are called *form factors* [14],[15]. To lowest order, $F_E = 1$ and $F_M = 0$. In principle, the form factors can be computed to any order in perturbation theory. Since F_E and F_M contain complete information about the influence of an electromagnetic field on the lepton, they should contain the lepton's gross electric and magnetic couplings [15]. To identify the electric charge of a lepton, we set $A^{cl}_\mu(x) = (\phi(\vec{\mathbf{x}}), \mathbf{0})$, and study the case of Coulomb scattering of a nonrelativistic fermion from a region of nonzero electrostatic potential. The amplitude becomes

$$i\mathcal{M} = -ie\overline{u}(p')\Gamma^0(p', p)u(p)\tilde{\phi}(\vec{\mathbf{q}}). \tag{3.21}$$

Taking the limit $\vec{\mathbf{q}} \to 0$ in the spinor matrix element, only the F_E contributes (considering only vector contributions). In the nonrelativistic limit,

$$\overline{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \approx 2m\xi'^\dagger\xi \tag{3.22}$$

the amplitude for electron scattering from an electrostatic field becomes

$$i\mathcal{M} = -ieF_E(0)\tilde{\phi}(\vec{\mathbf{q}}) \cdot 2m\xi'^\dagger\xi, \tag{3.23}$$

and we identify the Born approximation for scattering from a potential

$$V(\vec{\mathbf{x}}) = eF_E(0)\phi(\vec{\mathbf{x}}), \tag{3.24}$$

where $eF_E(0)$ is the physical electric charge of the electron. Since $F_E(0) = 1$ (the renormalization condition of the electric charge), radiative corrections to $F_E(q^2)$

should vanish at $q^2 = 0$. Since we are concerned primarily with the magnetic moment, we focus on the F_M term. Setting $A_\mu^{cl}(x) = (0, \vec{\mathbf{A}}^{cl}(\vec{\mathbf{x}}))$, the scattering amplitude takes the form

$$i\mathcal{M} = +ie\overline{u}(p')\left(\gamma^i F_E + \frac{i\sigma^{i\nu}q_\nu}{2m}F_M\right)u(p)\tilde{A}^i_{cl}(\vec{\mathbf{q}}). \tag{3.25}$$

Repeating the same analysis for a lepton scattering from a static vector potential, we arrive at the following results (Appendix E):

$$\begin{aligned}\overline{u}(p')\gamma^i u(p) &= 2m\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi,\\ \overline{u}(p')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p) &= 2m\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi.\end{aligned} \tag{3.26}$$

The complete form of the lepton-photon vertex function is

$$\overline{u}(p')\left(\gamma^i F_E + \frac{i\sigma^{i\nu}q_\nu}{2m}F_M\right)u(p) \overset{q\to 0}{\approx} 2m\xi'^\dagger\left[\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\Big(F_E(0) + F_M(0)\Big)\right]\xi. \tag{3.27}$$

Using the magnetic field described in momentum space

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}) = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}(\vec{\mathbf{x}}) \quad \Rightarrow \quad \tilde{B}^k(\vec{\mathbf{q}}) = i\epsilon^{ijk}q_j\tilde{A}^{cl}_i(\vec{\mathbf{q}}), \tag{3.28}$$

and inserting Eq(3.27) into the amplitude, we find

$$i\mathcal{M} = -i(2m)\cdot e\xi'^\dagger\left[\frac{-1}{2m}\sigma^k\Big(F_E(0) + F_M(0)\Big)\right]\xi\tilde{B}^k(\vec{\mathbf{q}}). \tag{3.29}$$

Immediately we identify the Born approximation to the scattering of the lepton from a potential well (magnetic moment interaction),

$$V(\vec{\mathbf{x}}) = -\langle\vec{\boldsymbol{\mu}}\rangle\cdot\vec{\mathbf{B}}(\vec{\mathbf{x}}), \tag{3.30}$$

where

$$\langle\vec{\boldsymbol{\mu}}\rangle = \frac{e}{m}\big[F_E(0) + F_M(0)\big]\xi'^\dagger\frac{\vec{\boldsymbol{\sigma}}}{2}\xi. \tag{3.31}$$

Comparing it with the standard form

$$\vec{\boldsymbol{\mu}} = g\left(\frac{e}{2m}\right)\vec{\mathbf{S}}, \tag{3.32}$$

where $\vec{\mathbf{S}}$ the fermion spin, the coefficient g, called *Lande g-factor*, is

$$g = 2\big[F_E(0) + F_M(0)\big] = 2 + 2F_M(0). \tag{3.33}$$

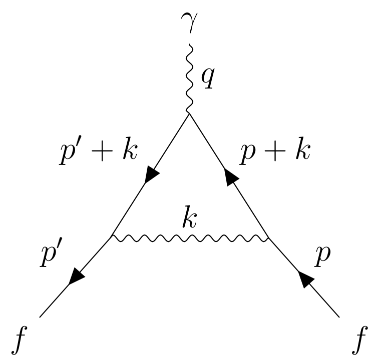
In the leading order of perturbation theory, $F_M(0) = 0$. It is the standard prediction of the Dirac equation. However, in higher orders $F_M(q^2)$ takes nonzero value, differing from the Dirac value. In the following subsections, we present the one-loop corrections to the anomalous magnetic moment of a lepton, with a virtual photon, Z and Higgs boson. These specific contributions were calculated in order to show the mathematical techniques needed for this task. Understanding the underlying concepts of these calculations is important, since in a Beyond Standard Model scenario, like the one presented in Chapter 4, one can manipulate the fermionic couplings and calculate the new contributions using similar techniques (at one-loop).

3.2 The One-loop QED contribution

For this kind of computation, we utilize the algebra of the gamma matrices

$$\begin{gathered} \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \\ \gamma^\mu \gamma_\mu = 4 \end{gathered} \tag{3.34}$$

where $g^{\mu\nu}$ is the Minkowski metric. The calculations are based on the textbooks of Schwartz [7] and Peskin, Schroeder [15], and as I explain later, I consider here the simple case of $q \to 0$,



where the virtual photon has momentum k. The vertex function is written as,

$$\Gamma^\mu = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{N^\mu}{D}, \tag{3.35}$$

where the numerator is

$$N^\mu = \gamma^\nu (\not{p}' + \not{k} + m)\gamma^\mu (\not{p} + \not{k} + m)\gamma_\nu, \tag{3.36}$$

and the denominator

$$D = \Big[(p' + k)^2 - m^2\Big]\Big[(p + k)^2 - m^2\Big]k^2, \tag{3.37}$$

with m being the mass of the virtual lepton in the loop, which in QED is the same as the on-shell leptons of the problem. The first step of the calculation is to contract dummy Lorentz indices and reduce the number of γ matrices using the on-shell condition

$$\begin{aligned} \not{p}u(p) &= mu(p), \\ \overline{u}(p')\not{p}' &= m\overline{u}(p'). \end{aligned} \tag{3.38}$$

So, we need to change N^μ into a form that contains γ^μ and $(p'^\mu + p^\mu)$ terms.

$$\begin{aligned} N^\mu &= \gamma^\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} + m)\gamma_\nu \\ &= \gamma^\nu\Big[(-\gamma^\mu\gamma^a + 2g^{\mu a})(p'_a + k_a) + m\gamma^\mu\Big](\not{p} + \not{k} + m)\gamma_\nu \\ &= \gamma^\nu\Big[-\gamma^\mu(\not{p} + \not{k}) + 2(p'^\mu + k^\mu) + m\gamma^\mu\Big](\not{p} + \not{k} + m)\gamma_\nu \\ &= \gamma^\nu\gamma^\mu\big[-\not{p}' - \not{k} + m\big]\big[(-\gamma_\nu\gamma_a + 2g_{\nu a})(p^a + k^a) + m\gamma_\nu\big] \\ &\qquad + 2(p'^\mu + k^\mu)\gamma^\nu\big[(-\gamma_\nu\gamma_a + 2g_{\nu a})(p^a + k^a) + m\gamma_\nu\big] \\ &= (-\gamma^\mu\gamma^\nu + 2g^{\mu\nu})\big[-\not{p}' - \not{k} + m\big]\big[-\gamma_\nu(\not{p} + \not{k}) + 2(p_\nu + k_\nu) + m\gamma_\nu\big] \\ &\qquad + 2(p'^\mu + k^\mu)\gamma^\nu\big[-\gamma_\nu(\not{p} + \not{k}) + 2(p_\nu + k_\nu) + m\gamma_\nu\big] \\ &= \gamma^\mu\gamma^\nu\big[\not{p}' + \not{k} - m\big]\big[-\gamma_\nu(\not{p} + \not{k}) + 2(p_\nu + k_\nu) + m\gamma_\nu\big] \\ &\quad + 2\big[m - \not{p}' - \not{k}\big]\big[-\gamma^\mu(\not{p} + \not{k}) + 2(p^\mu + k^\mu) + m\gamma^\mu\big] \\ &\quad + 2(p'^\mu + k^\mu)\big[-2(\not{p} + \not{k}) + 4m\big] \end{aligned}$$

Now, acting from both sides with the spinors like this: $\overline{u}(p')N^\mu u(p)$, we get,

$$\begin{aligned} N^\mu &= \gamma^\mu\gamma^\nu\big[\not{p}' + \not{k} - m\big]\big[-\gamma_\nu(m + \not{k}) + 2(p_\nu + k_\nu) + m\gamma_\nu\big] \\ &\quad - 2\not{k}\big[-\gamma^\mu(m + \not{k}) + 2(p^\mu + k^\mu) + m\gamma^\mu\big] \\ &\quad + 2(p'^\mu + k^\mu)\big[-2\not{k} + 2m\big] \\ &= \gamma^\mu\gamma^\nu\big[-(-\gamma_\nu\gamma_a + 2g_{\nu a})(p'^a + k^a) + m\gamma_\nu\big]\not{k} + 2\gamma^\mu\big[\not{p} + \not{k}\big]\big[\not{p}' + \not{k} - m\big] \\ &\quad + 2k_a(-\gamma^\mu\gamma^a + 2g^{\mu a})\not{k} - 4(p^\mu + k^\mu)\not{k} + 4(p'^\mu + k^\mu)(m - \not{k}) \\ &= \gamma^\mu\gamma^\nu\big[\gamma_\nu(\not{p}' + \not{k}) - 2(p'_\nu + k_\nu) + m\gamma_\nu\big]\not{k} + 2\gamma^\mu\big[\not{p} + \not{k}\big]\big[\not{p}' + \not{k} - m\big] \\ &\quad - 2\gamma^\mu k^2 + 4k^\mu\not{k} - 4(p^\mu + k^\mu)\not{k} + 4(p'^\mu + k^\mu)(m - \not{k}) \\ \big[\times u(p)\big] \quad &= \gamma^\mu\big[4(\not{p}' + \not{k}) - 2(\not{p}' + \not{k}) + 4m\big]\not{k} + 2\gamma^\mu\big[\not{p} + \not{k}\big]\big[\not{p}' + \not{k} - m\big] \\ &\quad - 2\gamma^\mu k^2 - 4p^\mu\not{k} + 4(p'^\mu + k^\mu)(m - \not{k}) \\ &= \gamma^\mu\Big\{2(\not{p}' + \not{k})\not{k} + 4m\not{k} + 2\big[\not{p} + \not{k}\big]\big[\not{p}' + \not{k} - m\big] - 2k^2\Big\} \\ &\quad + p^\mu\big\{-4\not{k}\big\} + p'^\mu\big\{4(m - \not{k})\big\} + k^\mu\big\{4(m - \not{k})\big\}, \end{aligned} \tag{3.39}$$

where we used $\not{k}\not{k} = k_a k_\beta \gamma^a \gamma^\beta = \frac{1}{4}k^2 g_{a\beta}\gamma^a\gamma^\beta = k^2$. The first and the third term of γ^μ can be reformed in the following way:

$$\begin{aligned}
\gamma^\mu\big[2(\not{p}' + \not{k})\not{k}\big] &= 2(-\gamma^a\gamma^\mu + 2g^{\mu a})p'_a\not{k} + 2\gamma^\mu k^2 \\
&\quad - 2\not{p}'\gamma^\mu\not{k} + 4p'^\mu\not{k} + 2\gamma^\mu k^2 \\
\big[\overline{u}(p') \times\,\big] \qquad &= -2m\gamma^\mu\not{k} + 4p'^\mu\not{k} + 2\gamma^\mu k^2 \\
&= \gamma^\mu\Big\{2(k^2 - m\not{k})\Big\} + p'^\mu\big\{4\not{k}\big\},
\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}
&\gamma^\mu\Big\{2\big[\not{p} + \not{k}\big]\big[\not{p}' + \not{k} - m\big]\Big\} = \\
&\qquad = \gamma^\mu\Big\{2(\not{p}\not{p}' + \not{p}\not{k} - m\not{p} + \not{k}\not{p}' + k^2 - m\not{k})\Big\} \\
\big[\times u(p)\big] &\qquad = \gamma^\mu\Big\{2\big[p^a p'^\beta(-\gamma_\beta\gamma_a + 2g_{a\beta}) + p^a k^\beta(-\gamma_\beta\gamma_a + 2g_{a\beta}) - m^2 \\
&\qquad\qquad + k^a p'^\beta(-\gamma_\beta\gamma_a + 2g_{a\beta}) + k^2 - m\not{k}\big]\Big\} \\
&\qquad = \gamma^\mu\Big\{2\big[-\not{p}'\not{p} + 2p\cdot p' - \not{k}\not{p} + 2p\cdot k - m^2 - \not{p}'\not{k} + 2p\cdot k + k^2 - m\not{k}\big]\Big\} \\
\big[\times u(p)\big] &\qquad = \gamma^\mu\Big\{2\big[-m\not{p}' - m\not{k} + 2(p\cdot p' + p\cdot k + p'\cdot k) - m^2 - \not{p}'\not{k} + k^2 - m\not{k}\big]\Big\} \\
&\qquad = 2(-\gamma^a\gamma^\mu + 2g^{a\mu})\big[-mp'_a - p'_a\not{k}\big] \\
&\qquad\quad + 2\gamma^\mu\big[-2m\not{k} + 2(p\cdot p' + p\cdot k + p'\cdot k) - m^2 + k^2\big] \\
&\qquad = 2\big[m\not{p}'\gamma^\mu + \not{p}'\gamma^\mu\not{k} - 2p'^\mu(m + \not{k})\big] \\
&\qquad\quad + 2\gamma^\mu\big[-2m\not{k} + 2(p\cdot p' + p\cdot k + p'\cdot k) - m^2 + k^2\big] \\
\big[\overline{u}(p') \times\,\big] &\qquad = 2\big[m^2\gamma^\mu + m\gamma^\mu\not{k} - 2p'^\mu(m + \not{k})\big] \\
&\qquad\quad + 2\gamma^\mu\big[-2m\not{k} + 2(p\cdot p' + p\cdot k + p'\cdot k) - m^2 + k^2\big] \\
&\qquad = \gamma^\mu\Big\{4(p\cdot p' + p\cdot k + p'\cdot k) - 2m\not{k} + 2k^2\Big\} + p'^\mu\Big\{-4(m + \not{k})\Big\}.
\end{aligned} \tag{3.41}$$

Now, we substitute Eq(3.40) and Eq(3.41) into Eq(3.39) to get

$$
\begin{aligned}
N^\mu &= \gamma^\mu\Big\{2(k^2 - m\not{k}) + 4m\not{k} + 4(p\cdot p' + p\cdot k + p'\cdot k) - 2m\not{k} + 2k^2 - 2k^2\Big\} \\
&\quad + p^\mu\Big\{-4\not{k}\Big\} + p'^\mu\Big\{4(m-\not{k}) + 4(-m-\not{k}) + 4\not{k}\Big\} + k^\mu\Big\{4(m-\not{k})\Big\} \\
N^\mu &= \gamma^\mu\Big\{2k^2 + 4(p\cdot p' + p\cdot k + p'\cdot k)\Big\} \\
&\quad + p^\mu\Big\{-4\not{k}\Big\} + p'^\mu\Big\{-4\not{k}\Big\} + k^\mu\Big\{4(m-\not{k})\Big\}.
\end{aligned}
\tag{3.42}
$$

At this stage, we are able to transform the denominator D using the Feynman-Schwinger parametrization (Appendix D):

$$
\frac{1}{a_1a_2a_3} = 2\int_0^1 dx\int_0^{1-x} dy\frac{1}{\big[xa_1 + ya_2 + (1-x-y)a_3\big]^3},
\tag{3.43}
$$

and D takes the form

$$
\begin{aligned}
\frac{1}{D} &= 2\int_0^1 dx\int_0^{1-x} dy\frac{1}{\big[x\big((p'+k)^2 - m^2\big) + y\big((p+k)^2 - m^2\big) + (1-x-y)k^2\big]^3} \\
&= 2\int_0^1 dx\int_0^{1-x} dy\frac{1}{d^3},
\end{aligned}
\tag{3.44}
$$

where

$$
\begin{aligned}
d &= x(p'^2 + k^2 + 2p'\cdot k) - xm^2 + y(p^2 + k^2 + 2p\cdot k) - ym^2 + (1-x-y)k^2 \\
&= k^2 + 2k\cdot(xp' + yp).
\end{aligned}
\tag{3.45}
$$

It is now time to change variables, $l^\mu = k^\mu + (xp'^\mu + yp^\mu)$. Considering the case of $q = p' - p \to 0$ we substitute the new variables on d, and we get

$$
\begin{aligned}
d &= l^2 + (xp'^\mu + yp^\mu)^2 - 2l\cdot(xp' + yp) + 2\big[l - (xp' + yp)\big]\cdot(xp' + yp) \\
&= l^2 - m^2(x^2 + y^2) + 2xyp'\cdot p \\
&= l^2 - m^2(x+y)^2.
\end{aligned}
\tag{3.46}
$$

The vertex function becomes

$$
\Gamma^\mu = -2ie^2\int_0^1 dx\int_0^{1-x}\int\frac{d^4l}{(2\pi)^4}\frac{N^\mu}{l^2 - m^2(x+y)^2}
\tag{3.47}
$$

and then we change variables in N^μ as well $\big[l^\mu = k^\mu + (xp'^\mu + yp^\mu)\big]$:

$$\begin{aligned}
N^\mu &= \gamma^\mu \Big\{ 2(l - xp' - yp)^2 + 4\big[p\cdot p' + p\cdot(l - xp' - yp) + p'\cdot(l - xp' - yp)\big] \Big\} \\
&\quad + p^\mu \Big\{ -4(\not{l} - x\not{p}' - y\not{p}) \Big\} + p'^\mu \Big\{ -4(\not{l} - x\not{p}' - y\not{p}) \Big\} \\
&\quad + (l^\mu - xp'^\mu - yp^\mu) \Big\{ 4\big[m - \not{l} + x\not{p}' + y\not{p}\big] \Big\} \\
\big[\overline{u}(p')\cdots u(p)\big] &= \gamma^\mu \Big\{ 2\big[l^2 + m^2(x+y)^2 - 2l\cdot(xp' + yp)\big] + 4\big[m^2 + l\cdot(p' + p) - 2m^2(x+y)\big] \Big\} \\
&\quad + p^\mu \Big\{ -4\big[\not{l} - m(x+y)\big] \Big\} + p'^\mu \Big\{ -4\big[\not{l} - m(x+y)\big] \Big\} \\
&\quad + (l^\mu - xp'^\mu - yp^\mu) \Big\{ 4\big[m\big(1 + (x+y)\big) - \not{l}\big] \Big\} \\
&= \gamma^\mu \Big\{ 2l^2 + m^2\big[2(x+y)^2 - 8(x+y) + 4\big] + 4l\cdot\big[(1-x)p' + (1-y)p\big] \Big\} \\
&\quad + p^\mu \Big\{ -4\big[\not{l} - m(x+y)\big] - 4y\big[m\big(1 + (x+y)\big) - \not{l}\big] \Big\} \\
&\quad + p'^\mu \Big\{ -4\big[\not{l} - m(x+y)\big] - 4x\big[m\big(1 + (x+y)\big) - \not{l}\big] \Big\} \\
&\quad + l^\mu \Big\{ 4\big[m\big(1 + (x+y)\big) - \not{l}\big] \Big\}.
\end{aligned} \tag{3.48}$$

Terms linear in l disappear in the integration as now the denominator D is an even function of l.

$$\begin{aligned}
N^\mu &= \gamma^\mu \Big\{ 2l^2 + m^2\big[2(x+y)^2 - 8(x+y) + 4\big] \Big\} \\
&\quad + p^\mu \Big\{ 4m\big[x - y(x+y)\big] \Big\} + p'^\mu \Big\{ 4m\big[y - x(x+y)\big] \Big\} + l^\mu \Big\{ -4\not{l} \Big\}.
\end{aligned} \tag{3.49}$$

Since we are interested in the magnetic contribution only, the form factor that is spin-associated is the one analogous to $(p'^\mu + p^\mu)$. For now, we can ignore the other terms of N^μ, and we shall deal with

$$N^\mu_{eff} = p^\mu \Big\{ 4m\big[x - y(x+y)\big] \Big\} + p'^\mu \Big\{ 4m\big[y - x(x+y)\big] \Big\}. \tag{3.50}$$

By symmetrizing N^{μ}_{eff}, we get,

$$\begin{aligned} N^{\mu}_{eff} &= 4m(p^{\mu}+p'^{\mu})\frac{1}{2}\big[x - xy - y^2 + y - x^2 - xy\big] \\ &= 2m(p^{\mu}+p'^{\mu})\big[(x+y) - (x+y)^2\big] \\ &= 2m(p^{\mu}+p'^{\mu})\Big[(x+y)\big(1-(x+y)\big)\Big]. \end{aligned} \tag{3.51}$$

As we can see, the momentum integral that we have to calculate has the following form

$$\int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n}, \quad \Delta = m^2(x+y)^2. \tag{3.52}$$

By applying a Wick rotation, $l_0 \to -il_0$, we change to Euclidean coordinates $l_E = (-il_0, l_i)$, and the integral becomes

$$\begin{aligned} \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n} &= \frac{i(-1)^n}{(2\pi)^4}\int d^4l_E \frac{1}{(l_E^2+\Delta)^n} \\ &= \frac{i(-1)^n}{(2\pi)^4}\int d\Omega_3 \int_0^{\infty} dl_E \frac{l_E^3}{(l_E^2+\Delta)^n} \\ &= \frac{i(-1)^n 2\pi^2}{(2\pi)^4}\int_0^{\infty} dl_E \frac{l_E^3}{(l_E^2+\Delta)^n} \\ &= \frac{i(-1)^n 2\pi^2}{(2\pi)^4}\int_0^{\infty} l_E dl_E \frac{l_E^2}{(l_E^2+\Delta)^n}. \end{aligned} \tag{3.53}$$

Making a simple change of variables: $\alpha = l_E^2 + \Delta \Rightarrow d\alpha = 2l_E dl_E$, the integral is equal to

$$\begin{aligned} \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n} &= \frac{i(-1)^n}{(4\pi)^2}\int_{\Delta}^{\infty} d\alpha \frac{\alpha-\Delta}{\alpha^n} \\ &= \frac{i(-1)^n}{(4\pi)^2}\int_{\Delta}^{\infty} d\alpha\big(\alpha^{1-n} - \Delta\alpha^{-n}\big) \\ &= \frac{i(-1)^n}{(4\pi)^2}\left[\frac{\alpha^{2-n}}{2-n} - \frac{\Delta\alpha^{1-n}}{1-n}\right]_{\Delta}^{\infty} \\ &= \frac{i(-1)^n}{(4\pi)^2}\left[-\frac{1}{2-n} + \frac{1}{1-n}\right]\frac{1}{\Delta^{n-2}} \\ \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^n} &= \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(2-n)(1-n)}\frac{1}{\Delta^{n-2}}. \end{aligned} \tag{3.54}$$

For $n = 3$,

$$\int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^3} = \frac{-i}{2(4\pi)^2}\frac{1}{\Delta}. \tag{3.55}$$

The effective vertex function then becomes

$$
\begin{aligned}
\Gamma^{\mu}_{eff} &= -2ie^2 \int_0^1 dx \int_0^{1-x} dy \frac{(-i)}{2(4\pi)^2} \frac{2m(p^{\mu}+p'^{\mu})(x+y)\big[1-(x+y)\big]}{m^2(x+y)^2} \\
&= -\frac{e^2}{8\pi^2 m}(p^{\mu}+p'^{\mu}) \int_0^1 dx \int_0^{1-x} dy \frac{1-(x+y)}{x+y}
\end{aligned}
\tag{3.56}
$$

with the integrals

$$
\begin{aligned}
\int_0^1 dx \int_0^{1-x} dy \frac{1-(x+y)}{x+y} &= \int_0^1 dx \int_x^1 dz \frac{1-z}{z} \\
&= \int_0^1 dx \big[lnz - z\big]_x^1 \\
&= \int_0^1 dx \big[-lnx - (1-x)\big] \\
&= -\int_0^1 dx lnx - \frac{1}{2} \\
&= -x\big[lnx - 1\big]_0^1 - \frac{1}{2} \\
\int_0^1 dx \int_0^{1-x} dy \frac{1-(x+y)}{x+y} &= \frac{1}{2}.
\end{aligned}
\tag{3.57}
$$

So, Γ^{μ}_{eff} becomes

$$
\Gamma^{\mu}_{eff} = -\frac{e^2}{16\pi^2 m}(p^{\mu}+p'^{\mu}) = -\frac{\alpha}{4\pi m}(p^{\mu}+p'^{\mu}) \tag{3.58}
$$

where $\alpha = e^2/4\pi$ is the fine structure constant. Utilizing the Gordon identity in the momentum space

$$
\overline{u}(p')\gamma^{\mu}u(p) = \overline{u}(p')\left[\frac{p^{\mu}+p'^{\mu}}{2m} + i\frac{\Sigma^{\mu\nu}q_{\nu}}{2m}\right]u(p) \tag{3.59}
$$

we recognize that the factor of $p^{\mu}+p'^{\mu}$ multiplied by $-2m$ is equal to the correction of the g factor,

$$
a = \frac{\alpha}{2\pi} \Rightarrow g = 2 + \frac{\alpha}{\pi}. \tag{3.60}
$$

This concludes the calculation of the 1-loop QED contribution to the anomalous magnetic moment of a lepton. However, if we proceed to calculate the integral containing the γ^{μ} term, we face the following form of an integral:

$$
\int \frac{d^4 l}{(2\pi)^4} \frac{l^2}{(l^2-\Delta)^n} = \frac{i(-1)^{n-1}}{(2\pi)^4} \int d^4 l_E \frac{l_E^2}{(l_E^2+\Delta)^n}. \tag{3.61}
$$

Switching to spherical coordinates, the integral takes the following form

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^n} &= \frac{i(-1)^{n-1}}{(2\pi)^4}\int d\Omega_3 \int_0^\infty dl_E \frac{l_E^5}{(l_E^2+\Delta)^n} \\ &= \frac{i(-1)^{n-1}}{(2\pi)^4}(2\pi^2)\int_0^\infty (l_E dl_E)\frac{l_E^4}{(l_E^2+\Delta)^n},\end{aligned} \tag{3.62}$$

and by changing variables $\alpha = l_E^2 + \Delta \Rightarrow d\alpha = 2l_E dl_E$ we get

$$\begin{aligned}\frac{i(-1)^{n-1}}{(4\pi)^2}\int_\Delta^\infty d\alpha \frac{(\alpha-\Delta)^2}{a^n} &= \frac{i(-1)^{n-1}}{(4\pi)^2}\int_\Delta^\infty d\alpha \frac{\alpha^2-2\alpha\Delta+\Delta^2}{a^n} \\ &= \frac{i(-1)^{n-1}}{(4\pi)^2}\int_\Delta^\infty d\alpha\Big(\alpha^{2-n}-2\alpha^{1-n}\Delta+\Delta^2\alpha^{-n}\Big) \\ &= \frac{i(-1)^{n-1}}{(4\pi)^2}\left[\frac{\alpha^{3-n}}{3-n}-\frac{2\Delta\alpha^{2-n}}{2-n}+\frac{\Delta^2\alpha^{1-n}}{1-n}\right]_\Delta^\infty \\ &= \frac{i(-1)^{n-1}}{(4\pi)^2}\left[-\frac{1}{3-n}+\frac{2}{2-n}-\frac{1}{1-n}\right]\frac{1}{\Delta^{n-3}} \\ &= \frac{i(-1)^{n-1}}{(4\pi)^2}\frac{2}{(n-1)(n-2)(n-3)}\frac{1}{\Delta^{n-3}}.\end{aligned} \tag{3.63}$$

It is obvious enough that for the value $n=3$, the integral in Eq(3.63) is divergent in any event. In order to make this integral finite, we need to replace in the photon propagator [15]

$$\frac{1}{k^2+i\epsilon} \to \frac{1}{k^2+i\epsilon} - \frac{1}{k^2-\Lambda^2+i\epsilon} \tag{3.64}$$

where Λ is a very large mass. Considering the second term as the propagator of a fictitious heavy photon, the numerator algebra of the integral in Eq(3.35) remains unchanged and the numerator is altered by

$$\Delta \to \Delta_\Lambda = (x+y)^2m^2 + (1-x-y)\Lambda^2 \tag{3.65}$$

Eventually, the integral in Eq(3.62), for $n=3$, is replaced with a convergent integral

$$\begin{aligned}\int \frac{d^4l}{(2\pi)^4}\left(\frac{l^2}{(l^2-\Delta)^3}-\frac{l^2}{(l^2-\Delta_\Lambda)^3}\right) &= \frac{i}{(4\pi)^2}\int_0^\infty dl_E^2\left(\frac{l_E^4}{(l_E^2-\Delta)^3}-\frac{l_E^4}{(l_E^2-\Delta_\Lambda)^3}\right) \\ &= \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right).\end{aligned} \tag{3.66}$$

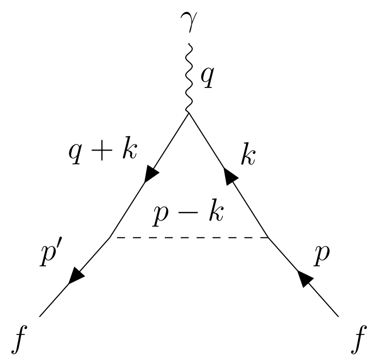
Then, the convergent terms are modified by terms of order Λ^{-2}. This prescription for rendering Feynman integrals finite (by introducing fictitious heavy particles) is known as *Pauli-Villars regularization.* Since this contribution corrects $F_E(q^2 = 0)$ (which should be fixed at the value 1), we make the following substitution

$$\delta F_E(q^2) \to \delta F_E(q^2) - \delta F_E(0) \tag{3.67}$$

which absorbs the divergence in a renormalization constant.

3.3 The One-loop Higgs boson contribution

Similar methods are used to calculate the 1-loop correction with a virtual scalar boson. In this case, the Standard Model's Higgs boson.



The 1-loop vertex correction to the anomalous magnetic moment of a fermion by the exchange of a virtual Higgs boson is

$$\begin{aligned}\overline{u}(p')\delta\Gamma^\mu u(p) &= \left(\frac{i\lambda}{\sqrt{2}}\right)^2 \int \frac{d^4k}{(2\pi)^4} \overline{u}(p') \frac{i}{(k-p)^2 - m_h^2} \frac{i}{\not{k} + \not{q} - m} \gamma^\mu \frac{i}{\not{k} - m} u(p) \\ &= \frac{i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \overline{u}(p') \frac{(\not{k} + \not{q} + m)\gamma^\mu(\not{k} + m)}{\left[(k-p)^2 - m_h^2\right]\left[(k+q)^2 - m^2\right]\left[k^2 - m^2\right]} u(p).\end{aligned} \tag{3.68}$$

Now, the denominator needs to be modified via the Feynman parametrization

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \int_0^{1-x} dx_2 \cdots \int_0^{1-x_1-\dots-x_{n-2}} \frac{(n-1)!}{\left[A_1 + x_1(A_2 - A_1) + \dots + x_{n-1}(A_n - A_1)\right]^n} \tag{3.69}$$

which, in the case of $n = 3$, we have

$$\frac{1}{A_1 A_2 A_3} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{2}{\left[A_1 + x_1(A_2 - A_1) + x_2(A_3 - A_1)\right]^3}. \tag{3.70}$$

In this specific occasion, the denominator is written as

$$
\begin{aligned}
D &= \underbrace{\left[k^2 - m^2\right]}_{A_1} + \underbrace{x\left[(k-p)^2 - m_h^2 - k^2 + m^2\right]}_{x_1(A_2 - A_1)} + \underbrace{y\left[(k+q)^2 - m^2 - k^2 + m^2\right]}_{x_2(A_3 - A_1)} \\
&= k^2 + (x-1)m^2 + xp^2 + yq^2 + 2k \cdot (-xp + yq) - xm_h^2.
\end{aligned}
\tag{3.71}
$$

We may now shift the momentum k, so that $l = k - xp + yq$ is the new variable under the integration. The denominator needs to depend on l^2:

$$
\begin{aligned}
l^2 &= k^2 + (-xp + yq)^2 + 2k \cdot (-xp + yq) \\
&= k^2 + x^2p^2 + y^2q^2 - 2xyq \cdot p + 2k \cdot (-xp + yq)
\end{aligned}
\tag{3.72}
$$

and

$$
l^2 - D = (1-x)m^2 + xm_h^2 + y(y-1)q^2 + x(x-1)p^2 - 2xyq \cdot p, \tag{3.73}
$$

where we name the above difference as Δ

$$
\Delta = (1-x)m^2 + xm_h^2 + y(y-1)q^2 + x(x-1)p^2 - 2xyq \cdot p, \tag{3.74}
$$

so that we can express the denominator as

$$
D = l^2 - \Delta. \tag{3.75}
$$

Following the same calculational methods as in the previous section, we utilize the following gamma matrices identities:

$$
\begin{aligned}
\gamma^\nu \gamma^\mu \gamma_\nu &= -2\gamma^\mu, \\
\gamma^\nu \gamma^\mu \gamma^\sigma \gamma_\nu &= 4g^{\mu\sigma}, \\
\gamma^\nu \gamma^\rho \gamma^\mu \gamma^\sigma \gamma_\nu &= -2\gamma^\sigma \gamma^\mu \gamma^\rho, \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu},
\end{aligned}
\tag{3.76}
$$

and apply them in the modification of the numerator N^μ

$$
\begin{aligned}
N^\mu &= (\not{k} + \not{q} + m)\gamma^\mu(\not{k} + m) \\
&= \big[\not{l} + x\not{p} + (1-y)\not{q} + m\big]\gamma^\mu\big[\not{l} + x\not{p} - y\not{q} + m\big] \\
&= \not{l}\gamma^\mu\not{l} + \underbrace{\not{l}\gamma^\mu(x\not{p} - y\not{q} + m)}_{\text{vanishes}} + x\not{p}\gamma^\mu(\underbrace{\not{l}}_{\text{vanishes}} + x\not{p} - y\not{q} + m) \\
&\quad + (1-y)\not{q}\gamma^\mu(\underbrace{\not{l}}_{\text{vanishes}} + x\not{p} - y\not{q} + m) + m\gamma^\mu(\underbrace{\not{l}}_{\text{vanishes}} + x\not{p} - y\not{q} + m) \\
[\ldots u(p)] \qquad &= -\frac{l^2}{2}\gamma^\mu + x(-\gamma^\mu\gamma^a + 2g^{\mu a})p_a\big[(1+x)m - y\not{q}\big] \\
&\quad + (1-y)\not{q}\gamma^\mu\big[(1+x)m - y\not{q}\big] + m\gamma^\mu\big[(1+x)m - y\not{q}\big] \\
&= -\frac{l^2}{2}\gamma^\mu + x(-\gamma^\mu\not{p} + 2p^\mu)\big[(1+x)m - y\not{q}\big] \\
&\quad + (1-y)(1+x)m\not{q}\gamma^\mu - y(1-y)\not{q}\gamma^\mu\not{q} + m^2(1+x)\gamma^\mu - my\gamma^\mu\not{q}
\end{aligned} \tag{3.77}
$$

where the terms proportional to l vanish due to

$$\int \frac{d^4l}{(2\pi)^4}\frac{l^\mu}{D^3} = 0.$$

From this point on, we utilize the following relations of gamma matrices when the spinors act on them from both sides:

$$
\begin{aligned}
\gamma^\mu\not{q} &= \gamma^\mu q_\beta\gamma^\beta = q_\beta(-\gamma^\beta\gamma^\mu + 2g^{\mu\beta}) = -\not{q}\gamma^\mu + 2q^\mu, \\
\overline{u}(p')\not{q}\gamma^\mu u(p) &= \overline{u}(p')(\not{p}' - \not{p})\gamma^\mu u(p) = \overline{u}(p')(m\gamma^\mu - \not{p}\gamma^\mu)u(p) \\
&= \overline{u}(p')(m\gamma^\mu + \gamma^\mu\not{p} - 2p^\mu)u(p) \\
&= \overline{u}(p')(2m\gamma^\mu - 2p^\mu)u(p), \\
\overline{u}(p')\not{q}u(p) &= \overline{u}(p')(\not{p}' - \not{p})u(p) = \overline{u}(p')(m - m)u(p) = 0.
\end{aligned} \tag{3.78}
$$

Therefore,

$$\begin{aligned}
N^\mu =& -\frac{l^2}{2}\gamma^\mu - mx(1+x)\gamma^\mu \not{p} + xy\gamma^\mu \not{p}\not{q} + 2mx(1+x)p^\mu - 2xyp^\mu \not{q} \\
&+ m(1-y)(1+x)\not{q}\gamma^\mu - y(1-y)\not{q}\underbrace{(-\gamma^a\gamma^\mu + 2g^{\mu a})q_a}_{-\not{q}\gamma^\mu + 2q^\mu} \\
&+ m^2(1+x)\gamma^\mu - my(-\not{q}\gamma^\mu + 2q^\mu) \\
\overline{u}(p')N^\mu u(p) =& -\frac{l^2}{2}\gamma^\mu - m^2x(1+x)\gamma^\mu + xy\gamma^\mu(-\gamma^a\gamma^\beta + 2g^{a\beta})q_a p_\beta \\
&+ 2mx(1+x)p^\mu + m(1-y)(1+x)(2m\gamma^\mu - 2p^\mu) \\
&+ y(1-y)q^2\gamma^\mu - 2y(1-y)q^\mu \not{q} + m^2(1+x)\gamma^\mu \\
&+ my\not{q}\gamma^\mu - 2myq^\mu \\
\overline{u}(p')N^\mu u(p) =& -\frac{l^2}{2}\gamma^\mu - m^2x(1+x)\gamma^\mu - xy\gamma^\mu \not{q}\not{p} + 2xy\gamma^\mu q\cdot p + 2mx(1+x)p^\mu \\
&+ 2m^2(1-y)(1+x)\gamma^\mu - 2m(1-y)(1+x)p^\mu + y(1-y)q^2\gamma^\mu \\
&+ m^2(1+x)\gamma^\mu + ym(2m\gamma^\mu - 2p^\mu) - 2myq^\mu \\
\overline{u}(p')N^\mu u(p) =& \Bigg[-\frac{l^2}{2} - m^2x(1+x) + 2xyq\cdot p + 2m^2(1-y)(1+x) + y(1-y)q^2 \\
&+ m^2(1+x) + 2m^2y\Bigg]\gamma^\mu - mxy(-\not{q}\gamma^\mu + 2q^\mu) \\
&+ 2m\Bigg[x(1+x) - (1-y)(1+x) - y\Bigg]p^\mu - 2myq^\mu.
\end{aligned} \tag{3.79}$$

The $q \cdot p$ product can be simplified to

$$\begin{aligned}
p' - p = q \Rightarrow & p' = p + q \\
& p'^2 = p^2 + q^2 + 2q\cdot p \\
& m^2 = m^2 + q^2 + 2q\cdot p \Rightarrow \\
\Rightarrow & q^2 = -2q\cdot p,
\end{aligned}$$

and substituting to the previous equation, we get

$$
\begin{aligned}
\overline{u}(p')N^{\mu}u(p) = & \Bigg[-\frac{l^2}{2} - m^2x(1+x) - xyq^2 + 2m^2(1-y)(1+x) + y(1-y)q^2 \\
& + m^2(1+x) + 2m^2y\Bigg]\gamma^{\mu} + mxy(2m\gamma^{\mu} - 2p^{\mu}) - 2mxyq^{\mu} \\
& + 2m\Bigg[x(1+x) - (1-y)(1+x) - y\Bigg]p^{\mu} - 2myq^{\mu} \\
= & \Bigg[-\frac{l^2}{2} + m^2\Bigg(-x(1+x) + 2(1-y)(1+x) + (1+x) + 2y + 2xy\Bigg) \\
& + q^2\Bigg(y(1-y) - xy\Bigg)\Bigg]\gamma^{\mu} + 2m\Bigg[-xy + x(1+x) - (1-y)(1+x) - y\Bigg]p^{\mu} \\
& - 2m\Bigg[y(1+x)\Bigg]q^{\mu} \\
= & \Bigg[-\frac{l^2}{2} + m^2(3-x)(1+x) + q^2(y - y^2 - xy)\Bigg]\gamma^{\mu} \\
& + 2m\Bigg[x^2 - 1\Bigg]p^{\mu} - 2m\Bigg[y(1+x)\Bigg]q^{\mu}.
\end{aligned}
\tag{3.80}
$$

All these modifications in the numerator were made in order to write it in the following, most useful form

$$N^{\mu} = A\gamma^{\mu} + B(p'+p)^{\mu} + C(p'-p)^{\mu}. \tag{3.81}$$

So, we write the p^{μ} and q^{μ} terms in Eq(3.80) as

$$2m\big\{(x^2-1)p^{\mu} - y(1+x)(p'-p)^{\mu}\big\} = 2m\Bigg\{\big[x^2-1+y(1+x)\big]p^{\mu} - y(1+x)p'^{\mu}\Bigg\}. \tag{3.82}$$

Generalizing the above factors of p, and p',

$$\Bigg[R_1(x,y) + R_2(x,y)\Bigg]p'^{\mu} + \Bigg[R_1(x,y) - R_2(x,y)\Bigg]p^{\mu}, \tag{3.83}$$

we solve the system of equations

$$
\begin{aligned}
R_1(x,y) + R_2(x,y) &= -y(1+x), \\
R_1(x,y) - R_2(x,y) &= x^2 - 1 + y(1+x),
\end{aligned}
\tag{3.84}
$$

to find the R_1, R_2 functions

$$R_1(x,y) = \frac{1}{2}(x^2-1), \quad R_2(x,y) = -\frac{1}{2}(x^2-1) - y(1+x). \tag{3.85}$$

Finally, the numerator takes the following form

$$\begin{aligned} N^\mu = &\left[-\frac{l^2}{2} + m^2(3-x)(1+x) + q^2(y - y^2 - xy) \right] \gamma^\mu \\ &+ m(x^2-1)(p'+p)^\mu + \underbrace{2R_2(x,y)q^\mu}_{\text{can be thrown away}} , \end{aligned} \tag{3.86}$$

where the term proportional to q^μ can be thrown away by Ward identity. At this point, we utilize the Gordon Identity,

$$\overline{u}(p') \frac{(p'+p)^\mu}{2m} u(p) = \overline{u}(p') \left[\gamma^\mu - i \frac{\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

to finally get the numerator of the integral in the proper form

$$N^\mu = \left[-\frac{l^2}{2} + m^2(x+1)^2 + q^2(y - y^2 - xy) \right] \gamma^\mu + 2m^2(1-x^2) i \frac{\sigma^{\mu\nu} q_\nu}{2m}. \tag{3.87}$$

We deal with the magnetic form factor only,

$$\delta F_M(q) = 2i\lambda^2 m^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4 l}{(2\pi)^4} \frac{1-x^2}{(l^2 - \Delta)^3}. \tag{3.88}$$

In the previous section, we proved the following integral

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2-\Delta)^3} = \frac{i(-1)^3}{(4\pi)^2} \frac{1}{(2-3)(1-3)} \frac{1}{\Delta^{3-2}} = \frac{-i}{2(4\pi)^2} \frac{1}{\Delta},$$

which can be substituted into Eq(3.88), to give

$$\begin{aligned} \delta F_M(q=0) &= 2i\lambda^2 m^2 \int_0^1 dx \int_0^{1-x} dy \frac{(-i)(1-x^2)}{2(4\pi)^2} \frac{1}{\Delta(q=0)} \\ &= \frac{\lambda^2}{(4\pi)^2} m^2 \int_0^1 dx \int_0^{1-x} dy \frac{1-x^2}{(1-x)m^2 + x m_h^2 - x(1-x)m^2} \\ &= \frac{\lambda^2}{(4\pi)^2} m^2 \int_0^1 dx \frac{(1-x^2)(1-x)}{m^2(1-x)^2 + x m_h^2} \\ &= \frac{\lambda^2}{(4\pi)^2} m^2 \int_0^1 dx \frac{(1-x)^2(1+x)}{m^2(1-x)^2 + x m_h^2} \\ \delta F_M(q=0) &= \frac{\lambda^2}{(4\pi)^2} \int_0^1 dx \frac{(1-x)^2(1+x)}{(1-x)^2 + x(m_h/m)^2} \end{aligned} \tag{3.89}$$

At this stage, we take into consideration that $m_h \gg m$, and we end up with the following result:

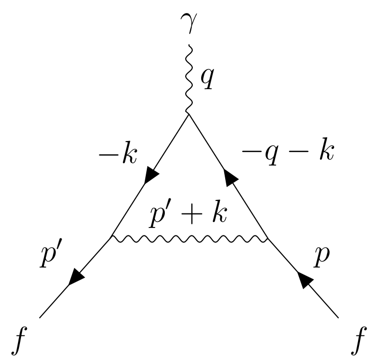
$$\delta F_M^h(q^2=0) \simeq \frac{\sqrt{2}G_F m^2}{8\pi^2}\frac{m^2}{m_h^2}\ln\left(\frac{m_h^2}{m^2}\right). \tag{3.90}$$

This is the one-loop contribution to the anomalous magnetic moment of a lepton when a virtual Higgs Boson is exchanged. The value for the muon lepton is approximately

$$\delta F_M^h(q^2=0) \approx 21.64\times 10^{-15}. \tag{3.91}$$

3.4 The One-loop Z boson contribution

In this case, the virtual boson that is exchanged is the SM's Z boson that couples to the neutral weak current.



The Γ vertex function in this interaction is written as follows,

$$\begin{aligned}\delta_Z\Gamma^\mu(q) =&\left(\frac{ig}{4c_w}\right)^2\int\frac{d^4k}{(2\pi)^4}\frac{-ig_{\rho\sigma}}{(p'+k)^2-m_Z^2}\gamma^\rho(4s_W^2-1-\gamma^5)\frac{i}{-\not{k}-m}\\ &\times\gamma^\mu\frac{i}{-\not{q}-\not{k}-m}\gamma^\sigma(4s_W^2-1-\gamma^5)\\ =&\frac{-ig^2}{16c_W^2}\int\frac{d^4k}{(2\pi)^4}\frac{\gamma_\sigma(4s_W^2-1-\gamma^5)(\not{k}-m)\gamma^\mu(\not{q}+\not{k}-m)\gamma^\sigma(4s_W^2-1-\gamma^5)}{[(p'+k)^2-m_Z^2][k^2-m^2][(q+k)^2-m^2]},\end{aligned} \tag{3.92}$$

where $c_W=\cos\theta_W$, $s_W=\sin\theta_w$, and θ_W is the Weinberg angle. Proceeding with the Feynman parametrization trick,

$$\frac{1}{A_1A_2A_3} = \int_0^1 dx_1\int_0^{1-x_1}dx_2\frac{2}{[A_1+x_1(A_2-A_1)+x_2(A_3-A_1)]^3}, \tag{3.93}$$

the denominator becomes:

$$
\begin{aligned}
D =& \underbrace{[k^2 - m^2]}_{A_1} + \underbrace{x[(q+k)^2 - m^2 - k^2 + m^2]}_{x(A_2 - A_1)} + \underbrace{y[(p'+k)^2 - m_Z^2 - k^2 + m^2]}_{y(A_3 - A_1)} \\
=& k^2 - m^2 + x[q^2 + k^2 + 2k \cdot q - k^2] + y[p'^2 + k^2 + 2k \cdot p' - m_Z^2 - k^2 + m^2] \\
=& k^2 + (y-1)m^2 + yp'^2 + xq^2 + 2k \cdot (yp' + xq) - ym_Z^2.
\end{aligned} \tag{3.94}
$$

The next step, just like in the previous contributions, is to change variables, by shifting the internal momentum to a new one. In fact, we shift to a momentum ideal enough to remove the k product with the momenta p' and q. With that said, the new momentum is $l = k + xq + yp'$. Now, we need to reform the denominator into a more manageable for the integral form. We need the square of l,

$$
\begin{aligned}
l^2 =& k^2 + (xq + yp')^2 + 2k \cdot (xq + yp') \\
=& k^2 + x^2 q^2 + y^2 p'^2 + 2xyq \cdot p' + 2k \cdot (xq + yp')
\end{aligned} \tag{3.95}
$$

and then

$$
\begin{aligned}
l^2 - D =& k^2 + x^2 q^2 + y^2 p'^2 + 2xyq \cdot p' + 2k \cdot (xq + yp') \\
& - [k^2 + (y-1)m^2 + yp'^2 + xq^2 + 2k \cdot (yp' + xq) - ym_Z^2] \\
=& (1-y)m^2 + ym_Z^2 + x(x-1)q^2 + y(y-1)p'^2 + 2xyq \cdot p' \\
=& \Delta,
\end{aligned} \tag{3.96}
$$

which gives us a simplified form of the denominator

$$
D = l^2 - \Delta. \tag{3.97}
$$

The numerator,

$$
\begin{aligned}
N^\mu =& \gamma_\sigma (4s_W^2 - 1 - \gamma^5)(\not{k} - m)\gamma^\mu(\not{q} + \not{k} - m)\gamma^\sigma(4s_W^2 - 1 - \gamma^5) \\
=& \gamma_\sigma (4s_W^2 - 1 - \gamma^5)(\not{k} - m)\gamma^\mu(\not{q} + \not{k} - m)\gamma^\sigma(4s_W^2 - 1 - \gamma^5) \\
& - \gamma_\sigma \gamma^5 (\not{k} - m)\gamma^\mu(\not{q} + \not{k} - m)\gamma^\sigma(4s_W^2 - 1 - \gamma^5).
\end{aligned} \tag{3.98}
$$

Omitting the terms proportional to γ^5, the numerator becomes

$$
N^\mu = \gamma_\sigma \big[(\not{k} + m)\gamma^\mu(\not{k} + \not{q} + m) + (4s_W^2 - 1 - \gamma^5)^2(\not{k} - m)\gamma^\mu(\not{k} + \not{q} - m)\big]\gamma^\sigma. \tag{3.99}
$$

We proceed by reshaping the numerator, while acting from the left and right with the spinors $\overline{u}(p')$ and $u(p)$ respectively.

$$\begin{aligned}
N^\mu =&(-\not{k}\gamma_\sigma + 2k_\sigma + \gamma_\sigma m)\gamma^\mu(\not{k}+\not{q}+m)\gamma^\sigma \\
&+ (4s_W^2-1)^2(-\not{k}\gamma_\sigma + 2k_\sigma - \gamma_\sigma m)\gamma^\mu(\not{k}+\not{q}-m)\gamma^\sigma \\
=&(-\not{k}+m)(-\gamma^\mu\gamma_\sigma + 2g^\mu_\sigma)(\not{k}+\not{q}+m)\gamma^\sigma + 2\gamma^\mu(\not{k}+\not{q}+m)\not{k} \\
&+ (4s_W^2-1)^2(-\not{k}-m)(-\gamma^\mu\gamma_\sigma + 2g^\mu_\sigma)(\not{k}+\not{q}-m)\gamma^\sigma + 2\gamma^\mu(\not{k}+\not{q}-m)\not{k} \\
=&(\not{k}-m)\gamma^\mu\big[-(\not{k}+\not{q})\gamma_\sigma + 2(k+q)_\sigma + m\gamma_\sigma\big]\gamma^\sigma - 2(\not{k}-m)(\not{k}+\not{q}+m)\gamma^\mu \\
&+ 2\gamma^\mu(\not{k}+\not{q}+m)\not{k} + 2\gamma^\mu(\not{k}+\not{q}-m)\not{k} \\
&+ (4s_W^2-1)^2(\not{k}+m)\gamma^\mu\big[-(\not{k}+\not{q})\gamma_\sigma + 2(k+q)_\sigma - m\gamma_\sigma\big]\gamma^\sigma \\
&- 2(4s_W^2-1)^2(\not{k}+m)(\not{k}+\not{q}-m)\gamma^\mu \\
=&(\not{k}-m)\gamma^\mu\big[-4(\not{k}+\not{q}) + 2(\not{k}+\not{q}) + 4m\big] - 2(\not{k}-m)(\not{k}+\not{q}+m)\gamma^\mu \\
&+ 4\gamma^\mu(k^2+\not{q}\not{k}) + (4s_W^2-1)^2(\not{k}+m)\gamma^\mu\big[-4(\not{k}+\not{q}) + 2(\not{k}+\not{q}) - 4m\big] \\
&- 2(4s_W^2-1)^2(\not{k}+m)(\not{k}+\not{q}-m)\gamma^\mu.
\end{aligned} \tag{3.100}$$

At this stage, we shall substitute the new variable l, and in the meantime, we ignore the terms linear in l, since the integral vanishes on these terms.

$$\begin{aligned}
N^\mu =&(\not{l}-x\not{q}-y\not{p}'-m)\gamma^\mu\big[-2\big(\not{l}+(1-x)\not{q}-y\not{p}'\big)+4m\big] \\
&- 2(\not{l}-x\not{q}-y\not{p}'-m)\big[\not{l}+(1-x)\not{q}-y\not{p}'+m\big]\gamma^\mu \\
&+ 4\gamma^\mu\big[(l-xq-yp')^2 + \not{q}(\not{l}-x\not{q}-y\not{p}')\big] \\
&+ (4s_W^2-1)^2(\not{l}-x\not{q}-y\not{p}'+m)\gamma^\mu\big[-2\big(\not{l}+(1-x)\not{q}-y\not{p}'\big)-4m\big] \\
&- 2(4s_W^2-1)^2(\not{l}-x\not{q}-y\not{p}'+m)\big[\not{l}+(1-x)\not{q}-y\not{p}'-m\big]\gamma^\mu \\
=&-2\not{l}\gamma^\mu\not{l} - (x\not{q}+y\not{p}'+m)\gamma^\mu\big[-2\big((1-x)\not{q}-y\not{p}'\big)+4m\big] \\
&- 2l^2\gamma^\mu + 2(x\not{q}+y\not{p}'+m)\big[(1-x)\not{q}-y\not{p}'+m\big]\gamma^\mu \\
&+ 4\gamma^\mu\big[l^2 + (xq+yp')^2 - \underbrace{2l\cdot(xq+yp')}_{vanishes} - xq^2 - y\not{q}\not{p}'\big] \\
&- 2(4s_W^2-1)^2\not{l}\gamma^\mu\not{l} + (4s_W^2-1)^2(x\not{q}+y\not{p}'-m)\gamma^\mu\big[2\big((1-x)\not{q}-y\not{p}'\big)+4m\big] \\
&- 2(4s_W^2-1)^2l^2\gamma^\mu + 2(4s_W^2-1)^2(x\not{q}+y\not{p}'-m)\big[(1-x)\not{q}-y\not{p}'-m\big]\gamma^\mu
\end{aligned}$$

$$
\begin{aligned}
\overline{u}(p')N^\mu =& l^2\gamma^\mu + 2\big[x\not{q} + (1+y)m\big]\gamma^\mu\big[(1-x)\not{q} - y\not{p}'\big] - 4m\big[x\not{q} + (1+y)m\big]\gamma^\mu \\
&- 2l^2\gamma^\mu + 2\big[x\not{q} + (1+y)m\big]\big[(1-x)\not{q} - y\not{p}' + m\big]\gamma^\mu \\
&+ 4\gamma^\mu\big[l^2 + x^2q^2 + y^2p'^2 + 2xyq\cdot p' - xq^2 - y\not{q}\not{p}'\big] \\
&+ (4s_W^2-1)^2 l^2\gamma^\mu + (4s_W^2-1)^2\big[x\not{q} - (1-y)m\big]\gamma^\mu\big[2\big((1-x)\not{q} - y\not{p}'\big) + 4m\big] \\
&- 2(4s_W^2-1)^2 l^2\gamma^\mu + 2(4s_W^2-1)^2\big[x\not{q} - (1-y)m\big]\big[(1-x)\not{q} - y\not{p}' - m\big]\gamma^\mu \\
\overline{u}(p')N^\mu u(p) =& \big[3l^2 - (4s_W^2-1)^2 l^2\big]\gamma^\mu + 2\big[x\not{q} + (1+y)m\big]\gamma^\mu\big[(1-x)\not{q} - y\not{p}'\big] \\
&- 4mx(2m\gamma^\mu - 2p^\mu) - 4m^2(1+y)\gamma^\mu + 2\big[x\not{q} + (1+y)m\big]\big[(1-x)\not{q} - y\not{p}' + m\big]\gamma^\mu \\
&+ 4\gamma^\mu\big[x(x-1)q^2 + y^2p'^2 - xyq^2 - y\not{q}\not{p}'\big] \\
&+ (4s_W^2-1)^2\big[x\not{q} - (1-y)m\big]\gamma^\mu\big[2\big((1-x)\not{q} - y\not{p}'\big) + 4m\big] \\
&+ 2(4s_W^2-1)^2\big[x\not{q} - (1-y)m\big]\big[(1-x)\not{q} - y\not{p}' - m\big]\gamma^\mu
\end{aligned} \tag{3.101}
$$

Now, we start to collect the terms proportional to γ^μ,

$$
\begin{aligned}
N^\mu =& \big\{l^2[3 - (4s_W^2-1)^2] - 4m^2(2x+1+y) + 4[q^2\big(x(x-1) - xy\big) + y^2p'^2]\big\}\gamma^\mu \\
&+ 2[x\not{q} + (1+y)m]\gamma^\mu[(1-x)\not{q} - y\not{p}'] + 8mxp^\mu \\
&+ 2[x\not{q} + (1+y)m][(1-x)\not{q} - y\not{p}' + m]\gamma^\mu - 4y\gamma^\mu\not{q}\not{p}' \\
&+ 2(4s_W^2-1)^2[x\not{q} - (1-y)m]\gamma^\mu[(1-x)\not{q} - y\not{p}' + 2m] \\
&+ 2(4s_W^2-1)^2[x\not{q} - (1-y)m][(1-x)\not{q} - y\not{p}' - m]\gamma^\mu,
\end{aligned} \tag{3.102}
$$

which we may ignore, since we focus only in the $(p+p')^\mu$ terms that will appear in our calculations.

$$
\begin{aligned}
N^\mu =& \big\{\dots\big\}\gamma^\mu + 2x(1-x)\not{q}\gamma^\mu\not{q} + 2(1+y)m(1-x)\gamma^\mu\not{q} - 2xy\not{q}\gamma^\mu\not{p}' \\
&- 2y(1+y)m\gamma^\mu\not{p}' + 8mxp^\mu + 2x(1-x)q^2\gamma^\mu - 2xy\not{q}\not{p}'\gamma^\mu + 2xm\not{q}\gamma^\mu \\
&+ 2(1+y)(1-x)m\not{q}\gamma^\mu - 2(1+y)ym\not{p}'\gamma^\mu + 2(1+y)m^2\gamma^\mu - 4y\gamma^\mu\not{q}\not{p}' \\
&+ 2(4s_W^2-1)^2\Big\{x(1-x)\not{q}\gamma^\mu\not{q} - xy\not{q}\gamma^\mu\not{p}' + 2mx\not{q}\gamma^\mu - (1-y)(1-x)m\gamma^\mu\not{q} \\
&+ (1-y)ym\gamma^\mu\not{p}' - 2(1-y)m^2\gamma^\mu + x(1-x)q^2\gamma^\mu - xy\not{q}\not{p}'\gamma^\mu - xm\not{q}\gamma^\mu \\
&- (1-y)(1-x)m\not{q}\gamma^\mu + (1-y)ym\not{p}'\gamma^\mu + (1-y)m^2\gamma^\mu\Big\}.
\end{aligned} \tag{3.103}
$$

Now, we take the time to remind the reader the following relations,

$$\begin{aligned}\gamma^\mu \not{q} =& -\not{q}\gamma^\mu + 2q^\mu,\\ \overline{u}(p')\not{q}\gamma^\mu u(p) =& \overline{u}(p')\big[2m\gamma^\mu - 2p^\mu\big]u(p),\\ \overline{u}(p')\not{q}u(p) =& 0,\end{aligned} \tag{3.104}$$

and apply them in our calculations:

$$\begin{aligned}N^\mu =& \big\{\dots\big\}\gamma^\mu + 2x(1-x)(-q^2\gamma^\mu + \underbrace{2\not{q}q^\mu}_{\overline{u}(p')\not{q}u(p)=0}) + 2(1+y)(1-x)m(-2m\gamma^\mu + 2p^\mu + 2q^\mu)\\ &- 2xy\not{q}(-\not{p}'\gamma^\mu + 2p'^\mu) - 2y(1+y)m(-\not{p}'\gamma^\mu + 2p'^\mu) + 8mxp^\mu + 2x(1-x)q^2\gamma^\mu\\ &- 2xy(-\not{p}'\not{q} + \underbrace{2p'\cdot q}_{-q^2})\gamma^\mu + 2xm(2m\gamma^\mu - 2p^\mu) + 2(1+y)(1-x)m(2m\gamma^\mu - 2p^\mu)\\ &- 2(1+y)ym^2\gamma^\mu + 2(1+y)m^2\gamma^\mu - 4y\gamma^\mu(-\not{p}'\not{q} + \underbrace{2q\cdot p'}_{-q^2})\\ &+ 2(4s_W^2-1)^2\Big\{x(1-x)(-q^2\gamma^\mu + \underbrace{2\not{q}q^\mu}_{vanishes}) - xy\not{q}(-\not{p}'\gamma^\mu + 2p'^\mu) + 2mx(2m\gamma^\mu - 2p^\mu)\\ &- (1-y)(1-x)m(-2m\gamma^\mu + 2p^\mu + 2q^\mu) + (1-y)ym(-\not{p}'\gamma^\mu + 2p'^\mu) - 2(1-y)m^2\gamma^\mu\\ &+ x(1-x)q^2\gamma^\mu - xy(-\not{p}'\not{q} + \underbrace{2p'\cdot q}_{-q^2})\gamma^\mu - xm(2m\gamma^\mu - 2p^\mu)\\ &- (1-y)(1-x)m(2m\gamma^\mu - 2p^\mu) + (1-y)ym^2\gamma^\mu + (1-y)m^2\gamma^\mu\Big\}.\end{aligned} \tag{3.105}$$

By collecting once more the terms proportional to γ^μ, and acting with the spinors, we get,

$$\begin{aligned}N^\mu =& \big\{\dots\big\}\gamma^\mu + 4(1+y)(1-x)m(p+q)^\mu + 2xy(-\not{p}'\not{q} + 2p'\cdot q)\gamma^\mu - \underbrace{4xy\not{q}p^\mu}_{\overline{u}(p')\not{q}u(p)=0}\\ &- 4y(1+y)mp'^\mu + 8mxp^\mu + 4xym\not{q}\gamma^\mu - 4xmp^\mu - 4(1+y)(1-x)mp^\mu\\ &+ 4y(-\not{p}'\gamma^\mu + 2p'^\mu)\not{q} + (4s_W^2-1)^2\big\{2xy(-\not{p}'\not{q} + 2q\cdot p')\gamma^\mu - \underbrace{4xy\not{q}p'^\mu}_{vanishes} + 8mxp^\mu\\ &- 4(1-y)(1-x)m(p+q)^\mu + 4(1-y)ymp'^\mu + 2xym\not{q}\gamma^\mu + 4xmp^\mu\\ &+ 4(1-y)(1-x)mp^\mu\big\}.\end{aligned} \tag{3.106}$$

Focusing now only on the p'^μ and p^μ terms:

$$\begin{aligned}
N^\mu =& p^\mu\big\{4mx - 4(4s_W^2-1)^2 mx\big\} + p'^\mu\big\{-4y(1+y)m + 4(4s_W^2-1)^2(1-y)ym\big\}\\
&+ q^\mu\big\{4(1+y)(1-x)m - 4(4s_W^2-1)^2(1-y)(1-x)m\big\}\\
&+ 4xymp^\mu - 8xymp^\mu - 8ymp^\mu - 8ymq^\mu + 4(4s_W^2-1)^2 xymp^\mu - 4(4s_W^2-1)^2 xymp^\mu\\
=& p^\mu\bigg\{4x\big[1-y-(4s_W^2-1)^2\big]m - 8ym\bigg\}\\
&+ p'^\mu\bigg\{-4y\big[1+y-(4s_W^2-1)^2(1-y)\big]m\bigg\}\\
&+ q^\mu\bigg\{4(1-x)\big[1+y-(4s_W^2-1)^2(1-y)\big]m - 8ym\bigg\},
\end{aligned} \tag{3.107}$$

and by substituting $q = p' - p$, we get:

$$\begin{aligned}
N^\mu =& p'^\mu\bigg\{4(1-x-y)\big[1+y-(4s_W^2-1)^2(1-y)\big]m - 8ym\bigg\}\\
&+ p^\mu\bigg\{4(4s_W^2-1)^2\big[(1-x)(1-y)-x\big]m - 4(1+y)m\bigg\}
\end{aligned} \tag{3.108}$$

By generalizing the above factors of p and p',

$$\begin{aligned}
&4m\big[R_1(x,y)+R_2(x,y)\big]p'^\mu + 4m\big[R_1(x,y)-R_2(x,y)\big]p^\mu =\\
&= 4mp'^\mu\bigg\{(1-x-y)\big[1+y-(4s_W^2-1)^2(1-y)\big] - 2y\bigg\}\\
&\quad + 4mp^\mu\bigg\{(4s_W^2-1)^2\big[(1-x)(1-y)-x\big] - (1+y)\bigg\},
\end{aligned} \tag{3.109}$$

we solve the following system of equations:

$$\begin{aligned}
R_1(x,y)+R_2(x,y) &= (1-x-y)\big[1+y-(4s_W^2-1)^2(1-y)\big] - 2y,\\
R_1(x,y)-R_2(x,y) &= -(1+y) + (4s_W^2-1)^2\big[(1-x)(1-y)-x\big],
\end{aligned} \tag{3.110}$$

in order to reform the numerator in the following way

$$N^\mu = R_1(x,y)(p+p')^\mu + R_2(x,y)q^\mu. \tag{3.111}$$

The $R_1(x,y)$ parameter is found to be

$$R_1(x,y) = \frac{1}{2}\bigg\{(1-x-y)\big[1+y-(4s_W^2-1)^2(1-y)\big] - 1 - 3y + (4s_W^2-1)^2\Big[(1-x)(1-y)-x\Big]\bigg\} \tag{3.112}$$

and the $R_2(x,y)q^\mu$ term vanishes due to the Ward identity. Substituting $R_1(x,y)$ in the numerator and utilizing the Gordon identity to complete the form of the magnetic form factor, the integral gives the following result:

$$\delta F_M^Z(q^2=0) = \frac{\sqrt{2}G_F m^2}{16\pi^2}\frac{(-1+4\sin^2\theta_w)^2-5}{3}. \tag{3.113}$$

The value for the muon lepton is approximately

$$\delta F_M^Z(q^2=0) \simeq -193.90(1)\times 10^{-11}. \tag{3.114}$$

4 See-saw lepton masses and muon $g-2$ from heavy vector-like leptons

In this chapter we review the model suggested in [3], where a vector-like lepton can be used to explain the small muon mass by a see-saw mechanism, based on lepton-specific two Higgs doublet models with a local $U(1)'$ symmetry. The radiative contributions that we calculated in the previous chapter can be used and applied here, by switching the gauge couplings to the ones introduced in this model.
A singlet $SU(2)_L$ vector-like lepton, E, is introduced, with charge -2 under the $U(1)'$ gauge symmetry. From now on we use the notation $particle\big(U(1)'charge\big)$, where as mentioned above $E(-2)$. There is also a dark Higgs field $\phi(-2)$ and a leptophilic Higgs doublet $H'(+2)$. It is assumed that the SM Higgs doublet H and the SM fermions are neutral under the $U(1)'$. In order to be consistent with suppressed Flavor Changing Neutral Currents (FCNCs) and obtain lepton masses from the VEV of the leptophilic Higgs doublet, a Z_2 parity is also imposed on the SM, and new fields as in Type-X (or lepton-specific) two Higgs doublet models (2HDM). The assignments for $U(1)'$ charges and Z_2 parities are given in the table below.

$U(1)'$ charges , Z_2 parities										
	q_L	u_R	d_R	l_L	l_R	H	H'	E_L	E_R	ϕ
$U(1)'$	0	0	0	0	0	0	+2	-2	-2	-2
Z_2	+	-	-	+	+	-	+	+	+	+

Table 1: $U(1)'$ charges and Z_2 parities of the particles.

The Lagrangian for the SM Yukawa couplings including Z', dark Higgs ϕ and the vector-like lepton is

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}\sin\xi F'_{\mu\nu}B^{\mu\nu} + |D_\mu\phi|^2 + |D_\mu H'|^2 - V\big(\phi, H, H'\big) + \mathcal{L}_{VLSM} \quad (4.1)$$

with

$$\mathcal{L}_{VLSM} = -y_d\bar{q}_L d_R H - y_u\bar{q}_L u_R\tilde{H} - M_E\overline{E}E - \lambda_E\phi\overline{E}_L l_R - y_E\bar{l}_L E_R H' + h.c. \quad (4.2)$$

Here, $\tilde{H} = i\sigma^2 H^*$, $F'_{\mu\nu} = \partial_\mu Z'_\nu - \partial_\nu Z'_\mu$, $B_{\mu\nu}$ is the field strength tensor for the SM hypercharge and the covariant derivatives

$$\begin{aligned} D_\mu\phi &= \big(\partial_\mu + 2ig_{Z'}Z'_\mu\big)\phi, \\ D_\mu H' &= \left(\partial_\mu - 2ig_{Z'}Z'_\mu - \frac{1}{2}ig_Y B_\mu - \frac{1}{2}ig\tau^i W^i_\mu\right)H'. \end{aligned} \quad (4.3)$$

The scalar potential $V\big(\phi, H, H'\big)$ for the singlet scalar ϕ, the leptophilic Higgs H', and the SM Higgs H, is given by

$$\begin{aligned} V\big(\phi, H, H'\big) =& \mu_1^2 H^\dagger H + \mu_2^2 H'^\dagger H' + \big(\mu_3 \phi H^\dagger H' + h.c.\big) \\ &+ \lambda_1 \big(H^\dagger H\big)^2 + \lambda_2 \big(H'^\dagger H'\big)^2 + \lambda_3 \big(H^\dagger H\big)\big(H'^\dagger H'\big) \\ &+ \mu_\phi^2 \phi^* \phi + \lambda_\phi \big(\phi^* \phi\big)^2 + \lambda_{H\phi} H^\dagger H \phi^* \phi + \lambda_{H'\phi} H'^\dagger H' \phi^* \phi. \end{aligned} \tag{4.4}$$

It is noted here that the quartic couplings for the Higgs doublets in 2HDMs are constrained, due to the fact that the second Higgs doublet $H'(+2)$ is charged under the $U(1)'$ symmetry, and the mixing mass term between the two Higgs doublets is generated after the $U(1)'$ symmetry is broken. In this model, the physical lepton masses and the lepton Yukawa couplings to the SM Higgs can be generated correctly due to the mixing with the vector-like lepton, via the see-saw mechanism for leptons. In general, one can introduce one vector-like lepton per generation for lepton masses without inducing the mixings between leptons.

Now, working out the symmetry breaking of the $U(1)'$ and the Electroweak symmetry, we find for $\langle H \rangle = \frac{1}{\sqrt{2}} \upsilon_1$, $\langle H' \rangle = \frac{1}{\sqrt{2}} \upsilon_2$, $\langle \phi \rangle = \upsilon_\phi$, the masses of the gauge bosons:

$$\begin{aligned} m_{Z'}^2 &= g_{z'}^2 \big(8\upsilon_\phi^2 + 4\upsilon_2^2\big), \qquad (\text{Appendix A} \to 4\upsilon_\phi^2) \\ m_Z &= \frac{1}{2}\sqrt{g^2 + g_Y^2}\,\upsilon, \\ m_W &= \frac{1}{2} g \upsilon, \end{aligned} \tag{4.5}$$

with $\upsilon = \sqrt{\upsilon_1^2 + \upsilon_2^2}$. I note here that my calculations (see Appendix A) show that $m_{Z'}^2 = g_{z'}^2 \big(4\upsilon_\phi^2 + 4\upsilon_2^2\big)$. An important thing to mention is that the VEV of the SM Higgs doublet H leads to quark masses and mixings, while the VEV of the extra Higgs doublet H' leads to the mixing between the SM leptons and the vector-like lepton. Due to the gauge kinetic mixing and the nonzero $U(1)'$ charge of the leptophilic Higgs, there is a mass mixing between Z and Z' gauge bosons, which must be suppressed to satisfy the electroweak precision data and the collider bounds.

A see-saw mechanism is used for generating small masses for charged leptons through the vector-like lepton. It is also used to identify the gauge and Yukawa interactions for the vector-like lepton. The muon is the lepton that mixes with the vector-like lepton in this model, and the mass terms for the lepton sector are the following:

$$\mathcal{L}_{L,mass} = -M_E \overline{E} E - (m_R \overline{E}_L l_R + m_L \bar{l}_L E_R + h.c.) \tag{4.6}$$

where $m_R = \lambda_E \upsilon_\phi$ and $m_L = \frac{1}{\sqrt{2}} y_E \upsilon_2$ are the mixing masses. After diagonalizing the mass matrix for leptons, the mass eigenvalues for leptons are,

$$m_{l_1,l_2}^2 = \frac{1}{2}\bigg(M_E^2 + m_L^2 + m_R^2 \mp \sqrt{(M_E^2 + m_L^2 - m_R^2)^2 + 4m_R^2 M_E^2}\bigg). \tag{4.7}$$

The rotation matrices that were used for the right-handed and the left-handed leptons are given by,

$$\begin{pmatrix} l_L \\ E_L \end{pmatrix} = \begin{pmatrix} \cos\theta_L & \sin\theta_L \\ -\sin\theta_L & \cos\theta_L \end{pmatrix} \begin{pmatrix} l_{1L} \\ l_{2L} \end{pmatrix}, \\ \begin{pmatrix} l_R \\ E_R \end{pmatrix} = \begin{pmatrix} \cos\theta_R & \sin\theta_R \\ -\sin\theta_R & \cos\theta_R \end{pmatrix} \begin{pmatrix} l_{1R} \\ l_{2R} \end{pmatrix}, \tag{4.8}$$

with the mixing angles given by

$$\sin(2\theta_R) = \frac{2M_E m_R}{m_{l_2}^2 - m_{l_1}^2}, \\ \sin(2\theta_L) = \frac{m_L^2}{m_{l_1} m_{l_2}} \sin 2\theta_R. \tag{4.9}$$

The authors by approximating $m_R, m_L \ll M_E$, they identify

$$m_{l_1}^2 \approx \frac{m_R^2 m_L^2}{M_E^2}, \tag{4.10}$$

whereas the mass squared for the vector-like lepton becomes

$$m_{l_2}^2 \approx M_E^2 + m_L^2 + m_R^2. \tag{4.11}$$

Therefore, a see-saw mechanism is at work for generating small masses for charged leptons due to heavy vector-like leptons, and for this purpose we need both the electroweak symmetry breaking with the leptophillic Higgs doublet for $m_L \neq 0$ and the $U(1)'$ symmetry breaking for $m_R \neq 0$ at the same time. In terms of the Yukawa couplings, m_{l_1} can be written as follows,

$$m_{l_1} \approx \frac{\lambda_E y_E \upsilon_\phi \upsilon_2}{\sqrt{2} M_E}. \tag{4.12}$$

Choosing $m_{l_1} = m_\mu$ for the muon mass and setting the following perturbativity conditions on the Yukawa couplings, $\lambda_E < 1$, $y_E < 1$, the upper limit on the vector-like mass is

$$M_E \simeq \frac{\lambda_E y_E \upsilon_\phi \upsilon_2}{\sqrt{2} m_\mu} < 6700\text{GeV}\left(\frac{\upsilon_\phi \upsilon_2}{10^3\text{GeV}^2}\right). \tag{4.13}$$

Notice that the vector-like lepton can be decoupled from the weak scale, while generating the muon mass and satisfying the perturbativity conditions. Moreover, for $m_R, m_L \ll M_E$, the lepton mixing angles become

$$\sin(2\theta_R) \approx \frac{2m_{l_1}}{m_L}, \\ \sin(2\theta_L) \approx \frac{2m_L}{m_{l_2}}, \tag{4.14}$$

and in general we can parametrize the small mixing mass parameters by $m_L \approx (\theta_L/\theta_R)^{1/2}\sqrt{m_{l_1}m_{l_2}}$ and $m_R \approx (\theta_R/\theta_L)^{1/2}\sqrt{m_{l_1}m_{l_2}}$. The authors proceed with the construction of the gauge and Yukawa interactions of the model. I present here the effective interactions of lepton to Z' and weak bosons that the authors of [3] have constructed,

$$\begin{aligned}\mathcal{L}_{L,eff} =& -2g_{Z'}Z'_\mu\big[c_R^2\overline{E}\gamma^\mu P_R E + s_R^2\bar{l}\gamma^\mu P_R l - s_R c_R(\overline{E}\gamma^\mu P_R l + \bar{l}\gamma^\mu P_R E)\\ &\quad + c_L^2\overline{E}\gamma^\mu P_L E + s_L^2\bar{l}\gamma^\mu P_L l - s_L c_L(\overline{E}\gamma^\mu P_L l + \bar{l}\gamma^\mu P_L E)\big]\\ &+ \frac{g}{2c_W}Z_\mu(\upsilon_l + \alpha_l)\big[(c_L^2 - 1)\bar{l}\gamma^\mu P_L l + s_L c_L(\overline{E}\gamma^\mu P_L l + \bar{l}\gamma^\mu P_L E) + s_L^2\overline{E}\gamma^\mu P_L E\big]\\ &+ \frac{g}{2c_W}Z_\mu(\upsilon_l - \alpha_l)\big[\overline{E}\gamma^\mu P_R E + c_L^2\overline{E}\gamma^\mu P_L E + s_L^2\bar{l}\gamma^\mu P_L l - s_L c_L(\overline{E}\gamma^\mu P_L l + \bar{l}\gamma^\mu P_L E)\big]\\ &+ \frac{g}{\sqrt{2}}W_\mu^-\left[c_L\bar{l}\gamma^\mu P_L\nu + s_L\overline{E}\gamma^\mu P_L\nu\right] + h.c. + \mathcal{L}_{L,\xi},\end{aligned} \tag{4.15}$$

with $s_{L/R} = \sin\theta_{L/R}$, $c_{L/R} = \cos\theta_{L/R}$, $\upsilon_l = \frac{1}{2}(-1+4s_W^2)$ and $\alpha_l = -\frac{1}{2}$. $\mathcal{L}_{L,\xi}$ contains the extra couplings due to gauge kinetic mixing, given by

$$\begin{aligned}\mathcal{L}_{L,\xi} =& Z'_\mu\left[e\xi c_\zeta c_W\bar{l}\gamma^\mu l + \frac{e}{2c_W s_W}(s_\zeta - t_\xi c_\zeta s_W)\big(\bar{l}\gamma^\mu(\upsilon_l - \alpha_l\gamma^5)l + \overline{\nu}\gamma^\mu P_L\nu\big)\right]\\ &+ \frac{e}{2c_W s_W}(c_\zeta - t_\xi s_\zeta s_W)Z_\mu\left[\bar{l}\gamma^\mu(\upsilon_l - \alpha_l\gamma^5)l + \overline{\nu}\gamma^\mu P_L\nu\right],\end{aligned} \tag{4.16}$$

where $c_\zeta = \cos\zeta$, $s_\zeta = \sin\zeta$, $t_\xi = \tan\xi$, and the mixing angle ζ between Z and Z' gauge bosons is given by

$$\tan(2\zeta) = \frac{2m_{12}^2(m_{Z_2}^2 - m_Z^2)}{(m_{Z_2}^2 - m_Z^2)^2 - m_{12}^4}, \tag{4.17}$$

with m_Z being the Z-boson mass in the SM, m_{Z_2} being the mass eigenvalue for the Z'-like gauge boson, and m_{12}^2 being the mixing mass,

$$m_{12}^2 = \frac{m_Z^2 s_W}{c_\xi}\left(s_\xi - \frac{4g_{Z'}\upsilon_2^2}{g_Y\upsilon^2}\right). \tag{4.18}$$

Eqs(4.15)-(4.18) are based on the papers [5], [6]. The paper continues with the Yukawa interactions of the model. Expanding the scalar fields around their VEVs we get,

$$H = \begin{pmatrix}\phi_1^+\\ \frac{1}{\sqrt{2}}(\upsilon_1 + \rho_1 + i\eta_1)\end{pmatrix}, H' = \begin{pmatrix}\phi_2^+\\ \frac{1}{\sqrt{2}}(\upsilon_2 + \rho_2 + i\eta_2)\end{pmatrix} \tag{4.19}$$

and

$$\phi = \upsilon_\phi + \frac{1}{\sqrt{2}}(\varphi + ia). \tag{4.20}$$

The two would-be neutral Goldstone bosons, G, G', and the CP-odd scalar A can be identified as

$$
\begin{aligned}
G =& \cos\beta\eta_1 + \sin\beta\eta_2, \\
G' =& \frac{1}{\sqrt{2\upsilon_\phi^2+\upsilon_2^2}}\left(\sqrt{2}\upsilon_\phi a - \upsilon\sin\beta\eta_2\right), \\
A =& N_A\left(\sin\beta\eta_1 - \cos\beta\eta_2 - \frac{\upsilon}{\sqrt{2}\upsilon_\phi}\sin\beta\cos\beta a\right)
\end{aligned}
\tag{4.21}
$$

with

$$
N_A = \frac{1}{\sqrt{1+\upsilon^2\sin^2\beta\cos^2\beta/(2\upsilon_\phi^2)}}. \tag{4.22}
$$

Here, $\upsilon_1 = \upsilon\cos\beta$, $\upsilon_2 = \upsilon\sin\beta$. Ignoring the mixing between the dark Higgs φ and $\rho_{1,2}$, we also obtain the mass eigenstates for CP-odd scalars, h and H, as

$$
\begin{aligned}
h =& \cos\alpha\rho_1 + \sin\alpha\rho_2, \\
H =& -\sin\alpha\rho_1 + \cos\alpha\rho_2
\end{aligned}
\tag{4.23}
$$

where α is the mixing angle between CP-even scalars. The would-be charged Goldstone boson G^+ and the charged Higgs H^+ are

$$
\begin{aligned}
G^+ =& \cos\beta\phi_1^+ + \sin\beta\phi_2^+, \\
H^+ =& \sin\beta\phi_1^+ - \cos\beta\phi_2^+.
\end{aligned}
\tag{4.24}
$$

Then, the authors proceed with the construction of the Yukawa interactions, where the Yukawa couplings depend on the values of the mixing angles α and β, and the model is complete in order to perform one-loop calculations on the anomalous magnetic moment. In this model, the vector-like lepton, Z' gauge boson as well as extra scalars contribute to the muon $g-2$ at one loop, as follows

$$
\Delta\alpha_\mu = \Delta\alpha_\mu^{Z',E} + \Delta\alpha_\mu^{Z,E} + \Delta\alpha_\mu^{Z',\mu} + \Delta\alpha_\mu^{h,E} + \Delta\alpha_\mu^{h,\mu} + \Delta\alpha_\mu^{H^-}. \tag{4.25}
$$

Assuming the approximations $m_{l_1} \approx m_\mu$ and $m_{l_2} \approx M_E$, we get the following contributions:

$$
\begin{aligned}
\Delta\alpha_\mu^{Z',E} &\approx \begin{cases} \frac{g_{Z'}^2 M_E m_\mu}{16\pi^2 m_{Z'}^2}(c_V^2 - c_A^2), & M_E \gg m_{Z'}, \\ \frac{g_{Z'}^2 M_E m_\mu}{4\pi^2 m_{Z'}^2}(c_V^2 - c_A^2), & m_\mu \ll M_E \ll m_{Z'}, \end{cases} \\
\Delta\alpha_\mu^{Z',\mu} &\approx \frac{g_{Z'}^2}{12\pi^2 m_{Z'}^2}(\upsilon_\mu'^2 - 5\alpha_\mu'^2), \quad m_{Z'} \gg m_\mu, \\
\Delta\alpha_\mu^{h,E} &\approx \frac{m_\mu^2}{48\pi^2 M_E^2}\left[|\upsilon_i^E|^2 + |\alpha_i^E|^2 + \frac{3M_E}{m_\mu}(|\upsilon_i^E|^2 - |\alpha_i^E|^2)\right], \quad M_E \gg m_{h_i}, \\
\Delta\alpha_\mu^{h,E} &\approx \frac{m_\mu^2}{24\pi^2 m_{h_i}^2}\left[|\upsilon_i^E|^2 + |\alpha_i^E|^2 + \frac{3M_E}{m_\mu}(|\upsilon_i^E|^2 - |\alpha_i^E|^2)\left(\ln\frac{m_{h_i}^2}{M_E^2} - \frac{3}{2}\right)\right], \quad M_E \ll m_{h_i}, \\
\Delta\alpha_\mu^{H^-} &\approx -\frac{m_\mu^2}{24\pi^2 m_{H^-}^2}|\upsilon_{H^-}|^2.
\end{aligned}
\tag{4.26}
$$

Choosing specific values for the parameters, we plot the contributions:

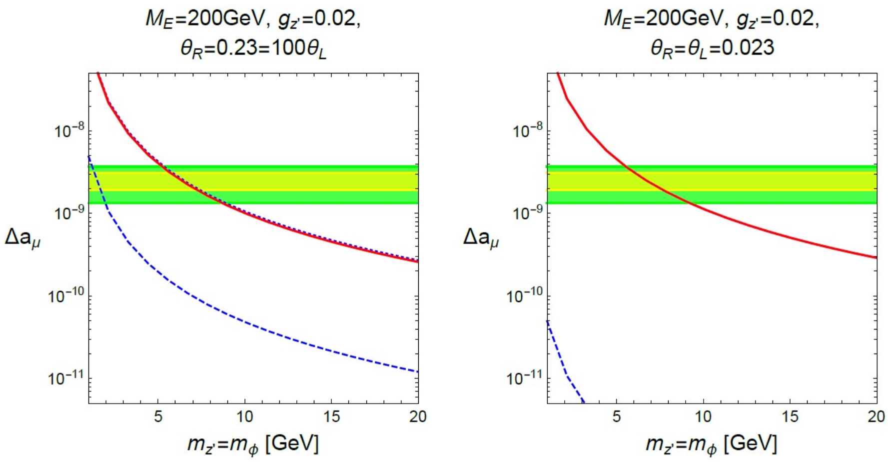


Figure 1: The one-loop contribution from Z' and vector-like lepton, the one-loop contribution from the dark Higgs φ and the vector-like lepton, and the combined one-loop results, are shown in blue dotted, blue dashed, red solid line, respectively. The yellow (green) bands indicate the deviations of the muon $g-2$ from the SM value within $1\sigma(2\sigma)$. $\theta_R = 0.23 = 100\theta_L$ were taken on left and $\theta_R = \theta_L = 0.023$ on right. For both plots, $M_E = 200GeV$, $g_{Z'} = 0.02$, $m_{Z'} = m_\varphi$ and $\upsilon_\varphi = \frac{m_{Z'}}{2\sqrt{2}g_{Z'}}$ were chosen.

In Figure 1, the Z' contributes positively to Δa_μ because the vectorial coupling is larger than the axial coupling in this model. On the other hand, the dark Higgs

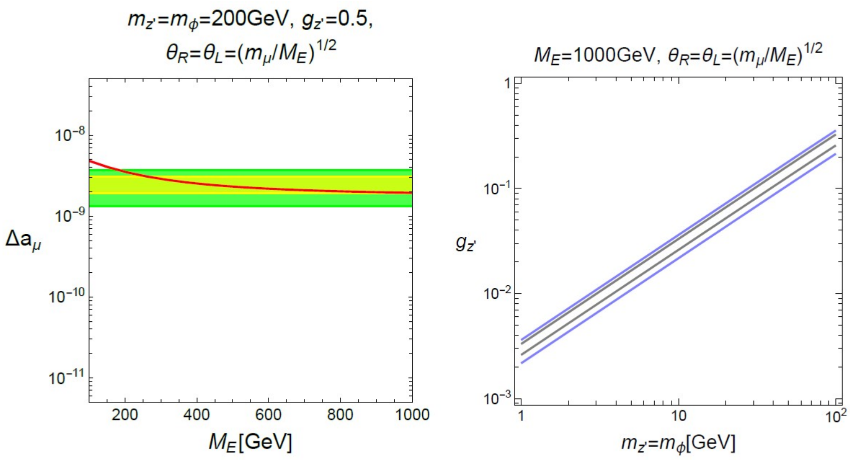


Figure 2: (Left) Δa_μ as a function of M_E in red line, in comparison to $1\sigma(2\sigma)$ bands for the deviation of the muon $g-2$ in yellow(green) for fixed values $m_{Z'} = m_\phi = 200GeV$ and $g_{Z'} = 0.5$, and the vector-like mixing angles to $\theta_R = \theta_L = \sqrt{m_\mu/M_E}$. (Right) The new contribution to the muon $g-2$ in the parameter space for $m_{Z'} = m_\phi$ versus $g_{Z'}$ within $1\sigma(2\sigma)$, shown between black(blue) lines, taking $M_E = 1000GeV$ and $\theta_R = \theta_L = \sqrt{m_\mu/M_E}$.

contributes negatively to Δa_μ, because the pseudo-scalar coupling is larger than the scalar coupling. For $m_{Z'} = m_\phi$ and $g_{Z'} = 0.02$, namely, $\lambda_\phi \simeq 2g_{Z'}^2 = 0.0008$ for $v_2 \ll v_\phi$, and $v_\phi = m_{Z'}/(2\sqrt{2}g_{Z'}) = 18m_{Z'}$, the Z' and dark Higgs masses around $5-8GeV$ are favored to explain the experimental value of the muon $g-2$. For either $\theta_R \gg \theta_L$ or $\theta_R = \theta_L$, the one-loop corrections with Z' and vector-like lepton contribute dominantly to the muon $g-2$. In Figure 2, for both plots, Z' loops with vector-like lepton give rise to a dominant contribution to the muon $g-2$ and become independent of vector-like lepton masses for $M_E \gg m_{Z'} = m_\phi$.

The one-loop diagrams with Z' and the vector-like lepton contribute to the branching ratios of $\mu \to e\gamma$, $\tau \to \mu\gamma$ and $\tau \to e\gamma$ as well, considering simultaneous mixings of the vector-like lepton with muon and other leptons. Furthermore, indirect constraints on the vector-like lepton come from electroweak precision data and Higgs data. The 2σ deviation of the experimental value of the ρ parameter from the theoretical value of SM strongly constrains the gauge couplings of the vector-like lepton and the Z' interactions. The modifications on the ρ parameter that come from the vector-like lepton and the Z' gauge boson, depend on the M_E and $m_{Z'}$ masses respectively, while also from the values of the mixing angles. For a heavy vector-like lepton with $M_E \gtrsim m_Z$, in order to satisfy the current bound on $\Delta\rho$, it is suffcient to choose either $|\sin\xi| \lesssim 10^{-2}$ and $\sin\beta \lesssim 0.1\sqrt{g_Y/g_{Z'}}$ for $m_{Z'} \ll m_Z$ unless there is a cancellation.

The bounds are less severe for $m_{Z'} \gg m_Z$, due to the overall suppression by $m_Z^2/m_{Z'}^2$.

Regarding the particle production at the colliders, the vector-like lepton can be produced by the Drell-Yann processes with off-shell γ^* and Z^* in the s-channels at LEP and LHC. Since the vector-like lepton is an $SU(2)_L$ singlet in this model, the Drell-Yann production cross section with Z^* is suppressed by $\sin^4\theta_w$. The decay channels for each particle depend on the values of their masses. For instance, for $M_E > m_Z, M_{Z'}$ the VL lepton can decay by $E \to W\nu, Zl, Z'l, \varphi l$ while Z' decays dominantly into a pair of muon and anti-muon, and since φ has comparable mass as the Z' mass, it decays into a pair of muon and anti-muon as well. If $M_E < m_{Z'} < 2M_E$, the Z' gauge boson can decay by $Z' \to E\bar{l}, \overline{E}l$, becoming dominant decay channels. In this case, for $E \to Zl$, there can be at least two leptons in the final state from the Z' decay. For $m_{Z'} > 2ME$, the Z' gauge boson can decay dominantly by $Z' \to \overline{E}E$, leading to at least four leptons in the final state.

5 $R_{K^{(*)}}$ and the origin of Yukawa couplings

The beyond Standard Model scenarios that include extra Z' bosons after the spontaneous symmetry breaking of $U(1)'$ symmetries, can also be studied and developed in ways that explain the origin of the Yukawa couplings. In this chapter, I review one of the many models that exist in the literature. The author of the paper [9] considers the case of an additional vector-like fourth family and also induce flavourful Z' couplings. The couplings of the SM generations mix with the fourth vector-like family, and this leads to the production of the Yukawa couplings.

5.1 Introduction

The violation of $\mu - e$ universality in semi-leptonic B decays has motivated the theoretical physics community to consider non-universal Z' models. Regarding the measurement of $R_{K^{(*)}}$, a number of phenomenological analyses of these data, favour a new physics operator of the form $\overline{b}_L\gamma^\mu s_L\overline{\mu}_L\gamma_\mu\mu_L$, or of the form, $\overline{b}_L\gamma^\mu s_L\overline{\mu}\gamma_\mu\mu$, each with a coefficient Λ^{-2} where $\Lambda \sim 31.5$TeV, or some linear combination of these two operators. For example, in a flavourful Z' model, the new physics operator will arise from tree-level Z' exchange, where the Z' must dominantly couple to $\mu\mu$ over ee, and must also have the quark flavour changing coupling b_Ls_L which must dominate over b_Rs_R.
The author investigates the possible connection between the experimental signal for new physics in $R_{K^{(*)}}$ and the origin of fermion Yukawa couplings. The Standard Model is considered an effective theory at the electroweak scale, resulting from some theory at some higher scale(s) which may be as low as the TeV scale. All fermion Yukawa couplings must result from higher dimension operators so that the effective Yukawa couplings of the SM can be expressed in terms of the left-handed fermion electroweak doublets $\psi_i = L_i, Q_i$, where $i = 1, 2, 3$, and the CP-conjugated right-handed electroweak singlets $\psi^c_j = u^c_j, d^c_j, e^c_j, \nu^c_j$

$$\mathcal{L}^{Yuk}_{eff} = \left(\frac{\langle\phi_i\rangle}{\Lambda^{\psi}_{i,n}}\right)^n \left(\frac{\langle\phi_j\rangle}{\Lambda^{\psi^c}_{j,m}}\right)^m H\psi_i\psi^c_j + h.c. \tag{5.1}$$

Eq(5.1) involves new SM singlet fields ϕ_i which develop VEVs, leading to effective Yukawa couplings suppressed by powers of $\langle\phi_i\rangle/\Lambda$. This scenario also involves a massive Z' under which the three SM families ψ_i have zero charge, and which only couple to it via the same singlet fields ϕ_i which have non-zero charge under the associated $U(1)'$ gauge group,

$$\mathcal{L}^{Z'}_{eff} = \left(\frac{\langle\phi_i\rangle}{\Lambda'^{\psi}_{i,n}}\right)^n \left(\frac{\langle\phi_j\rangle}{\Lambda'^{\psi^c}_{j,m}}\right)^m g' Z'_\mu \psi^\dagger_i\gamma^\mu\psi_j + \left(\psi \to \psi^c\right), \tag{5.2}$$

where g' is the $U(1)'$ gauge coupling. The various Λ and Λ' may be simply related. The key feature of this scenario is that the same numerator factors of $\langle\phi_i\rangle$ control both the Yukawa couplings in Eq(5.1) and the Z' couplings in Eq(5.2). Another key feature of the above scenario is that the Z' is also generated by the VEVs of $\langle\phi_i\rangle$, so that $M_{Z'} \approx g'\langle\phi_i\rangle$.
In the scenario of Eqs(5.1, 5.2), in the limit that $\langle\phi_i\rangle = 0$, there are no Yukawa couplings and also no couplings of SM fermions to the Z' since it is assumed they are not charged under the associated $U(1)'$ gauge group. When $\langle\phi_i\rangle/\Lambda$ are switched on then Yukawa couplings and small non-universal and flavour dependent couplings of SM fermions to the Z' are generated simultaneously, as well as the Z' mass itself. The above framework provides a link between flavour changing observables and the origin of Yukawa couplings.

5.2 The Model

The model involves three chiral families $\psi_i(0)$, $\psi_i^c(0)$, plus a forth vector-like family consisting of $\psi_4(1)$, $\psi_4^c(1)$ plus the conjugate representations $\overline{\psi_4}(-1)$, $\overline{\psi_4^c}(-1)$, where the $U(1)'$ charges are shown in parentheses. The gauged $U(1)'$ is broken by the singlet scalars $\phi(1)$, with VEVs around the TeV scale, yielding a massive Z' at this scale. Since the Higgs doublets H(-1) are charged under the $U(1)'$, this forbids all Yukawa couplings, except those which couple the first three families to the fourth family.

$U(1)'$ charges								
	ψ_i	ψ_i^c	ψ_4	ψ_4^c	$\overline{\psi}_4$	$\overline{\psi^c}_4$	$H_{u,d}$	ϕ
$U(1)'$	0	0	1	1	-1	-1	-1	1

Table 2: $U(1)'$ charges of the particle spectrum.

The model here involves a gauged $U(1)'$ resulting in effective Yukawa and flavourful Z' couplings as in Eqs (5.1) and (5.2) which are related. A welcome consequence of this is that, it is not required an additional Z_2 symmetry to forbid renormalisable Yukawa couplings. Two Higgs doublets are required H_u, H_d, both with negative $U(1)'$ charge. The allowed renormalisable Yukawa couplings and explicit masses allowed by $U(1)'$ are,

$$\mathcal{L}^{ren} = y_{i4}^{\psi} H\psi_i\psi_4^c + y_{4i}^{\psi} H\psi_4\psi_i^c + x_i^{\psi}\phi\psi_i\overline{\psi_4} + x_i^{\psi^c}\phi\psi_i^c\overline{\psi_4^c} + M_4^{\psi}\psi_4\overline{\psi_4} + M_4^{\psi^c}\psi_4^c\overline{\psi_4^c} \quad (5.3)$$

plus h.c., where x, y are dimensionless coupling constants ideally of order unity, while M are explicit mass terms of order a few TeV.

5.3 Mass insertion approximation

Although the usual Yukawa couplings $y_{ij}^{\psi} H \psi_i \psi_j^c$ are forbidden for $i, j = 1, 2, 3$ (since H are charged under $U(1)'$) effective 3×3 Yukawa couplings may be generated by the two mass insertion diagrams (up to an irrelevant sign),

$$\mathcal{L}_{eff}^{Yuk} = \frac{x_j^{\psi^c} \langle \phi \rangle}{M_4^{\psi^c}} y_{i4}^{\psi} H \psi_i \psi_j^c + \frac{x_i^{\psi} \langle \phi \rangle}{M_4^{\psi}} y_{4j}^{\psi} H \psi_i \psi_j^c + h.c. \tag{5.4}$$

The model also involves a massive Z' under which the three SM families ψ_i, ψ_i^c have zero $U(1)'$ charge. Although the usual Z' couplings $g' Z'_{\mu} \psi_i^{\dagger} \gamma^{\mu} \psi_j$ are forbidden for $i, j = 1, 2, 3$, the fourth vector-like family has non-zero $U(1)'$ charge, and effective Z' couplings may be generated by the two mass insertion diagrams,

$$\mathcal{L}_{eff}^{Z'} = \frac{x_i^{\psi} \langle \phi \rangle}{M_4^{\psi}} \frac{x_j^{\psi} \langle \phi \rangle}{M_4^{\psi}} g' Z'_{\mu} \psi_i^{\dagger} \gamma^{\mu} \psi_j + \frac{x_i^{\psi^c} \langle \phi \rangle}{M_4^{\psi^c}} \frac{x_j^{\psi^c} \langle \phi \rangle}{M_4^{\psi^c}} g' Z'_{\mu} \psi_i^{c\dagger} \gamma^{\mu} \psi_j^c. \tag{5.5}$$

In this effective theory, Yukawa and Z' couplings are both controlled by the same physics, in this case the VEVs $\langle \phi \rangle$ and the fourth family vector-like masses M_4^{ψ} and $M_4^{\psi^c}$. The mass of the Z' is given by $M_{Z'} = g' \langle \phi \rangle$, which is the same scale at which the Yukawa couplings are generated. While the Yukawa couplings are generated at first order, the Z' couplings are generated at second order in the mass insertion approximation.
There is a such a Yukawa matrix as in Eq (5.4) for each of the four charged sectors $\psi = u, d, e, \nu$. In the case of neutrinos, this refers to the Dirac Yukawa matrix, and there will be a further Majorana mass matrix for the singlet neutrinos $M_{ij}^{\nu^c} \nu_i^c \nu_j^c$. Since nothing prevents the Majorana masses $M_{ij}^{\nu^c}$ being arbitrarily large, well above the $U(1)'$ breaking scale, this will lead to a conventional seesaw mechanism for small neutrino masses. On the other hand it is assumed that the vector-like masses M_4^{ψ} and $M_4^{\psi^c}$ to be close to the $U(1)'$ breaking scale of order the TeV scale.

5.4 The 5×5 Matrix and the $R_{K^{(*)}}$ anomaly

Since the large top quark Yukawa coupling y_t is not present at renormalisable level, it must also arise from mixing with the fourth vector-like family. In the case $y_t \sim y_{33}^u \sim 1$, the mass insertion approximation breaks down, and that motivates us to go beyond this approximation. The masses and couplings in Eq(5.3) are arranged into

5×5 matrices, one for each sector $\psi = u, d, e, \nu$,

$$M^{\psi} = \begin{pmatrix} 0 & 0 & 0 & y^{\psi}_{14}H & x^{\psi}_{1}\phi \\ 0 & 0 & 0 & y^{\psi}_{24}H & x^{\psi}_{2}\phi \\ 0 & 0 & 0 & y^{\psi}_{34}H & x^{\psi}_{3}\phi \\ y^{\psi}_{41}H & y^{\psi}_{42}H & y^{\psi}_{34}H & 0 & M^{\psi}_{4} \\ x^{\psi^c}_{1}\phi & x^{\psi^c}_{2}\phi & x^{\psi^c}_{3}\phi & M^{\psi^c}_{4} & 0 \end{pmatrix} \tag{5.6}$$

in the $\left(\psi_1, \psi_2, \psi_3, \psi_4, \overline{\psi^c_4}\right)$ basis (rows) and $\left(\psi^c_1, \psi^c_2, \psi^c_3, \psi^c_4, \overline{\psi_4}\right)$ (columns). There are three distinct mass scales in these matrices: the Higgs VEVs $\langle H \rangle$, the ϕ VEVs $\langle \phi \rangle$ and the vector-like fourth family masses M^{ψ}_4, $M^{\psi^c}_4$. If all these mass scales are of the same order then the correct procedure is to diagonalise the full 5×5 matrices in each of the charge sectors (apart from neutrinos which must be treated differently due to the Majorana masses and the seesaw mechanism). Then unitary violation will play a role. In the approximation $\langle H \rangle \ll \langle \phi \rangle$ (physically $M_Z \ll M_{Z'}$), it will not be necessary to diagonalise the full matrix in one step.
The author then proceeds with the transformation of the 5×5 matrix into a convenient basis for quarks, while considering some restrictions on the x and y parameters. After some manipulations, the effective 3×3 Yukawa matrices for the quarks are obtained from the 5×5 matrices, and then considers a basis where one can decouple the heavy fourth family. We present here some of the steps that were followed and studied. Choosing these particular values for the quark couplings $x^Q_{1,2} = 0$, $y^u_{41,42} = 0$ and $y^d_{41,42} = 0$ and then rotating the first and second families to set $x^{u^c}_1 = 0$, $x^{d^c}_1 = 0$, and $y^u_{14} = 0$, the effective 3×3 Yukawa matrices for the quarks $y^u_{ij} H_u Q_i u^c_j$, $y^d_{ij} H_d Q_i d^c_j$ obtained from the 5×5 matrices are given by

$$\begin{aligned} y^u_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & y^u_{24}x^{u^c}_2 & y^u_{24}x^{u^c}_3 \\ 0 & y^u_{34}x^{u^c}_2 & y^u_{34}x^{u^c}_3 \end{pmatrix} \frac{\langle \phi \rangle}{M^{u^c}_4} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x^Q_3 y^u_{43} \end{pmatrix} \frac{\langle \phi \rangle}{M^Q_4}, \\ y^d_{ij} &= \begin{pmatrix} 0 & y^d_{14}x^{d^c}_2 & y^d_{14}x^{d^c}_3 \\ 0 & y^d_{24}x^{d^c}_2 & y^d_{24}x^{d^c}_3 \\ 0 & y^d_{34}x^{d^c}_2 & y^d_{34}x^{d^c}_3 \end{pmatrix} \frac{\langle \phi \rangle}{M^{d^c}_4} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x^Q_3 y^d_{43} \end{pmatrix} \frac{\langle \phi \rangle}{M^Q_4}. \end{aligned} \tag{5.7}$$

In order to go beyond the mass insertion approximation $\langle \phi \rangle \ll M^{\psi}_4$ we return to the full 5×5 mass matrices, by assuming $\langle H \rangle \ll \langle \phi \rangle$ to switch off the Higgs VEVs all together in the first instance, and obtain an effective SM after integrating out the heavy fourth family. Starting from a basis in the following form:

$$M^{\psi} = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & & \tilde{y}'^{\psi}_{\alpha\beta} H & & 0 \\ & & & & \tilde{M}^{\psi}_4 \\ 0 & 0 & 0 & \tilde{M}^{\psi^c}_4 & 0 \end{pmatrix}, \tag{5.8}$$

the $\tilde{y}'^{\psi}_{\alpha\beta}$ are the 4×4 upper block Yukawa matrices in this basis. The zeros in the fifth row and column achieved by rotating the first four families by the following 4×4 unitary transformations

$$V_Q = V^Q_{34} V^Q_{24} V^Q_{14}, \quad V_{u^c} = V^{u^c}_{34} V^{u^c}_{24} V^{u^c}_{14}, \quad V_{d^c} = V^{d^c}_{34} V^{d^c}_{24} V^{d^c}_{14} \tag{5.9}$$

which are parametrized by a single angle θ_{i4} that describes the mixing between the ith chiral family and the 4th vector-like family. The SM Yukawa matrices correspond to the remaining 3×3 upper blocks of the 4×4 Yukawa matrices. The Yukawa matrices of the SM, $\tilde{y}'^{\psi}_{ij}$ correspond to the remaining 3×3 upper blocks of the 4×4 Yukawa matrices, $\tilde{y}'^{\psi}_{\alpha\beta}$. The three undecoupled families in this basis contain admixtures of the original fourth vector-like family due to the mixing. The 4×4 Yukawa couplings are

$$\tilde{y}'^{u}_{\alpha\beta} = V_Q \tilde{y}^u_{\alpha\beta} V^{\dagger}_{u^c}, \quad \tilde{y}'^{d}_{\alpha\beta} = V_Q \tilde{y}^d_{\alpha\beta} V^{\dagger}_{d^c} \tag{5.10}$$

and $\tilde{y}^u_{\alpha\beta}$, $\tilde{y}^d_{\alpha\beta}$ are identified with the 4×4 upper blocks of the transformed matrix in Eq(5.6), with the specific values that were chosen for the couplings in this basis. The effective SM Yukawa couplings for the quarks then correspond to the 3×3 upper blocks of $\tilde{y}'^{u}_{\alpha\beta}$ and $\tilde{y}'^{u}_{\alpha\beta}$:

$$y^u_{ij} H_u Q_i u^c_j, \quad y^d_{ij} H_d Q_i d^c_j, \text{ with } y^u_{ij} \equiv \tilde{y}'^{u}_{ij}, \quad y^d_{ij} \equiv \tilde{y}'^{d}_{ij}, (i,j = 1,2,3). \tag{5.11}$$

The effective SM Yukawa couplings have non-zero elements due to the mixing, even though originally they were all zero. This is the origin of flavour in the low energy effective SM theory. The analysis now may proceed depending on the phenomenology that one is interested in, followed by the respective approximation on the values of the mixing angles and the masses. We remark here that after the mixing with the fourth family, the three light families have induced non-universal and flavor violating couplings to the Z', which depend on the mixing angles of the 4×4 unitary matrices in Eq(5.9).

For closing remarks, the anomalies in $R_{K^{(*)}}$ can be explained by the b and s quark couplings to Z'. The CKM matrix for the quarks may be constructed in the usual way, by diagonalising the Yukawa matrices, to yield the 3×3 CKM matrix

$V_{CKM} = V_{u_L} V_{d_L}^\dagger$. Assuming that $M_4^Q \ll M_4^{d^c} \ll M_4^{u^c}$ implies that the CKM mixing originates predominantly from the down sector, thus $V_{CKM} \approx V_{d_L}^\dagger$. This means that the off-diagonal quark coupling is generated with $g_{bs} \approx g'(s_{34}^Q)^2 V_{ts}$. Combining this result with the effective 4-fermion operator with left-handed muon, b-quark and s-quark that was mentioned in the introduction of this chapter, the Z' coupling to bs leads to an additional tree-level contribution to $B_s - \overline{B}_s$ mixing due to the effective operator arising from Z' exchange at tree level:

$$\Delta\mathcal{L}_{eff} \supset -\frac{g_{bs}^2}{2M_{Z'}^2}\left(\overline{s}_L \gamma^\mu b_L\right)^2 + h.c. \tag{5.12}$$

If we take the milder $B_s - \overline{B}_s$ mixing bound then this constrains

$$\frac{|g_{bs}|}{M_{Z'}} \lesssim \frac{1}{150TeV} \tag{5.13}$$

and since g_{bs} is known in this model ($g_{bs} \sim -1/50$ for $s_{34}^Q \sim 1/\sqrt{2}$, $g' \sim 1$ and $V_{ts} \sim -0.04$), this leads to a lower bound on the Z' mass in this model

$$M_{Z'} \gtrsim 3TeV. \tag{5.14}$$

Since the Higgs doublets are charged under $U(1)'$, they will induce $Z - Z'$ mixing which will affect the SM prediction of M_W/M_Z, leading to corrections to the well determined parameter ρ.

6 A different approach to the origins of the Yukawa couplings

In this chapter, I present a thought of mine, that was born during the time period I was studying about the Yukawa couplings. As we can see from the theory of the SM, that was developed and has been confirmed over and over for the last fifty years, there is no mechanism that can produce terms which fermions can interact with the Higgs scalar boson, and one can simply add these terms into the Lagrangian. Physicists are familiar with the gauge symmetries, they study and apply them on a daily basis on their models. For instance, if we are dealing with a local $U(1)$ symmetry, the covariant derivative acting on a singlet scalar field ϕ, has the form:

$$D_\mu \phi = \big(\partial_\mu - igA_\mu\big)\phi \tag{6.1}$$

where A_μ is the massless gauge field that the theory needs in order to be gauge invariant under the $U(1)$ symmetry, and g the coupling constant. This seems to be a really effective and important way to introduce vector-fields in one theory, since after the spontaneous symmetry breaking of the symmetry, the vector field now has not only kinetic terms, but even interaction terms with the scalar boson (including potential mass term, however, in a symmetry like $SU(2)_L \times U(1)_Y$ the photon acquires no mass after SSB). That bears the question: "How do fermions enter the theory? Is there a way to produce/introduce fermions in the Lagrangian not by hand, but with a similar kind of mechanism?" Maybe the Yukawa couplings are generated from a different kind of symmetry breaking. Let's consider a scenario which the covariant derivative is given by,

$$D_\mu = \partial_\mu - i\big(g \cdot \rho(x)\big)\epsilon_\mu \tag{6.2}$$

where $(g \cdot \rho)\epsilon_\mu$ needs to transform as a vector field and ϵ_μ is an arbitrary four-vector. Notice that the second term in the above equation is similar to the second term in Eq(6.1). The only difference is that instead of introducing a pure vector-field in the covariant derivative, we introduce $(g \cdot \rho)\epsilon_\mu$ which plays the role of one. Now, we have to check the dimension of each quantity through dimensional analysis. Since $[D_\mu] = M$, this means that:

$$[D_\mu] = M \quad \Rightarrow \quad [g \cdot \rho\epsilon_\mu] \overset{!}{=} M. \tag{6.3}$$

Since ϵ_μ is a four-vector, its dimension is $[\epsilon_\mu] = M^{-1}$. At this point, we may assume that ρ^2 has dimension M, so if we denote $\rho^2 \equiv \tilde{\rho}$, then the $\tilde{\rho}$ field is a scalar field. This means that ρ has dimensions of a spinor, $[\rho] = M^{\frac{1}{2}}$, and in order for the Eq(6.3) to hold true, the constant g must have dimension of a spinor as well, $[g] = M^{\frac{3}{2}}$:

$$[\epsilon_\mu] = M^{-1}, \quad [\rho] = M^{\frac{1}{2}}, \quad [g] = M^{\frac{3}{2}}. \tag{6.4}$$

Since g and ρ are spinors, we introduce them to the covariant derivative as a spinor product of Weyl spinor components [13]:

$$g \cdot \rho \equiv g^a \rho_a = g^a \epsilon_{ab} \rho^b = -\rho^b \epsilon_{ab} g^a = \rho^b \epsilon_{ba} g^a = \rho \cdot g, \tag{6.5}$$

where ϵ_{ab} is the antisymmetric symbol which lowers/raises the spinor indices:

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1, \quad \epsilon^{11} = \epsilon^{22} = \epsilon_{11} = \epsilon_{22} = 0. \tag{6.6}$$

The complex conjugate of Eq(6.5) on the other hand is

$$(g \cdot \rho)^* = \rho^\dagger \cdot g^\dagger = g^\dagger \cdot \rho^\dagger. \tag{6.7}$$

In case this product does not make sense, since $[g] = M^{3/2}$ and $[\rho] = M^{1/2}$, we can easily introduce another parameter in order to acquire a dimension $M^{1/2}$, meaning that $[g] = M^{3/2} = M^{2/2+1/2}$. After introducing all these quantities, we return back to our Lagrangian. In order to determine the interactions of this theory under the $U(1)$ symmetry, one needs to calculate $|D_\mu \phi|^2$. The scalar field ϕ undergoes an SSB, and after acquiring a VEV, it takes the following form

$$\phi = \frac{1}{\sqrt{2}}(\upsilon + \varphi) \tag{6.8}$$

and the interactions as mentioned are produced from

$$|D_\mu \phi|^2_\rho = \frac{1}{2}\left[\partial_\mu + i(g \cdot \rho)^* \epsilon_\mu\right](\upsilon + \varphi)\left[\partial^\mu - i(g \cdot \rho)\epsilon^\mu\right](\upsilon + \varphi). \tag{6.9}$$

Notice that the mixed terms in this case will not cancel each other because g and ρ contain *Grassmann numbers* as components, and also there won't be any term proportional to $\rho^2 = \rho \cdot \rho = \tilde{\rho}$. So we need to make new assumptions about the elements of g and ρ. We may use Grassmann numbers for the elements such that $\rho^i = (\rho^i)^*$ and $g^i = (g^i)^*$. These particular Grassmann numbers are called *super-real.* Now, starting from the spinor product

$$g \cdot \rho = g^a \rho_a = g^a \epsilon_{ab} \rho^b = g^2 \rho^1 - g^1 \rho^2 \tag{6.10}$$

and since $(g \cdot \rho)^* = g \cdot \rho$, the square of the spinor product becomes:

$$\begin{aligned}
(g \cdot \rho)^2 &= (g^1 \rho^2 - g^2 \rho^1)(g^1 \rho^2 - g^2 \rho^1) \\
&= g^1 \rho^2 g^1 \rho^2 - g^1 \rho^2 g^2 \rho^1 - g^2 \rho^1 g^1 \rho^2 + g^2 \rho^1 g^2 \rho^1 \\
&= -\underbrace{g^1 g^1}_{=0} \underbrace{\rho^2 \rho^2}_{=0} + g^1 g^2 \rho^2 \rho^1 + g^2 g^1 \rho^1 \rho^2 - \underbrace{g^2 g^2}_{=0} \underbrace{\rho^1 \rho^1}_{=0} \\
&= g^1 g^2 (\rho^2 \rho^1 - \rho^1 \rho^2) \\
&= g^1 g^2 (\rho \cdot \rho).
\end{aligned} \tag{6.11}$$

This result can be rewritten in the following form

$$(g \cdot \rho)^2 = \tilde{g}^2 \tilde{\rho}(x) \tag{6.12}$$

where $\tilde{g}^2 = g^1 g^2$, $[\tilde{g}] = M^{\frac{3}{2}}$ and $[\tilde{\rho}] = M$. Let's substitute this result into $|D_\mu \phi|_\rho^2$ and calculate it correctly.

$$\begin{aligned} |D_\mu \phi|_\rho^2 &= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(g \cdot \rho)^2 \epsilon^2 (\upsilon + \varphi)^2 \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\tilde{g}^2 \epsilon^2) \tilde{\rho}(x)(\upsilon + \varphi)^2 \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}(\tilde{g}^2 \epsilon^2) \tilde{\rho}\big(\upsilon^2 + 2\upsilon\varphi + \varphi^2\big). \end{aligned} \tag{6.13}$$

Since $\tilde{\rho}$ has dimension of a scalar field, we may assume that, at a higher energy than the SSB of the $U(1)$, the $U(1)_\rho$ which $\tilde{\rho}$ is charged under, undergoes an SSB as well. In the unitary gauge, $\tilde{\rho}$ becomes:

$$\tilde{\rho} = \frac{1}{\sqrt{2}}(\tilde{\upsilon} + \lambda \overline{\psi}\psi), \tag{6.14}$$

where ψ is a Dirac (SM) fermion, $\tilde{\upsilon}$ is the VEV of $\tilde{\rho}$, and λ is a constant of dimension $[\lambda] = M^{-2}$. Finally, we have:

$$\begin{aligned} |(D_\mu \phi)_\rho|^2 &= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2\sqrt{2}}(\tilde{g}^2 \epsilon^2)(\tilde{\upsilon} + \lambda \overline{\psi}\psi)(\upsilon^2 + 2\upsilon\varphi + \varphi^2) \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} g'^2 \big[\underbrace{\tilde{\upsilon}\upsilon^2}_{constant} + 2\tilde{\upsilon}\upsilon\varphi + \tilde{\upsilon}\varphi^2 + \lambda \upsilon^2 \overline{\psi}\psi + 2\lambda\upsilon\overline{\psi}\psi\varphi + \lambda\overline{\psi}\psi\varphi^2 \big] \\ &= \frac{1}{2}(\partial_\mu \varphi)^2 + g'^2 \tilde{\upsilon}\upsilon\varphi + \frac{1}{2}\tilde{m}^2 \varphi^2 - m_\psi \overline{\psi}\psi - y_\psi \overline{\psi}\psi\varphi + \frac{1}{2} g'^2 \lambda \overline{\psi}\psi\varphi^2, \end{aligned} \tag{6.15}$$

where we have omitted the constant term from the Lagrangian. This theory seems to work, with the right dimensions of the quantities. As we can see, this mechanism provides us with a term that reminds us of the Yukawa coupling term in the SM:

$$-y_\psi \overline{\psi}\psi\varphi, \quad y_\psi = -g'^2 \lambda \upsilon \quad \text{(Yukawa coupling)}. \tag{6.16}$$

We also have the mass term of the fermion ψ:

$$-m_\psi \overline{\psi}\psi, \quad m_\psi = -\frac{1}{2} g'^2 \lambda \upsilon \quad \text{(mass of } \psi\text{)}. \tag{6.17}$$

Notice the minus sign that we need in order for these terms to be consistent with the SM Lagrangian. This means that λ needs to have negative values only, meaning that $\lambda < 0$. A source term appears as well, $g'^2 \tilde{\upsilon}\upsilon\varphi$, while also the term

$$\frac{1}{2} g'^2 \lambda \overline{\psi}\psi\varphi^2 \tag{6.18}$$

which could be interpreted as a new interaction, not included in the SM, a *Higgs-fermion scattering term*! Notice also that the Higgs gains an extra mass contribution due to $\tilde{\rho}$'s VEV:

$$\frac{1}{2}\tilde{m}^2\varphi^2, \quad \tilde{m} = g'^2\tilde{\upsilon} \quad \text{(extra mass contribution for } \varphi\text{)}. \tag{6.19}$$

The full Lagrangian would include the terms that provides the interactions of the scalar φ with the gauge boson of $U(1)$ as well:

$$\mathcal{L} = |D_\mu\phi|^2 + |D_\mu\phi|^2_\rho - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.20}$$

where $D_\mu\phi = (\partial_\mu - ig_A A_\mu)\phi$, $(D_\mu\phi)_\rho = (\partial_\mu - ig\cdot\rho\epsilon_\mu)\phi$, and $V(\phi)$ the scalar potential. Thus far we have achieved on generating the Yukawa couplings in the Abelian symmetry breaking, by imposing the existence of another exotic gauge field $g\cdot\rho\epsilon_\mu$. The product $\rho\cdot\rho$ provides us the scalar $\tilde{\rho}$ field, which its VEV breaks the $U(1)_\rho$ at a higher energy scale than the symmetry breaking of $U(1)$ by the VEV of ϕ. This new complex kind of SSB provides the theory with a fermion and its Yukawa interactions with φ as well. Since we have established this theoretical background, we are left with the kinetic and potential terms of the $\tilde{\rho}$ field. By definition, $\tilde{\rho} = \rho\cdot\rho$, and the square of this term is

$$(\rho\cdot\rho)^2 = (\rho^2\rho^1 - \rho^1\rho^2)^2 = (2\rho^2\rho^1)^2 = 0, \tag{6.21}$$

due to the property of the Grassmann numbers, their square is equal to zero. This automatically forbids the $\tilde{\rho}$ field to have kinetic terms! Furthermore, the Higgs potential contains quadratic and quartic terms, while on the other hand $V(\tilde{\rho})$ must have at most terms proportional to $|\tilde{\rho}|$. A probable exotic potential would be of the form of a continuous piecewise function:

$$V(\tilde{\rho}) = \begin{cases} V_0 & ,\tilde{\rho} = 0 \\ b|\tilde{\rho} - \tilde{\upsilon}| & ,|\tilde{\rho}| > 0 \end{cases} \tag{6.22}$$

which is $U(1)_\rho$ symmetric. At some energy scale, a mechanism must deviate the field, boosting it for a rapid moment to acquire a VEV, and thus break the $U(1)_\rho$ symmetry. Leaving this discussion for a later time, we try to apply what we have learned so far to the SM Lagrangian. The Higgs field is an $SU(2)_L$ doublet

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \upsilon + H(x) \end{pmatrix}. \tag{6.23}$$

So we have to construct the covariant derivative, but now the exotic SSB mechanism needs to generate 9 fermions, 3 leptons and 6 quarks (we ignore the mass of neutrinos for the time being). The most general form for the exotic covariant derivative would be:

$$(D_\mu\phi)_\rho^{l,q} = \left[\partial_\mu - i\sum_\kappa (g_\kappa^{l,q}\cdot\rho_\kappa^{l,q})\epsilon_{\kappa\mu}^{l,q}\right]\phi \tag{6.24}$$

where l, q denote the covariant derivatives for leptons and quarks respectively, and $\kappa = 1, 2, \ldots$ is a summation index. However, we face a problem here. The square of this term will include a huge amount of terms and parameters by increasing the maximum value of κ. Let's consider a simpler choice where leptons and quarks are generated by the same covariant derivative

$$(D_\mu\phi)_\rho = \left[\partial_\mu - i\sum_\kappa (g\cdot\rho_\kappa)\epsilon_\mu\right]\phi \tag{6.25}$$

and the ρ fields have the same coupling constant g, with only one ϵ_μ four-vector. If we consider three ρ fields, then the square of the covariant derivative includes a term proportional to

$$\begin{aligned}\sum_{\kappa,\lambda}(g\cdot\rho_\kappa)(g\cdot\rho_\lambda) &= \sum_{\kappa,\lambda}(g^1\rho_\kappa^2 - g^2\rho_\kappa^1)(g^1\rho_\lambda^2 - g^2\rho_\lambda^1)\\ &= \sum_{\kappa,\lambda}(-g^1\rho_\kappa^2 g^2\rho_\lambda^1 - g^2\rho_\kappa^1 g^1\rho_\lambda^2)\\ &= \sum_{\kappa,\lambda} g^1 g^2(\rho_\kappa^2\rho_\lambda^1 - \rho_\kappa^1\rho_\lambda^2)\end{aligned} \tag{6.26}$$

and generates nine $\tilde{\rho}$ scalar fields, which after obtaining a VEV, they give rise to nine fermion masses and their corresponding Yukawa interactions with the Higgs field H. However, each unique combination of $\rho_\kappa^2\rho_\lambda^1 - \rho_\kappa^1\rho_\lambda^2$ corresponds to a unique fermion. We remind the reader that the mass of the fermion is proportional to the λ parameter Eq(6.17). This means that for the same g coupling constant for all ρ fields, the λ parameters from

$$\tilde{\rho}_\alpha = \frac{1}{\sqrt{2}}(\tilde{\upsilon}_\alpha + \lambda_\alpha\overline{\psi}_\alpha\psi_\alpha) \tag{6.27}$$

where $\alpha = 1, 2, \ldots, 9$, are the ones that give each fermion a different mass value! This mechanism might explain the reason why, i.e. the electron is approximately 200 lighter than the muon. It is because their $\tilde{\rho}$ fields have different λ parameters, that results from the mixing of the ρ Grassmann fields. We may connect the VEV of each field $\tilde{\rho}$ with the cosmological epoch that each fermion appeared from, or maybe the λ parameters also determine in some way the order that fermions decouple from the relativistic gas at the early times of the universe. But this result comes from the simplest choice that only the ρ field has a κ index. Depending on the values of κ and which quantity has index, we may construct much more complex combinations, for instance some ϵ^{μ}_{κ}'s might be orthogonal to each other or considering different covariant derivatives for the leptons and the quarks, we might get different descriptions for the same physics, or even add new fermionic fields to the theory!

This mechanism, despite producing the desirable terms of the fermion masses and Yukawa couplings, it has also its own flaws. This mechanism alone does not determine which fermion obtains a leptonic or baryonic number, unless the sum of the $U(1)_\rho$ quantum numbers add to a value that determines somehow the species of the fermion. Additionally, this mechanism needs a modification to include, for instance, the up-type quarks, since the Higgs doublet obtains each VEV in the lower component. Most importantly, it does not include SUSY, which is the most important candidate for Beyond Standard Model scenarios and New Physics!

This mechanism is currently at an early stage of its development, and surely some significant modifications are needed in order to respect the observable measurements and the phenomenology of the SM.

7 Conclusions

In this thesis, we have studied the Z' models in order to explain phenomenologically the discrepancy in the muon anomalous magnetic moment between the theory and the experiment, and also apply this theoretical framework to discover the origin of the Yukawa couplings.

Regarding the first model, by calculating some of the one-loop contributions to the anomalous magnetic moment, we try to utilize our knowledge in a BSM scenario. By extending the particle spectrum with an extra vector-like lepton, it is presented the role of a seesaw mechanism in lepton-specific 2HDMs with a local $U(1)'$ symmetry that generates the muon mass. The non-decoupling effects of the vector-like lepton can be used to explain the muon $g-2$ anomaly at one-loop due to the light gauge boson Z' and the light dark Higgs boson φ. The electroweak precision and Higgs data constrain the vector-like lepton relatively weakly, but the collider bounds on the vector-like lepton can be significant due to multi-lepton signatures. In order to probe the parameter space with the light Z' and dark Higgs for explaining the muon $g-2$ anomaly, it is important to look for the light resonances with muon channels together with the vector-like lepton at LHC and future collider experiments.

In the second model, we studied an explicit renormalizable model with a vector-like family as an ultraviolet completion of the theory. Only the vector-like family is charged under the additional $U(1)'$ group. After the spontaneous symmetry breaking at the TeV scale, the mixing between the fourth family and the three chiral families then provides the effective Yukawa couplings, as well as the non-universal effective couplings involving the three light families.

Finally, in the last chapter I suggest a mechanism that generates the Yukawa couplings in the theory of the SM. An exotic gauge boson for each fermion production needs to be included, and by utilizing the Grassmann algebra, the desired terms to the Lagrangian are introduced, with the inclusion of a scalar field source term and an effective Higgs-fermion scattering term. Being at an early stage of its development, this mechanism needs to be modified in order to respect the observable measurements and the phenomenology of the SM.

A Masses of the Gauge Bosons

The scalar potential contains the following terms of ϕ, H and H':

$$
\begin{aligned}
V\big(\phi, H, H'\big) =& \mu_1^2 H^\dagger H + \mu_2^2 H'^\dagger H' + \big(\mu_3 \phi H^\dagger H' + h.c.\big) \\
&+ \lambda_1 \big(H^\dagger H\big)^2 + \lambda_2 \big(H'^\dagger H'\big)^2 + \lambda_3 \big(H^\dagger H\big)\big(H'^\dagger H'\big) \\
&+ \mu_\phi^2 \phi^* \phi + \lambda_\phi \big(\phi^* \phi\big)^2 + \lambda_{H\phi} H^\dagger H \phi^* \phi + \lambda_{H'\phi} H'^\dagger H' \phi^* \phi.
\end{aligned}
\tag{A.1}
$$

and the VEVs of each field are $\langle H \rangle = \frac{1}{\sqrt{2}} \upsilon_1$, $\langle H' \rangle = \frac{1}{\sqrt{2}} \upsilon_2$, $\langle \phi \rangle = \upsilon_\phi$. In order to obtain the masses of the gauge bosons, we need to calculate the square of the kinetic terms, which contain the covariant derivative D_μ for each scalar field. Starting with $D_\mu \phi$ we get,

$$
\begin{aligned}
|D_\mu \phi|^2 &= \Big[\big(\partial_\mu + 2ig_{Z'} Z'_\mu\big)\phi\Big]^2 \\
&= \Big[\big(\partial_\mu + 2ig_{Z'} Z'_\mu\big)\frac{1}{\sqrt{2}}\big(\upsilon_\phi + \varphi\big)\Big]^2 \\
&= \Big[\frac{1}{\sqrt{2}}\big(\partial_\mu \varphi\big) + \frac{1}{\sqrt{2}} 2ig_{Z'} Z'_\mu \big(\upsilon_\phi + \varphi\big)\Big]^2 \\
&= \frac{1}{2}\big(\partial_\mu \varphi\big)^2 + 2g_{Z'}^2 \big(Z'_\mu\big)^2 \big(\upsilon_\phi + \varphi\big)^2 \\
&= \frac{1}{2}\big(\partial_\mu \varphi\big)^2 + 2g_{Z'}^2 \upsilon_\phi^2 \big(Z'_\mu\big)^2 + 4g_{Z'}^2 \upsilon_\phi \big(Z'_\mu\big)^2 \varphi + 2g_{Z'}^2 \big(Z'_\mu\big)^2 \varphi^2,
\end{aligned}
\tag{A.2}
$$

and we acquire the mass term $2g_{Z'}^2 \upsilon_\phi^2 \big(Z'_\mu\big)^2$. Following the same procedure on $|D_\mu H'|^2$ as well, we have,

$$
|D_\mu H'|^2 = \Big[\big(\partial_\mu - 2ig_{Z'} Z'_\mu - \frac{1}{2} ig_Y B_\mu - \frac{1}{2} ig\tau^i W_\mu^i\big) H'\Big]^2, \tag{A.3}
$$

where in the unitary gauge

$$
H' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon_2 + h' \end{pmatrix}. \tag{A.4}
$$

Expanding (A.3):

$$
\begin{aligned}
|D_\mu H'|^2 &= \left[\left(\partial_\mu - 2ig_{Z'}Z'_\mu - \frac{1}{2}ig_Y B_\mu - \frac{1}{2}ig\tau^i W^i_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \upsilon_2+h'\end{pmatrix}\right]^2 \\
&= \frac{1}{2}\Big[\big(0,\partial_\mu h'\big) + 2ig_{Z'}\big(0, Z'_\mu(\upsilon_2+h')\big) + \frac{1}{2}ig_Y\big(0, B_\mu(\upsilon_2+h')\big) \\
&\quad + \frac{1}{2}ig\begin{pmatrix}W^3_\mu & W^1_\mu - iW^2_\mu\\ W^1_\mu + iW^2_\mu & -W^3_\mu\end{pmatrix}\big(0,\upsilon_2+h'\big)\Big]\times \\
&\quad \Big[\begin{pmatrix}0\\ \partial^\mu h'\end{pmatrix} - 2ig_{Z'}\begin{pmatrix}0\\ Z'^\mu(\upsilon_2+h')\end{pmatrix} - \frac{1}{2}ig_Y\begin{pmatrix}0\\ B^\mu(\upsilon_2+h')\end{pmatrix} \\
&\quad - \frac{1}{2}ig\begin{pmatrix}W^{3\mu} & W^{1\mu} - iW^{2\mu}\\ W^{1\mu} + iW^{2\mu} & -W^{3\mu}\end{pmatrix}\begin{pmatrix}0\\ \upsilon_2+h'\end{pmatrix}\Big] \\
&= \frac{1}{2}\big(\partial_\mu h'\big)^2 + 2g^2_{Z'}\big(Z'_\mu\big)^2\big(\upsilon_2+h'\big)^2 + \frac{1}{8}g_Y\big(B_\mu\big)^2\big(\upsilon_2+h'\big)^2 \\
&\quad + \frac{1}{8}g^2\Big((W^1_\mu + iW^2_\mu)(\upsilon_2+h'), -W^3_\mu(\upsilon_2+h')\Big)\begin{pmatrix}(W^{1\mu}+iW^{2\mu})(\upsilon_2+h')\\ -W^{3\mu}(\upsilon_2+h')\end{pmatrix} \\
&\quad + \text{interactions} \\
&= \frac{1}{2}\big(\partial_\mu h'\big)^2 + 2g^2_{Z'}\big(Z'_\mu\big)^2\big(\upsilon_2+h'\big)^2 + \frac{1}{8}g_Y\big(B_\mu\big)^2\big(\upsilon_2+h'\big)^2 \\
&\quad + \frac{1}{8}g^2\Big[(W^1_\mu)^2 + (W^2_\mu)^2 + (W^3_\mu)^2\Big](\upsilon_2+h')^2 \\
&\quad + 2g_{Z'}g_Y Z'_\mu B^\mu(\upsilon_2+h')^2 + 2g_{Z'}gZ'_\mu W^{3\mu}(\upsilon_2+h')^2 + 2g_Y g B_\mu W^{3\mu}(\upsilon_2+h')^2 \\
&\quad + \text{interactions.}
\end{aligned}
\tag{A.5}
$$

The B_μ and W^i_μ terms can be transformed to the SM $Z, W^\pm$ and photon. However, we notice that the last three terms in (A.5) are mixing terms, which the first two are between the Z' and the B, W^3 respectively.
In conclusion, starting from the $|D_\mu\phi|^2$ and $|D_\mu H'|^2$ we obtain the masses for each boson of the theory, $m^2_{Z'} = g^2_{z'}\big(4\upsilon^2_\phi + 4\upsilon^2_2\big)$, $m_Z = \frac{1}{2}\sqrt{g^2+g^2_Y}\upsilon$, $m_W = \frac{1}{2}g\upsilon$. The Z'_μ mixes with B_μ and W^3_μ, and it is either analogous to the VEV of H', υ_2, or to the h' itself (interaction). The physical photon, $W^\pm$, Z, are obtained from the spontaneous symmetry breaking of the SM H, with an extra contribution from the VEV of H', υ_2, meaning that the VEV υ factor of the masses mentioned above is actually $\upsilon = \sqrt{\upsilon^2_1 + \upsilon^2_2}$.

B See-saw mechanism and calculations

Here is an attempt to extract the results in the see-saw mechanism of [3], precisely the Eq (4.7). From Eq(4.6) we can write down the mass matrix as,

$$\mathcal{M} = \begin{pmatrix} 0 & m_L \\ m_R & M_E \end{pmatrix}, \tag{B.1}$$

and Eq(4.6) becomes

$$\mathcal{L}_{L,mass} = -\big(\bar{l}_L, \overline{E}_L\big)\mathcal{M} \begin{pmatrix} l_R \\ E_R \end{pmatrix}. \tag{B.2}$$

Starting with the $\mathcal{M}^2$

$$\mathcal{M}^2 = \begin{pmatrix} m_R^2 & m_R M_E \\ m_R M_E & m_L^2 + M_E^2 \end{pmatrix} \tag{B.3}$$

and then, by calculating the following

$$\begin{aligned} &\det\big(\mathcal{M}^2 - \lambda I\big) = 0 \Rightarrow \\ &(m_R^2 - \lambda)(m_L^2 + M_E^2 - \lambda) - m_R^2 M_E^2 = 0 \Rightarrow \\ &\lambda^2 - \lambda(m_R^2 + m_L^2 + M_E^2) + m_R^2 m_L^2 = 0 \end{aligned} \tag{B.4}$$

we reach to the following eigenvalues:

$$\lambda_{1,2} = \frac{1}{2}\bigg(m_R^2 + m_L^2 + M_E^2 \pm \sqrt{(m_R^2 + m_L^2 + M_E^2)^2 - 4m_R^2 m_L^2}\bigg) \tag{B.5}$$

Then, we manipulate the terms under the square root

$$\begin{aligned} &(m_R^2 + m_L^2 + M_E^2)^2 - 4m_R^2 m_L^2 = \\ &= m_R^4 + (m_L^2 + M_E^2)^2 + 2m_R^2 m_L^2 + 2m_R^2 M_E^2 - 4m_R^2 m_L^2 \\ &= m_R^4 + m_L^4 + M_E^4 + 2m_L^2 M_E^2 + 2m_R^2 m_L^2 + 2m_R^2 M_E^2 - 4m_R^2 m_L^2 \\ &= m_R^4 + m_L^4 + M_E^4 + 2m_L^2 M_E^2 - 2m_R^2 m_L^2 + 2m_R^2 M_E^2 \\ &= M_E^4 + (m_L^2 - m_R^2)^2 + 2m_L^2 M_E^2 + 2m_R^2 M_E^2 + (2m_R^2 M_E^2 - 2m_R^2 M_E^2) \\ &= (M_E^2 + m_L^2 - m_R^2)^2 + 4m_R^2 M_E^2 \end{aligned} \tag{B.6}$$

and by substituting Eq(B.6) into Eq(B.5) we get

$$\lambda_{1,2} = \frac{1}{2}\bigg(m_R^2 + m_L^2 + M_E^2 \mp \sqrt{(M_E^2 + m_L^2 - m_R^2)^2 + 4m_R^2 M_E^2}\bigg) \tag{B.7}$$

where I changed the signs in order to match the eigenvalues of Eq(4.7).

C Proof of Gordon Identity

The Σ tensor is expressed as,

$$\Sigma^{\mu\nu} = \frac{i}{2}\big[\gamma^{\mu}, \gamma^{\nu}\big]. \tag{C.1}$$

Then, acting on $\frac{i\Sigma^{\mu\nu}q_{\nu}}{2m}$ from left and right side with $\overline{u}(p'), u(p)$ respectively, we get,

$$\begin{aligned}
&\overline{u}\frac{i\Sigma^{\mu\nu}q_{\nu}}{2m}u(p) = \frac{i}{2m}\frac{i}{2}\overline{u}(p')\big[\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}\big](p'_{\nu} - p_{\nu})u(p) \\
&= \frac{i}{2m}\frac{i}{2}\overline{u}(p')\big[(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})p'_{\nu} - (\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})p_{\nu}\big]u(p) \\
&= \frac{i}{2m}\frac{i}{2}\overline{u}(p')\big[(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} - 2\gamma^{\nu}\gamma^{\mu})p'_{\nu} - (2\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})p_{\nu}\big]u(p) \\
&= \frac{i}{2m}\frac{i}{2}\overline{u}(p')\big[(2g^{\mu\nu} - 2\gamma^{\nu}\gamma^{\mu})p'_{\nu} - (2\gamma^{\mu}\gamma^{\nu} - 2g^{\mu\nu})p_{\nu}\big]u(p) \\
&= \frac{i}{2m}\frac{i}{2}\overline{u}(p')\big[2(p'^{\mu} - \not{p}'\gamma^{\mu}) - 2(\gamma^{\mu}\not{p} - p^{\mu})\big]u(p).
\end{aligned} \tag{C.2}$$

We now use the Dirac equation,

$$\begin{aligned}
&\frac{-1}{2m}\overline{u}(p')\big[(p'^{\mu} - m\gamma^{\mu}) - (\gamma^{\mu}m - p^{\mu})\big]u(p) = \\
&= \frac{-1}{2m}\overline{u}(p')\big[(p'^{\mu} + p^{\mu}) - 2m\gamma^{\mu}\big]u(p) \\
&= \overline{u}(p')\left[\gamma^{\mu} - \frac{p'^{\mu} + p^{\mu}}{2m}\right]u(p).
\end{aligned} \tag{C.3}$$

Inserting this into the Gordon identity:

$$\overline{u}(p')\left[\frac{p'^{\mu} + p^{\mu}}{2m} + \frac{i\Sigma^{\mu\nu}q_{\nu}}{2m}\right] = \overline{u}(p')\gamma^{\mu}u(p) \tag{C.4}$$

D Proof of Feynman-Schwinger parametrization

In the case of two denominators:

$$\begin{aligned}
&\int_0^1 \frac{dx}{\big[xa_1 + (1-x)a_2\big]^2} = \frac{1}{a_1 - a_2}\int_0^1 \frac{dx(a_1 - a_2)}{\big[x(a_1 - a_2) + a_2\big]^2} \\
&= \frac{1}{a_1 - a_2}\int_{a_2}^{a_1}\frac{d\rho}{\rho^2} = \frac{1}{a_1 - a_2}\big[\frac{1}{a_2} - \frac{1}{a_1}\big] = \frac{1}{a_1 a_2}.
\end{aligned} \tag{D.1}$$

In the case of three denominators:

$$\begin{aligned}
&\int_0^1 dx \int_0^{1-x} dy \frac{1}{\big[xa_1 + ya_2 + (1-x-y)a_3\big]^3} = \\
&= \frac{1}{a_2 - a_3} \int_0^1 dx \int_0^{1-x} \frac{dy(a_2 - a_3)}{\big[x(a_1 - a_3) + y(a_2 - a_3) + a_3\big]^3} \\
&= \frac{1}{a_2 - a_3} \int_0^1 dx \int_{xa_1+(1-x)a_3}^{xa_1+(1-x)a_2} \frac{d\rho}{\rho^3} \\
&= -\frac{1}{2(a_2 - a_3)} \int_0^1 dx \left[\frac{1}{\big[xa_1 + (1-x)a_2\big]^2 - \big[xa_1 + (1-x)a_3\big]^2}\right] \\
&= -\frac{1}{2(a_2 - a_3)} \left[\frac{1}{a_1 a_2} - \frac{1}{a_1 a_3}\right] \\
&= \frac{1}{2a_1 a_2 a_3}.
\end{aligned} \tag{D.2}$$

More generally,

$$\frac{1}{a_1 \cdots a_n} = (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \frac{1}{\Big[a_1 + \cdots + x_{n-1}(a_n - a_1)\Big]^n} \tag{D.3}$$

E Lande g factor and Form Factors

Considering that the 4-vector potential is related to a static magnetic field, we have,

$$A_\mu^{cl}(x) = \Big(0, \vec{\mathbf{A}}(\vec{\mathbf{x}})\Big). \tag{E.1}$$

Therefore, the Feynman amplitude becomes:

$$\begin{aligned}
i\mathcal{M} &= -ie\overline{u}(p')\Gamma^i u(p)\tilde{A}_i^{cl}(q) \\
&= -ie\overline{u}(p')\left[\gamma^i F_E(q^2) + i\frac{\sigma^{i\nu}q_\nu}{2m}F_M(q^2)\right]u(p)\tilde{A}_i^{cl}(q).
\end{aligned}$$

Considering the explicit form of the spinors:

$$u(p) = \frac{\not{p} + m}{\sqrt{2m(E+m)}}u(0) = \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\xi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\xi(0) \end{pmatrix} \tag{E.2}$$

where $\xi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for spin $\frac{1}{2}$ and $\xi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for spin $-\frac{1}{2}$, and also

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{E.3}$$

The first term in the amplitude can be written as:

$$\begin{aligned}
\overline{u}(p')\gamma^i u(p) &= u^\dagger(p')\gamma^0\gamma^i u(p) = u^\dagger(p') \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} u(p) \\
&= \begin{pmatrix} \xi'^\dagger(0)\sqrt{\frac{E'+m}{2m}}, & \xi'^\dagger(0)\frac{\vec{\sigma}\cdot\vec{p}'}{\sqrt{2m(E'+m)}} \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\xi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\xi(0) \end{pmatrix} \\
&= \begin{pmatrix} \xi'^\dagger(0)\frac{\vec{\sigma}\cdot\vec{p}'}{\sqrt{2m(E'+m}}\sigma^i, \xi'^\dagger(0)\sqrt{\frac{E'+m}{2m}}\sigma^i \end{pmatrix} \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\xi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\xi(0) \end{pmatrix} \\
&= \frac{1}{2m}\left[\sqrt{\frac{E+m}{E'+m}}\xi'^\dagger(0)\vec{\sigma}\cdot\vec{p}'\sigma^i\xi(0) + \sqrt{\frac{E'+m}{E+m}}\xi'^\dagger(0)\sigma^i\vec{\sigma}\cdot\vec{p}\xi(0)\right]
\end{aligned}$$

In the non-relativistic limit, $E \approx m \approx E'$, this term is equal to:

$$\begin{aligned}
\overline{u}(p')\gamma^i u(p) &= \frac{1}{2m}\Big(\xi'^\dagger(0)\vec{\sigma}\cdot\vec{p}'\sigma^i\xi(0) + \xi'^\dagger(0)\sigma^i\vec{\sigma}\cdot\vec{p}\xi(0)\Big) \\
&= \frac{1}{2m}\xi'^\dagger(0)\Big[\sigma^j\sigma^i p'_j + \sigma^i\sigma^j p_j\Big]\xi(0),
\end{aligned}$$

and using the identity $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma_k$, we get,

$$\begin{aligned}
\overline{u}(p')\gamma^i u(p) &= \frac{1}{2m}\xi'^\dagger(0)\Big[\big(\delta^{ij} + i\epsilon^{ijk}\sigma_k\big)p'_j + \big(\delta^{ij} + i\epsilon^{ijk}\sigma_k\big)p_j\Big]\xi(0) \\
&= \frac{1}{2m}\xi'^\dagger(0)\Big[p'^i - i\epsilon^{ijk}\sigma_k P'_j + p^i + i\epsilon^{ijk}\sigma_k p_j\Big]\xi(0) \\
&= \frac{1}{2m}\xi'^\dagger(0)\Big[\big(p'+p\big)^i - i\epsilon^{ijk}\sigma_k\big(p'-p\big)_j\Big]\xi(0).
\end{aligned}$$

The term that interests us is the one linear in q (nonrelativistic limit):

$$\overline{u}(p')\gamma^i u(p) = -\frac{i}{2m}\xi'^\dagger(0)\epsilon^{ijk}q_j\sigma_k\xi(0). \tag{E.4}$$

Taking now the non-relativistic limit of the second term in the amplitude, and assuming the photon energy to be negligibly small, we have:

$$\frac{i}{2m}\overline{u}(p')\sigma^{i\nu}q_\nu u(p) \approx \frac{i}{2m}\overline{u}(p')\sigma^{ij}q_j u(p),$$

where $\sigma^{ij} = \frac{i}{2}\left[\gamma^i, \gamma^j\right] = \epsilon^{ijk}\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}$, and we get:

$$\begin{aligned}
\frac{i}{2m}\overline{u}(p')\sigma^{i\nu}q_\nu u(p) &\approx \frac{i}{2m}u^\dagger(p')\begin{pmatrix}I & 0 \\ 0 & -I\end{pmatrix}\begin{pmatrix}\sigma_k & 0 \\ 0 & \sigma_k\end{pmatrix}u(p)\epsilon^{ijk}q_j \\
&\approx \frac{i}{2m}u^\dagger(p')\begin{pmatrix}\sigma_k & 0 \\ 0 & -\sigma_k\end{pmatrix}\epsilon^{ijk}q_j u(p) \\
&\approx \frac{i}{2m}\left(\xi'^\dagger(0)\sqrt{\tfrac{E'+m}{2m}},\quad \xi'^\dagger(0)\tfrac{\vec{\sigma}\cdot\vec{p'}}{\sqrt{2m(E'+m)}}\right)\begin{pmatrix}\sigma_k & 0 \\ o & -\sigma_k\end{pmatrix}\epsilon^{ijk}q_j\begin{pmatrix}\sqrt{\frac{E+m}{2m}}\xi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\xi(0)\end{pmatrix} \\
&\approx \frac{i}{2m}\left(\sqrt{\tfrac{E'+m}{2m}}\xi'^\dagger(0)\epsilon^{ijk}q_j\sigma_k,\quad -\xi'^\dagger(0)\tfrac{\vec{\sigma}\cdot\vec{p'}}{\sqrt{2m(E'+m)}}\epsilon^{ijk}q_j\sigma_k\right)\begin{pmatrix}\sqrt{\frac{E+m}{2m}}\xi(0) \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(E+m)}}\xi(0)\end{pmatrix} \\
&\approx \frac{i}{2m}\Bigg[\frac{\sqrt{E'+m}\sqrt{E+m}}{2m}\xi'^\dagger(0)\epsilon^{ijk}q_j\sigma_k\xi(0) \\
&- \xi'^\dagger(0)\frac{(\vec{\sigma}\cdot\vec{p'})(\vec{\sigma}\cdot\vec{p})}{2m\sqrt{(E'+m)(E+m)}}\epsilon^{ijk}q_j\sigma_k\xi(0)\Bigg] \\
&\approx \frac{i}{2m}\xi'^\dagger(0)\left[\frac{\sqrt{E'+m}\sqrt{E+m}}{2m}\epsilon^{ijk}q_j\sigma_k - \frac{|\vec{p'}||\vec{p}|}{2m\sqrt{(E'+m)(E+m)}}\epsilon^{ijk}q_j\sigma_k\right]\xi(0).
\end{aligned}$$

Ignoring terms not linear in q, we end up with:

$$\frac{i}{2m}\overline{u}(p')\sigma^{i\nu}q_\nu u(p) \approx -\frac{i}{2m}\xi'^\dagger(0)\sigma_k\xi(0)\epsilon^{ijk}q_j, \tag{E.5}$$

and the amplitude becomes:

$$\begin{aligned}
i\mathcal{M} &= -ie\overline{u}(p')\left[\gamma^i F_1(q^2) + i\frac{\sigma^{i\nu}q_\nu}{2m}F_2(q^2)\right]u(p)\tilde{A}_i^{cl}(\vec{\mathbf{q}}) \\
&\approx -ie\xi'^\dagger(0)\left[-\frac{i}{2m}\sigma_k\big(F_1(0)+F_2(0)\big)\right]\xi(0)\left(i\epsilon^{ijk}q_j\tilde{A}_i^{cl}(\vec{\mathbf{q}})\right).
\end{aligned} \tag{E.6}$$

We can now plug in magnetic field described in momentum space,

$$\vec{\mathbf{B}}(\vec{\mathbf{x}}) = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}(\vec{\mathbf{x}}) \Rightarrow \tilde{B}^k(\vec{\mathbf{q}}) = i\epsilon^{ijk} q_j \tilde{A}_i^{cl}(\vec{\mathbf{q}}). \tag{E.7}$$

The amplitude becomes:

$$\begin{aligned} i\mathcal{M} &\approx ie\xi'^{\dagger}(0)\Big[-\frac{i}{2m}\sigma_k\Big(F_1(0)+F_2(0)\Big)\Big]\xi(0)\tilde{B}^k(\vec{\mathbf{q}}) \\ &\approx \frac{e}{2m}2\Big[F_1(0)+F_2(0)\Big]\xi'^{\dagger}(0)\frac{\sigma_k}{2}\xi(0)\tilde{B}^k(\vec{\mathbf{q}}). \end{aligned}$$

Identifying

$$\xi'^{\dagger}(0)\frac{\sigma_k}{2}\xi(0) = \langle S \rangle_k, \tag{E.8}$$

we get

$$i\mathcal{M} \approx \frac{e}{2m}2\Big[F_1(0)+F_2(0)\Big]\langle S\rangle_k \tilde{B}^k(\vec{\mathbf{q}}). \tag{E.9}$$

Comparing it with the result of the classical limit, the Hamiltonian density is written as:

$$\mathcal{H} = -\langle\vec{\boldsymbol{\mu}}\rangle \cdot \vec{\mathbf{B}} = -\frac{e}{2m}g\langle\vec{\mathbf{S}}\rangle \cdot \vec{\mathbf{B}}. \tag{E.10}$$

Then, in first order we make the following comparison:

$$i\mathcal{M}^{cl}_{NR} = \langle\vec{\boldsymbol{\mu}}\rangle \cdot \vec{\mathbf{B}}(\vec{\mathbf{q}}) = \frac{e}{2m}g\langle\vec{\mathbf{S}}\rangle \cdot \vec{\mathbf{B}}(\vec{\mathbf{q}}), \tag{E.11}$$

$$i\mathcal{M}_{NR} = \frac{e}{2m}2\Big[F_1(0)+F_2(0)\Big]\langle\vec{\mathbf{S}}\rangle \cdot \vec{\mathbf{B}}(\vec{\mathbf{q}}) \tag{E.12}$$

and the quantum corrected magnetic moment is

$$\langle\vec{\boldsymbol{\mu}}\rangle = \frac{e}{2m}g\langle\vec{\mathbf{S}}\rangle = \frac{e}{2m}2\Big[F_1(0)+F_2(0)\Big]\langle\vec{\mathbf{S}}\rangle, \tag{E.13}$$

meaning that the Lande g factor in first order correction becomes

$$g = 2\Big[F_1(0)+F_2(0)\Big]. \tag{E.14}$$

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