

The Bellman function of the dyadic maximal operator in connection with the Dyadic Carleson Imbedding Theorem and related inequalities

Eleftherios N. Nikolidakis

November 19, 2019

Abstract

We provide an alternative proof of the expression of the Bellman function of the dyadic maximal operator in connection with the Dyadic Carleson Imbedding Theorem, which appears in [10]. We also state and prove a sharp integral inequality for this operator in connection with the above Bellman function, and give an application.

1 Introduction

The dyadic maximal operator on \mathbb{R}^n is a useful tool in analysis and is defined by

$$\mathcal{M}_d\phi(x) = \sup \left\{ \frac{1}{|S|} \int_S |\phi(u)| \, du : x \in S, \, S \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}, \quad (1.1)$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$, where $|\cdot|$ denotes the Lebesgue measure on \mathbb{R}^n , and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, for $N = 0, 1, 2, \dots$. It is well known that it satisfies the following weak type (1,1) inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_d\phi > \lambda\}} |\phi(u)| \, du, \quad (1.2)$$

for every $\phi \in L^1(\mathbb{R}^n)$, and every $\lambda > 0$, from which it is easy to get the following L^p -inequality

$$\|\mathcal{M}_d\phi\|_p \leq \frac{p}{p-1} \|\phi\|_p, \quad (1.3)$$

for every $p > 1$, and every $\phi \in L^p(\mathbb{R}^n)$. It is easy to see that the weak type inequality (1.2) is the best possible. For refinements of this inequality consult [15].

It has also been proved that (1.3) is best possible (see [2] and [3] for general martingales and [36] for dyadic ones). An approach for studying the behaviour of

this maximal operator in more depth is the introduction of the so-called Bellman functions which play the role of generalized norms of \mathcal{M}_d . Such functions related to the L^p -inequality (1.3) have been precisely identified in [8], [10] and [20]. For the study of the Bellman functions of \mathcal{M}_d , we use the notation $\text{Av}_E(\psi) = \frac{1}{|E|} \int_E \psi$, whenever E is a Lebesgue measurable subset of \mathbb{R}^n of positive measure and ψ is a real valued measurable function defined on E . We fix a dyadic cube Q and define the localized maximal operator $\mathcal{M}'_d \phi$ as in (1.1) but with the dyadic cubes S being assumed to be contained in Q . Then for every $p > 1$ we let

$$B_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (\mathcal{M}'_d \phi)^p : \text{Av}_Q(\phi) = f, \text{Av}_Q(\phi^p) = F \right\}, \quad (1.4)$$

where ϕ is nonnegative in $L^p(Q)$ and the variables f, F satisfying $0 < f^p \leq F$. By a scaling argument it is easy to see that (1.4) is independent of the choice of Q (so we may choose Q to be the unit cube $[0, 1]^n$). In [10], the function (1.4) has been precisely identified for the first time. The proof has been given in a much more general setting of tree-like structures on probability spaces.

More precisely we consider a non-atomic probability space (X, μ) and let \mathcal{T} be a family of measurable subsets of X , that has a tree-like structure similar to the one in the dyadic case (the exact definition will be given in Section 2). Then we define the dyadic maximal operator associated to \mathcal{T} , by

$$\mathcal{M}_{\mathcal{T}} \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}, \quad (1.5)$$

for every $\phi \in L^1(X, \mu)$, $x \in X$.

This operator is related to the theory of martingales and satisfies essentially the same inequalities as \mathcal{M}_d does. Now we define the corresponding Bellman function of four variables of $\mathcal{M}_{\mathcal{T}}$, by

$$B_p^{\mathcal{T}}(f, F, L, k) = \sup \left\{ \int_K [\max(\mathcal{M}_{\mathcal{T}} \phi, L)]^p d\mu : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^p d\mu = F, K \subseteq X \text{ measurable with } \mu(K) = k \right\}, \quad (1.6)$$

the variables f, F, L, k satisfying $0 < f^p \leq F$, $L \geq f$, $k \in (0, 1]$. The exact evaluation of (1.6) is given in [10], for the cases where $k = 1$ or $L = f$. In the first case the author (in [10]) precisely identifies the function $B_p^{\mathcal{T}}(f, F, L, 1)$ by evaluating it in a first stage for the case where $L = f$. That is he precisely identifies $B_p^{\mathcal{T}}(f, F, f, 1)$ (in fact $B_p^{\mathcal{T}}(f, F, f, 1) = F \omega_p(\frac{f^p}{F})^p$, where $\omega_p : [0, 1] \rightarrow [1, \frac{p}{p-1}]$ is the inverse function H_p^{-1} , of $H_p(z) = -(p-1)z^p + pz^{p-1}$). Then using several calculus argument he provides the evaluation of $B_p^{\mathcal{T}}(f, F, L, 1)$ for every $L \geq f$. Now in [20] the authors give a direct proof of the evaluation of $B_p^{\mathcal{T}}(f, F, L, 1)$ by using alternative methods. In fact they prove a sharp symmetrization principle that holds for the dyadic maximal operator, which is stated as Theorem 2.1 (see Section 2).

In the second case, where $L = f$, the author (in [10]) uses the evaluation of $B_p^T(f, F, f, 1)$ and provides a proof of the more general $B_p^T(f, F, f, k)$, $k \in (0, 1]$. We write from now on this function as $B_p^T(f, F, k)$. This function is related to the Dyadic Carleson Imbedding Theorem and in fact, as is proved in [10], the following is true

$$B_p^T(f, F, k) = \sup \left\{ \sum_{I \in \mathcal{T}} \lambda_I (Av_I(\phi))^p, \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^p \, d\mu = F, \right. \\ \text{and the nonnegative } \lambda_I \text{'s satisfy } \sum_{J \in \mathcal{T}: J \subseteq I} \lambda_J \leq \mu(I) \text{ for every } I \in \mathcal{T} \\ \left. \text{and } \sum_{I \in \mathcal{T}} \lambda_I = k \right\}. \quad (1.7)$$

As an immediate step for the evaluation of $B_p^T(f, F, k)$ in [10], it is provided an alternative expression for this function. This is stated in the following theorem

Theorem 1.1. *The following is true*

$$B_p^T(f, F, k) = \sup \left\{ \left(F - \frac{(f-B)^p}{(1-k)^{p-1}} \right) \omega_p \left(\frac{B^p}{k^{p-1} \left(F - \frac{(f-B)^p}{(1-k)^{p-1}} \right)} \right) : \right. \\ \left. \text{for all } B \in [0, f] \text{ such that } h_k(B) \leq F \right\}, \quad (1.8)$$

where h_k is defined by $h_k(B) = \frac{(f-B)^p}{(1-k)^{p-1}} + \frac{B^p}{k^{p-1}}$.

After proving the above theorem, the author in [10], precisely evaluated $B_p^T(f, F, k)$, by using a chain of calculus arguments. In Section 3 we provide an alternative proof of Theorem 1.1. Now in view of the symmetrization principle that appears in [20] (see Theorem 2.1 below) we conclude that

$$B_p^T(f, F, k) = \sup \left\{ \int_0^k \left(\frac{1}{t} \int_0^t g \right)^p dt : \text{where } g : (0, 1] \longrightarrow \mathbb{R}^+ \text{ is} \right. \\ \left. \text{non-increasing, } \int_0^1 g = f, \int_0^1 g^p = F \right\}. \quad (1.9)$$

In Section 4 we prove the following

Theorem 1.2. *There exists a function $g = g_k : (0, 1] \longrightarrow \mathbb{R}^+$ non-increasing and continuous, satisfying $\int_0^1 g = f$ and $\int_0^1 g^p = F$, for which the supremum in (1.9) is attained.*

In fact by using the methods that appear in [17], one can see that the function g_k that satisfies the statement of the above theorem should be unique. Moreover we explicitly construct the function g_k , mentioned above.

In Section 5 we provide a 3-parameter inequality for the operator that we study, which is connected with the above two theorems. In fact in [10] is proved a 1-parameter inequality which states that, for every $\phi \in L^p(X, \mu)$ satisfying $\int_X \phi \, d\mu = f$ and $\int_X \phi^p \, d\mu = F$, the inequality

$$F \geq \frac{1}{(\beta + 1)^{p-1}} f^p + \frac{(p-1)\beta}{(\beta + 1)^p} \int_X (M_{\mathcal{T}}\phi)^p \, d\mu, \quad (1.10)$$

is true for every value of the parameter β and sharp for one that depends on f and F . This gives as a consequence the evaluation of $B_p^{\mathcal{T}}(f, F, f, 1)$. In this paper we prove an inequality that connects in a sharp way the L^p -integral of ϕ on X and K , and also the L^p -integral of $M_{\mathcal{T}}\phi$, on X and K , where K is an arbitrary measurable subset of X . More precisely we prove the following

Theorem 1.3. *Let $\beta \geq \gamma \geq 0$, and K an arbitrary measurable subset of X , with measure $k \in (0, 1]$. Then for every $\phi \in L^p(X, \mu)$ such that $\int_X \phi \, d\mu = f$ and $\int_X \phi^p \, d\mu = F$ the following inequality is true*

$$F \geq \left[1 - \frac{1}{(1 + \gamma)^{p-1}} \right] \int_K \phi^p \, d\mu + \frac{(p-1)\beta}{(\beta + 1)^p} \int_X (M_{\mathcal{T}}\phi)^p \, d\mu - \frac{(p-1)\gamma}{(\beta + 1)^p} \int_K (M_{\mathcal{T}}\phi)^p \, d\mu + \frac{f^p}{(\beta + 1)^{p-1}}. \quad (1.11)$$

Moreover (1.11) is sharp in the sense that for each $k \in (0, 1]$ there exist $\beta, \gamma \geq 0$ such that $\beta \geq \gamma$, a sequence of measurable $(K_n)_{n \in \mathbb{N}}$ subsets of X with $\lim_n \mu(K_n) = k$, and a sequence $(\phi_n)_{n \in \mathbb{N}}$ of non-negative functions in $L^p(X, \mu)$, satisfying the above integral conditions, giving equality in (1.11) in the limit.

Note that if we set $\gamma = 0$ in (1.11) we get (1.10). We also get the same conclusion if we let k tend to zero. In fact Theorem 1.3 gives, under the above mentioned integral conditions for ϕ , the best possible connection of the quantities $\int_X (M_{\mathcal{T}}\phi)^p \, d\mu$ and $\int_K (M_{\mathcal{T}}\phi)^p \, d\mu$, where $\mu(K) = k \in (0, 1]$ is given.

It is obvious that the inequality (1.11) connects in the best possible way the quantities $\int_X (M_{\mathcal{T}}\phi)^p \, d\mu$ and $\int_K (M_{\mathcal{T}}\phi)^p \, d\mu$ (along with f, F), and this is done for an arbitrary K measurable subset of X .

We also need to mention that the extremizers for the standard Bellman function $B_p^{\mathcal{T}}(f, F, f, 1)$ has been studied in [16], and in [18] for the case $0 < p < 1$. We note also that further study of the dyadic maximal operator can be seen in [19, 20] where symmetrization principles for this operator are presented, while other approaches for the determination of certain Bellman functions are given in [26, 27, 31, 32, 33].

There are several problems in Harmonic Analysis where Bellman functions naturally arise. Such problems (including the dyadic Carleson Imbedding Theorem and weighted inequalities) are described in [14] (see also [12, 13]).

We should mention also that the exact computation of a Bellman function is a difficult task which is connected with the deeper structure of the corresponding Harmonic Analysis problem. Thus far several Bellman functions have been computed (see [2, 9, 11, 25, 27, 31, 32, 33]). In [26] L. Slavin, A. Stokolos and V. Vasyunin linked the Bellman function computation to solving certain PDE's of the Monge-Ampère type, and in this way they obtained an alternative proof for the evaluation of the Bellman functions related to the dyadic maximal operator.

Also in [33], using the Monge-Ampère equation approach, a more general Bellman function than the one related to the dyadic Carleson Imbedding Theorem has been precisely evaluated thus generalizing the corresponding result in [10]. For more recent developments we refer to [1, 6, 7, 23, 24, 28, 29, 37]. Additional results can be found in [34, 35] while for the study of general theory of maximal operators one can consult [4, 5] and [30]. Also in [22] one can find other approaches for the study of the dyadic maximal operator.

In this paper, as in our previous ones we use Bellman functions as a mean to get in deeper understanding of the corresponding maximal operators and we are not using the standard techniques as Bellman dynamics and induction, corresponding PDE's, obstacle conditions etc. Instead our methods being different from the Bellman function technique, we rely on the combinational structure of these operators. For such approaches, which enables us to study and solve problems as the one which is described in this article one can see [8, 9, 10, 11, 16, 17, 18, 19, 20, 21].

Acknowledgement: I would like to thank Anastasios D. Delis for the idea of splitting the sets A_I^j s. This was a motivation for me to state and prove Theorem 1.3.

2 Preliminaries

Let (X, μ) be a nonatomic probability space. We give the following

Definition 2.1. A set \mathcal{T} of measurable subsets of X will be called a tree if the following conditions are satisfied:

- i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.
- ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that
 - a) the elements of $C(I)$ are pairwise disjoint subsets of I .
 - b) $I = \cup C(I)$.
- iii) $\mathcal{T} = \cup_{m \geq 0} \mathcal{T}_{(m)}$, where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \cup_{I \in \mathcal{T}_{(m)}} C(I)$.
- iv) We have $\lim_{m \rightarrow \infty} \left(\sup_{I \in \mathcal{T}_{(m)}} \mu(I) \right) = 0$

v) The tree \mathcal{T} differentiates $L^1(X, \mu)$. That is for every $\phi \in L^1(X, \mu)$ it is true that

$$\lim_{\substack{x \in I \in \mathcal{T} \\ \mu(I) \rightarrow 0}} \frac{1}{\mu(I)} \int_I \phi \, d\mu = \phi(x),$$

for μ -almost every $x \in X$.

Then we define the dyadic maximal operator corresponding to \mathcal{T} by

$$M_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in \mathcal{T} \right\}, \quad (2.1)$$

for every $\phi \in L^1(X, \mu)$, $x \in X$.

We give the following which appears in [10].

Lemma 2.1. *Let $p > 1$ be fixed. Then the function $\omega_p : [0, 1] \rightarrow [1, \frac{p}{p-1}]$, defined as the inverse of $H_p(z) = pz^{p-1} - (p-1)z^p$, is strictly decreasing, and if we define U_p on $(0, 1]$, by $U_p(x) = \frac{\omega_p(x)^p}{x}$, we have that U_p is also strictly decreasing.*

Lemma 2.2. *For every $I \in \mathcal{T}$ and every α such that $0 < \alpha < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise disjoint subsets of I such that*

$$\mu \left(\bigcup_{J \in \mathcal{F}(I)} J \right) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1 - \alpha)\mu(I).$$

Definition 2.2. Let $\phi : (X, \mu) \rightarrow \mathbb{R}^+$. Then $\phi^* : (0, 1] \rightarrow \mathbb{R}^+$ is defined as the unique non-increasing, left continuous and equimeasurable to ϕ function on $(0, 1]$.

There are several formulas that express ϕ^* , in terms of ϕ . One of them is as follows:

$$\phi^*(t) = \inf \left(\{y > 0 : \mu(\{x \in X : \phi(x) > y\}) < t\} \right),$$

for every $t \in (0, 1]$. An equivalent formulation of the non increasing rearrangement can be given by

$$\phi^*(t) = \sup_{e \subseteq X, \mu(e) \geq t} \left[\inf_{x \in e} \phi(x) \right],$$

for any $t \in (0, 1]$.

In [21] one can see the following symmetrization principle for the dyadic maximal operator $M_{\mathcal{T}}$.

Theorem 2.1. *Let $g : (0, 1] \rightarrow \mathbb{R}^+$ be non-increasing and G_1, G_2 be non-decreasing and non-negative functions defined on $[0, +\infty)$. Then the following is true, for any $k \in (0, 1]$*

$$\begin{aligned} \sup \left\{ \int_K G_1(M_{\mathcal{T}}\phi) G_2(\phi) \, d\mu : \phi^* = g \text{ and } \mu(K) = k \right\} = \\ = \int_0^k G_1 \left(\frac{1}{t} \int_0^t g \right) G_2(g(t)) \, dt. \end{aligned}$$

We also state the following, which is a standard fact in the theory of real functions.

Lemma 2.3. *Let $g_1, g_2 : (0, 1] \rightarrow \mathbb{R}^+$ be non-increasing functions, such that*

$$\int_0^1 G(g_1(t)) dt \leq \int_0^1 G(g_2(t)) dt$$

for every $G : [0, +\infty) \rightarrow [0, +\infty)$ non-decreasing. Then the inequality $g_1(t) \leq g_2(t)$ holds almost everywhere on $(0, 1]$

We now state some facts that appear in [10]. Fix $k \in (0, 1)$, $p > 1$ and consider the function

$$h_k(B) = \frac{(f - B)^p}{(1 - k)^{p-1}} + \frac{B^p}{k^{p-1}}, \quad (2.2)$$

defined for $B \in [0, f]$.

We also define

$$\mathcal{R}_k(B) = \left(F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right) \omega_p \left(\frac{B^p}{k^{p-1} \left(F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right)} \right)^p, \quad (2.3)$$

for B such that $B \in [0, f]$ and $h_k(B) \leq F$. Note that $\mathcal{R}_k(B)$ is defined for all $B \in [0, f]$ for which $h_k(B) \leq F$ or equivalently:

$$\frac{(f - B)^p}{(1 - k)^{p-1}} + \frac{B^p}{k^{p-1}} \leq F \iff 0 \leq \frac{B^p}{k^{p-1} \left[F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right]} \leq 1$$

so that (2.3) makes sense in view of the definition of ω_p .

Then as one can see in [10] the domain of \mathcal{R}_k is an interval $[p_0(f, F, k), p_1(f, F, k)]$. We state the following from [10] (For completeness reasons we provide also the first statement of the lemma that appears right below).

Lemma 2.4. *i) For every $\theta \in [0, 1]$ the equation*

$$\sigma(z) = -(p - 1)z^p + (p - 1 + k)z^{p-1} - \theta \left[1 + (1 - k) \left(\frac{p - 1}{z} - p \right) \right] = 0$$

has a unique solution in the interval $\left[1, 1 + \frac{k}{p-1} \right]$ which is denoted by $\omega_{p,k}(\theta)$.

ii) The function \mathcal{R}_k defined on $[p_0(f, F, k), p_1(f, F, k)]$ attains its absolute maximum at the unique interior point $B_0 \in \left(kf, \min \left(\frac{pk}{p - 1 + k}, p_1(f, F, k) \right) \right)$

such that $\frac{f(1 - k)}{f - B_0} = \omega_{p,k} \left(\frac{f^p}{F} \right)$. Moreover B_0 satisfies

$$H_p \left(\frac{B_0}{k} \frac{1 - k}{f - B_0} \right) = \frac{B_0^p}{k^{p-1} \left[F - \frac{(f - B_0)^p}{(1 - k)^{p-1}} \right]}$$

3 The Bellman function $B_p(f, F, k)$

Lemma 3.1. *For any $\phi : (X, \mu) \rightarrow \mathbb{R}^+$ integrable, the following inequality is true*

$$(M_{\mathcal{T}}\phi)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) du, \text{ for every } t \in (0, 1].$$

Proof. By Theorem 2.1 we have for any $G : [0, +\infty) \rightarrow [0, +\infty)$ non-decreasing

$$\int_X G(M_{\mathcal{T}}\phi) d\mu \leq \int_0^1 G\left(\frac{1}{t} \int_0^t \phi^*(u) du\right) dt. \quad (3.1)$$

Since G is non-decreasing we have that

$$[G(M_{\mathcal{T}}\phi)]^*(t) = G[(M_{\mathcal{T}}\phi)^*](t), \text{ for almost every } t \in (0, 1].$$

Thus $\int_0^1 G[(M_{\mathcal{T}}\phi)^*](t) dt = \int_0^1 [G(M_{\mathcal{T}}\phi)]^*(t) dt = \int_X G(M_{\mathcal{T}}\phi) d\mu \leq \int_0^1 G\left(\frac{1}{t} \int_0^t \phi^*(u) du\right) dt$, by (3.1). Thus by Lemma 2.3 we immediately conclude that

$$(M_{\mathcal{T}}\phi)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) du, \quad (3.2)$$

almost everywhere on $(0, 1]$. Since now $(M_{\mathcal{T}}\phi)^*(t)$ is left continuous, we conclude that (3.2) should hold everywhere on $(0, 1]$, and in this way we derive the proof of our Lemma. \square

There is also a second, simpler proof of Lemma 3.1 which we present right below

2nd proof of Lemma 3.1.

Suppose that we are given $\phi : (X, \mu) \rightarrow \mathbb{R}^+$ integrable and $t \in (0, 1]$ fixed. We set $A = \frac{1}{t} \int_0^t \phi^*(u) du$. Then obviously $A \geq \int_0^1 \phi^*(u) du = f$, by the fact that ϕ^* is non-increasing on $(0, 1]$. We consider the set $E = \{M_{\mathcal{T}}\phi > A\} \subseteq X$. Then by the weak type inequality (1.2) for $M_{\mathcal{T}}\phi$, we have that

$$\begin{aligned} \mu(E) &< \frac{1}{A} \int_E |\phi| d\mu \Rightarrow \\ A &= \frac{1}{t} \int_0^t \phi^*(u) d\mu < \frac{1}{\mu(E)} \int_E \phi d\mu \leq \frac{1}{\mu(E)} \int_0^{\mu(E)} \phi^*(u) du, \end{aligned} \quad (3.3)$$

where the last inequality in (3.3) holds due to the definition of ϕ^* . Since ϕ^* is non-increasing we must have from (3.3), that $\mu(E) < t$. But $\mu(E) = |\{(M_{\mathcal{T}}\phi)^*(t) > A\}|$ since $(M_{\mathcal{T}}\phi)$ and $(M_{\mathcal{T}}\phi)^*$ are equimeasurable. But since $(M_{\mathcal{T}}\phi)^*$ is non-increasing and because of the fact that $\mu(E) < t$ we conclude that $\{(M_{\mathcal{T}}\phi)^* > A\} = (0, \gamma)$ for some $\gamma < t$. Thus $t \notin \{(M_{\mathcal{T}}\phi)^* > A\} \Rightarrow (M_{\mathcal{T}}\phi)^*(t) \leq A = \frac{1}{t} \int_0^t \phi^*(u) du$, which is the desired result. \square

We are now in position to state and prove

Lemma 3.2. *Let $\phi : (X, \mu) \rightarrow \mathbb{R}^+$ be such that $\int_X \phi d\mu = f$ and $\int_X \phi^p d\mu = F$ where $0 < f^p \leq F$. Suppose also that we are given a measurable subset K of X such that $\mu(K) = k$, where k is fixed such that $k \in (0, 1]$. Then the following inequality is true:*

$$\int_K (M_{\mathcal{T}}\phi)^p d\mu \leq \int_0^k [\phi^*(u)]^p du \cdot \omega_p \left(\frac{\left(\int_0^k \phi^*(u) du \right)^p}{k^{p-1} \int_0^k [\phi^*(u)]^p du} \right)^p.$$

Proof. We obviously have that

$$\int_K (M_{\mathcal{T}}\phi)^p d\mu \leq \int_0^k [(M_{\mathcal{T}}\phi)^*]^p(t) dt. \quad (3.4)$$

We evaluate the right-hand side of (3.4). We have:

$$\int_0^k [(M_{\mathcal{T}}\phi)^*]^p dt \leq \int_0^k \left(\frac{1}{t} \int_0^t \phi^*(u) du \right)^p dt \quad (3.5)$$

by using Lemma 3.1. Additionally

$$\begin{aligned} \int_0^k \left(\frac{1}{t} \int_0^t \phi^*(u) du \right)^p dt &= \int_{\lambda=0}^{+\infty} p\lambda^{p-1} \left| \left\{ t \in (0, k] : \frac{1}{t} \int_0^t \phi^* \geq \lambda \right\} \right| d\lambda = \\ &= \int_{\lambda=0}^{f_k} + \int_{\lambda=f_k}^{+\infty} p\lambda^{p-1} \left| \left\{ t \in (0, k] : \frac{1}{t} \int_0^t \phi^* \geq \lambda \right\} \right| d\lambda, \end{aligned} \quad (3.6)$$

where the first equation is justified by a use of Fubini's theorem and f_k is defined by $f_k = \frac{1}{k} \int_0^k \phi^*(u) du > f = \int_0^1 \phi^*(u) du$.

The first integral in (3.6) is obviously equal to $k(f_k)^p = \frac{1}{k^{p-1}} \left(\int_0^k \phi^* \right)^p$. We suppose now that $\lambda > f_k$ is fixed. Then there exists $\alpha(\lambda) \in (0, k]$ such that $\frac{1}{\alpha(\lambda)} \int_0^{\alpha(\lambda)} \phi^*(u) du = \lambda$. Note that, without loss of generality, we assume that $\phi^*(0^+) = +\infty$ (the case $\phi^*(0^+) < +\infty$ can be handled similarly). As a consequence $\left\{ t \in (0, k] : \frac{1}{t} \int_0^t \phi^*(u) du \geq \lambda \right\} = (0, \alpha(\lambda)]$, thus

$$\left| \left\{ t \in (0, k] : \frac{1}{t} \int_0^t \phi^*(u) du \geq \lambda \right\} \right| = \alpha(\lambda).$$

So the second integral in (3.6) equals

$$\int_{\lambda=f_k}^{+\infty} p\lambda^{p-1} \alpha(\lambda) d\lambda = \int_{\lambda=f_k}^{+\infty} p\lambda^{p-1} \frac{1}{\lambda} \left(\int_0^{\alpha(\lambda)} \phi^*(u) du \right) d\lambda,$$

by the definition of $\alpha(\lambda)$. The last now integral, equals

$$\int_{\lambda=f_k}^{+\infty} p\lambda^{p-2} \left(\int_{\left\{ t \in (0, k] : \frac{1}{t} \int_0^t \phi^* \geq \lambda \right\}} \phi^*(u) du \right) d\lambda = \int_{t=0}^k \frac{p}{p-1} \phi^*(t) [\lambda^{p-1}]_{f_k}^{\frac{1}{t} \int_0^t \phi^*} dt, \quad (3.7)$$

by a use of Fubini's theorem. As a consequence (3.6) gives

$$\begin{aligned} \int_0^k \left(\frac{1}{t} \int_0^t \phi^\star(u) \, du \right)^p \, dt &= -\frac{1}{p-1} \frac{1}{k^{p-1}} \left(\int_0^k \phi^\star \right)^p + \\ &\quad \frac{p}{p-1} \int_0^k \phi^\star(t) \left(\frac{1}{t} \int_0^t \phi^\star \right)^{p-1} \, dt. \end{aligned} \quad (3.8)$$

Then by Hölder's inequality, applied in the second integral on the right side of (3.8), we have that

$$\begin{aligned} \int_0^k \left(\frac{1}{t} \int_0^t \phi^\star \right)^p \, dt &\leq -\frac{1}{p-1} \frac{1}{k^{p-1}} \left(\int_0^k \phi^\star \right)^p + \\ &\quad \frac{p}{p-1} \left(\int_0^k [\phi^\star]^p \right)^{\frac{1}{p}} \left[\int_0^k \left(\frac{1}{t} \int_0^t \phi^\star \right)^p \, dt \right]^{\frac{(p-1)}{p}}. \end{aligned} \quad (3.9)$$

We set now

$$J(k) = \int_0^k \left(\frac{1}{t} \int_0^t \phi^\star \right)^p \, dt, \quad A(k) = \int_0^k [\phi^\star]^p \quad \text{and} \quad B(k) = \int_0^k \phi^\star.$$

Then we conclude by (3.9) that

$$\begin{aligned} J(k) &\leq -\frac{1}{p-1} \frac{1}{k^{p-1}} [B(k)]^p + \frac{p}{p-1} [A(k)]^{\frac{1}{p}} [J(k)]^{\frac{(p-1)}{p}} \Rightarrow \\ \frac{J(k)}{A(k)} &\leq -\frac{1}{p-1} \left(\frac{[B(k)]^p}{k^{p-1} A(k)} \right) + \frac{p}{p-1} \left[\frac{J(k)}{A(k)} \right]^{\frac{(p-1)}{p}}. \end{aligned} \quad (3.10)$$

We set now in (3.10) $\Lambda(k) = \left[\frac{J(k)}{A(k)} \right]^{\frac{1}{p}}$, thus we get

$$\begin{aligned} \Lambda(k)^p &\leq -\frac{1}{p-1} \left(\frac{[B(k)]^p}{k^{p-1} [A(k)]} \right) + \frac{p}{p-1} \Lambda(k)^{p-1} \Rightarrow \\ p[\Lambda(k)]^{p-1} - (p-1)[\Lambda(k)]^p &\geq \frac{(\int_0^k \phi^\star)^p}{k^{p-1} \int_0^k [\phi^\star]^p} \Rightarrow \\ H_p(\Lambda(k)) &\geq \frac{(\int_0^k \phi^\star)^p}{k^{p-1} \int_0^k [\phi^\star]^p} \Rightarrow \Lambda(k) \leq \omega_p \left(\frac{(\int_0^k \phi^\star)^p}{k^{p-1} \int_0^k [\phi^\star]^p} \right) \Rightarrow \\ J(k) &\leq \int_0^k [\phi^\star]^p \omega_p \left(\frac{(\int_0^k \phi^\star)^p}{k^{p-1} \int_0^k [\phi^\star]^p} \right)^p. \end{aligned} \quad (3.11)$$

At last by (3.4), (3.5) and (3.11) we derive the proof of our Lemma. \square

We fix now $k \in (0, 1]$, and $K \subseteq X$ measurable such that $\mu(K) = k$. Then if $A = A(k)$, $B = B(k)$ are defined as in the proof of Lemma 3.2 we conclude that

$$\int_K (M_{\mathcal{T}}\phi)^p d\mu \leq A \omega_p \left(\frac{B^p}{k^{p-1}A} \right). \quad (3.12)$$

Note now that the A, B must satisfy the following conditions

- i) $B^p \leq k^{p-1}A$, because of Hölder's inequality for ϕ^* on the interval $(0, k]$.
- ii) $A \leq F$ and $B \leq f$,
- iii) $(f - B)^p \leq (1 - k)^{p-1}(F - A)$, because of Hölder's inequality for ϕ^* on the interval $[k, 1]$.

From all the above we conclude the following

Corollary 3.1.

$$B_p^{\mathcal{T}}(f, F, k) \leq \sup \left\{ A \omega_p \left(\frac{B^p}{k^{p-1}A} \right)^p : A, B \text{ satisfy i), ii) and iii) above} \right\}.$$

For the next Lemma we fix $0 < k < 1$ and we consider the function $h_k(B)$, defined by (2.2), for $0 \leq B \leq f$. Now by Lemma 2.1 and the condition iii) for A, B we immediately conclude the following

Corollary 3.2.

$$B_p^{\mathcal{T}}(f, F, k) \leq \sup \left\{ \left(F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right) \omega_p \left(\frac{B^p}{k^{p-1} \left(F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right)} \right)^p : \right. \\ \left. \text{for all } B \in [0, f] \text{ such that } h_k(B) \leq F \right\}. \quad (3.13)$$

We now prove that we have equality in 3.13. Fix $k \in (0, 1]$ and a B which satisfy the conditions stated in Corollary 3.2. We set $A = F - \frac{(f - B)^p}{(1 - k)^{p-1}}$ and we fix also a $\delta \in (0, 1)$.

We use now Lemma 2.2 to pick a family $\{I_1, I_2, \dots\}$ of pairwise disjoint elements of \mathcal{T} such that $\sum_j \mu(I_j) = k$ and since $\frac{B^p}{k^{p-1}} \leq A$, using the value of $B_p^{\mathcal{T}}(f, F, f, 1)$ which is evaluated in [20], for each j we choose a non-negative $\phi_j \in L^p \left(I_j, \frac{1}{\mu(I_j)} \mu \right)$ such that

$$\int_{I_j} \phi^p d\mu = \frac{A}{k} \mu(I_j), \quad \int_{I_j} \phi d\mu = \frac{B}{k} \mu(I_j), \quad (3.14)$$

$$\text{and } \int_{I_j} (\mathcal{M}_{\mathcal{T}(I_j)}(\phi_j))^p d\mu \geq \delta \frac{A}{k} \omega_p \left(\frac{B^p}{k^{p-1}A} \right)^p \mu(I_j), \quad (3.15)$$

where $\mathcal{T}(I_j)$ is the subtree of \mathcal{T} , defined by

$$\mathcal{T}(I_j) = \{I \in \mathcal{T} : I \subseteq I_j\}.$$

Next we choose $\psi \in L^p(X \setminus K, \mu)$ such that $\int_{X \setminus K} \psi^p d\mu = F - A > 0$ and $\int_{X \setminus K} \psi d\mu = f - B > 0$ which, in view of the value of A , must be in fact constant and equal to $\frac{f-B}{1-k} = \left(\frac{(F-A)}{(1-k)}\right)^{\frac{1}{p}}$. Here K stands for $K = \cup I_j \subseteq X$. Then we define $\phi = \psi \chi_{X \setminus K} + \sum_j \phi_j \chi_{I_j}$, and we obviously have

$$\int_X \phi^p d\mu = F \quad \text{and} \quad \int_X \phi d\mu = f. \quad (3.16)$$

Additionally we must have by (3.15) that

$$\begin{aligned} \int_K (M_{\mathcal{T}} \phi)^p d\mu &\geq \delta A \omega_p \left(\frac{B^p}{k^{p-1} A} \right)^p = \\ &\delta \left(F - \frac{(f-B)^p}{(1-k)^{p-1}} \right) \omega_p \left(\frac{B^p}{k^{p-1} \left(F - \frac{(f-B)^p}{(1-k)^{p-1}} \right)} \right)^p. \end{aligned} \quad (3.17)$$

Letting $\delta \rightarrow 1^-$ we obtain equality in 3.13, thus proving Theorem 1.1.

Corollary 3.3. *In the statement of Corollary 3.1 we have equality.*

Proof. Immediate, since we have equality on (3.13), and the right side of (3.13) is greater or equal than the right side of the inequality that is stated on Corollary 3.1. \square

4 Construction of the function g_k

We now proceed to prove Theorem 1.2.

Proof. As it has been proved in Corollary 3.2, it is true that:

$$B_p^T(f, F, k) = \sup \{ \mathcal{R}_k(B) : 0 \leq B \leq f, \text{ and } h_k(B) \leq F \}$$

where $\mathcal{R}_k(B)$, $h_k(B)$ are defined as in Section 2. By Lemma 2.4 ii), we see that the value B_0 satisfies the following equation

$$\omega_p(Z_0) = \frac{B_0}{k} \frac{1-k}{f-B_0} \quad (4.1)$$

where Z_0 is given by

$$Z_0 = \frac{B_0^p}{k^{p-1} \left(F - \frac{(f-B_0)^p}{(1-k)^{p-1}} \right)}.$$

We search for a function of the form

$$g_k(t) = \begin{cases} A_1 t^{-1+\frac{1}{a}}, & t \in (0, k] \\ c, & t \in (k, 1] \end{cases} \quad (4.2)$$

for some constants a, c, A_1 , which satisfies the properties

$$B_p^\mathcal{T}(f, F, k) = \int_0^k \left(\frac{1}{t} \int_0^t g_k \right)^p dt, \quad (4.3)$$

and

$$\int_0^1 g_k = f, \quad \int_0^1 g_k^p = F \quad (4.4)$$

Concerning the first equation in 4.4, we have

$$\begin{aligned} \int_0^1 g_k = f &\Leftrightarrow \int_0^k g_k + \int_k^1 g_k = f \Leftrightarrow \\ &\Leftrightarrow \int_0^k g_k + c(1-k) = f. \end{aligned} \quad (4.5)$$

We set $c = \frac{f-B_0}{1-k}$, in order to ensure that

$$\int_0^k g_k = B_0. \quad (4.6)$$

Note that (4.6) is (in view of (4.2)) equivalent to

$$\int_0^k A_1 t^{-1+\frac{1}{a}} dt = B_0 \Leftrightarrow A_1 = \frac{B_0 k^{-1/a}}{a}, \quad (4.7)$$

so that we found A_1 , in terms of a . We search now for a value of a such that the second equation in (4.4) is true. Thus we should have

$$\begin{aligned} A_1^p \int_0^k t^{-p+\frac{p}{a}} dt &= F - \frac{(f-B_0)^p}{(1-k)^{p-1}} \Leftrightarrow \\ \frac{B_0^p k^{-p/a}}{a^p} \frac{1}{1+\frac{p}{a}-p} k^{1-p+p/a} &= F - \frac{(f-B_0)^p}{(1-k)^{p-1}} \Leftrightarrow \\ \frac{B_0^p}{k^{p-1}} \frac{1}{p a^{p-1} - (p-1)a^p} &= F - \frac{(f-B_0)^p}{(1-k)^{p-1}} \Leftrightarrow \\ \frac{B_0^p}{k^{p-1} H_p(a)} = F - \frac{(f-B_0)^p}{(1-k)^{p-1}} &\Leftrightarrow H_p(a) = \frac{B_0^p}{k^{p-1} \left(F - \frac{(f-B_0)^p}{(1-k)^{p-1}} \right)} \Leftrightarrow \\ H_p(a) = Z_0 &\Leftrightarrow a = \omega_p(Z_0) \in \left[1, \frac{p}{p-1} \right] \end{aligned} \quad (4.8)$$

□

As a consequence, if A_1, a are given by (4.7) and (4.8) respectively, equations (4.4) are true. Note now that for every $t \in (0, k]$ we have that

$$\frac{1}{t} \int_0^t g_k = a g_k(t), \quad \forall t \in (0, k].$$

Thus

$$\begin{aligned} \int_0^k \left(\frac{1}{t} \int_0^t g_k \right)^p dt &= a^p \int_0^k g_k^p = \\ &= \left(F - \frac{(f - B_0)^p}{(1 - k)^{p-1}} \right) \omega_p \left(\frac{B_0^p}{k^{p-1} \left(F - \frac{(f - B_0)^p}{(1 - k)^{p-1}} \right)} \right)^p. \end{aligned} \quad (4.9)$$

By Theorem 1.1 and Lemma 2.4 ii), the right side of (4.9) equals $B_p^T(f, F, k)$. We need only to prove that g_k is continuous on $t_0 = k$. It is enough to show that

$$\frac{f - B_0}{1 - k} = A_1 k^{-1 + \frac{1}{a}} \Leftrightarrow A_1 k^{-1 + \frac{1}{a}} = \left(\frac{B_0}{k} \frac{1 - k}{f - B_0} \right)^{-1} \frac{B_0}{k} \quad (4.10)$$

By (4.1) and (4.8), $a = \omega_p(Z_0) = \frac{B_0}{k} \frac{1 - k}{f - B_0}$. Thus (4.10) is equivalent to $A_1 k^{-1 + \frac{1}{a}} = a^{-1} \frac{B_0}{k}$, which is just (4.7). Theorem 1.2 is now proved.

5 A multiparameter inequality for $M_{\mathcal{T}}$

We begin by describing a linearization of the dyadic maximal operator, as it was introduced in [10]. First we give the notion of the \mathcal{T} -good function. Let $\phi \in L^1(X, \mu)$ be a non-negative function and for any $I \in \mathcal{T}$, set $\text{Av}_I(\phi) = \frac{1}{\mu(I)} \int_I \phi d\mu$. We will say that ϕ is \mathcal{T} -good, if the set

$$\mathcal{A}_\phi = \{x \in X : \mathcal{M}_{\mathcal{T}}\phi(x) > \text{Av}_I(\phi) \text{ for all } I \in \mathcal{T} \text{ such that } x \in I\}$$

has μ -measure zero.

For example one can define, for any $m \geq 0$, and $\lambda_I \geq 0$ for each $I \in \mathcal{T}_{(m)}$ (the m -level of the tree \mathcal{T}), the following function

$$\phi = \sum_{I \in \mathcal{T}_{(m)}} \lambda_I \chi_I,$$

where χ_I denotes the characteristic function of I . It is an easy matter to show that ϕ is \mathcal{T} -good.

Suppose that we are given a \mathcal{T} -good function ϕ . For any $x \in X \setminus \mathcal{A}_\phi$ (that is for μ -almost all $x \in X$), we denote by $I_\phi(x)$ the largest element in the non empty set

$$\{I \in \mathcal{T} : x \in I \text{ and } \mathcal{M}_{\mathcal{T}}\phi(x) = \text{Av}_I(\phi)\}.$$

We also define for any $I \in \mathcal{T}$

$$A(\phi, I) = \{x \in X \setminus \mathcal{A}_\phi : I_\phi(x) = I\}, \text{ and we set}$$

$$S_\phi = \{I \in \mathcal{T} : \mu(A(\phi, I)) > 0\} \cup \{X\}.$$

It is obvious that $\mathcal{M}_\mathcal{T}\phi = \sum_{I \in S_\phi} \text{Av}_I(\phi) \chi_{A(\phi, I)}$, μ -almost everywhere.

We also define the following correspondence $I \rightarrow I^*$ with respect to S_ϕ : I^* is the smallest element of $\{J \in S_\phi : I \subsetneq J\}$. This is defined for every $I \in S_\phi$ except X . It is clear that the family of sets $\{A(\phi, I) : I \in S_\phi\}$ consists of pairwise disjoint sets and its union has full measure on X , since $\mu(\cup_{J \in S_\phi} A(\phi, J)) = 0$.

We give without proof a lemma (appearing in [10]) which describes the properties of the class S_ϕ , and those of the sets $A(\phi, I)$, $I \in S_\phi$.

Lemma 5.1. *i) If $I, J \in S_\phi$ then either $A(\phi, J) \cap I = \emptyset$ or $J \subseteq I$.*

ii) If $I \in S_\phi$, then there exists $J \in C(I)$ such that $J \notin S_\phi$.

iii) For every $I \in S_\phi$ we have that

$$I \approx \bigcup_{S_\phi \ni J \subseteq I} A(\phi, J).$$

iv) For every $I \in S_\phi$ we have that

$$A(\phi, I) = I \setminus \bigcup_{J \in S_\phi : J^* = I} J, \text{ and thus}$$

$$\mu(A(\phi, I)) = \mu(I) - \sum_{J \in S_\phi : J^* = I} \mu(J).$$

Here by writing $A \approx B$, we mean that A, B are measurable subsets of X such that $\mu(A \setminus B) = \mu(B \setminus A) = 0$.

From the above lemma we immediately get that

$$\text{Av}_I(\phi) = \frac{1}{\mu(I)} \sum_{J \in S_\phi : J \subseteq I} \int_{A(\phi, J)} \phi \, d\mu,$$

for any $I \in S_\phi$. We are now in position to prove the first part of Theorem 1.3, that is the validity of 1.11.

Proof. We begin by considering a \mathcal{T} -good function ϕ , satisfying $\int_X \phi \, d\mu = f$ and $\int_X \phi^p \, d\mu = F$. Let K be a measurable subset of X , with $\mu(K) = k \in (0, 1]$ and β, γ such that $\beta > \gamma > 0$.

By Lemma 5.1 we get that $F = \int_X \phi^p \, d\mu = \sum_{I \in S_\phi} \int_{A_I} \phi^p$, where we write A_I for the set $A(\phi, I)$, $I \in S_\phi$. We split the set A_I in two measurable subsets B_I, Γ_I for any $I \in S_\phi$, where $\mu(B_I), \mu(\Gamma_I) > 0$. The choice of B_I, Γ_I will be given in the sequel. Write $\mu(A_I) = a_I$, for $I \in S_\phi$. For any $I \in S_\phi$ we search for a constant $\tau_I > 0$ for which

$$\mu(I) \tau_I - (\beta + 1) \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) - (\gamma + 1) \mu(B_I) = \mu(\Gamma_I), \quad (5.1)$$

Then (5.1) in view of Lemma 5.1 is equivalent to

$$\begin{aligned} \mu(I) \tau_I - (\beta + 1) (\mu(I) - \mu(A_I)) - (\gamma + 1) \mu(B_I) &= \mu(\Gamma_I) \Leftrightarrow \\ [\tau_I - (\beta + 1)] \mu(I) + (\beta + 1) \mu(B_I) + (\beta + 1) \mu(\Gamma_I) - (\gamma + 1) \mu(B_I) &= \mu(\Gamma_I) \Leftrightarrow \\ [\tau_I - (\beta + 1)] \mu(I) + \beta \mu(\Gamma_I) &= (\gamma - \beta) \mu(B_I), \end{aligned} \quad (5.2)$$

We let $\mu(\Gamma_I) = k_I a_I$, for some $k_I \in (0, 1)$, so $\mu(B_I) = (1 - k_I) a_I$. Thus (5.2) becomes

$$\begin{aligned} [\tau_I - (\beta + 1)] \mu(I) &= (\gamma - \beta)(1 - k_I) a_I - \beta k_I a_I \Leftrightarrow \\ [\tau_I - (\beta + 1)] \mu(I) &= \gamma(1 - k_I) a_I - \beta a_I, \end{aligned} \quad (5.3)$$

We now set $p_I = \frac{a_I}{\mu(I)}$, for any $I \in S_\phi$. Thus (5.3) gives

$$\begin{aligned} \tau_I - (\beta + 1) &= \gamma(1 - k_I) p_I - \beta p_I \Leftrightarrow \\ \tau_I &= ((\beta + 1) - \beta p_I) + (1 - k_I) \gamma p_I, \end{aligned} \quad (5.4)$$

Note that this choice of $\tau_I, I \in S_\phi$, immediately gives $\tau_I > 0$, since $\beta > \gamma > 0$ and $0 < p_I \leq 1$ for any $I \in S_\phi$.

We write now

$$\begin{aligned} F &= \sum_{I \in S_\phi} \int_{A_I} \phi^p d\mu = \sum_{I \in S_\phi} \int_{B_I} \phi^p d\mu + \sum_{I \in S_\phi} \int_{\Gamma_I} \phi^p d\mu \geq \\ &\geq \sum_{I \in S_\phi} \frac{\left(\int_{B_I} \phi d\mu \right)^p}{\mu(B_I)^{p-1}} + \sum_{I \in S_\phi} \frac{\left(\int_{\Gamma_I} \phi d\mu \right)^p}{\mu(\Gamma_I)^{p-1}}, \end{aligned} \quad (5.5)$$

in view of Hölder's inequality. We denote the first and the second sum on the right of (5.5) by Σ_1, Σ_2 respectively. Then by (5.1) Lemma 5.1 iv) we have the following

$$\begin{aligned} \Sigma_2 &= \sum_{I \in S_\phi} \frac{1}{\mu(\Gamma_I)^{p-1}} \left(\int_I \phi d\mu - \sum_{\substack{J \in S_\phi \\ J^* = I}} \int_J \phi d\mu - \int_{B_I} \phi d\mu \right)^p = \\ &= \sum_{I \in S_\phi} \frac{\left(\mu(I) y_I - \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) y_J - \int_{B_I} \phi d\mu \right)^p}{\left(\tau_I \mu(I) - (\beta + 1) \sum_{\substack{J \in S_\phi \\ J^* = I}} \mu(J) - (\gamma + 1) \mu(B_I) \right)^{p-1}}, \end{aligned} \quad (5.6)$$

where $y_I = Av_I(\phi)$, for every $I \in S_\phi$. Now because of Hölder's inequality in the form

$$\frac{(\lambda_1 + \lambda_2 + \dots + \lambda_\nu)^p}{(\mu_1 + \mu_2 + \dots + \mu_\nu)^{p-1}} \leq \frac{\lambda_1^p}{\mu_1^{p-1}} + \frac{\lambda_2^p}{\mu_2^{p-1}} + \dots + \frac{\lambda_\nu^p}{\mu_\nu^{p-1}}, \quad (5.7)$$

where $p > 1$, $\mu_i > 0$ and $\lambda_i \geq 0$, for $i = 1, 2, \dots, \nu$, we have in view of (5.6) that:

$$\begin{aligned} \Sigma_2 \geq & \sum_{I \in S_\phi} \frac{(\mu(I)y_I)^p}{(\tau_I \mu(I))^{p-1}} - \sum_{I \in S_\phi} \sum_{\substack{J \in S_\phi \\ J^* = I}} \frac{(\mu(J)y_J)^p}{((\beta + 1)\mu(J))^{p-1}} - \\ & - \sum_{I \in S_\phi} \frac{1}{(\gamma + 1)^{p-1}} \frac{\left(\int_{B_I} \phi d\mu \right)^p}{\mu(B_I)^{p-1}}. \end{aligned} \quad (5.8)$$

By (5.8) we obtain

$$\begin{aligned} \Sigma_1 + \Sigma_2 \geq & \left(1 - \frac{1}{(1 + \gamma)^{p-1}} \right) \Sigma_1 + \sum_{I \in S_\phi} \mu(I) \frac{y_I^p}{\tau_I^{p-1}} - \sum_{\substack{I \in S_\phi \\ I \neq X}} \mu(I) \frac{y_I^p}{(\beta + 1)^{p-1}} = \\ = & \left(1 - \frac{1}{(1 + \gamma)^{p-1}} \right) \Sigma_1 + \frac{y_X^p}{\tau_X^{p-1}} + \sum_{\substack{I \in S_\phi \\ I \neq X}} \mu(I) y_I^p \left(\frac{1}{\tau_I^{p-1}} - \frac{1}{(\beta + 1)^{p-1}} \right) = \\ = & \left(1 - \frac{1}{(1 + \gamma)^{p-1}} \right) \Sigma_1 + \frac{f^p}{\tau_X^{p-1}} + \\ + & \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{a_I}{p_I} \left(\frac{1}{((\beta + 1 - \beta p_I) + (1 - k_I)\gamma p_I)^{p-1}} - \frac{1}{(\beta + 1)^{p-1}} \right) y_I^p. \end{aligned} \quad (5.9)$$

Note that in (5.9) we have used the properties of the correspondence $I \longrightarrow I^*$, on S_ϕ .

We denote now Σ_3 the sum on the right of (5.9). Then

$$\Sigma_3 \geq \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{1}{p_I} \left[\frac{\beta p_I - (1 - k_I)\gamma p_I}{(\beta + 1)^p} (p - 1) \right] a_I y_I^p, \quad (5.10)$$

because of the inequality

$$\frac{1}{((\beta + 1) - s)^{p-1}} - \frac{1}{(\beta + 1)^{p-1}} \geq \frac{(p - 1)s}{(\beta + 1)^p}, \quad (5.11)$$

which is true for any $\beta > 0$, and $s \in [0, \beta]$, by the mean value theorem on derivatives. Note that since $\beta > \gamma > 0$, we have that the quantity $s = \beta p_I - (1 - k_I)\gamma p_I$ is positive and less than β so (5.11) applies in Σ_3 , and gives (5.10).

Thus

$$\begin{aligned}
\Sigma_3 &\geq (p-1) \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{\beta - \gamma}{(\beta + 1)^p} a_I y_I^p + (p-1) \sum_{\substack{I \in S_\phi \\ I \neq X}} \frac{k_I \gamma}{(\beta + 1)^p} a_I y_I^p = \\
&= (p-1) \frac{\beta - \gamma}{(\beta + 1)^p} \sum_{I \in S_\phi} a_I y_I^p - (p-1) \frac{\beta - \gamma}{(\beta + 1)^p} a_X y_X^p + \\
&\quad + (p-1) \frac{\gamma}{(\beta + 1)^p} \sum_{I \in S_\phi} k_I a_I y_I^p - \frac{(p-1)\gamma}{(\beta + 1)^p} k_X a_X y_X^p = \\
&= (p-1) \frac{\beta - \gamma}{(\beta + 1)^p} \int_X (M_{\mathcal{T}} \phi)^p d\mu + \\
&\quad + (p-1) \frac{\gamma}{(\beta + 1)^p} \int_\Gamma (M_{\mathcal{T}} \phi)^p d\mu - \frac{(p-1)}{(\beta + 1)^p} ((\beta - \gamma) a_X + \gamma k_X a_X) f^p, \quad (5.12)
\end{aligned}$$

where we have set $\Gamma = \bigcup_{I \in S_\phi} \Gamma_I$.

By (5.9) and (5.12) we get

$$\begin{aligned}
\Sigma_1 + \Sigma_2 &\geq \left(1 - \frac{1}{(1 + \gamma)^{p-1}}\right) \Sigma_1 + (p-1) \frac{\beta - \gamma}{(\beta + 1)^p} \int_X (M_{\mathcal{T}} \phi)^p d\mu + \\
&\quad + (p-1) \frac{\gamma}{(\beta + 1)^p} \int_\Gamma (M_{\mathcal{T}} \phi)^p d\mu + \lambda_4, \quad (5.13)
\end{aligned}$$

where

$$\lambda_4 = \frac{f^p}{\tau_X^{p-1}} - \frac{(p-1)}{(\beta + 1)^p} ((\beta - \gamma) a_X + \gamma k_X a_X) f^p. \quad (5.14)$$

By definition of τ_X , (5.14) gives

$$\begin{aligned}
\lambda_4 &= f^p \left[\frac{1}{((\beta + 1) - \beta p_X + (1 - k_X) \gamma p_X)^{p-1}} - (p-1) \frac{(\beta - \gamma) a_X + \gamma a_X k_X}{(\beta + 1)^p} \right] = \\
&= f^p \left(\frac{1}{((\beta + 1) - \delta)^{p-1}} - (p-1) \frac{\delta}{(\beta + 1)^p} \right), \quad (5.15)
\end{aligned}$$

where $\delta = (\beta - \gamma) a_X + \gamma a_X k_X$ (note that we used that $p_X = a_X$).

Now because of the inequality (5.11) we have that

$$\frac{1}{((\beta + 1) - s)^{p-1}} - (p-1) \frac{s}{(\beta + 1)^p} \geq \frac{1}{(\beta + 1)^{p-1}}, \forall s \in [0, \beta]$$

and note that $\delta \in (0, \beta)$, by the definition of δ .

So (5.15) gives $\lambda_4 \geq \frac{f^p}{(\beta+1)^{p-1}}$. Then, by (5.13) we have

$$\begin{aligned} \Sigma_1 + \Sigma_2 &\geq \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \Sigma_1 + (p-1) \frac{\beta-\gamma}{(\beta+1)^p} \int_X (M_{\mathcal{T}}\phi)^p d\mu + \\ &\quad + (p-1) \frac{\gamma}{(\beta+1)^p} \int_{\Gamma} (M_{\mathcal{T}}\phi)^p d\mu + \frac{f^p}{(\beta+1)^{p-1}} = \\ &\quad = \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \Sigma_1 + \\ &\quad + \frac{(p-1)}{(\beta+1)^p} \left[\beta \int_X (M_{\mathcal{T}}\phi)^p d\mu - \gamma \int_B (M_{\mathcal{T}}\phi)^p d\mu \right] + \frac{f^p}{(\beta+1)^{p-1}}, \end{aligned} \quad (5.16)$$

where $B = \bigcup_{I \in S_{\phi}} B_I = X \setminus \Gamma$.

Now (5.16) gives

$$\begin{aligned} \Sigma_2 + \frac{1}{(1+\gamma)^{p-1}} \Sigma_1 &\geq \\ &\geq \frac{f^p}{(\beta+1)^{p-1}} + \frac{(p-1)}{(\beta+1)^p} \left[\beta \int_X (M_{\mathcal{T}}\phi)^p d\mu - \gamma \int_B (M_{\mathcal{T}}\phi)^p d\mu \right] \Rightarrow \\ &\quad \frac{1}{(1+\gamma)^{p-1}} (\Sigma_1 + \Sigma_2) + \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \Sigma_2 \geq \\ &\geq \frac{f^p}{(\beta+1)^{p-1}} + \frac{(p-1)}{(\beta+1)^p} \left[\beta \int_X (M_{\mathcal{T}}\phi)^p d\mu - \gamma \int_B (M_{\mathcal{T}}\phi)^p d\mu \right]. \end{aligned} \quad (5.17)$$

But $F \geq \Sigma_1 + \Sigma_2$, and $\Sigma_2 \leq \int_{\Gamma} \phi^p d\mu$, so that we conclude from (5.17) that

$$\begin{aligned} \frac{F}{(1+\gamma)^{p-1}} + \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \int_{\Gamma} \phi^p d\mu &\geq \frac{f^p}{(\beta+1)^{p-1}} + \\ &\quad \frac{(p-1)}{(\beta+1)^p} \left[\beta \int_X (M_{\mathcal{T}}\phi)^p d\mu - \gamma \int_B (M_{\mathcal{T}}\phi)^p d\mu \right] \end{aligned} \quad (5.18)$$

Now from (5.18) we immediately get

$$\begin{aligned} \frac{1}{(1+\gamma)^{p-1}} \int_B \phi^p d\mu + \int_{\Gamma} \phi^p d\mu &\geq \\ &\geq \frac{f^p}{(\beta+1)^{p-1}} + \frac{(p-1)}{(\beta+1)^p} \left[\beta \int_X (M_{\mathcal{T}}\phi)^p d\mu - \gamma \int_B (M_{\mathcal{T}}\phi)^p d\mu \right]. \end{aligned} \quad (5.19)$$

But the left side of (5.19) equals $F - \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \int_B \phi^p d\mu$, so that (5.19) becomes

$$\begin{aligned} F &\geq \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \int_B \phi^p d\mu + \frac{f^p}{(\beta+1)^{p-1}} + \\ &\quad + \frac{(p-1)\beta}{(\beta+1)^p} \int_X (M_{\mathcal{T}}\phi)^p d\mu - \frac{(p-1)\gamma}{(\beta+1)^p} \int_B (M_{\mathcal{T}}\phi)^p d\mu. \end{aligned} \quad (5.20)$$

Inequality (5.20) is in fact true for every choice of B , since every measurable subset B , of X can be written as $B = \bigcup_{I \in S_\phi} B_I$, where $B_I = B \cap A_I$. Then setting $\Gamma_I = A_I \setminus B_I$ and following the above proof, we obtain the validity of (5.20). Theorem 1.3 is thus proved for any ϕ which is \mathcal{T} -good function (replace B by K). Note that in the above proof we have used the fact that $\mu(B_I) > 0$, for every $I \in S_\phi$, but this can be applied (by using the fact that (X, μ) is nonatomic) to prove (5.20) even if $\mu(B_I) = 0$, for some $I \in S_\phi$. Now if $\phi \in L^p(X, \mu)$ is arbitrary, we consider the sequence $(\phi_m)_m$, where $\phi_m = \sum_{J \in \mathcal{T}_{(m)}} Av_J(\phi) \chi_J$, and we set $\Phi_m = \sum_{J \in \mathcal{T}_{(m)}} \max \{Av_I(\phi) : J \subseteq I \in \mathcal{T}\} \chi_I$.

Then since $Av_J(\phi) = Av_I(\phi_m)$, for any $J \in \mathcal{T}$ for which $J \subseteq I \in \mathcal{T}_{(m)}$, we immediately see that $\Phi_m = M_{\mathcal{T}}\phi_m$.

Obviously $\int_X \phi_m d\mu = \int_X \phi d\mu = f$, and we can easily see that $F_m = \int_X \phi_m^p d\mu \leq \int_X \phi^p d\mu = F$. That is $\phi_m \in L^p(X, \mu), \forall m \in \mathbb{N}$.

Additionally Φ_m converges monotonically to $M_{\mathcal{T}}\phi$. Now ϕ_m is \mathcal{T} -good for any $m \in \mathbb{N}$, so that (5.20) is true, for ϕ_m , and for any $B \subseteq X$ measurable. Since $M_{\mathcal{T}}\phi_m$ increases to $M_{\mathcal{T}}\phi$ on X , we get

$$\lim_m \int_X (M_{\mathcal{T}}\phi_m)^p d\mu = \int_X (M_{\mathcal{T}}\phi)^p d\mu,$$

and

$$\lim_m \int_B (M_{\mathcal{T}}\phi_m)^p d\mu = \int_B (M_{\mathcal{T}}\phi)^p d\mu,$$

while by the construction of ϕ_m , and the fact that the tree \mathcal{T} differentiates $L^1(X, \mu)$ we obtain that $\phi_m \rightarrow \phi$, μ -a.e on X . Now since $\phi_m \leq M_{\mathcal{T}}\phi_m \leq M_{\mathcal{T}}\phi$ and $M_{\mathcal{T}}\phi \in L^p(X, \mu)$ (because $\phi \in L^p(X, \mu)$), we have, using the dominated convergence theorem that $\lim_m \int_X \phi_m^p = F$ and $\lim_m \int_B \phi_m^p = \int_B \phi^p d\mu$. From all these facts we deduce the validity of (5.20) for general $\phi \in L^p(X, \mu)$. For $\beta = \gamma > 0$, (5.20) remains true by continuity reasons. \square

6 Sharpness of inequality (1.11) and applications

Let $h : (0, 1] \rightarrow \mathbb{R}^+$ be an arbitrary non-increasing function such that $\int_0^1 h = f$ and $\int_0^1 h^p = F$. Let $k \in (0, 1]$ and fix a non atomic probability space (X, μ) , equipped with a tree structure \mathcal{T} , such that \mathcal{T} differentiates $L^p(X, \mu)$. By the proof of Theorem 2.1 (see [20]), we can construct a family $(\phi_\alpha)_{\alpha \in (0, 1]}$, of non-negative measurable functions defined on (X, μ) , and a family $(K_\alpha)_{\alpha \in (0, 1]}$ of measurable subsets of X , such that the following hold: $\phi_\alpha^* = h, \forall \alpha \in (0, 1]$, $\lim_{\alpha \rightarrow 0+} \int_{K_\alpha} (M_{\mathcal{T}}\phi_\alpha)^p d\mu = \int_0^k \left(\frac{1}{t} \int_0^t h\right)^p dt$, $\lim_{\alpha \rightarrow 0+} \int_{K_\alpha} \phi_\alpha^p d\mu = \int_0^k h^p$ and $\lim_{\alpha \rightarrow 0+} \mu(K_\alpha) = k$. If we apply the inequality (1.11), for ϕ_α and K_α , for any $\alpha \in (0, 1]$, we get:

$$F \geq \left(1 - \frac{1}{(1+\gamma)^{p-1}}\right) \int_{K_\alpha} \phi_\alpha^p d\mu + \frac{f^p}{(\beta+1)^{p-1}} + \frac{(p-1)\beta}{(\beta+1)^p} \int_X (M_{\mathcal{T}}\phi_\alpha)^p d\mu - \frac{(p-1)\gamma}{(\beta+1)^p} \int_{K_\alpha} (M_{\mathcal{T}}\phi_\alpha)^p d\mu, \quad (6.1)$$

for any $\beta \geq \gamma > 0$.

Obviously $\int_X \phi_\alpha d\mu = f$ and $\int_X \phi_\alpha^p d\mu = F$, since $\phi_\alpha^* = h, \forall \alpha \in (0, 1]$. Letting $\alpha \rightarrow 0^+$, we immediately see by (6.1) that

$$\frac{(p-1)\beta}{(\beta+1)^p} \int_0^1 \left(\frac{1}{t} \int_0^t h\right)^p dt \leq \frac{(p-1)\gamma}{(\beta+1)^p} \int_0^k \left(\frac{1}{t} \int_0^t h\right)^p dt + F - \frac{f^p}{(\beta+1)^{p-1}} + \left(\frac{1}{(1+\gamma)^{p-1}} - 1\right) \int_0^k h^p. \quad (6.2)$$

Set now $\delta = \delta_k = \left(\frac{\int_0^k (\frac{1}{t} \int_0^t h)^p dt}{\int_0^k h^p}\right)^{\frac{1}{p}}$. Obviously $1 \leq \delta < \frac{p}{p-1}$ and $\delta = 1 \Leftrightarrow h$ is constant on $(0, k]$. We assume that $\beta > \delta - 1$. We wish, for any such β , to minimize the right side of (6.2), with respect to $\gamma \in (0, \beta)$. For this purpose we define

$$G_\beta(\gamma) = \frac{(p-1)\gamma}{(\beta+1)^p} \int_0^k \left(\frac{1}{t} \int_0^t h\right)^p dt + \frac{1}{(1+\gamma)^{p-1}} \int_0^k h^p,$$

for $\gamma \in (0, \beta]$. Note that

$$G'_\beta(\gamma) = \frac{(p-1)}{(\beta+1)^p} \int_0^k \left(\frac{1}{t} \int_0^t h\right)^p dt - \frac{(p-1)}{(\gamma+1)^p} \int_0^k h^p.$$

Then $G'_\beta(\gamma) = 0 \Leftrightarrow \frac{\beta+1}{\gamma+1} = \delta \Leftrightarrow \gamma = \frac{\beta+1}{\delta} - 1$. Since $\beta > \delta - 1$, if we set $\gamma_0 = \frac{\beta+1}{\delta} - 1$ we have that $\gamma_0 \in (0, \beta]$. We easily get now that $\min \{G_\beta(\gamma) : \gamma \in (0, \beta]\} = G_\beta(\gamma_0)$. Replacing the value γ_0 into (6.2) for any $\beta > \delta - 1$, and using the definition of δ we get

$$\int_0^1 \left(\frac{1}{t} \int_0^t h\right)^p dt \leq \frac{1}{\beta} \left(\frac{\beta+1}{\delta} - 1\right) \delta^p \int_0^k h^p + \frac{(\beta+1)^p}{(p-1)\beta} F - \frac{(\beta+1)}{(p-1)\beta} f^p + \frac{(\beta+1)^p}{(p-1)\beta} \left(-1 + \left(\frac{\delta}{\beta+1}\right)^{p-1}\right) \int_0^k h^p, \quad (6.3)$$

$\forall \beta > \delta - 1$.

Now the right side of (6.3), equals

$$\begin{aligned}
& \frac{(\beta+1)}{\beta(p-1)} \left(p\delta^{p-1} \int_0^k h^p - f^p \right) - \\
& - \left(\frac{\beta+1}{\beta} - 1 \right) \delta^p \int_0^k h^p + \frac{(\beta+1)^p}{(p-1)\beta} \left(F - \int_0^k h^p \right) = \\
& = \frac{(\beta+1)}{\beta(p-1)} \left(p\delta^{p-1} \int_0^k h^p - (p-1)\delta^p \int_0^k h^p - f^p \right) + \\
& \quad + \delta^p \int_0^k h^p + \frac{(\beta+1)^p}{(p-1)\beta} \int_k^1 h^p = \\
& = \frac{(\beta+1)}{\beta(p-1)} \left(H_p(\delta) \int_0^k h^p - f^p \right) + \delta^p \int_0^k h^p + \frac{(\beta+1)^p}{(p-1)\beta} \int_k^1 h^p = \\
& \quad = \int_0^k \left(\frac{1}{t} \int_0^t h \right)^p dt + \Lambda(\beta)
\end{aligned}$$

where

$$\Lambda(\beta) = \frac{(\beta+1)^p}{(p-1)\beta} \int_k^1 h^p + \frac{(\beta+1)}{(p-1)\beta} \left(H_p(\delta) \int_0^k h^p - f^p \right) \quad (6.4)$$

Assume also that δ satisfies

$$\delta \leq \omega_p \left(\frac{f^p}{F} \right). \quad (6.5)$$

We wish to find the minimum value of $\Lambda(\beta)$, for $\beta > \delta - 1$, when δ satisfies (6.5).

It is a simple matter to show that

$$\Lambda'(\beta) = -\frac{H_p(\beta+1)}{(p-1)\beta^2} \int_k^1 h^p - \frac{1}{(p-1)\beta^2} \left(H_p(\delta) \int_0^k h^p - f^p \right).$$

We solve now the equation $\Lambda'(\beta) = 0 \Leftrightarrow$

$$H_p(\beta+1) = \frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p}. \quad (6.6)$$

Note that the right side of (6.6) is less or equal than $H_p(\delta)$, that is

$$H_p(\delta) \geq \frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \quad (6.7)$$

Indeed (6.7) is equivalent to $H_p(\delta)F \geq f^p \Leftrightarrow \delta \leq \omega_p \left(\frac{f^p}{F} \right)$, which is true in view of the assumption that we made on δ .

Now H_p , defined on $[1, +\infty)$ satisfies the following: $H_p(1) = 1$, H_p is strictly decreasing, and $\lim_{x \rightarrow +\infty} H_p(x) = -\infty$.

Thus there exists a unique value $\beta_0 > \delta - 1$ for which, we have equality in (6.6). That is

$$H_p(\beta_0 + 1) = \frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \quad (6.8)$$

As is easily seen for this value of β_0 we have that $\min_{\delta-1 < \beta < +\infty} \Lambda(\beta) = \Lambda(\beta_0)$, thus (6.3) and (6.4) give in view of the above calculations that

$$\int_k^1 \left(\frac{1}{t} \int_0^t h \right)^p dt \leq \frac{(\beta_0 + 1)^p}{(p-1)\beta_0} \int_k^1 h^p + \frac{\beta_0 + 1}{(p-1)\beta_0} \left(H_p(\delta) \int_0^k h^p - f^p \right). \quad (6.9)$$

It is not difficult now to show, that the right side of (6.9) equals

$$\int_k^1 h^p \omega_p \left(\frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \right)^p, \quad \text{where } \omega_p : (-\infty, 1] \longrightarrow [1, +\infty)$$

is the inverse of $H_p : H_p^{-1}$. Thus (6.9) states that for any $h : (0, 1] \longrightarrow \mathbb{R}^+$ non-increasing, with $\int_0^1 h = f$, $\int_0^1 h^p = F$ and any $k \in (0, 1]$ for which $\delta_k \leq \omega_p \left(\frac{f^p}{F} \right)$, we have:

$$\int_k^1 \left(\frac{1}{t} \int_0^t h \right)^p dt \leq \int_k^1 h^p \omega_p \left(\frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \right)^p. \quad (6.10)$$

Note that (6.10) is sharp since if we consider the function $h = g_1$ (that is g_k for $k = 1$ - see Section 4), we get by the properties that g_1 satisfies, that $\int_0^1 h = f$, $\int_0^1 h^p = F$ and $\frac{1}{t} \int_0^t h = \omega_p \left(\frac{f^p}{F} \right) h(t)$, $\forall t \in (0, 1]$.

Thus for any $k \in (0, 1]$ we have $\delta_k = \omega_p \left(\frac{f^p}{F} \right)$ and then the right side of (6.10) equals:

$$\begin{aligned} \int_k^1 h^p \omega_p \left(\frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \right)^p &= \int_k^1 h^p \omega_p \left(\frac{f^p}{F} \frac{F - \int_0^k h^p}{\int_k^1 h^p} \right)^p = \\ &= \int_k^1 h^p \omega_p \left(\frac{f^p}{F} \right)^p = \int_k^1 \left(\frac{1}{t} \int_0^t h \right)^p dt \end{aligned}$$

so we have equality in (6.10) for this choice of h . Using Theorem 6.1, the sharpness of (6.10), and the calculus arguments that are given right above we conclude, by choosing $\beta = \delta - 1$ and letting γ tend to zero, the sharpness of inequality (1.11), for any $k \in (0, 1]$.

In fact we have proved that the sharpness of inequality (1.11) is reduced to the sharpness of (1.10), for $\beta = \omega_p \left(\frac{f^p}{F} \right) - 1$. We gave the above proof in order to reach the inequality (6.10) which importance is seen below.

Let $h : (0, 1] \rightarrow \mathbb{R}^+$ be a non-increasing function, satisfying $\int_0^1 h = f$ and $\int_0^1 h^p = F$, then the set of k 's belonging on $(0, 1)$ for which $\delta_k \leq \omega_p\left(\frac{f^p}{F}\right)$ is a non empty open subset of $(0, 1]$

This is obviously true for $h = g_1$, while if $h \neq g_1$ we have that

$$\int_0^1 \left(\frac{1}{t} \int_0^t h \right)^p dt < F \omega_p \left(\frac{f^p}{F} \right)^p,$$

because g_1 is the unique non-increasing function on $(0, 1]$ for which we get $\int_0^1 g_1 = f$, $\int_0^1 g_1^p = F$ and $\int_0^1 \left(\frac{1}{t} \int_0^t g_1 \right)^p dt = F \omega_p \left(\frac{f^p}{F} \right)^p$, (see [17]).

Thus considering such a $k \in (0, 1]$, we get by (6.10), that for any $h : (0, 1] \rightarrow \mathbb{R}^+$ non-increasing with $\int_0^1 h = f$, the inequality

$$H_p(\delta'_k) \geq \left(\frac{f^p - H_p(\delta) \int_0^k h^p}{\int_k^1 h^p} \right) \quad (6.11)$$

is true, where $\delta'_k = \left(\frac{\int_k^1 \left(\frac{1}{t} \int_0^t h \right)^p dt}{\int_k^1 h^p} \right)^{\frac{1}{p}}$, or that

$$H_p(\delta_k) \int_0^k h^p + H_p(\delta'_k) \int_k^1 h^p \geq f^p, \quad (6.12)$$

for any h and $k \in (0, 1]$ as above.

Inequality (6.12) and its sharpness gives us even more information for the geometric behaviour of $\mathcal{M}_{\mathcal{T}}$ because of the appearance of the free parameter $k \in (0, 1]$ which is invoked under the condition $\delta_k \leq \omega_p\left(\frac{f^p}{F}\right)$. Note at last that in view of Theorem 2.1, the validity and sharpness of inequality (6.12) for $k = 1$, gives the results in [10], that is the determination of the Bellman function of two variables for $\mathcal{M}_{\mathcal{T}}$.

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Nikolidakis Eleftherios
 Assistant Professor
 Department of Mathematics
 Panepistimioupolis, University of Ioannina, 45110
 Greece
 E-mail address: enikolid@uoi.gr