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OBSTRUCTIONS TO ISOMETRIC IMMERSIONS

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Dedicated to Athina Onti, Niki Onti, Christos Patlias and Niki Patlia

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Introduction

According to Nash's embedding theorem we know that every Riemannian manifold can be isometrically immersed into a Euclidean space with sufficiently high codimension. However, there are results that impose restrictions on the existence of isometric immersions with low codimension. To this direction, a classical result due to Chern and Kuiper [16], Otsuki [51], states that a compact n-dimensional Riemannian manifold of non-positive sectional curvature cannot be isometrically immersed in the Euclidean space $\mathbb{R}^{2n-1}$. On the other hand, Moore [41–44] investigated topological restrictions on isometric immersions of positively curved Riemannian manifolds into Euclidean space with low codimension. In particular, he proved in [43] that every compact n-dimensional Riemannian manifold with positive sectional curvature that admits an isometric immersion into $\mathbb{R}^{n+2}$, is homeomorphic to the sphere $\mathbb{S}^n$. That result was improved by Ziller and Florit in [31], where they proved that M^n is actually diffeomorphic to $\mathbb{S}^n$. Moreover, in [42, 44], Moore proved that every compact n-dimensional Riemannnian manifold with constant curvature that admits an isometric immersion into $\mathbb{R}^{2n-1}$, is isometric to the round sphere.

Shiohama and Xu were concerned in [58] with the investigation of integral curvature bounds in terms of the Betti numbers for compact submanifolds of the Euclidean space. In particular, they considered the $L^{n/2}$-norm of a certain tensor $\widetilde{R}$ of a compact n-dimensional Riemannian manifold (M^n, g), which measures how far (M^n, g) deviates from having constant sectional curvature. They were able to give a lower bound in terms of the Betti numbers in the case where (M^n, g) admits an isometric immersion as a hypersurface into the Euclidean space $\mathbb{R}^{n+1}$. Their proof strongly uses the fact that the ambient space is the Euclidean one. In Section 3.4, we extend their result for compact hypersurfaces in spheres or in the hyperbolic space. This result is contained in [49].

For higher codimension, they posed the following:

Problem. If (M^n, g), $n \geq 3$, is a compact n-dimensional Riemannian manifold that admits an isometric immersion into the Euclidean space $\mathbb{R}^{2n-1}$, does there exist a positive constant $\varepsilon(n)$ depending only on n such that if

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM < \varepsilon(n),$$

then M^n is homeomorphic to the sphere $\mathbb{S}^n$?

In Chapter 2, we provide integral curvature bounds concerning the $L^{n/2}$-norm of the tensor $\widetilde{R}$ in terms of the Betti numbers, for compact submanifolds of the Euclidean space with low codimension. As a consequence, we obtain partial answers to the aforementioned problem and extend previous results given in [67]. Moreover, we obtain topological obstructions for δ-pinched immersions and intrinsic obstructions for compact minimal submanifolds of the sphere with pinched second fundamental form. These results are contained in [50].

In conformal geometry, the fundamental tensor is the Weyl tensor and his role is similar to the one of the curvature tensor in Riemannian geometry. Several authors have worked on the question of how certain conditions on the Weyl tensor affect the geometry and the topology of Riemannian manifolds (cf. [10, 29]). Schouten's theorem asserts that the vanishing of the Weyl tensor of a Riemannian n-manifold M^n is equivalent to the fact that M^n is conformally flat, i.e., it is locally is conformally diffeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$, with the canonical metric, if $n \geq 4$. The $L^{n/2}$-norm of the Weyl tensor, which is a conformal invariant, measures how far a compact Riemannian manifold deviates from being conformally flat. There are plenty of papers that investigate the effect of restrictions on the $L^{n/2}$-norm of the Weyl tensor to both geometric and topological properties (cf. [3, 33, 34, 39, 56, 62]).

In Chapter 3, we provide a universal lower bound for the $L^{n/2}$-norm of the Weyl tensor in terms of the Betti numbers for compact n-dimensional Riemannian manifolds that are conformally immersed in $\mathbb{R}^{n+1}$. As a consequence, we are able to determine the homology of compact almost conformally flat hypersurfaces. Moreover, we obtain many applications: For instance, we provide a necessary condition for a compact Riemannian manifold to allow a conformal immersion as a hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. These results are contained in [49].

In Chapter 4, we provide bounds in terms of the Betti numbers for the $L^{n/2}$-norm of the Weyl tensor for compact n-dimensional Riemannian manifolds that admit conformal immersions into the Euclidean space with low codimension. As a consequence, we obtain topological obstructions for δ-pinched immersions and intrinsic obstructions for compact minimal submanifolds of the sphere with pinched second fundamental form. These results are contained in [50].

The study of isometric immersions of Riemannian manifolds with constant sectional curvature into space forms is a basic topic in submanifold theory that goes back to Cartan [8,9]. Several interesting results towards the classification of these immersions have been obtained ever since (see [21–23, 41]). A natural generalization of the concept of Riemannian manifolds with constant sectional curvature is the notion of manifolds with constant Ricci curvature, namely Einstein manifolds. Fialkow [30] and Thomas [65] initiated the study of isometric immersions of Einstein manifolds into space forms. Indeed, after the early work of Fialkow and Thomas, Ryan [55] gave a local classification of Einstein hypersurfaces in any space form. In arbitrary codimension, Di Scala [28] proved that Einstein real Kähler submanifolds of a Euclidean space are totally geodesic provided that they are minimal. The same conclusion still holds for minimal Einstein submanifolds with flat normal bundle in the Euclidean space (see [47]). In Chapter 5, we provide a complete classification of Einstein submanifolds in space forms with flat normal bundle and parallel mean curvature vector field. This result is contained in [48] and extends a previous result due to Dajczer and Tojeiro [21] for isometric immersions of Riemannian manifolds with constant sectional curvature.

A remarkable class of submanifolds in space forms are those that enjoy the property of being holonomic. An isometric immersion $f\colon (M^n, g) \to \mathbb{Q}_c^N$ of a Riemannian manifold into a space form of constant sectional curvature c is said to be *holonomic* if M^n carries a global system of orthogonal coordinates such that at any point the coordinate vector fields diagonalize its second fundamental form $\alpha\colon TM \times TM \to N_f M$ with values in the normal bundle.

There are several conditions that imply that a submanifold of a space form has to be locally holonomic. By locally we mean holonomic along connected components of an open dense subset of the manifold. For instance, this is the case of any isometric immersion $f\colon M_c^n \to \mathbb{Q}_c^N$ of a Riemannian manifold with the same constant sectional curvature as the ambient space form provided that the index of relative nullity vanishes at any point; cf. [24]. Recall that the *index of relative nullity* $\nu(x)$ of an isometric immersion $f\colon (M^n, g) \to \mathbb{Q}_c^N$ at $x \in M^n$ is the dimension of the *relative nullity subspace* $\Delta(x) \subset T_x M$ given by

$$\Delta(x) = \{X \in T_x M : \alpha(X, Y) = 0 \text{ for all } Y \in T_x M\}.$$

Isometric immersions $f\colon M_c^n \to \mathbb{Q}_{\tilde{c}}^{n+p}$ with sectional curvatures $c < \tilde{c}$ and in the least possible codimension $p = n - 1$ have flat normal bundle and thus are always locally holonomic. This was already known to Cartan [8,9] who made an exhaustive study of the subject and determined the degree of generality of such submanifolds. Moreover, being holonomic is also necessarily the case for $c > \tilde{c}$ but now under the extra condition that the submanifold is free of weak-umbilic points; see [41].

In Chapter 5, we show that results discussed above still hold for isometric immersions of the larger class of Einstein manifolds. In fact, this turns out to be the case even in the presence of a constant positive index of relative nullity; thus in the case of submanifolds of manifolds with the same constant sectional curvature the restriction mentioned above can be removed. As an application, when assuming that the index of relative nullity of the immersion is a positive constant, we conclude that the submanifold has the structure of a generalized cylinder over a submanifold with flat normal bundle. These results are contained in [27].

ORGANIZATION OF THE THESIS

- CHAPTER 1: We recall some basic facts of the theory of submanifolds and give a brief overview of the background material needed for the rest of the thesis.

- CHAPTER 2: We investigate the $L^{n/2}$-norm of the tensor
$$\widetilde{R} = R - \frac{\text{scal}}{n(n-1)} R_1,$$
for compact n-dimensional Riemannian manifolds (M^n, g) that admit isometric immersions into the Euclidean space with low codimension, where R and scal denote the curvature tensor and the scalar curvature of the metric g, respectively, and $R_1 = (1/2) g \owedge g$, where $\owedge$ stands for the Kulkarni-Nomizu product. In particular, we provide bounds concerning that norm of the tensor $\widetilde{R}$ in terms of the Betti numbers (Theorem 2.1). As a consequence, we obtain topological obstructions for δ-pinched immersions and intrinsic obstructions for compact minimal submanifolds of the sphere with pinched second fundamental form.

- CHAPTER 3: We prove a universal lower bound for the $L^{n/2}$-norm of the Weyl tensor $\mathcal{W}$ in terms of the Betti numbers for compact n-dimensional Riemannian manifolds that are conformally immersed as hypersurfaces in the Euclidean space (Theorem 3.1). As a consequence, we determine the homology of almost conformally flat hypersurfaces. Furthermore, we provide a necessary condition for a compact Riemannian manifold to admit an isometric minimal immersion as a hypersurface in the round sphere and extend a result due to Shiohama and Xu [58] for compact hypersurfaces in any space form.

- CHAPTER 4: We investigate the $L^{n/2}$-norm of $\mathcal{W}$ for compact n-dimensional Riemannian manifolds that admit conformal immersions into the Euclidean space with low codimension $k > 1$. In particular, we provide bounds concerning that norm of $\mathcal{W}$ in terms of the Betti numbers (Theorem 4.1). As a consequence, we obtain topological obstructions for δ-pinched immersions and intrinsic obstructions for compact minimal submanifolds of the sphere with pinched second fundamental form.

- CHAPTER 5: We prove that Einstein submanifolds with flat normal bundle in space forms are (at least locally) holonomic. As an application, when assuming that the index of relative nullity of the immersion is a positive constant we conclude that the submanifold has the structure of a generalized cylinder over a submanifold with flat normal bundle. Moreover, we provide a complete classification for such submanifolds, under the additional hypothesis that the mean curvature vector field is also parallel.

CHAPTER 1

Background material

In this chapter we set up the notation and give a brief overview of the background material needed for the rest of the thesis. In particular, Sections 1.1 and 1.3 are needed for Chapters 2, 3 and 4, and Section 1.4 for Chapter 5.

1.1 Algebraic preliminaries

Let V and W be finite dimensional real vector spaces equipped with non-degenerate inner products which, by abuse of notation, are both denoted by $\langle\cdot,\cdot\rangle$. The inner product of V is assumed to be positive definite. We denote by $\mathrm{Hom}(V \times V, W)$ the space of all bilinear forms with values in W and by $\mathrm{Sym}(V \times V, W)$ its subspace that consists of all symmetric bilinear forms. The space $\mathrm{Sym}(V \times V, W)$ can be viewed as a complete metric space with respect to the usual Euclidean norm $\|\cdot\|$.

1.1.1 The Kulkarni-Nomizu product

The *Kulkarni-Nomizu product* [35, p. 341] of two bilinear forms $\phi, \psi \in \mathrm{Hom}(V \times V, \mathbb{R})$ is the $(0,4)$-tensor $\phi \owedge \psi \colon V \times V \times V \times V \to \mathbb{R}$ defined by

$$\begin{aligned} \phi \owedge \psi(x_1, x_2, x_3, x_4) &= \phi(x_1, x_3)\psi(x_2, x_4) + \phi(x_2, x_4)\psi(x_1, x_3) \\ &\quad -\phi(x_1, x_4)\psi(x_2, x_3) - \phi(x_2, x_3)\psi(x_1, x_4). \end{aligned}$$

Using the inner product of W, we extend the *Kulkarni-Nomizu product* to bilinear forms $\beta, \gamma \in \mathrm{Hom}(V \times V, W)$ as the $(0,4)$-tensor $\beta \owedge \gamma \colon V \times V \times V \times V \to \mathbb{R}$ defined by

$$\begin{aligned} \beta \owedge \gamma(x_1, x_2, x_3, x_4) = {} & \langle \beta(x_1, x_3), \gamma(x_2, x_4)\rangle - \langle \beta(x_1, x_4), \gamma(x_2, x_3)\rangle \\ & + \langle \beta(x_2, x_4), \gamma(x_1, x_3)\rangle - \langle \beta(x_2, x_3), \gamma(x_1, x_4)\rangle. \end{aligned}$$

1.1.2 Flat bilinear forms

A bilinear form $\beta \in \mathrm{Hom}(V \times V, W)$ is called *flat* with respect to the inner product of W if
$$\langle \beta(x_1, x_3), \beta(x_2, x_4) \rangle - \langle \beta(x_1, x_4), \beta(x_2, x_3) \rangle = 0,$$
for all $x_1, x_2, x_3, x_4 \in V$, or, equivalently, if $\beta \owedge \beta = 0$.

Associated to each bilinear form β is the *nullity space* $\mathcal{N}(\beta)$ defined by
$$\mathcal{N}(\beta) = \{x \in V \ : \ \beta(x, y) = 0 \ \text{ for all } \ y \in V\}.$$

For flat bilinear forms we have the following result due to Moore [42, Proposition 2, p. 93].

Lemma 1.1. *Let $\beta \in \mathrm{Sym}(V \times V, W)$ be a flat bilinear form with respect to a Lorentzian inner product of W. If $\dim V > \dim W$ and $\beta(x, x) \neq 0$ for all non-zero $x \in V$, then there is a non-zero isotropic vector $e \in W$ and a bilinear form $\phi \in \mathrm{Sym}(V \times V, \mathbb{R})$ such that*
$$\dim \mathcal{N}(\beta - e\phi) \geq \dim V - \dim W + 2.$$

Moreover, by using Lemma 1.1, we have the following result [67, Lemma 2.1].

Lemma 1.2. *Let $\beta \in \mathrm{Sym}(V \times V, W)$, where V and W are both equipped with positive definite inner products and $\dim W \leq \dim V - 2$. If $\beta \owedge \beta = \mu \langle \cdot, \cdot \rangle \owedge \langle \cdot, \cdot \rangle$ for some $\mu \neq 0$, then $\mu > 0$ and there exist a unit vector $\xi \in W$ and a subspace $V_1 \subset V$ such that*
$$\dim V_1 \geq \dim V - \dim W + 1$$
and
$$\beta(x, y) = \sqrt{\mu}\, \langle x, y \rangle \xi \ \text{ for all } \ x \in V_1 \ \text{ and } \ y \in V.$$

1.2 Isometric immersions

Let (M^n, g) be a Riemannian manifold with Levi-Civitá connection ∇ and let $\mathscr{X}(M)$ denote the set of smooth local vector fields of M^n.

The $(1,3)$-curvature tensor
$$\mathrm{R}\colon \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$
is defined by
$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By means of g, R, the $(0,4)$-curvature tensor

$$R\colon \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M)$$

is defined by

$$R(X,Y,Z,W) = -g(\mathrm{R}(X,Y)Z,W).$$

The *sectional curvature* $K(X \wedge Y)$ at the point $x \in M^n$ and along the plane spanned by the orthonormal vectors $X, Y \in T_xM$ is defined by

$$K(X \wedge Y) = R(X,Y,X,Y).$$

We have the following [5, Lemma 3.4, p. 96]:

Lemma 1.3. *The Riemannian manifold* (M^n, g) *has constant sectional curvature* c *if and only if* $R = cR_1$, *where*

$$R_1\colon \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to C^\infty(M;\mathbb{R})$$

is the $(0,4)$*-tensor defined by*

$$R_1 = \frac{1}{2} g \owedge g.$$

A complete and simply-connected N-dimensional Riemannian manifold with constant sectional curvature c is called a *space form* and is denoted by $\mathbb{Q}^N_c$. It is well known that $\mathbb{Q}^N_c$ is the Euclidean space $\mathbb{R}^N$, the sphere $\mathbb{S}^N$ with radius $1/\sqrt{c}$ or the hyperbolic space $\mathbb{H}^N$ according to whether $c = 0$, $c > 0$ or $c < 0$, respectively.

The *Ricci tensor* is defined by

$$\mathrm{Ric}(X,Y) = \operatorname{trace} R(X,\cdot,Y,\cdot), \quad X, Y \in \mathscr{X}(M).$$

The *Ricci curvature* in the direction of a unit vector field $X \in \mathscr{X}(M)$ is defined by

$$\mathrm{Ric}(X) = \mathrm{Ric}(X,X).$$

The *scalar curvature* $\mathrm{scal} \in C^\infty(M^n, \mathbb{R})$ is defined by

$$\mathrm{scal} = \operatorname{trace} \mathrm{Ric}\,.$$

It is a standard fact that the $(0,4)$-curvature tensor R admits the following orthogonal decomposition (cf. [35, Chapter 8D])

$$R = \mathcal{W} + \frac{\mathrm{scal}}{2n(n-1)} g \owedge g + \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{scal}}{n} g\right) \owedge g, \tag{1.1}$$

which is in addition irreducible with respect to the (simultaneous) action of the orthogonal group $\mathrm{O}(n)$ on the four arguments of R. The $(0,4)$-tensor $\mathcal{W}$ is known as the *Weyl tensor* of (M^n, g). Equation (1.1) is equivalent to

$$R = \mathcal{W} + \mathcal{L} \owedge g, \tag{1.2}$$

where

$$\mathcal{L} = \frac{1}{n-2}\Big(\operatorname{Ric} - \frac{\operatorname{scal}}{2(n-1)} g\Big).$$

The $(0,2)$-tensor $\mathcal{L}$ is known as the *Schouten tensor* of (M^n, g).

Now, we recall some basic facts of the theory of submanifolds in $\mathbb{Q}_c^N$. At first, we denote by $\langle \cdot, \cdot \rangle$ and $\tilde{\nabla}$ the Riemannian metric and the Levi-Civitá connection of $\mathbb{Q}_c^N$ respectively.

A differentiable map $f \colon M^n \to \mathbb{Q}_c^N$ is called an *immersion* if the differential $f_*(x) \colon T_x M^n \to T_{f(x)} \mathbb{Q}_c^N$ is injective for all $x \in M^n$. An immersion f is said to be an *isometric immersion* if, moreover,

$$g(X, Y) = \langle f_*(x) X, f_*(x) Y \rangle$$

for all $x \in M^n$ and $X, Y \in T_x M$. The number $p = N - n$ is called the *codimension* of f and we refer to f as *submanifold* of $\mathbb{Q}_c^N$. The immersion f is called a *hypersurface* if $p = 1$.

Given an isometric immersion $f \colon (M^n, g) \to \mathbb{Q}_c^N$ we denote by $f^* T\mathbb{Q}_c^N$ the induced bundle over M^n whose fiber at every point $x \in M^n$ is the tangent space $T_{f(x)} \mathbb{Q}_c^N$. The orthogonal complement of $f_*(x) T_x M$ in $T_{f(x)} \mathbb{Q}_c^N$ is called the *normal space* of f at x and is denoted by $N_f M(x)$. The *normal bundle* $N_f M$ of f is the vector subbundle of the induced bundle $f^* T\mathbb{Q}_c^N$ whose fiber at every point $x \in M^n$ is $N_f M(x)$. In the sequel, the set of smooth sections of the normal bundle $N_f M$ is denoted by $\Gamma(N_f M)$.

The Levi-Civitá connection of $\mathbb{Q}_c^N$ induces a connection on the induced bundle $f^* T\mathbb{Q}_c^N$, which by abuse of notation is denoted again by $\tilde{\nabla}$. Given vector fields $X, Y \in \mathscr{X}(M)$ we decompose

$$\tilde{\nabla}_X f_* Y = \big(\tilde{\nabla}_X f_* Y\big)^{\top} + \big(\tilde{\nabla}_X f_* Y\big)^{\perp} \tag{1.3}$$

with respect to the orthogonal decomposition

$$f^* T\mathbb{Q}_c^N = f_* TM \oplus N_f M.$$

One can easily verify that

$$f_*^{-1}\big(\tilde\nabla_X f_* Y\big)^\top = \nabla_X Y,$$

where ∇ is the Levi-Civitá connection of (M^n, g). Moreover, the map

$$\alpha\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \Gamma(N_f M)$$

defined by

$$\alpha(X,Y) = \big(\tilde\nabla_X f_* Y\big)^\perp.$$

is called the *second fundamental form* of f.

1.2.1 The Gauss and Weingarten formulas

From (1.3) we obtain the following first basic formula of the theory of submanifolds, known as the *Gauss formula*

$$\tilde\nabla_X f_* Y = \nabla_X Y + \alpha(X,Y), \;\; X, Y \in \mathscr{X}(M). \tag{1.4}$$

For every $\xi \in N_f M(x)$ the endomorphism $A_\xi\colon T_x M \to T_x M$ defined by

$$g(A_\xi X, Y) = \langle \alpha(X,Y), \xi \rangle.$$

is called the *shape operator* of f at $x \in M^n$ in the direction ξ. Now, the second basic formula, known as the *Weingarten formula*, is

$$\tilde\nabla_X \xi = -f_* A_\xi X + \nabla_X^\perp \xi, \;\; \xi \in \Gamma(N_f M), \; X \in \mathscr{X}(M),$$

where $\nabla^\perp$ is the *normal connection* of f.

The *mean curvature vector* of f at $x \in M^n$ is the normal vector defined by

$$\mathcal{H}(x) = \frac{1}{n}\sum_{i=1}^{n} \alpha(X_i, X_i)$$

where $X_1, \ldots, X_n$ is an orthonormal basis of $T_x M$. The immersion f is called *minimal* at $x \in M^n$ if $\mathcal{H}(x) = 0$. We say that f is a *minimal immersion* if the mean curvature vector vanishes identically.

1.2.2 The Gauss, Codazzi and Ricci equations

By using the Gauss and Weingarten formulas, we derive the compatibility equations of an isometric immersion into $\mathbb{Q}_c^N$. The first one, is known as the *Gauss equation* and is written as

$$R = cR_1 + \frac{1}{2}\alpha \owedge \alpha.$$

The second one, is known as the *Codazzi equation* and is

$$(\nabla_X^\perp \alpha)(Y,Z) = (\nabla_Y^\perp \alpha)(X,Z), \;\; X,Y,Z \in \mathscr{X}(M).$$

The last one, is known as the *Ricci equation* and is

$$\langle R^\perp(X,Y)\xi,\eta\rangle = g([A_\xi, A_\eta]X,Y), \;\; X,Y \in TM, \; \xi,\eta \in N_fM.$$

where $R^\perp$ is the curvature tensor of the normal bundle N_fM.

1.2.3 Submanifolds with flat normal bundle

An isometric immersion $f\colon (M^n,g) \to \mathbb{Q}_c^N$ is said to have *flat normal bundle* if the curvature tensor $R^\perp$ of the normal connection $\nabla^\perp$ of N_fM vanishes. In this case, it is a standard fact (see [53]) that at any point $x \in M^n$ there exists a set of unique pairwise distinct normal vectors $\eta_i(x) \in N_fM(x)$, $1 \le i \le s = s(x)$, called the *principal normals* of f at x. Moreover, there is an associated orthogonal splitting of the tangent space as

$$T_xM = E_1(x) \oplus \cdots \oplus E_s(x),$$

where

$$E_i(x) = \big\{X \in T_xM : \alpha(X,Y) = g(X,Y)\eta_i(x) \text{ for all } Y \in T_xM\big\}. \tag{1.5}$$

Hence, the second fundamental form of f at x acquires the form

$$\alpha(X,Y) = \sum_{i=1}^{s} g(X^i,Y^i)\eta_i(x), \; X,Y \in T_xM, \tag{1.6}$$

where X^i denotes the $E_i(x)$-component of X.

The function

$$x \in M^n \to s(x) \in \{1,2,\dots,n\},$$

is lower semi-continuous. Hence, if G_k denotes the interior of the subset

$$\{x \in M^n : s(x) = k\},$$

then $\cup_{k=1}^{n} G_k$ is open and dense in M^n. On each G_k the maps

$$x \in M^n \mapsto \eta_i(x),\ 1 \le i \le k,$$

are smooth vector fields called the *principal normal vector fields* of f and the distributions

$$x \in M^n \mapsto E_i(x),\ 1 \le i \le k,$$

are smooth.

The Codazzi equation on G_k is easily seen to yield

$$g(\nabla_X Y, Z)(\eta_i - \eta_j) = g(X,Y)\nabla^{\perp}_{Z}\eta_i \tag{1.7}$$

and

$$g(\nabla_X V, Z)(\eta_j - \eta_l) = g(\nabla_V X, Z)(\eta_j - \eta_i), \tag{1.8}$$

for all $X, Y \in E_i, Z \in E_j$, and $V \in E_l$, where $1 \le i \ne j \ne l \ne i \le k$.

An isometric immersion f with flat normal bundle is called *proper* if $s(x) = k$ is constant on M^n.

1.3 Total curvature and Morse theory

In this section, we will recall some well known facts on the total curvature and how Morse theory provides restrictions on the Betti numbers.

Let $f\colon (M^n, g) \to \mathbb{R}^{n+k}$ be an isometric immersion of a compact, connected and oriented n-dimensional Riemannian manifold into the $(n+k)$-dimensional Euclidean space $\mathbb{R}^{n+k}$ equipped with the usual inner product $\langle\cdot,\cdot\rangle$. The unit normal bundle is defined by

$$UN_f = \left\{(x,\xi) \in N_f M \ :\ \|\xi\| = 1\right\}.$$

The *generalized Gauss map* $\nu\colon UN_f \to \mathbb{S}^{n+k-1}$ is defined by $\nu(x,\xi) = \xi$, where $\mathbb{S}^{n+k-1}$ is the unit $(n+k-1)$-dimensional sphere of $\mathbb{R}^{n+k}$. For each $u \in \mathbb{S}^{n+k-1}$, we consider the height function $h_u\colon M^n \to \mathbb{R}$ defined by

$$h_u(x) = \langle f(x), u\rangle.$$

Since h_u has a degenerate critical point if and only if u is a critical point of the generalized Gauss map, by Sard's theorem there exists a subset $E \subset \mathbb{S}^{n+k-1}$ of measure zero such that h_u is a Morse function for all $u \in \mathbb{S}^{n+k-1} \smallsetminus E$. For each $u \in \mathbb{S}^{n+k-1} \smallsetminus E$, we denote by $\mu_i(u)$ the number of critical points of h_u of index i.

We also set $\mu_i(u) = 0$ for any $u \in E$. Following Kuiper [36], we define the *total curvature of index i of f* by

$$\tau_i(f) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n+k-1})} \int_{\mathbb{S}^{n+k-1}} \mu_i(u)\, d\mathbb{S},$$

where $d\mathbb{S}$ denotes the volume element of the sphere $\mathbb{S}^{n+k-1}$.

Let $\beta_i(M^n;\mathbb{F}) = \dim_{\mathbb{F}} H_i(M^n;\mathbb{F})$ be the i-th Betti number of M^n over an arbitrary coefficient field $\mathbb{F}$, where $H_i(M^n;\mathbb{F})$ is the i-th homology group. Then, due to weak Morse inequalities [40, Theorem 5.2, p. 29] we have that $\mu_i(u) \geq \beta_i(M^n;\mathbb{F})$, for all $u \in \mathbb{S}^{n+k-1} \setminus E$. By integrating over $\mathbb{S}^{n+k-1}$, we obtain

$$\tau_i(f) \geq \beta_i(M^n;\mathbb{F}). \tag{1.9}$$

On the unit normal bundle UN_f there is a natural volume element denoted by $d\Sigma$. In fact, if dV is a $(k-1)$-form on UN_f such that its restriction to the fiber of the unit normal bundle at (x,ξ) is the volume element of the unit $(k-1)$-sphere of the normal space of f at x, then $d\Sigma = dM \wedge dV$, where dM is the volume element of M^n with respect to the metric g. Furthermore, we have

$$\nu^*(d\mathbb{S}) = G(x,\xi)d\Sigma,$$

where $G(x,\xi) = (-1)^n \det A_\xi$ is the *Lipschitz-Killing curvature* at $(x,\xi) \in UN_f$.

The *total absolute curvature* $\tau(f)$ in the sense of Chern and Lashof [17] is defined by

$$\tau(f) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n+k-1})} \int_{UN_f} |\nu^*(d\mathbb{S})| = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n+k-1})} \int_{UN_f} |\det A_\xi| d\Sigma.$$

We have the following result (see [17, 18]).

Theorem 1.4. *Let $f\colon M^n \to \mathbb{R}^{n+k}$ be an isometric immersion of a compact Riemannian manifold M^n. Then the total absolute curvature of f satisfies*

$$\tau(f) \geq \sum_{i=0}^{n} \beta_i(M^n;\mathbb{F}).$$

Shiohama and Xu considered for each $0 \leq i \leq n$ the subset U^iN_f of the unit normal bundle of f defined by

$$U^iN_f = \{(x,\xi) \in UN_f \ : \ \mathrm{Index}(A_\xi) = i\}$$

and proved (cf. [58, Lemma, p. 381]) that

$$\int_{U^iN_f} |\det A_\xi|\, d\Sigma = \int_{\mathbb{S}^{n+k-1}} \mu_i(u)\, d\mathbb{S}. \tag{1.10}$$

1.4 Extrinsic products of immersions

In this section, we recall the notion of extrinsic products of immersions in any space form (cf. [66]) that will be used in the proof of Theorem 5.6.

A map $f\colon M^n \to \mathbb{R}^N$ from a product manifold $M^n = \Pi_{i=1}^k M_i$ is called the *extrinsic product of immersions* $f_i\colon M_i \to \mathbb{R}^{m_i}$, $1 \le i \le k$, if there exist an orthogonal decomposition

$$\mathbb{R}^N = \Pi_{i=0}^k \mathbb{R}^{m_i},$$

with $\mathbb{R}^{m_0}$ possibly trivial, such that f is given by

$$f(x) = (v, f_1(x_1), \dots, f_k(x_k))$$

for all $x = (x_1, \dots, x_k) \in M^n$ and $v \in \mathbb{R}^{m_0}$.

A map $f\colon M^n \to \mathbb{S}^N(r) \subset \mathbb{R}^{N+1}$ from a product manifold $M^n = \Pi_{i=1}^k M_i$ into the sphere

$$\mathbb{S}^N(r) = \left\{ x \in \mathbb{R}^{N+1} : \|x\| = r \right\},$$

is called the *extrinsic product of immersions*

$$f_i\colon M_i \to \mathbb{S}^{m_i-1}(r_i) \subset \mathbb{R}^{m_i},\ 1 \le i \le k,$$

if there exist an orthogonal decomposition

$$\mathbb{R}^{N+1} = \Pi_{i=0}^k \mathbb{R}^{m_i},$$

with $\mathbb{R}^{m_0}$ possibly trivial, such that f is given by

$$f(x) = (v, f_1(x_1), \dots, f_k(x_k))$$

for all $x = (x_1, \dots, x_k) \in M^n$ with $v \in \mathbb{R}^{m_0}$ and

$$\|v\|^2 + \sum_{i=1}^k r_i^2 = r^2.$$

We now consider extrinsic products in the hyperbolic space of sectional curvature $-1/r^2$, $r > 0$,

$$\mathbb{H}^N(r) = \left\{ x = (x_0, \dots, x_N) \in \mathbb{L}^{N+1} : \langle x, x \rangle = -r^2,\ x_0 > 0 \right\},$$

where $\mathbb{L}^{N+1}$ denotes the Lorentz space of dimension $N+1$. In this case, there are three different types of extrinsic products called hyperbolic, elliptic and parabolic.

A map $f\colon M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1}$ from a product manifold $M^n = \Pi_{i=1}^k M_i$ is called the *extrinsic product of hyperbolic type* of immersions $f_1, \ldots, f_k$ if there exist an orthogonal decomposition

$$\mathbb{L}^{N+1} = \mathbb{L}^{m_1} \times \Pi_{i=2}^{k+1} \mathbb{R}^{m_i},$$

with $\mathbb{R}^{m_{k+1}}$ possibly trivial, and immersions

$$f_1\colon M_1 \to \mathbb{H}^{m_1-1}(r_1) \subset \mathbb{L}^{m_1}$$

and

$$f_i\colon M_i \to \mathbb{S}^{m_i-1}(r_i) \subset \mathbb{R}^{m_i},\ 2 \leq i \leq k,$$

such that f is given by

$$f(x) = (f_1(x_1), \ldots, f_k(x_k), v)$$

for all $x = (x_1, \ldots, x_k) \in M^n$ with $v \in \mathbb{R}^{m_{k+1}}$ and

$$-r_1^2 + \sum_{i=2}^{k} r_i^2 + \|v\|^2 = -r^2.$$

A map $f\colon M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1}$ from a product manifold $M^n = \Pi_{i=1}^k M_i$ is called the *extrinsic product of elliptic type* of immersions $f_1, \ldots, f_k$ if there exist an orthogonal decomposition

$$\mathbb{L}^{N+1} = \Pi_{i=1}^k \mathbb{R}^{m_i} \times \mathbb{L}^{m_{k+1}},$$

a vector $v \in \mathbb{L}^{m_{k+1}}$, and immersions

$$f_i\colon M_i \to \mathbb{S}^{m_i-1}(r_i) \subset \mathbb{R}^{m_i},\ 1 \leq i \leq k,$$

such that f is given by

$$f(x) = (f_1(x_1), \ldots, f_k(x_k), v)$$

for all $x = (x_1, \ldots, x_k) \in M^n$ with

$$\sum_{i=1}^{k} r_i^2 + \langle v, v \rangle = -r^2.$$

Finally, a map $f\colon M^n \to \mathbb{H}^N(r) \subset \mathbb{L}^{N+1}$ from a product manifold $M^n = \Pi_{i=1}^k M_i$ is called the *extrinsic product of parabolic type* of immersions $f_1, \ldots, f_k$ if there exists $s \in \{1, \ldots, k\}$, an orthogonal decomposition

$$\mathbb{L}^{N+1} = \mathbb{L}^{l+1} \times \Pi_{i=s+1}^{k+1} \mathbb{R}^{m_i},$$

with $\mathbb{R}^{m_{k+1}}$ possibly trivial, and immersions

$$f_i\colon M_i \to \mathbb{R}^{m_i},\ 1 \le i \le s,$$

and

$$f_j\colon M_j \to \mathbb{S}^{m_j-1}(r_j) \subset \mathbb{R}^{m_j},\ s+1 \le j \le k \text{ if } s < k,$$

such that f is given by

$$f(x) = \big(\mathsf{i}(f_1(x_1), \dots, f_s(x_s)), f_{s+1}(x_{s+1}), \dots, f_k(x_k), v\big)$$

for all $x = (x_1, \dots, x_k) \in M^n$ with $v \in \mathbb{R}^{m_{k+1}}$. Here

$$\mathsf{i}\colon \Pi_{i=1}^s \mathbb{R}^{m_i} = \mathbb{R}^{l-1} \to \mathbb{H}^l(r_1) \subset \mathbb{L}^{l+1}$$

denotes an umbilical inclusion with

$$-r_1^2 + \sum_{i=2}^k r_i^2 + \|v\|^2 = -r^2.$$

Let $f\colon (M^n, g) \to \mathbb{Q}_c^N$ be an isometric immersion of a Riemannian manifold. If $M^n = \Pi_{i=1}^k M_i$ is a product manifold then the second fundamental form α of f is said to be *adapted* to the product structure of M^n if

$$\alpha(X_i, X_j) = 0 \text{ for all } X_i \in TM_i,\ X_j \in TM_j \text{ with } 1 \le i \ne j \le k,$$

where the tangent bundles TM_i are identified with the corresponding tangent distributions to M^n. The next result, which is due to Moore [45] for the case $c = 0$ and to Molzan [46, 54] for the case $c \neq 0$, shows that products of isometric immersions are characterized by this property among isometric immersions of Riemannian products.

Theorem 1.5. *Let $f\colon (M^n, g) \to \mathbb{Q}_c^N$ be an isometric immersion of a Riemannian product manifold $M^n = \Pi_{i=1}^k M_i$ with adapted second fundamental form. Then f is an extrinsic product of isometric immersions.*

CHAPTER 2

Integral curvature and topological obstructions to isometric immersions

Let (M^n, g) be a compact n-dimensional Riemannian manifold. The $L^{n/2}$-norm of the tensor

$$\widetilde{R} = R - \frac{\text{scal}}{n(n-1)} R_1,$$

measures how far (M^n, g) deviates from having constant sectional curvature. Shiohama and Xu [58] gave a lower bound in terms of the Betti numbers in the case where (M^n, g) admits an isometric immersion as a hypersurface into the Euclidean space $\mathbb{R}^{n+1}$. For higher codimension, they posed the following:

Problem. If (M^n, g), $n \geq 3$, is a compact n-dimensional Riemannian manifold that admits an isometric immersion into the Euclidean space $\mathbb{R}^{2n-1}$, does there exist a positive constant $\varepsilon(n)$ depending only on n such that if

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM < \varepsilon(n),$$

then M^n is homeomorphic to the sphere $\mathbb{S}^n$?

In this chapter, we provide integral curvature bounds concerning the $L^{n/2}$-norm of the tensor $\widetilde{R}$ in terms of the Betti numbers, for compact submanifolds of the Euclidean space with low codimension (Theorem 2.1). As a consequence, we obtain partial answers to the aforementioned problem and extend previous results given in [67]. All manifolds under consideration are assumed to be without boundary, connected and oriented.

2.1 The main result

Let $f\colon (M^n, g) \to \mathbb{R}^{n+k}$ be an isometric immersion. Let $H = \|\mathcal{H}\|$ be the mean curvature of f and let S be the squared norm of the second fundamental form. We set for $\delta > 0$

$$S_\delta = S - \delta n^2 H^2$$

and

$$S_{\delta_+} = \max\{S_\delta, 0\}.$$

The idea of the proof of the following main result in this chapter, is to relate the $L^{n/2}$-norm of the tensor $\widetilde{R}$ with the Betti numbers using Morse theory, results of Chern-Lashof [17, 18] and the Gauss equation. To this aim we prove an algebraic inequality for symmetric bilinear forms (see Prop. 2.4).

Theorem 2.1. *Given an integer $n \geq 4$ and $1/n < \delta < 1$, there exists a positive constant $c(n,\delta)$ such that if (M^n, g) is a compact n-dimensional Riemannian manifold that admits an isometric immersion into $\mathbb{R}^{n+k}$ with codimension $2 \leq k \leq n/2$, then*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_{\delta_+}^{n/2} dM \geq c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M^n; \mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_{\delta_+}^{n/2} dM < c(n,\delta), \tag{2.1}$$

then M^n has the homotopy type of a CW-complex with no cells of dimension $k \leq i \leq n-k$. If in addition $k = 2$, then the fundamental group $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ generators, and M^n is homeomorphic to $\mathbb{S}^n$ if $\pi_1(M^n)$ is finite. Furthermore,

(i) If the scalar curvature of M^n is everywhere non-positive, then

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_{\delta}^{n/2} dM \geq c(n,\delta) \sum_{i=0}^{n} \beta_i(M^n; \mathbb{F}).$$

(ii) If the scalar curvature is everywhere non-positive and

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_{\delta}^{n/2} dM < 3c(n,\delta),$$

then M^n is homeomorphic to $\mathbb{S}^n$.

Remark 2.2. In the case where (2.1) is satisfied, the homology groups of the manifold M^n satisfy the condition $H_i(M^n;\mathbb{F}) = 0$ for $k \leq i \leq n-k$.

Remark 2.3. The presence of the integral

$$\int_{M^n} S_{\delta_+}^{n/2} dM$$

in Theorem 2.1 is essential since the algebraic inequality (see Proposition 2.4) fails if we drop the corresponding term.

2.2 The algebraic auxiliary result

Let V and W be finite dimensional real vector spaces with $\dim V = n$ and $\dim W = k$, both endowed with positive definite inner products which, by abuse of notation, are both denoted by $\langle \cdot, \cdot \rangle$. For each $\beta \in \mathrm{Sym}(V \times V, W)$ we define the map

$$\beta^\sharp \colon W \to \mathrm{End}(V), \;\; \xi \mapsto \beta^\sharp(\xi)$$

such that

$$\langle \beta^\sharp(\xi)x, y \rangle = \langle \beta(x,y), \xi \rangle \;\text{ for all }\; x, y \in V,$$

where $\mathrm{End}(V)$ denotes the set of all selfadjoint endomorphisms of V.

If $2 \leq k \leq n/2$, then for each $\beta \in \mathrm{Sym}(V \times V, W)$ we denote by $\Phi(\beta)$ the subset of the unit $(k-1)$-sphere $\mathbb{S}^{k-1}$ in W given by

$$\Phi(\beta) = \left\{ u \in \mathbb{S}^{k-1} : k \leq \mathrm{Index}\, \beta^\sharp(u) \leq n - k \right\},$$

where $\mathrm{Index}\, \beta^\sharp(u)$ is the number of negative eigenvalues of $\beta^\sharp(u)$.

We define the map $\mathsf{scal} \colon \mathrm{Sym}(V \times V, W) \to \mathbb{R}$ by

$$\mathsf{scal}(\beta) = \text{trace } \mathsf{Ric}(\beta),$$

where

$$\mathsf{Ric}(\beta)(x,y) = \text{trace } \mathsf{R}(\beta)(\cdot, x, \cdot, y), \;\; x, y \in V \;\text{ and }\; \mathsf{R}(\beta) = \frac{1}{2} \beta \owedge \beta.$$

The following proposition is crucial for the proof of Theorem 2.1.

Proposition 2.4. *Given integers $2 \leq k \leq n/2$ and $1/n < \lambda < 1$, there exists a constant $\varepsilon(n,k,\lambda) > 0$ such that the following inequality holds*

$$\frac{1}{4} \left\| \beta \owedge \beta - \frac{\mathsf{scal}(\beta)}{n(n-1)} \langle \cdot, \cdot \rangle \owedge \langle \cdot, \cdot \rangle \right\|^2 + \left(\|\beta\|^2 - \lambda |\text{ trace } \beta|^2 \right)_+^2$$

$$\geq \varepsilon(n,k,\lambda)\Big(\int_{\Lambda(\beta)} |\det \beta^\sharp(u)| d\mathbb{S}_u\Big)^{4/n} \tag{2.2}$$

for any $\beta \in \mathrm{Sym}(V \times V, W)$*, where*

$$\Lambda(\beta) = \begin{cases} \Phi(\beta), & \textit{if } \mathsf{scal}(\beta) > 0 \\ \mathbb{S}^{k-1}, & \textit{if } \mathsf{scal}(\beta) \leq 0 \end{cases}$$

and

$$\big(\|\beta\|^2 - \lambda|\operatorname{trace}\ \beta|^2\big)_+^2 = \max\Big\{\big(\|\beta\|^2 - \lambda|\operatorname{trace}\ \beta|^2\big)^2, 0\Big\}.$$

Proof: We consider the functions $\phi_\lambda, \psi\colon \mathrm{Sym}(V \times V, W) \to \mathbb{R}$ defined by

$$\phi_\lambda(\beta) = \frac{1}{4}\Big\|\beta \owedge \beta - \frac{\mathsf{scal}(\beta)}{n(n-1)}\langle\cdot,\cdot\rangle \owedge \langle\cdot,\cdot\rangle\Big\|^2 + \big(\|\beta\|^2 - \lambda|\operatorname{trace}\ \beta|^2\big)_+^2$$

and

$$\psi(\beta) = \int_{\Lambda(\beta)} |\det \beta^\sharp(u)| d\mathbb{S}_u.$$

We shall prove that ϕ_λ attains a positive minimum on the level set

$$\Sigma_{n,k} = \{\beta \in \mathrm{Sym}(V \times V, W) : \psi(\beta) = 1\}.$$

Let $\{\beta_m\}$ be a sequence in $\Sigma_{n,k}$ such that

$$\lim_{m\to\infty} \phi_\lambda(\beta_m) = \inf \phi_\lambda(\Sigma_{n,k}) \geq 0.$$

Since $\beta_m \neq 0$, for all $m \in \mathbb{N}$, we may write $\beta_m = \|\beta_m\|\hat{\beta}_m$, where $\|\hat{\beta}_m\| = 1$.

We claim that the sequence $\{\beta_m\}$ is bounded. Assume to the contrary that there exists a subsequence of $\{\beta_m\}$, which by abuse of notation is again denoted by $\{\beta_m\}$, such that

$$\lim_{m\to\infty} \|\beta_m\| = +\infty.$$

We may assume, by taking a subsequence if necessary, that $\{\hat{\beta}_m\}$ converges to some $\hat{\beta} \in \mathrm{Sym}(V \times V, W)$ with $\|\hat{\beta}\| = 1$. Using that ϕ_λ is homogeneous of degree 4, we have

$$\phi_\lambda(\hat{\beta}_m) = \frac{\phi_\lambda(\beta_m)}{\|\beta_m\|^4}.$$

Thus,

$$\lim_{m\to\infty} \phi_\lambda(\hat{\beta}_m) = 0$$

and consequently $\phi_\lambda(\hat\beta) = 0$, or equivalently

$$\hat\beta \owedge \hat\beta = \frac{\mathrm{scal}(\hat\beta)}{n(n-1)} \langle\cdot,\cdot\rangle \owedge \langle\cdot,\cdot\rangle$$

with

$$1 \leq \lambda |\operatorname{trace}\ \hat\beta|^2 = \lambda(\mathrm{scal}(\hat\beta) + 1). \tag{2.3}$$

Since $\lambda < 1$, equation (2.3) implies that $\mathrm{scal}(\hat\beta) > 0$. According to Lemma 1.2, there exists a unit vector $\hat\xi \in W$ and subspace $\hat V_1 \subset V$ with

$$\dim \hat V_1 \geq n - k + 1$$

such that

$$\hat\beta(x,y) = \Big(\frac{\mathrm{scal}(\hat\beta)}{n(n-1)}\Big)^{1/2} \langle x,y\rangle \hat\xi \ \text{ for all } \ x \in \hat V_1 \ \text{ and } \ y \in V. \tag{2.4}$$

Moreover, since $\{\beta_m\}$ is contained in $\Sigma_{n,k}$, there exists an open subset $\hat{\mathcal{U}}_m \subset \mathbb{S}^{k-1}$ such that

$$\hat{\mathcal{U}}_m \subset \Lambda(\hat\beta_m) \text{ and } \det \hat\beta_m^\sharp(u) \neq 0, \text{ for all } u \in \hat{\mathcal{U}}_m \text{ and } m \in \mathbb{N}.$$

From $\mathrm{scal}(\hat\beta) > 0$, we have that $\mathrm{scal}(\hat\beta_m) > 0$ and so $\hat{\mathcal{U}}_m \subset \Phi(\hat\beta_m)$ for m large enough.

Let $\{\hat u_m\}$ be a sequence such that $\hat u_m \in \hat{\mathcal{U}}_m$ for all $m \in \mathbb{N}$. We may assume that $\{\hat u_m\}$ is convergent and set

$$\hat u = \lim_{m\to\infty} \hat u_m.$$

Since

$$\lim_{m\to\infty} \hat\beta_m^\sharp(\hat u_m) = \hat\beta^\sharp(\hat u)$$

and $\hat u_m \in \hat{\mathcal{U}}_m$, we have that Index $\hat\beta^\sharp(\hat u) \leq n-k$. From (2.4) we obtain $\langle \hat\xi, \hat u\rangle \geq 0$. We claim that $\langle \hat\xi, \hat u\rangle = 0$. Indeed, if $\langle \hat\xi, \hat u\rangle > 0$, then (2.4) implies that $\hat\beta^\sharp(\hat u)$ has at least $n-k+1$ positive eigenvalues and so $\hat\beta_m^\sharp(\hat u_m)$ has at least $n-k+1$ positive eigenvalues for m large enough. Since $\det \hat\beta_m^\sharp(u) \neq 0$ for all $u \in \hat{\mathcal{U}}_m$ it follows that $\hat\beta_m^\sharp(\hat u_m)$ has at most $k-1$ negative eigenvalues. Therefore, Index $\hat\beta_m^\sharp(\hat u_m) \leq k-1$ which is a contradiction, since $\hat u_m \in \hat{\mathcal{U}}_m$.

Thus, we have that

$$\langle \lim_{m\to\infty} \hat u_m, \hat\xi\rangle = 0$$

for any convergent sequence $\{\hat u_m\}$ such that $\hat u_m \in \hat{\mathcal{U}}_m$ for any m.

Since $\hat{\mathcal{U}}_m$ is open, we may choose convergent sequences $\{\hat{u}_m^{(1)}\}, \ldots, \{\hat{u}_m^{(k)}\}$ in $\hat{\mathcal{U}}_m$ such that $\hat{u}_m^{(1)}, \ldots, \hat{u}_m^{(k)}$ span W for any $m \in \mathbb{N}$. From (2.4) and

$$\langle \lim_{m\to\infty} \hat{u}_m^{(a)}, \hat{\xi} \rangle = 0$$

for all $a \in \{1, \ldots, k\}$, we have that the restriction of $\hat{\beta}_m$ to $\hat{V}_1 \times \hat{V}_1$ satisfies

$$\lim_{m\to\infty} \hat{\beta}_m|_{\hat{V}_1\times\hat{V}_1} = 0$$

and consequently

$$\lim_{m\to\infty} (\hat{\beta}_m \owedge \hat{\beta}_m)|_{\hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1} = 0. \tag{2.5}$$

Using

$$\lim_{m\to\infty} \phi_\lambda(\hat{\beta}_m) = 0,$$

(2.5) and the inequality

$$\left\| \left(\hat{\beta}_m \owedge \hat{\beta}_m - \frac{\mathsf{scal}(\hat{\beta}_m)}{n(n-1)} \langle\cdot,\cdot\rangle \owedge \langle\cdot,\cdot\rangle \right)|_{\hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1} \right\|^2 \leq 4\phi_\lambda(\hat{\beta}_m),$$

we obtain that $\mathsf{scal}(\hat{\beta}) = 0$, which contradicts (2.3). Thus, the sequence $\{\beta_m\}$ is bounded. We may assume that

$$\lim_{m\to\infty} \beta_m = \beta \in \mathrm{Sym}(V \times V, W).$$

We claim that $\phi_\lambda(\beta) > 0$. Arguing indirectly, we assume that $\phi_\lambda(\beta) = 0$. Then

$$\beta \owedge \beta = \frac{\mathsf{scal}(\beta)}{n(n-1)} \langle\cdot,\cdot\rangle \owedge \langle\cdot,\cdot\rangle$$

and

$$\|\beta\|^2 \leq \lambda |\operatorname{trace} \beta|^2 = \lambda(\mathsf{scal}(\beta) + \|\beta\|^2). \tag{2.6}$$

We observe that $\beta \neq 0$. Indeed, if $\beta = 0$, then $\beta^\sharp(u) = 0$ for all $u \in \mathrm{S}^{k-1}$. Since $\beta_m \in \Sigma_{n,k}$, there exists $\xi_m \in \Lambda(\beta_m)$ such that

$$|\det \beta_m^\sharp(\xi_m)| \mathrm{Vol}(\Lambda(\beta_m)) = 1 \text{ for all } m \in \mathbb{N}. \tag{2.7}$$

We may assume that ξ_m converges to some ξ, by passing to a subsequence if necessary. Then

$$\lim_{m\to\infty} \beta_m^\sharp(\xi_m) = \beta^\sharp(\xi) = 0,$$

which contradicts (2.7). Therefore $\beta \neq 0$.

Now, from (2.6) we obtain that $\mathsf{scal}(\beta) \neq 0$. Lemma 1.2 implies that $\mathsf{scal}(\beta) > 0$ and there exists a unit vector $\xi \in W$ and a subspace $V_1 \subset V$ with

$$\dim V_1 \geq n - k + 1$$

such that

$$\beta(x,y) = \left(\frac{\mathsf{scal}(\beta)}{n(n-1)}\right)^{1/2} \langle x,y\rangle\xi \text{ for all } x \in V_1 \text{ and } y \in V. \tag{2.8}$$

Since $\beta_m \in \Sigma_{n,k}$, there exists an open subset $\mathcal{U}_m \subset \mathbb{S}^{k-1}$ such that

$$\mathcal{U}_m \subset \Lambda(\beta_m) \text{ and } \det \beta_m^\sharp(u) \neq 0 \text{ for all } u \in \mathcal{U}_m \text{ and } m \in \mathbb{N}.$$

Moreover, we have that $\mathsf{scal}(\beta_m) > 0$ and so $\mathcal{U}_m \subset \Phi(\beta_m)$ for m large enough.

Let $\{u_m\}$ be a sequence with $u_m \in \mathcal{U}_m$ for all $m \in \mathbb{N}$. We may assume that u_m is convergent and set

$$u = \lim_{m\to\infty} u_m.$$

Since

$$\lim_{m\to\infty} \beta_m^\sharp(u_m) = \beta^\sharp(u)$$

and $u_m \in \mathcal{U}_m$ it follows that Index $\beta^\sharp(u) \leq n-k$. From (2.8) we get $\langle\xi,u\rangle \geq 0$. We claim that $\langle\xi,u\rangle = 0$. Indeed, if $\langle\xi,u\rangle > 0$ then (2.8) implies that $\beta^\sharp(u)$ has at least $n-k+1$ positive eigenvalues and so $\beta_m^\sharp(u_m)$ has at least $n-k+1$ positive eigenvalues for m large enough. Since $\det \beta_m^\sharp(u) \neq 0$ for all $u \in \mathcal{U}_m$ and $m \in \mathbb{N}$, we obtain that $\beta_m^\sharp(u_m)$ has at most $k-1$ negative eigenvalues, which contradictics that $u_m \in \mathcal{U}_m$ for all $m \in \mathbb{N}$.

Thus,

$$\langle \lim_{m\to\infty} u_m, \xi\rangle = 0$$

for any convergent sequence $\{u_m\}$ such that $u_m \in \mathcal{U}_m$ for any m. Since $\mathcal{U}_m$ is open, we may choose convergent sequences $\{u_m^{(1)}\},\dots,\{u_m^{(k)}\}$ in $\mathcal{U}_m$ such that $u_m^{(1)},\dots,u_m^{(k)}$ span W for all $m \in \mathbb{N}$. From (2.8) and

$$\langle \lim_{m\to\infty} u_m^{(a)}, \xi\rangle = 0$$

for all $a \in \{1,\dots,k\}$, we have that

$$\lim_{m\to\infty} \beta_m|_{V_1\times V_1} = 0$$

and consequently

$$\lim_{m\to\infty} (\beta_m \owedge \beta_m)|_{V_1\times V_1\times V_1\times V_1} = 0. \tag{2.9}$$

Using that

$$\lim_{m\to\infty} \phi_\lambda(\beta_m) = 0,$$

(2.9) and the inequality

$$\left\| \left(\beta_m \owedge \beta_m - \frac{\mathsf{scal}(\beta_m)}{n(n-1)}\langle\cdot,\cdot\rangle \owedge \langle\cdot,\cdot\rangle\right)|_{V_1\times V_1\times V_1\times V_1}\right\|^2 \leq 4\phi_\lambda(\beta_m),$$

we obtain that $\mathsf{scal}(\beta) = 0$, which contradicts (2.6). Thus, ϕ_λ attains a positive minimum $\varepsilon(n,k,\lambda) = \phi_\lambda(\beta)$ on $\Sigma_{n,k}$ which depends only on n, k and λ.

Let $\beta \in \mathrm{Sym}(V \times V, W)$. Assume that $\psi(\beta) \neq 0$ and set

$$\gamma = \frac{\beta}{(\psi(\beta))^{1/n}}.$$

Clearly $\gamma \in \Sigma_{n,k}$ and consequently $\phi_\lambda(\gamma) \geq \varepsilon(n,k,\lambda)$. The desired inequality follows from the homogeneity of ϕ_λ. ∎

Remark 2.5. In the case where $\lambda \leq 1/n$, arguing as in the proof of Proposition 2.4, we have that there exists a positive constant $d(n,k)$ such that

$$\left(\|\beta\|^2 - \lambda|\,\mathrm{trace}\;\beta|^2\right)^2 \geq \left(\|\beta\|^2 - \frac{1}{n}|\,\mathrm{trace}\;\beta|^2\right)^2 \geq d(n,k)\Big(\int_{\mathbb{S}^{k-1}} |\det \beta^\sharp(u)| d\mathbb{S}_u\Big)^{4/n}$$

for any $\beta \in \mathrm{Sym}(V \times V, W)$.

However, for $1/n < \lambda < 1$ inequality (2.2) fails by dropping the first term of the left hand side. Indeed, let $n \geq 7$, $k = 2$ and $\{\xi_1, \xi_2\}$ be an orthonormal basis of W. Consider $\beta \in \mathrm{Sym}(V \times V, W)$ defined by

$$\beta(x,y) = \langle Ax, y\rangle \xi_1,$$

where

$$A = \mathrm{diag}(a, a, -a, \dots, -a),\ a > 0.$$

For any

$$\frac{n}{(n-4)^2} \leq \lambda < 1$$

we have that

$$\left(\|\beta\|^2 - \lambda|\,\mathrm{trace}\;\beta|^2\right)_+ = 0.$$

Moreover, $\Phi(\beta) = \mathbb{S}^1 \smallsetminus \{\pm\xi_2\}$ and this shows that inequality (2.2) cannot hold if we drop the first term of the left hand side.

2.3 Proof of Theorem 2.1

Let $f\colon (M^n, g) \to \mathbb{R}^{n+k}$ be an isometric immersion with second fundamental form α and shape operator A_ξ with respect to ξ, where $(x,\xi) \in UN_f$. Using the Gauss equation and applying Proposition 2.4 for $V = T_xM$, $W = N_fM(x)$ and $\beta = \alpha(x)$, we have

$$\left(\|\widetilde{R}\|^2 + S^2_{\delta_+}\right)^{n/4}(x) \geq (\varepsilon(n,k,\delta))^{n/4} \int_{\Lambda(\alpha(x))} |\det A_\xi| dV_\xi$$

for all $x \in M^n$. Integrating over M^n, using (1.10) and recalling the definition of the i-th total absolute curvature, we obtain

$$\int_{M^n} (\|\widetilde{R}\|^2 + S^2_{\delta_+})^{n/4} dM \geq (\varepsilon(n,k,\delta))^{n/4} \mathrm{Vol}(\mathbb{S}^{n+k-1}) \sum_{i=k}^{n-k} \tau_i(f). \tag{2.10}$$

Observe that

$$(\|\widetilde{R}\|^2 + S^2_{\delta_+})^{n/4}(x) \leq 2^{(n-4)/4} (\|\widetilde{R}\|^{n/2} + S^{n/2}_{\delta_+})(x) \tag{2.11}$$

for all $x \in M^n$. Thus, from (2.10) and (1.9) we obtain

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S^{n/2}_{\delta_+} dM \geq c(n,\delta) \sum_{i=k}^{n-k} \tau_i(f) \geq c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M;\mathbb{F}), \tag{2.12}$$

where

$$c(n,\delta) = \min_{2 \leq k \leq n/2} \left\{ 2 \Big(\frac{\varepsilon(n,k,\delta)}{2} \Big)^{n/4} \mathrm{Vol}(\mathbb{S}^{n+k-1}) \right\}.$$

Now, assume that

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S^{n/2}_{\delta_+} dM < c(n,\delta).$$

It follows directly from (2.12) that

$$\sum_{i=k}^{n-k} \tau_i(f) < 1.$$

Thus, there exists $u \in \mathbb{S}^{n+k-1}$ such that the height function h_u is a Morse function whose number of critical points of index i satisfies $\mu_i(u) = 0$ for any $k \leq i \leq n-k$. That M^n has the homotopy type of a CW-complex with no cells of dimension $k \leq i \leq n-k$ follows from the fundamental theorem of Morse theory (cf. [40, Theorem 3.5] or [11, Theorem 4.10]).

If $k = 2$, there are no 2-cells and thus by the cellular approximation theorem we have that the inclusion of the 1-skeleton $X^{(1)} \hookrightarrow M^n$ induces isomorphism between the fundamental groups. Therefore, $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ elements and $H_1(M^n;\mathbb{Z})$ is a free abelian group on $\beta_1(M^n;\mathbb{Z})$ generators. If $\pi_1(M^n)$ is finite, then $\pi_1(M^n) = 0$ and hence $H_1(M^n;\mathbb{Z}) = 0$. From Poincaré duality and the universal coefficient theorem it follows that $H_{n-1}(M^n;\mathbb{Z}) = 0$. Thus, M^n is a simply connected homology sphere and hence a homotopy sphere. By the generalized Poincaré conjecture (Smale $n \geq 5$, Freedman $n = 4$) M^n is homeomorphic to $\mathbb{S}^n$.

If the scalar curvature is everywhere non-positive, then the Gauss equation yields $S \geq \delta n^2 H^2$. Using Proposition 2.4 and the Gauss equation we have

$$(\|\widetilde{R}\|^2 + S_\delta^2)^{n/4}(x) \geq (\varepsilon(n,k,\delta))^{n/4} \int_{\mathbb{S}_x^{k-1}} |\det A_\xi| dV_\xi$$

for all $x \in M^n$. Integrating over M^n, we obtain

$$\int_{M^n} (\|\widetilde{R}\|^2 + S_\delta^2)^{n/4} dM \geq (\varepsilon(n,k,\delta))^{n/4} \int_{UN_f} |\det A_\xi| d\Sigma.$$

Bearing in mind the definition of the total absolute curvature and (2.11), we have

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_\delta^{n/2} dM \geq 2(\varepsilon(n,k,\delta)/2)^{n/4} \mathrm{Vol}(\mathbb{S}^{n+k-1})\tau(f) \geq c(n,\delta)\tau(f). \tag{2.13}$$

The desired inequality follows from Theorem 1.4.

If the scalar curvature is everywhere non-positive and

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM + \int_{M^n} S_\delta^{n/2} dM < 3c(n,\delta),$$

then (2.13) yields $\tau(f) < 3$. Thus there exists a height function which is a Morse function with exactly two critical points and M^n is homeomorphic to $\mathbb{S}^n$ by Reeb's theorem. ∎

2.4 Applications

In this section, we derive some applications of Theorem 2.1.

2.4.1 Minimal submanifolds in spheres

Minimal submanifolds with pinched second fundamental form have been studied by Simons [61], Chern, do Carmo, Kobayashi [15] and Leung [38], among others. We provide intrinsic obstructions for minimal submanifolds in spheres with sufficiently pinched second fundamental form.

Corollary 2.6. *Let $f\colon (M^n, g) \to \mathbb{S}^{n+k-1}$ be an isometric minimal immersion of a compact Riemannian manifold M^n with $2 \le k \le n/2$. If the squared norm of the second fundamental form satisfies $S \le n(\delta n - 1)$ for some $1/n < \delta < 1$, then*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM \geq c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M^n;\mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM < c(n,\delta),$$

then M^n has the homotopy type of a CW-complex with no cells of dimension $k \le i \le n-k$. Furthermore if $k = 2$, then $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ generators, and M^n is homeomorphic to $\mathbb{S}^n$ if $\pi_1(M^n)$ is finite.

Proof: We consider the immersion $\widetilde{f} = i \circ f$, where $i\colon \mathbb{S}^{n+k-1} \hookrightarrow \mathbb{R}^{n+k}$ is the totally umbilical inclusion. The proof follows directly from Theorem 2.1. ∎

2.4.2 δ-pinched immersions

An isometric immersion is called *δ-pinched* if the inequality $S \le \delta n^2 H^2$ holds everywhere, in which case we obtain $\delta \ge 1/n$. The integral

$$\int_{M^n} S_{\delta_+}^{n/2} dM$$

measures how far an immersion deviates from being δ-pinched. The geometry and topology of δ-pinched immersions have been studied by several authors (see [1, 2, 4, 12–14, 64]) in the case where $\delta = 1/(n-1)$. We note that Shiohama and Xu [59, 60] gave a topological lower bound of the above integral in the case where $\delta = 1/n$.

By using Theorem 2.1 we provide information on δ-pinched immersions for any $1/n < \delta < 1$. Indeed, the following corollary follows immediately from Theorem 2.1 and gives an intrinsic obstruction to δ-pinched immersions.

Corollary 2.7. *If a compact Riemannian manifold (M^n, g) admits an isometric δ-pinched immersion into $\mathbb{R}^{n+k}$ with $2 \le k \le n/2$ and $1/n < \delta < 1$, then*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM \geq c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M^n;\mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM < c(n,\delta)$$

then M^n has the homotopy type of a CW-complex with no cells of dimension $k \le i \le n-k$. Furthermore if $k = 2$, then $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ generators, and M^n is homeomorphic to $\mathbb{S}^n$ if $\pi_1(M^n)$ is finite.

2.4.3 Further results

The next results provide partial answers to the problem raised by Shiohama and Xu.

Corollary 2.8. *If a compact Riemannian manifold* (M^n, g) *admits an isometric immersion* f *into* $\mathbb{R}^{n+k}$ *with* $2 \leq k \leq n/2$ *such that*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM < \lambda\, c(n,\delta)$$

and

$$\int_{M^n} S_{\delta_+}^{n/2} dM \leq (1-\lambda)\, c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M^n;\mathbb{F})$$

for some $0 < \lambda < 1$ *and* $1/n < \delta < 1$*, then* f *is* δ*-pinched and* M^n *has the homotopy type of a CW-complex with no cells of dimension* $k \leq i \leq n-k$*. Furthermore,*

(i) If $k = 2$*, then* $\pi_1(M^n)$ *is a free group on* $\beta_1(M^n;\mathbb{Z})$ *generators, and* M^n *is homeomorphic to* $\mathbb{S}^n$ *if* $\pi_1(M^n)$ *is finite.*

(ii) If $H > 0$ *everywhere and* $\delta = 1/(n-1)$*, then* M^n *is diffeomorphic to* $\mathbb{S}^n$*.*

Proof: Our assumptions and Theorem 2.1 imply that $\beta_i(M^n;\mathbb{F}) = 0$, $k \leq i \leq n-k$. Hence, f is δ-pinched and the rest of the proof follows from Theorem 2.1. Moreover, if $H > 0$ everywhere and f is $1/(n-1)$-pinched, then a result due to Andrews and Baker [1] implies that M^n is diffeomorphic to $\mathbb{S}^n$. ∎

Corollary 2.9. *If a compact Riemannian manifold* (M^n, g) *admits an isometric immersion into* $\mathbb{R}^{n+k}$ *with* $2 \leq k \leq n/2$ *such that*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} dM \leq \lambda\, c(n,\delta) \sum_{i=k}^{n-k} \beta_i(M^n;\mathbb{F})$$

and

$$\int_{M^n} S_{\delta_+}^{n/2} dM < (1-\lambda)\, c(n,\delta)$$

for some $0 < \lambda < 1$ *and* $1/n < \delta < 1$*, then* M^n *is isometric to a constant curvature sphere.*

Proof: Our assumptions and Theorem 2.1 imply that $\widetilde{R} = 0$. It follows from Shur's lemma that M^n is a space form. According to a result due to Chern, Otsuki and Kuiper (cf. [37, Corollary 4.8]) the sectional curvature must be positive. Appealing to Moore [41, Proposition 4], M^n is isometric to a constant curvature sphere. ∎

CHAPTER 3

Almost conformally flat hypersurfaces

In conformal geometry, the fundamental tensor is the Weyl tensor and his role is similar to the one of the curvature tensor in Riemannian geometry. Several authors have worked on the question how certain conditions on the Weyl tensor affect the geometry and the topology of Riemannian manifolds (cf. [10, 29]). Schouten's theorem asserts that the vanishing of the Weyl tensor of a Riemannian n-manifold M^n is equivalent to the fact that M^n is conformally flat, i.e., locally is conformally diffeomorphic to an open subset of the Euclidean space $\mathbb{R}^n$, with the canonical metric, if $n \geq 4$. The $L^{n/2}$-norm of the Weyl tensor, which is a conformal invariant, measures how far a compact Riemannian manifold deviates from being conformally flat. There are plenty of papers that investigate the effect of restrictions on the $L^{n/2}$-norm of the Weyl tensor to both geometric and topological properties (cf. [3, 33, 34, 39, 56, 62]).

The investigation of conformally flat manifolds from the submanifold point of view was initiated by Cartan [7]. Moore [42] studied conformally flat submanifolds of the Euclidean space and proved that such submanifolds have the homotopy type of a CW-complex with no cells of dimension $k < i < n - k$ for low codimension k. Later do Carmo, Dajczer and Mercuri [6] studied the case of hypersurfaces in $\mathbb{R}^{n+1}$ and they proved that diffeomorphically every such hypersurface M^n is a sphere $\mathbb{S}^n$ with $\beta_1(M^n;\mathbb{Z})$ handles attached, where $\beta_1(M^n;\mathbb{Z})$ is the first Betti number of M^n. Moreover, they showed that geometrically every such hypersurface locally consists of nonumbilic submanifolds of $\mathbb{R}^{n+1}$ that are foliated by complete round $(n-1)$-spheres and are joined through their boundaries to the following three types of umbilic submanifolds of $\mathbb{R}^{n+1}$: (i) an open piece of an n-sphere or an n-plane bounded by round $(n-1)$-sphere, (ii) a round $(n-1)$-sphere, (iii) a point.

It is therefore natural to address the following:

Question. Let (M^n, g) be a compact n-dimensional Riemannian manifold that admits a conformal immersion into the Euclidean space $\mathbb{R}^{n+1}$. What can be said about the topology of M^n if (M^n, g) is almost conformally flat, in the sense that the $L^{n/2}$-norm of the Weyl tensor is sufficiently small?

In this chapter we provide an answer to the above question by giving a universal lower bound for the $L^{n/2}$-norm of the Weyl tensor in terms of the Betti numbers for compact n-dimensional Riemannian manifolds that are conformally immersed in $\mathbb{R}^{n+1}$. As a consequence, we are able to determine the homology of compact almost conformally flat hypersurfaces. Furthermore, the result provides a necessary condition for a compact Riemannian manifold to be conformally immersed as a hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. All manifolds under consideration are assumed to be without boundary, connected and oriented.

3.1 The main result

We prove the following:

Theorem 3.1. *Given $n \geq 4$, there exists a positive constant $c(n)$, depending only on n, such that if (M^n, g) is a compact n-dimensional Riemannian manifold that admits a conformal immersion in $\mathbb{R}^{n+1}$, then the Weyl tensor $\mathcal{W}$ associated to g satisfies*

$$\int_{M^n} \|\mathcal{W}\|^{n/2} \, dM \geq c(n) \sum_{i=2}^{n-2} \beta_i(M^n; \mathbb{F}). \tag{3.1}$$

In particular, if

$$\int_{M^n} \|\mathcal{W}\|^{n/2} \, dM < c(n) \tag{3.2}$$

then M^n has the homotopy type of a CW-complex with no cells of dimension i for $2 \leq i \leq n-2$ and the fundamental group $\pi_1(M^n)$ is a free group on $\beta_1(M^n; \mathbb{Z})$ generators. Moreover, if $\pi_1(M^n)$ is finite then M^n is homeomorphic to the sphere $\mathbb{S}^n$.

Remark 3.2. In the case where (3.2) is satisfied, the homology groups of M^n must satisfy the condition $H_i(M^n; \mathbb{F}) = 0$ for all $2 \leq i \leq n-2$.

Remark 3.3. The assumption on the codimension in Theorem 3.1 is essential, as the following example shows. We consider the manifold

$$M^n = \mathbb{S}^1(1) \times \mathbb{S}^1(1) \times \mathbb{S}^{n-2}(r), \; n \geq 4,$$

equipped with the product metric g, where $\mathbb{S}^{n-2}(r)$ is the $(n-2)$-dimensional round sphere of radius r. Since M^n is isometrically immersed into the sphere

$\mathbb{S}^{n+2}(\sqrt{2+r^2})$ in the obvious way, it follows that M^n admits a conformal immersion into $\mathbb{R}^{n+2}$. A long but straightforward computation yields that

$$\|\mathcal{W}\|^2 = \frac{\varepsilon(n)}{r^4},$$

where $\varepsilon(n)$ is a positive constant depending only on n. By integration we obtain

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM = \frac{a(n)}{r^2},$$

where $a(n)$ is a positive constant depending only on n. Since the second Betti number of M^n is non-zero, it follows that M^n does not satisfy inequality (3.1) for r large enough.

Remark 3.4. In Theorem 3.1, the ambient space $\mathbb{R}^{n+1}$ can be replaced by the round sphere $\mathbb{S}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$. Indeed, this follows from the fact that $\mathbb{S}^{n+1} \setminus \{\text{point}\}$ and $\mathbb{H}^{n+1}$ are conformally equivalent to the Euclidean space and that the $L^{n/2}$-norm of the Weyl tensor is conformally invariant.

Remark 3.5. There is an abundance of Riemannian manifolds that do not satisfy inequality (3.1), and therefore do not admit a conformal immersion as hypersurfaces in $\mathbb{R}^{n+1}$. For instance, let the manifold

$$M^n = N^m \times \mathbb{S}^{n-m}(r), \;\; 2 \leq m \leq n-2,$$

be equipped with the product metric g, where N^m is a compact flat m-dimensional Riemannian manifold. A direct computation yields that

$$\|\mathcal{W}\|^2 = \frac{\varepsilon(n)}{r^4},$$

where $\varepsilon(n)$ is a positive constant depending only on n. By integration we obtain

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM = \frac{\varepsilon(n)^{n/4}}{r^m}\mathrm{Vol}(\mathbb{S}^{n-m}(1))\mathrm{Vol}(N^m).$$

Since, the $(n-m)$-th Betti number of M^n is non-zero, we obtain that M^n does not satisfy inequality (3.1) for r large enough.

3.2 Algebraic auxiliary results

Let V be a finite dimensional real vector space equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$. We define the map

$$\mathsf{W}\colon \mathrm{Sym}(V\times V, \mathbb{R}) \to \mathrm{Hom}(V\times V\times V\times V, \mathbb{R})$$

by

$$\mathsf{W}(\beta) = \mathsf{R}(\beta) - \mathsf{L}(\beta) \owedge \langle \cdot, \cdot \rangle,$$

where

$$\mathsf{L}(\beta) = \frac{1}{n-2}\Big(\mathsf{Ric}(\beta) - \frac{\mathsf{scal}(\beta)}{2(n-1)}\langle \cdot, \cdot \rangle\Big).$$

Moreover, to each $\beta \in \mathrm{Sym}(V \times V, \mathbb{R})$ we assign a self-adjoint endomorphism $\beta^\sharp$ of V defined by

$$\langle \beta^\sharp(x), y \rangle = \beta(x, y), \;\; x, y \in V.$$

We have the following:

Lemma 3.6. *Let* $\dim V = n \geq 4$ *and* $\beta \in \mathrm{Sym}(V \times V, \mathbb{R})$*. Then* $\mathsf{W}(\beta) = 0$ *if and only if* $\beta^\sharp$ *has an eigenvalue of multiplicity at least* $n-1$*.*

Proof: Let $\beta \in \mathrm{Sym}(V \times V, \mathbb{R})$ be a symmetric bilinear form such that $\mathsf{W}(\beta) = 0$. We endow $\mathbb{R}^3$ with the Lorentzian inner product $\langle\langle \cdot, \cdot \rangle\rangle$ given by

$$\langle\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle\rangle = x_1 y_1 + x_2 y_3 + x_3 y_2,$$

and define the symmetric bilinear form $\widetilde{\beta}\colon V \times V \to \mathbb{R}^3$ by

$$\widetilde{\beta}(x, y) = \big(\beta(x, y), \langle x, y \rangle, -\mathsf{L}(\beta)(x, y)\big).$$

Since $\mathsf{W}(\beta) = 0$ it follows that $\widetilde{\beta}$ is flat with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. From Lemma 1.1, we know that there exists a non-zero isotropic vector $e = (t_1, t_2, t_3) \in \mathbb{R}^3$ and a symmetric bilinear form $\phi : V \times V \to \mathbb{R}$ such that

$$\dim \mathcal{N}(\widetilde{\beta} - \phi e) \geq n - 1.$$

By setting $V_1 = \mathcal{N}(\widetilde{\beta} - e\phi)$, we have that

$$\widetilde{\beta}(x, y) = \phi(x, y) e,$$

or equivalently

$$\beta(x, y) = \phi(x, y) t_1, \;\; \langle x, y \rangle = \phi(x, y) t_2 \;\text{ and }\; \mathsf{L}(\beta)(x, y) = -\phi(x, y) t_3$$

for all $x \in V_1$ and $y \in V$. Therefore,

$$\beta(x, y) = \langle x, y \rangle \lambda$$

for all $x \in V_1$ and $y \in V$, where $\lambda = t_1/t_2$. Hence, λ is an eigenvalue of $\beta^\sharp$ with multiplicity at least $n-1$.

Conversely, assume that

$$\beta^\sharp e_i = \lambda e_i, \;\; 1 \leq i \leq n-1 \;\text{ and }\; \beta^\sharp e_n = \mu e_n,$$

where $e_1, \ldots, e_n$ is an orthonormal basis of V. A long but straightforward computation then yields $\mathsf{W}(\beta) = 0$. ∎

The following proposition is crucial for the proof of Theorem 3.1.

Proposition 3.7. *Given $n \geq 4$, there exists a positive constant $\varepsilon(n)$, depending only on n, such that the following inequality holds*

$$\|\mathsf{W}(\beta)\|^2 \geq \varepsilon(n)|\det \beta^\sharp|^{4/n}$$

for all $\beta \in \mathrm{Sym}(V \times V, \mathbb{R}) \smallsetminus (E_+ \cup E_-)$, where V is an n-dimensional vector space equipped with a positive definite inner product,

$$E_\pm = \{\beta \in \mathrm{Sym}(V \times V, \mathbb{R}) \ : \ \mathcal{E}_\pm(\beta) \geq n-1\},$$

and $\mathcal{E}_+(\beta)$ (respectively, $\mathcal{E}_-(\beta)$) is the number of positive (respectively, negative) eigenvalues of $\beta^\sharp$, each one counted with its multiplicity.

Proof: Let $\beta \in \mathrm{Sym}(V \times V, \mathbb{R})$ and let $e_1, \dots, e_n$ be an orthonormal basis of V that diagonalizes $\beta^\sharp$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. By an easy computation we obtain

$$\|\mathsf{W}(\beta)\|^2 = 4\sum_{i<j}\left(\lambda_i\lambda_j - \frac{1}{n-2}\Big((\lambda_i+\lambda_j)\sum_{k=1}^n \lambda_k - (\lambda_i^2+\lambda_j^2) - \frac{\sum_{k\neq l}\lambda_k\lambda_l}{n-1}\Big)\right)^2.$$

We consider the functions $\phi, \psi \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\phi(x) = 4\sum_{i<j}\left(x_ix_j - \frac{1}{n-2}\Big(\sigma_1(x)(x_i+x_j) - (x_i^2+x_j^2) - \frac{2\sigma_2(x)}{n-1}\Big)\right)^2$$

and

$$\psi(x) = \Pi_{i=1}^n x_i,$$

where

$$\sigma_1(x) = \sum_{i=1}^n x_i, \ \sigma_2(x) = \sum_{i<j} x_ix_j \ \text{ and } \ x = (x_1, \dots, x_n).$$

In order to prove the desired inequality, it is sufficient to show that there exists a positive constant $\varepsilon(n)$, depending only on n, such that

$$\phi(x) \geq \varepsilon(n)\psi(x) \ \text{ for all } \ x \in U_n,$$

where $U_n = \mathbb{R}^n \smallsetminus (K_+^n \cup K_-^n)$ and K_+^n (respectively, K_-^n) is the subset of points in $\mathbb{R}^n$ with at least $n-1$ positive (respectively, negative) coordinates.

At first we are going to prove that ϕ attains a positive minimum on the level set

$$\Sigma_n = \{x \in U_n \ : \ \psi(x) = \varepsilon\},$$

where $\varepsilon = \pm 1$. Since $\phi(\Sigma_n)$ is bounded from below, there exists a sequence $\{z_m\}$ in Σ_n such that

$$\lim_{m\to\infty} \phi(z_m) = \inf \phi(\Sigma_n) \geq 0.$$

We write $z_m = \rho_m a_m$, where $\rho_m = \|z_m\|$ and a_m lies in the unit $(n-1)$-dimensional sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

We claim that $\{z_m\}$ is bounded. Suppose, to the contrary, that there exists a subsequence of $\{z_m\}$, which by abuse of notation is again denoted by $\{z_m\}$, such that

$$\lim_{m\to\infty} \rho_m = +\infty.$$

Since $a_m \in \mathbb{S}^{n-1}$ we may assume, by taking a subsequence if necessary, that

$$\lim_{m\to\infty} a_m = a \in \mathbb{S}^{n-1}.$$

From the homogeneity of ϕ and ψ we obtain

$$\phi(a_m) = \frac{\phi(z_m)}{\rho_m^4} \quad \text{and} \quad \rho_m = \frac{1}{|\psi(a_m)|^{1/n}}.$$

Hence,

$$\lim_{m\to\infty} \phi(a_m) = 0$$

and thus $\phi(a) = 0$. Lemma 3.6 implies that at least $n-1$ coordinates of a are equal. On the other hand, $a_m \in U_n$ and since U_n is closed we have $a \in U_n$. Therefore, $n-1$ coordinates of a vanish. After reenumeration if necessary, we may suppose that $a = (\varepsilon, 0, \dots, 0)$. We set

$$a_m = (a_{m,1}, \dots, a_{m,n}) \quad \text{and} \quad \eta_m = \Big(\sum_{i=2}^{n} a_{m,i}^2\Big)^{1/2}.$$

Since $\psi(a_m) \neq 0$, we may write

$$(a_{m,2}, \dots, a_{m,n}) = \eta_m \theta_m,$$

where

$$\theta_m = (\theta_{m,2}, \dots, \theta_{m,n})$$

lies in the unit $(n-2)$-dimensional sphere $\mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$. Then from

$$\phi(z_m) = \rho_m^4 \phi(a_m)$$

we have that

$$\phi(z_m) \geq$$
$$4\rho_m^4 \sum_{2\leq i<j} \left(a_{m,i}a_{m,j} - \frac{1}{n-2}\Big(\sigma_1(a_m)(a_{m,i}+a_{m,j}) - (a_{m,i}^2 + a_{m,j}^2) - \frac{2\sigma_2(a_m)}{n-1}\Big)\right)^2.$$

We observe that

$$\sigma_1(a_m) = a_{m,1} + \eta_m \sum_{i=2}^n \theta_{m,i} \text{ and } \sigma_2(a_m) = \eta_m \widetilde{\sigma}_2(\theta_m),$$

where

$$\widetilde{\sigma}_2(\theta_m) = a_{m,1}\sum_{j=2}^n \theta_{m,j} + \eta_m \sum_{2\leq i<j} \theta_{m,i}\theta_{m,j}.$$

Therefore, we have

$$\phi(z_m) \geq 4\rho_m^4 \eta_m^2 \delta_m, \tag{3.3}$$

where

$$\delta_m = \sum_{2\leq i<j} \left(\eta_m\theta_{m,i}\theta_{m,j} - \frac{1}{n-2}\Big(\sigma_1(a_m)(\theta_{m,i}+\theta_{m,j}) - \eta_m(\theta_{m,i}^2 + \theta_{m,j}^2) - \frac{2\widetilde{\sigma}_2(\theta_m)}{n-1}\Big)\right)^2.$$

Moreover, we obtain

$$\begin{aligned}
\rho_m^4\eta_m^2 &= \frac{\eta_m^2}{|\psi(a_m)|^{4/n}} \\
&= \frac{\eta_m^2}{|a_{m,1}|^{4/n}\left|\Pi_{i=2}^n a_{m,i}\right|^{4/n}} \\
&= \frac{1}{|a_{m,1}|^{4/n}\,\eta_m^{\frac{2(n-2)}{n}}\left|\Pi_{i=2}^n \theta_{m,i}\right|^{4/n}}.
\end{aligned}$$

By passing if necessary to a subsequence, we may assume that

$$\lim_{m\to\infty} \theta_m = (\bar{\theta}_2, \ldots, \bar{\theta}_n) \in \mathbb{S}^{n-2}.$$

Clearly the above yields

$$\lim_{m\to\infty} \rho_m^4\eta_m^2 = +\infty.$$

Using

$$\lim_{m\to\infty} \sigma_1(a_m) = \varepsilon \text{ and } \lim_{m\to\infty} \widetilde{\sigma}_2(\theta_m) = \varepsilon \sum_{k=2}^n \bar{\theta}_k,$$

we find that

$$\lim_{m\to\infty} \delta_m = \frac{1}{(n-2)^2} \sum_{2\le i<j} \left(\bar{\theta}_i + \bar{\theta}_j - \frac{2\sum_{k=2}^{n} \bar{\theta}_k}{n-1} \right)^2.$$

We claim that

$$\lim_{m\to\infty} \delta_m \neq 0.$$

To the contrary, assume that

$$\lim_{m\to\infty} \delta_m = 0.$$

This implies that $\bar{\theta}_2 = \cdots = \bar{\theta}_n \neq 0$, which for large m contradicts the fact that $a_m \in U_n$. Thus, by taking limits in (3.3) we reach a contradiction and this proves the claim that the sequence $\{z_m\}$ is bounded.

By passing if necessary to a subsequence, we have

$$\lim_{m\to\infty} z_m = z \in \Sigma_n.$$

Since Σ_n does not contain any zeros of ϕ it follows that

$$\min \phi(\Sigma_n) = \lim_{m\to\infty} \phi(z_m) = \phi(z) > 0.$$

Hence the function ϕ attains a positive minimum $\varepsilon(n) = \phi(z)$ on Σ_n, which obviously depends only on n.

Now, let $x \in U_n$. Assume that $\psi(x) \neq 0$ and set

$$\widetilde{x} = \frac{x}{|\psi(x)|^{1/n}}.$$

Clearly $\widetilde{x} \in \Sigma_n$ and consequently $\phi(\widetilde{x}) \geq \varepsilon(n)$. Since ϕ is homogeneous of degree 4, the desired inequality is obviously fulfilled. In the case where $\psi(x) = 0$, the inequality is trivial. ∎

Remark 3.8. The constant $\varepsilon(n)$ that appears in Proposition 3.7 is not computed explicitly here, although one can apply the Lagrange multiplier method to compute it.

3.3 Proof of Theorem 3.1

Let $f\colon (M^n, g) \to \mathbb{R}^{n+1}$ be a conformal immersion with unit normal bundle UN_f and shape operator A_ξ with respect to ξ, where $(x, \xi) \in UN_f$. Using the Gauss equation and the definition of W, it follows that the Weyl tensor $\mathcal{W}_{\tilde{g}}$ with respect to the metric $\tilde{g}$ induced by f is given by

$$\mathcal{W}_{\tilde{g}}(x) = \mathsf{W}(\alpha_f(x)).$$

From Proposition 3.7 we obtain

$$\|\mathcal{W}_{\tilde{g}}(x)\|^2 \geq \varepsilon(n)|\det A_\xi(x)|^{4/n},$$

for all $(x, \xi) \in U^i N_f$ and $2 \leq i \leq n-2$. By integrating over UN_f, we have that

$$\int_{UN_f} \|\mathcal{W}_{\tilde{g}}\|^{n/2}\, d\Sigma \geq (\varepsilon(n))^{n/4} \sum_{i=2}^{n-2} \int_{U^i N_f} |\det A_\xi|\, d\Sigma. \tag{3.4}$$

Using (1.10), it follows that

$$\sum_{i=2}^{n-2} \int_{U^i N_f} |\det A_\xi|\, d\Sigma = \sum_{i=2}^{n-2} \int_{\mathbb{S}^n} \mu_i(u)\, d\mathbb{S} = \mathrm{Vol}(\mathbb{S}^n) \sum_{i=2}^{n-2} \tau_i(f),$$

where $d\mathbb{S}$ is the volume element of the unit n-dimensional sphere $\mathbb{S}^n$. Therefore, from (3.4) and bearing in mind (1.9), we obtain

$$\int_{UN_f} \|\mathcal{W}_{\tilde{g}}\|^{n/2}\, d\Sigma \geq (\varepsilon(n))^{n/4}\mathrm{Vol}(\mathbb{S}^n) \sum_{i=2}^{n-2} \tau_i(f) \geq (\varepsilon(n))^{n/4}\mathrm{Vol}(\mathbb{S}^n) \sum_{i=2}^{n-2} \beta_i(M^n; \mathbb{F}). \tag{3.5}$$

Observe that

$$\int_{M^n} \|\mathcal{W}_{\tilde{g}}\|^{n/2}\, dM_{\tilde{g}} = \frac{1}{2} \int_{UN_f} \|\mathcal{W}_{\tilde{g}}\|^{n/2}\, d\Sigma.$$

Thus, from (3.5) and the fact that the $L^{n/2}$-norm of the Weyl tensor is conformally invariant, we have that

$$\int_{M^n} \|\mathcal{W}_g\|^{n/2}\, dM_g \geq c(n) \sum_{i=2}^{n-2} \beta_i(M^n; \mathbb{F}),$$

where the constant $c(n)$ is given by

$$c(n) = \frac{1}{2}(\varepsilon(n))^{n/4}\mathrm{Vol}(\mathbb{S}^n).$$

Now, assume that

$$\int_{M^n} \|\mathcal{W}_g\|^{n/2}\, dM_g < c(n).$$

Then, it follows from (3.5) that

$$\sum_{i=2}^{n-2} \tau_i(f) < 1.$$

Thus, there exists $u \in \mathbb{S}^n$ such that the height function $h_u \colon M^n \to \mathbb{R}$ is a Morse function whose number of critical points of index i satisfies $\mu_i(u) = 0$ for any $2 \leq i \leq n-2$. From the fundamental theorem of Morse theory (cf. [40, Theorem 3.5, p. 20] or [11, Theorem 4.10, p. 84]), it follows that M^n has the homotopy type of a CW-complex with no cells of dimension i for $2 \leq i \leq n-2$. Hence, the homology groups satisfy $H_i(M^n;\mathbb{Z}) = 0$ for all $2 \leq i \leq n-2$. Moreover, since there are no 2-cells, we conclude by the cellular approximation theorem that the inclusion of the 1-skeleton $X^{(1)} \hookrightarrow M^n$ induces isomorphism between the fundamental groups. Therefore, the fundamental group $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ elements and $H_1(M^n;\mathbb{Z})$ is a free abelian group on $\beta_1(M^n;\mathbb{Z})$ generators. In particular, if $\pi_1(M^n)$ is finite, then $\pi_1(M^n) = 0$ and hence $H_1(M^n;\mathbb{Z}) = 0$. From Poincaré duality and the universal coefficient theorem it follows that $H_{n-1}(M^n;\mathbb{Z}) = 0$. Thus, M^n is a simply connected homology sphere and hence a homotopy sphere. By the generalized Poincaré conjecture (Smale [63] for $n \geq 5$ and Freedman [32] for $n = 4$) we deduce that M^n is homeomorphic to $\mathbb{S}^n$. ∎

3.4 Applications

In this section we derive some applications of Theorem 3.1.

3.4.1 An obstruction to conformal immersions

Chern and Simons provided in [19, Theorem 6.4, p. 65] a necessary condition for a compact 3-dimensional Riemannian manifold to admit a conformal immersion into $\mathbb{R}^4$. Theorem 3.1 allows us to give another such condition for any dimension.

Corollary 3.9. *Let (M^n, g), $n \geq 4$, be a compact n-dimensional Riemannian manifold. A necessary condition that M^n admit a conformal immersion in $\mathbb{R}^{n+1}$ is inequality (3.1).*

3.4.2 Compact minimal hypersurfaces in spheres

We have the following result for compact minimal hypersurfaces in spheres.

Theorem 3.10. *Given $n \geq 4$, there exists a positive constant $c_1(n)$, depending only on n, such that if (M^n, g) is a compact n-dimensional Riemannian manifold that admits an isometric minimal immersion into the unit $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$, then*

$$\int_{M^n} \| \operatorname{Ric} - (n-1)g\|^{n/2}\, dM \geq c_1(n) \sum_{i=2}^{n-2} \beta_i(M^n;\mathbb{F}).$$

In particular, if

$$\int_{M^n} \|\operatorname{Ric} - (n-1)g\|^{n/2}\, dM < c_1(n),$$

then M^n has the homotopy type of a CW-complex with no cells of dimension i for $2 \leq i \leq n-2$ and the fundamental group $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ elements. Moreover, if $\pi_1(M^n)$ is finite then M^n is homeomorphic to the sphere $\mathbb{S}^n$.

Proof: Let $f\colon (M^n, g) \to \mathbb{S}^{n+1}$ be an isometric minimal immersion of a compact n-dimensional Riemannian manifold, with shape operator A. Since f is minimal, the Ricci tensor of M^n is given by

$$\operatorname{Ric}(X,Y) = (n-1)g(X,Y) - g(A^2X, Y), \;\; X, Y \in TM.$$

Let $\{e_1, \ldots, e_n\}$ a local orthonormal frame such that $Ae_i = \lambda e_i$, $1 \leq i \leq n$, where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of f. Using the Gauss equation, from (1.1) we obtain that

$$\begin{aligned}\|\mathcal{W}\|^2 &= \sum_{i,j,k,l} \|\mathcal{W}(e_i, e_j, e_k, e_l)\|^2 \\ &= 4\sum_{i<j}\left(\lambda_i\lambda_j + \frac{\lambda_i^2 + \lambda_j^2}{n-2} - \frac{\|A\|^2}{(n-1)(n-2)}\right)^2.\end{aligned}$$

After a straightforward computation, we find that

$$\|\mathcal{W}\|^2 = \gamma(n)\|A\|^4 - \delta(n)\|\operatorname{Ric} - (n-1)g\|^2, \tag{3.6}$$

where

$$\gamma(n) = \frac{2(n^2 - 3n + 5)}{(n-1)(n-2)}$$

and

$$\delta(n) = \frac{2(n+1)}{n-2}.$$

From the Cauchy-Schwartz inequality we have that

$$\|A\|^4 \leq n\|A^2\|^2 = n\|\operatorname{Ric} - (n-1)g\|^2.$$

Therefore, from (3.6) we obtain

$$\|\mathcal{W}\|^2 \leq a(n)\|\operatorname{Ric} - (n-1)g\|^2,$$

where $a(n) = n\gamma(n) - \delta(n)$. By integrating over M^n, it follows that

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM \leq \big(a(n)\big)^{n/4} \int_{M^n} \|\operatorname{Ric} - (n-1)g\|^{n/2}\, dM. \tag{3.7}$$

The proof follows from (3.7), Theorem 3.1 applied to the composition of f with the stereographic projection and the fact that the $L^{n/2}$-norm of the Weyl tensor is conformally invariant. Clearly,

$$c_1(n) = \frac{c(n)}{a(n)^{n/4}},$$

where $c(n)$ is the constant that appears in Theorem 3.1. This completes the proof. ∎

As an immediate application of Theorem 3.10, we obtain an obstruction for a Riemannian metric to be realized on a compact minimal hypersurface of the sphere.

Corollary 3.11. *A compact n-dimensional Riemannian manifold* (M^n, g), $n \geq 4$, *that satisfies*

$$\int_{M^n} \| \operatorname{Ric} - (n-1)g\|^{n/2} \, dM < c_1(n) \sum_{i=2}^{n-2} \beta_i(M^n; \mathbb{F}),$$

cannot admit an isometric minimal immersion into the unit sphere $\mathbb{S}^{n+1}$.

3.4.3 Almost constant curvature hypersurfaces

We recall (see Chapter 2) that Shiohama and Xu [58] gave a lower bound in terms of the Betti numbers for the $L^{n/2}$-norm of the $(0,4)$-tensor

$$\widetilde{R} = R - \frac{\text{scal}}{n(n-1)} R_1$$

of compact hypersurfaces in $\mathbb{R}^{n+1}$. In fact, they proved that

$$\int_{M^n} \|\widetilde{R}\|^{n/2} \, dM \geq C_n \sum_{i=1}^{n-1} \beta_i(M^n; \mathbb{F}),$$

where C_n is a universal positive constant depending only on n. Their proof strongly uses the fact that the ambient space is the Euclidean one.

Using Theorem 3.1, we are able to extend their result for compact hypersurfaces in spheres or in the hyperbolic space.

Theorem 3.12. *If a compact n-dimensional Riemannian manifold* (M^n, g), $n \geq 4$, *admits an isometric immersion into the sphere* $\mathbb{S}^{n+1}$ *or the hyperbolic space* $\mathbb{H}^{n+1}$, *then*

$$\int_{M^n} \|\widetilde{R}\|^{n/2} \, dM \geq c(n) \sum_{i=2}^{n-2} \beta_i(M^n; \mathbb{F}),$$

where $c(n)$ is the constant that appears in Theorem 3.1. In particular, if

$$\int_{M^n} \|\widetilde{R}\|^{n/2}\, dM < c(n),$$

then M^n has the homotopy type of a CW-complex with no cells of dimension i for $2 \leq i \leq n-2$ and the fundamental group $\pi_1(M^n)$ is a free group on $\beta_1(M^n;\mathbb{Z})$ elements. Moreover, if $\pi_1(M^n)$ is finite then M^n is homeomorphic to the sphere $\mathbb{S}^n$.

Proof: Assume that (M^n, g) admits an isometric immersion into the sphere $\mathbb{S}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$. Using the orthogonal decomposition of the curvature tensor R in (1.1) and recalling the fact that

$$R_1 = \frac{1}{2} g \owedge g,$$

we obtain

$$\left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^2 = \|\mathcal{W}\|^2 + \left\| \frac{1}{n-2} \Big(\text{Ric} - \frac{\text{scal}}{n} g \Big) \owedge g \right\|^2 \geq \|\mathcal{W}\|^2.$$

Thus, it follows that

$$\int_{M^n} \left\| R - \frac{\text{scal}}{n(n-1)} R_1 \right\|^{n/2} dM \geq \int_{M^n} \|\mathcal{W}\|^{n/2}\, dM. \tag{3.8}$$

Now the proof follows from (3.8), Theorem 3.1 and Remark 3.4. ∎

CHAPTER 4

The $L^{n/2}$-norm of the Weyl tensor in low codimension

In this Chapter we prove an inequality concerning the $L^{n/2}$-norm of the Weyl tensor for compact n-dimensional Riemannian manifolds that allow conformal immersions in the Euclidean space with low codimension > 1. This extends our investigation presented in Chapter 3 to higher codimension. All manifolds under consideration are assumed to be without boundary, connected and oriented.

4.1 The main result

Following the notation given in Chapter 2, we prove the following:

Theorem 4.1. *Given $n \geq 6$ and $1/n < \delta < 1$, there exists a positive constant $c_1(n,\delta)$ such that if (M^n, g) is a compact Riemannian manifold that admits a conformal immersion into $\mathbb{R}^{n+k}$ with codimension $2 \leq k \leq [(n-2)/2]$, then*

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM + \int_{M^n} S_{\delta_+}^{n/2} dM \geq c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n; \mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM + \int_{M^n} S_{\delta_+}^{n/2} dM < c_1(n,\delta),$$

then M^n has the homotopy type of a CW-complex with no cells of dimension $k < i < n-k$.

4.2 Algebraic auxiliary results

Let V and W be finite dimensional real vector spaces both equipped with positive definite inner products, which by abuse of notation are both denoted by $\langle\cdot,\cdot\rangle$. The following lemma is in fact contained in [42]. For the sake of completeness we give a short proof.

Lemma 4.2. *Let $\beta \in \mathrm{Sym}(V \times V, W)$ and $\dim W < \dim V - 2$. If $\mathsf{W}(\beta) = 0$, then there exists a vector $\xi \in W$ and a subspace $V_1 \subset V$ such that*

$$\dim V_1 \geq \dim V - \dim W$$

and

$$\beta(x,y) = \langle x,y\rangle\xi \text{ for all } x \in V_1 \text{ and } y \in V.$$

Proof: We endow $\widetilde{W} = W \oplus \mathbb{R}^2$ with the Lorentzian inner product $\langle\langle\cdot,\cdot\rangle\rangle$ given by

$$\langle\langle(\xi,(s_1,s_2)),(\eta,(t_1,t_2))\rangle\rangle = \langle\xi,\eta\rangle + s_1t_2 + s_2t_1$$

and define the symmetric bilinear form $\widetilde{\beta}\colon V \times V \to \widetilde{W}$ by

$$\widetilde{\beta}(x,y) = \big(\beta(x,y), \langle x,y\rangle, -\mathsf{L}(\beta)(x,y)\big).$$

Since $\mathsf{W}(\beta) = 0$ it follows that $\widetilde{\beta}$ is flat with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. Lemma 1.1 implies that there exists a non-zero isotropic vector $e = (\eta,(s,t)) \in \widetilde{W}$ and a symmetric bilinear form $\phi : V \times V \to \mathbb{R}$ such that

$$\dim \mathcal{N}(\widetilde{\beta} - \phi e) \geq \dim V - \dim W.$$

By setting $V_1 = \mathcal{N}(\widetilde{\beta} - e\phi)$, we have that $\widetilde{\beta}(x,y) = \phi(x,y)e$, or equivalently

$$\beta(x,y) = \phi(x,y)\eta,\ \ \langle x,y\rangle = s\phi(x,y) \text{ and } \mathsf{L}(\beta)(x,y) = -t\phi(x,y)$$

for all $x \in V_1$ and $y \in V$. Therefore, $\beta(x,y) = \langle x,y\rangle\xi$ where $\xi = (1/s)\eta$. ∎

Let $\dim V = n$ and $\dim W = k$. If $2 \leq k \leq [(n-2)/2]$, then for each $\beta \in \mathrm{Sym}(V \times V, W)$ we define the subset $\Omega(\beta)$ of the unit $(k-1)$-sphere $\mathbb{S}^{k-1}$ in W given by

$$\Omega(\beta) = \left\{u \in \mathbb{S}^{k-1} : k < \mathrm{Index}\,\beta^\sharp(u) < n-k\right\}.$$

The following proposition is crucial for the proof of Theorem 4.1.

Proposition 4.3. *Given integers $2 \le k \le [(n-2)/2]$ and $1/n < \lambda < 1$, there exists a constant $\varepsilon_1(n,k,\lambda) > 0$ such that the following inequality holds*

$$\|\mathsf{W}(\beta)\|^2 + \big(\|\beta\|^2 - \lambda|\operatorname{trace}\ \beta|^2\big)_+^2 \ge \varepsilon_1(n,k,\lambda)\Big(\int_{\Omega(\beta)} |\det \beta^\sharp(u)| d\mathbb{S}_u\Big)^{4/n} \tag{4.1}$$

for all $\beta \in \operatorname{Sym}(V \times V, W)$.

Proof: We consider the functions $\phi_\lambda, \psi \colon \operatorname{Sym}(V \times V, W) \to \mathbb{R}$ defined by

$$\phi_\lambda(\beta) = \|\mathsf{W}(\beta)\|^2 + \big(\|\beta\|^2 - \lambda|\operatorname{trace}\ \beta|^2\big)_+^2$$

and

$$\psi(\beta) = \int_{\Omega(\beta)} |\det \beta^\sharp(u)| d\mathbb{S}_u.$$

It is sufficient to show that ϕ_λ attaints a positive minimum on the level set

$$\Sigma_{n,k} = \{\beta \in \operatorname{Sym}(V \times V, W) : \psi(\beta) = 1\}.$$

Let $\{\beta_m\}$ be a sequence in $\Sigma_{n,k}$ such that

$$\lim_{m\to\infty} \phi_\lambda(\beta_m) = \inf \phi_\lambda(\Sigma_{n,k}) \ge 0.$$

Since $\beta_m \ne 0$ for all $m \in \mathbb{N}$, we may write $\beta_m = \|\beta_m\|\hat{\beta}_m$, where $\|\hat{\beta}_m\| = 1$.

We claim that the sequence $\{\beta_m\}$ is bounded. Assume to the contrary that there exists a subsequence of $\{\beta_m\}$, which by abuse of notation is again denoted by $\{\beta_m\}$, such that

$$\lim_{m\to\infty} \|\beta_m\| = +\infty.$$

We may assume that $\{\hat{\beta}_m\}$ converges to some $\hat{\beta} \in \operatorname{Sym}(V \times V, W)$ with $\|\hat{\beta}\| = 1$. From the homogeneity of ϕ_λ, we obtain

$$\phi_\lambda(\hat{\beta}_m) = \frac{\phi_\lambda(\beta_m)}{\|\beta_m\|^4}.$$

Thus,

$$\lim_{m\to\infty} \phi_\lambda(\hat{\beta}_m) = 0$$

and consequently $\phi_\lambda(\hat{\beta}) = 0$, or equivalently $\mathsf{W}(\hat{\beta}) = 0$ with

$$1 \le \lambda|\operatorname{trace}\ \hat{\beta}|^2 = \lambda(\mathsf{scal}(\hat{\beta}) + 1). \tag{4.2}$$

Lemma 4.2 implies that there exist a vector subspace $\hat{V}_1 \subset V$ with

$$\dim \hat{V}_1 \ge n - k$$

and a vector $\hat{\xi} \in W$ such that

$$\hat{\beta}(x,y) = \langle x,y\rangle\hat{\xi} \text{ for all } x \in \hat{V}_1 \text{ and } y \in V. \tag{4.3}$$

Since $\beta_m \in \Sigma_{n,k}$, there exists an open subset $\hat{\mathcal{U}}_m \subset \mathbb{S}^{k-1}$ such that

$$\hat{\mathcal{U}}_m \subset \Omega(\hat{\beta}_m) \text{ and } \det \hat{\beta}^\sharp_m(u) \neq 0 \text{ for all } u \in \hat{\mathcal{U}}_m \text{ and } m \in \mathbb{N}.$$

Let $\{\hat{u}_m\}$ be a sequence such that $\hat{u}_m \in \hat{\mathcal{U}}_m$ for all $m \in \mathbb{N}$. We may assume that $\{\hat{u}_m\}$ is convergent and set

$$\hat{u} = \lim_{m\to\infty} \hat{u}_m.$$

Since

$$\lim_{m\to\infty} \hat{\beta}^\sharp_m(\hat{u}_m) = \hat{\beta}^\sharp(\hat{u})$$

and $\hat{u}_m \in \hat{\mathcal{U}}_m$ it follows that Index $\hat{\beta}^\sharp(\hat{u}) < n-k$. From (4.3) we obtain $\langle\hat{\xi},\hat{u}\rangle \geq 0$. We claim that $\langle\hat{\xi},\hat{u}\rangle = 0$. Indeed, if $\langle\hat{\xi},\hat{u}\rangle > 0$ then (4.3) implies that $\hat{\beta}^\sharp(\hat{u})$ has at least $n-k$ positive eigenvalues and so $\hat{\beta}^\sharp_m(\hat{u}_m)$ has at least $n-k$ positive eigenvalues for m large enough. Since $\det \hat{\beta}^\sharp_m(u) \neq 0$ for all $u \in \hat{\mathcal{U}}_m$ and $m \in \mathbb{N}$, we have that $\hat{\beta}^\sharp_m(\hat{u}_m)$ has at most k negative eigenvalues, which contradicts that $\hat{u}_m \in \hat{\mathcal{U}}_m$ for all $m \in \mathbb{N}$.

Thus,

$$\langle \lim_{m\to\infty} \hat{u}_m, \hat{\xi}\rangle = 0$$

for any convergent sequence $\{\hat{u}_m\}$ such that $\hat{u}_m \in \hat{\mathcal{U}}_m$ for all $m \in \mathbb{N}$. We may choose convergent sequences $\{\hat{u}^{(1)}_m\},\ldots,\{\hat{u}^{(k)}_m\}$ in $\hat{\mathcal{U}}_m$ such that $\hat{u}^{(1)}_m,\ldots,\hat{u}^{(k)}_m$ span W for all $m \in \mathbb{N}$. Using (4.3) and

$$\langle \lim_{m\to\infty} \hat{u}^{(a)}_m, \hat{\xi}\rangle = 0$$

for all $a \in \{1,\ldots,k\}$ we have that

$$\lim_{m\to\infty} \hat{\beta}_m|_{\hat{V}_1\times\hat{V}_1} = 0$$

and consequently

$$\lim_{m\to\infty} \mathsf{R}(\hat{\beta}_m)|_{\hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1} = 0 \tag{4.4}$$

and

$$\lim_{m\to\infty} \mathsf{L}(\hat{\beta}_m)|_{\hat{V}_1\times\hat{V}_1} = -\frac{\mathsf{scal}(\hat{\beta})}{2(n-1)(n-2)} \langle\cdot,\cdot\rangle|_{\hat{V}_1\times\hat{V}_1}. \tag{4.5}$$

From

$$\lim_{m\to\infty} \phi_\lambda(\hat{\beta}_m) = 0,$$

(4.4), (4.5) and the inequality

$$\|\mathsf{W}(\hat{\beta}_m)|_{\hat{V}_1\times\hat{V}_1\times\hat{V}_1\times\hat{V}_1}\|^2 \leq \phi_\lambda(\hat{\beta}_m),$$

we obtain that $\mathsf{scal}(\hat{\beta}) = 0$, which contradicts (4.2). Thus, the sequence $\{\beta_m\}$ is bounded. We may assume that

$$\lim_{m\to\infty} \beta_m = \beta \in \mathrm{Sym}(V \times V, W).$$

We claim that $\phi_\lambda(\beta) > 0$. Arguing indirectly, we assume that $\phi_\lambda(\beta) = 0$. Then, we have $\mathsf{W}(\beta) = 0$ with

$$\|\beta\|^2 \leq \lambda|\,\mathrm{trace}\ \beta|^2 = \lambda(\mathsf{scal}(\beta) + \|\beta\|^2). \tag{4.6}$$

Lemma 4.2 implies that there exist a vector subspace $V_1 \subset V$ with

$$\dim V_1 \geq n - k$$

and a vector $\xi \in W$ such that

$$\beta(x,y) = \langle x,y\rangle\xi \ \text{ for all } x \in V_1 \text{ and } y \in V. \tag{4.7}$$

We observe that $\beta \neq 0$. Indeed, if $\beta = 0$ then $\beta^\sharp(u) = 0$ for all $u \in \mathbb{S}^{k-1}$. Since $\beta_m \in \Sigma_{n,k}$, there exists $\xi_m \in \Omega(\beta_m)$ such that

$$|\det \beta_m^\sharp(\xi_m)|\mathrm{Vol}(\Omega(\beta_m)) = 1 \ \text{ for all } \ m \in \mathbb{N}. \tag{4.8}$$

We may assume that the sequence $\{\xi_m\}$ converges to some $\xi \in \mathbb{S}^{k-1}$. Then we have that

$$\lim_{m\to\infty} \beta_m^\sharp(\xi_m) = \beta^\sharp(\xi) = 0,$$

which contradicts (4.8). Therefore $\beta \neq 0$.

From $\beta_m \in \Sigma_{n,k}$ we have that there exists an open subset $\mathcal{U}_m \subset \mathbb{S}^{k-1}$ such that

$$\mathcal{U}_m \subset \Omega(\beta_m) \text{ and } \det \beta_m^\sharp(u) \neq 0 \text{ for all } u \in \mathcal{U}_m \text{ and } m \in \mathbb{N}.$$

Let $\{u_m\}$ be a sequence with $u_m \in \mathcal{U}_m$ for all $m \in \mathbb{N}$. We may assume that u_m is convergent and set

$$u = \lim_{m\to\infty} u_m.$$

Since

$$\lim_{m\to\infty} \beta_m^\sharp(u_m) = \beta^\sharp(u)$$

and $u_m \in \mathcal{U}_m$, it follows that Index $\beta^\sharp(u) < n - k$. From (4.7) we have $\langle \xi, u\rangle \geq 0$. We claim that $\langle \xi, u\rangle = 0$. Indeed, if $\langle \xi, u\rangle > 0$, then (4.7) implies that $\beta^\sharp(u)$

has at least $n-k$ positive eigenvalues and so $\beta_m^\sharp(u_m)$ has at least $n-k$ positive eigenvalues for m large enough. Since $\det \beta_m^\sharp(u) \neq 0$ for all $u \in \mathcal{U}_m$ and $m \in \mathbb{N}$, we have that $\beta_m^\sharp(u_m)$ has at most k negative eigenvalues, which contradicts that $u_m \in \mathcal{U}_m$ for all $m \in \mathbb{N}$.

Thus,

$$\langle \lim_{m\to\infty} u_m, \xi \rangle = 0$$

for any convergent sequence $\{u_m\}$ such that $u_m \in \mathcal{U}_m$ for all $m \in \mathbb{N}$. We may choose convergent sequences $\{u_m^{(1)}\}, \dots, \{u_m^{(k)}\}$ in $\mathcal{U}_m$ such that $u_m^{(1)}, \dots, u_m^{(k)}$ span W for all $m \in \mathbb{N}$. Using (4.7) and

$$\langle \lim_{m\to\infty} u_m^{(a)}, \xi \rangle = 0$$

for all $a \in \{1, \dots, k\}$, we obtain that

$$\lim_{m\to\infty} \beta_m|_{V_1\times V_1} = 0$$

and consequently

$$\lim_{m\to\infty} \mathsf{R}(\beta_m)|_{V_1\times V_1\times V_1\times V_1} = 0 \tag{4.9}$$

and

$$\lim_{m\to\infty} \mathsf{L}(\beta_m)|_{V_1\times V_1} = -\frac{\mathsf{scal}(\beta)}{2(n-1)(n-2)} \langle \cdot, \cdot \rangle|_{V_1\times V_1} . \tag{4.10}$$

From

$$\lim_{m\to\infty} \phi_\lambda(\beta_m) = 0,$$

(4.9), (4.10) and the inequality

$$\|\mathsf{W}(\beta_m)|_{V_1\times V_1\times V_1\times V_1}\|^2 \leq \phi_\lambda(\beta_m),$$

we obtain that $\mathsf{scal}(\beta) = 0$, which contradicts (4.6). Thus, ϕ_λ attains a positive minimum $\varepsilon_1(n,k,\lambda) = \phi_\lambda(\beta)$ on $\Sigma_{n,k}$ which depends only on n, k and λ.

Let $\beta \in \mathrm{Sym}(V \times V, W)$. Assume that $\psi(\beta) \neq 0$ and set

$$\gamma = \frac{\beta}{(\psi(\beta))^{1/n}}.$$

Clearly $\gamma \in \Sigma_{n,k}$ and consequently $\phi_\lambda(\gamma) \geq \varepsilon_1(n,k,\lambda)$. The desired inequality follows from the homogeneity of ϕ_λ. ∎

Remark 4.4. Arguing as in Remark 2.5 we have that for $1/n < \lambda < 1$ inequality (4.1) fails if we drop the first term of the left hand side.

4.3 Proof of Theorem 4.1

Let $f\colon (M^n, g) \to \mathbb{R}^{n+k}$ be a conformal immersion with second fundamental form α and shape operator A_ξ with respect to ξ, where $(x, \xi) \in UN_f$. Using the Gauss equation, it follows that the Weyl tensor $\mathcal{W}_{\tilde{g}}$ with respect to the induced metric $\tilde{g}$ of f is given by $\mathcal{W}_{\tilde{g}}(x) = \mathsf{W}(\alpha(x))$. From Proposition 4.3, we have

$$\left(\|\mathcal{W}_{\tilde{g}}\|^2 + S^2_{\delta_+}\right)^{n/4}(x) \geq (\varepsilon_1(n,k,\delta))^{n/4} \int_{\Omega(\alpha(x))} |\det A_\xi| dV_\xi$$

for all $x \in M^n$. By integrating and using (1.10), we obtain

$$\int_{M^n} \left(\|\mathcal{W}_{\tilde{g}}\|^2 + S^2_{\delta_+}\right)^{n/4} dM \geq (\varepsilon_1(n,k,\delta))^{n/4} \mathrm{Vol}(\mathbb{S}^{n+k-1}) \sum_{i=k+1}^{n-k-1} \tau_i(f). \tag{4.11}$$

Observe that

$$\left(\|\mathcal{W}_{\tilde{g}}\|^2 + S^2_{\delta_+}\right)^{n/4}(x) \leq 2^{(n-4)/4} \left(\|\mathcal{W}_{\tilde{g}}\|^{n/2} + S^{n/2}_{\delta_+}\right)(x)$$

for all $x \in M^n$. From (4.11), (1.9) and since the $L^{n/2}$-norm of the Weyl tensor is conformally invariant, we have that

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM + \int_{M^n} S^{n/2}_{\delta_+} dM \geq c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \tau_i(f) \geq c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n; \mathbb{F}), \tag{4.12}$$

where

$$c_1(n,\delta) = \min_{2 \leq k \leq [(n-2)/2]} \left\{ 2 \Big(\frac{\varepsilon_1(n,k,\delta)}{2} \Big)^{n/4} \mathrm{Vol}(\mathbb{S}^{n+k-1}) \right\}.$$

Now assume that

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM + \int_{M^n} S^{n/2}_{\delta_+} dM < c_1(n,\delta).$$

It follows from (4.12) that

$$\sum_{i=k+1}^{n-k-1} \tau_i(f) < 1.$$

Thus, there exists $u \in \mathbb{S}^{n+k-1}$ such that the height function h_u is a Morse function whose number of critical points of index i satisfies $\mu_i(u) = 0$ for any $k < i < n-k$. Clearly M^n has the homotopy type of a CW-complex with no cells of dimension $k < i < n-k$. ∎

4.4 Applications

In this section we derive some applications of Theorem 4.1.

4.4.1 Minimal conformal immersions in spheres

Corollary 4.5. *If a compact Riemannian manifold* (M^n, g), $n \geq 8$, *admits a conformal minimal immersion into the unit* $(n+k-1)$*-dimensional sphere* $\mathbb{S}^{n+k-1}$ *with* $3 \leq k \leq [(n-2)/2]$ *and* $S \leq n(\delta n - 1)$ *for some* $1/n < \delta < 1$*, then*

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM \geq c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n; \mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM < c_1(n,\delta)$$

then M^n *has the homotopy type of a CW-complex with no cells of dimension* i *for* $k < i < n-k$.

Proof: We consider the immersion $\widetilde{f} = i \circ f$, where $i \colon \mathbb{S}^{n+k-1} \hookrightarrow \mathbb{R}^{n+k}$ is the totally umbilical inclusion. The proof follows directly from Theorem 4.1. ∎

4.4.2 δ-pinched immersions

Theorem 4.1 provides information on δ-pinched immersions (see subsection 2.4.2) for any $1/n < \delta < 1$. Indeed, the following corollary follows immediately from Theorem 4.1 and gives an intrinsic obstruction to δ-pinched immersions.

Corollary 4.6. *If a compact Riemannian manifold* (M^n, g) *admits a conformal* δ*-pinched immersion into* $\mathbb{R}^{n+k}$ *with* $2 \leq k \leq [(n-2)/2]$ *and* $1/n < \delta < 1$*, then*

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM \geq c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n; \mathbb{F}).$$

Moreover, if

$$\int_{M^n} \|\mathcal{W}\|^{n/2} dM < c_1(n,\delta)$$

then M^n *has the homotopy type of a CW-complex with no cells of dimension* $k < i < n-k$.

4.4.3 Further results

Corollary 4.7. *If a compact Riemannian manifold (M^n, g) admits a conformal immersion into $\mathbb{R}^{n+k}$ with $2 \leq k \leq [(n-2)/2]$ such that*

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM \leq \lambda\, c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n;\mathbb{F})$$

and

$$\int_{M^n} S_{\delta_+}^{n/2}\, dM < (1-\lambda)\, c_1(n,\delta),$$

for some $0 < \lambda < 1$ and $1/n < \delta < 1$, then (M^n, g) is conformally flat and has the homotopy type of a CW-complex with no cells of dimension i for $k < i < n-k$.

Proof: Our assumptions and Theorem 4.1 imply that $\beta_i(M^n;\mathbb{F}) = 0$ for $k < i < n-k$. Hence, f is conformally flat and the rest of the proof follows from Theorem 4.1. ∎

Corollary 4.8. *If a compact Riemannian manifold (M^n, g) admits a conformal immersion f into $\mathbb{R}^{n+k}$ with $2 \leq k \leq [(n-2)/2]$ such that*

$$\int_{M^n} \|\mathcal{W}\|^{n/2}\, dM < \lambda\, c_1(n,\delta)$$

and

$$\int_{M^n} S_{\delta_+}^{n/2}\, dM \leq (1-\lambda)\, c_1(n,\delta) \sum_{i=k+1}^{n-k-1} \beta_i(M^n;\mathbb{F}),$$

for some $0 < \lambda < 1$ and $1/n < \delta < 1$, then f is δ-pinched and M^n has the homotopy type of a CW-complex with no cells of dimension i for $k < i < n-k$. Furthermore, if $H > 0$ everywhere and $\delta = 1/(n-1)$, then M^n is diffeomorphic to $\mathbb{S}^n$.

Proof: Our assumptions and Theorem 4.1 imply that $\beta_i(M^n;\mathbb{F}) = 0$ for $k < i < n-k$. Hence, f is δ-pinched and the rest of the proof follows from Theorem 4.1. Moreover, if $H > 0$ everywhere and f is $1/(n-1)$-pinched, then a result due to Andrews and Baker [1] implies that M^n is diffeomorphic to $\mathbb{S}^n$. ∎

CHAPTER 5

Einstein submanifolds with flat normal bundle in space forms

A well known result asserts that any isometric immersion with flat normal bundle of a Riemannian manifold with constant sectional curvature into a space form is (at least locally) holonomic, under appropriate assumptions [22, 24]. In this chapter, we show that this conclusion remains valid for the larger class of Einstein manifolds. As an application, when assuming that the index of relative nullity of the immersion is a positive constant we conclude that the submanifold has the structure of a generalized cylinder over a submanifold with flat normal bundle. Moreover, we provide a complete classification of Einstein submanifolds in space forms with flat normal bundle and parallel mean curvature vector field.

5.1 The general case

A remarkable class of submanifolds in space forms are those that enjoy the property of being holonomic. An isometric immersion $f\colon (M^n, g) \to \mathbb{Q}_c^N$ of a Riemannian manifold into a space form of constant sectional curvature c is said to be *holonomic* if M^n carries a global system of orthogonal coordinates such that at any point the coordinate vector fields diagonalize its second fundamental form $\alpha\colon TM \times TM \to N_f M$ with values in the normal bundle.

Among several interesting facts regarding holonomic submanifolds, we recall that they are a natural play-ground for the Ribaucour transformation [24]. As an application of the so called vectorial Ribaucour transformation as given in [26], one can locally parametrically generate any proper holonomic submanifold in terms of a set of smooth functions whose Hessians are all diagonal with respect to the coordinate vector fields of a given orthogonal system of coordinates; see [25] for details.

There are several conditions that imply that a submanifold of a space form has to be locally holonomic. By locally we mean along connected components of an open dense subset of the manifold. For instance, this is the case of any isometric immersion $f\colon M^n_c \to \mathbb{Q}^N_c$ with flat normal bundle of a manifold with the same constant sectional curvature as the ambient space form provided that index of relative nullity vanishes at any point; see [24] for a more general result. Recall that the *index of relative nullity* $\nu(x)$ of $f\colon (M^n,g) \to \mathbb{Q}^N_c$ at $x \in M^n$ is the dimension of the *relative nullity subspace* $\Delta(x) \subset T_xM$ given by

$$\Delta(x) = \{X \in T_xM : \alpha(X,Y) = 0 \text{ for all } Y \in T_xM\}.$$

Isometric immersions $f\colon M^n_c \to \mathbb{Q}^{n+p}_{\tilde{c}}$ with sectional curvatures $c < \tilde{c}$ and in the least possible codimension $p = n-1$ have flat normal bundle and thus are always locally holonomic. This was already known to Cartan [8, 9] who made an exhaustive study of the subject and determined the degree of generality of such submanifolds. Moreover, being holonomic is also necessarily the case for $c > \tilde{c}$ but now under the extra condition that the submanifold is free of weak-umbilic points; see [41].

In this section, we show that results discussed above still hold for isometric immersions of the larger class of Einstein manifolds. In fact, this turns out to be the case even in the presence of a constant positive index of relative nullity, thus in the case of submanifolds of manifolds with the same constant sectional curvature the restriction mentioned above can be removed.

5.1.1 The main result

We prove the following:

Theorem 5.1. *Any isometric immersion $f\colon (M^n,g) \to \mathbb{Q}^N_c$ with flat normal bundle and proper of an Einstein manifold is locally holonomic.*

5.1.2 Auxiliary lemmas

For an isometric immersion $f\colon (M^n,g) \to \mathbb{Q}^N_c$ a straightforward computation of the Ricci tensor using the Gauss equation for f gives

$$\mathrm{Ric}(X,Y) = c(n-1)g(X,Y) + n\langle\alpha(X,Y),\mathcal{H}\rangle - \sum_{j=1}^{n}\langle\alpha(X,X_j),\alpha(Y,X_j)\rangle \tag{5.1}$$

where $X_1,\ldots,X_n$ is a local orthonormal tangent frame.

For submanifolds with flat normal we have pointwise the following two facts.

Lemma 5.2. *The Riemannian manifold* (M^n, g) *is Einstein with* $\mathrm{Ric} = \lambda g$ *if and only if the vector fields*

$$\hat{\eta}_i = \eta_i - \frac{n}{2}\mathcal{H}, \ \ 1 \le i \le k,$$

satisfy that

$$\|\hat{\eta}_i\|^2 = \frac{n^2}{4}\|\mathcal{H}\|^2 + c(n-1) - \lambda, \ \ 1 \le i \le k, \tag{5.2}$$

where η_i*,* $1 \le i \le k$*, are the principal normals of* f*.*

Proof: It follows easily from (5.1) using (1.6) that

$$\mathrm{Ric}(X,Y) = c(n-1)g(X,Y) + n\langle\alpha(X,Y),\mathcal{H}\rangle - \sum_{i=1}^{s} g(X^i, Y^i)\|\eta_i\|^2 \tag{5.3}$$

for all $X, Y \in TM$. From (1.6) and (5.3) we have that $\mathrm{Ric} = \lambda g$ is equivalent to

$$c(n-1) - \lambda = \|\eta_i\|^2 - n\langle\mathcal{H}, \eta_i\rangle, \ \ 1 \le i \le k,$$

and this in turn is equivalent to (5.2). ∎

Lemma 5.3. *The vectors* $\eta_i - \eta_j$ *and* $\eta_i - \eta_\ell$ *are linearly independent if* $i \ne j \ne \ell \ne i$*.*

Proof: Assume to the contrary that

$$\eta_i - \eta_j = \mu(\eta_i - \eta_\ell)$$

for some $\mu \ne 0$. Then

$$(1-\mu)\hat{\eta}_i = \hat{\eta}_j - \mu\hat{\eta}_\ell$$

yields

$$\|\hat{\eta}_i\|^2 - 2\mu\|\hat{\eta}_i\|^2 + \mu^2\|\hat{\eta}_i\|^2 = \|\hat{\eta}_j\|^2 - 2\mu\langle\hat{\eta}_j, \hat{\eta}_\ell\rangle + \mu^2\|\hat{\eta}_\ell\|^2.$$

We have from (5.2) that the $\hat{\eta}_j$'s are of equal length. Thus $\langle\hat{\eta}_j, \hat{\eta}_\ell\rangle = \|\hat{\eta}_j\|\|\hat{\eta}_\ell\|$ and hence $\hat{\eta}_j = \hat{\eta}_\ell$, and that is a contradiction since the principal normals are pairwise distinct. ∎

5.1.3 Proof of Theorem 5.1

It is a standard fact that in order to conclude holonomicity it suffices to show is that the distributions $E_j^\perp$ are integrable for $1 \le j \le k$. First, we observe that it follows from (1.7) that the E_i's are integrable. Thus, it is sufficient to argue for the case $k \ge 3$. In fact, it suffices to show that if $X \in E_i$ and $Y \in E_j$ then $[X,Y] \in E_\ell^\perp$ if $i \ne j \ne \ell \ne i$. We have from (1.8) that

$$g(\nabla_X Y, Z)(\eta_\ell - \eta_j) = g(\nabla_Y X, Z)(\eta_\ell - \eta_i)$$

for any $Z \in E_\ell$. We obtain using Lemma 5.3 that

$$g(\nabla_X Y, Z) = g(\nabla_Y X, Z) = 0,$$

and this completes the proof. ∎

5.1.4 An application

Let $h\colon (L^{n-s}, \tilde{g}) \to \mathbb{Q}_c^N$, $1 \leq s \leq n-1$, be an isometric immersion carrying a parallel flat normal subbundle $\mathcal{L} \subset N_h L$ of rank s. The *generalized cylinder* determined by the subbundle $\pi : \mathcal{L} \to L^{n-s}$ is the n-dimensional submanifold $f\colon M^n \to \mathbb{Q}_c^N$ parametrized (at regular points) by means of the exponential map of $\mathbb{Q}_c^N$ as

$$\mathcal{L} \ni \gamma \mapsto \exp_{h(\pi(\gamma))} \gamma.$$

We have that $\gamma \in \mathcal{L}$ is a regular point if and only if

$$P = I - A_\gamma^h$$

is nonsingular where A_γ^h stands for the shape operator of h corresponding to γ. Also

$$N_f M = \mathcal{L}^\perp,$$

up to parallel identification along the fibers of $\mathcal{L}$ that are contained in the relative nullity subspaces of f. Moreover, the relation between the second fundamental forms of f and h is given by

$$\alpha_f(X, Y) = (\alpha_h(X, PY))_{\mathcal{L}^\perp}$$

for all $X, Y \in TL$. It follows that f has flat normal bundle if and only if h has flat normal bundle.

The following result obtained in [25] asserts that any submanifold with a relative nullity distribution $x \in M^n \mapsto \Delta(x)$ of constant dimension whose *conullity* distribution $x \in M^n \mapsto \Delta^\perp(x)$ is integrable has to be a generalized cylinder.

Proposition 5.4. *Let $h\colon (L^{n-s}, \tilde{g}) \to \mathbb{Q}_c^N$ be an isometric immersion carrying a parallel flat normal subbundle $\mathcal{L} \subset N_h L$ of rank s such that*

$$\{Y \in T_x L : (\alpha_h(Y, X))_{\mathcal{L}^\perp} = 0 \ \textit{for all} \ \ X \in T_x L\} = \{0\}$$

for any point $x \in M^n$. Then the generalized cylinder over h determined by $\mathcal{L}$ has relative nullity of constant dimension s and integrable conullity.

Conversely, any submanifold $f\colon (M^n, g) \to \mathbb{Q}_c^N$ *with relative nullity distribution* Δ *of constant dimension* s *and integrable conullity arises this way locally. This means that* $\mathcal{L} = \Delta|_L$ *is a parallel flat normal subbundle of* $h = f|_L$ *for any integral leaf* L^{n-s} *of the conullity and* f *is locally an open neighborhood of* $h(L)$ *in the generalized cylinder over* h *determined by* $\mathcal{L}$*.*

We have the following consequence of Theorem 5.1.

Corollary 5.5. *Let* $f\colon (M^n, g) \to \mathbb{Q}_c^N$ *be an isometric immersion with flat normal bundle of an Einstein manifold that is proper and has constant index of relative nullity* $\nu = s \geq 1$*. Then* f *is locally a generalized cylinder over a submanifold* $g\colon L^{n-s} \to \mathbb{Q}_c^N$ *with flat normal bundle.*

Proof: Notice that if the index of relative nullity is $\nu \geq 1$ at any point and $\nu > 1$ at some point then it has to be constant since f is proper. The proof follows easily from Theorem 5.6 and Proposition 5.4. ∎

5.2 The case of parallel mean curvature

5.2.1 The main result

In [21] Dajczer and Tojeiro provided a classification of isometric immersions of Riemannian manifolds with constant sectional curvature into $\mathbb{Q}_c^N$ with flat normal bundle and parallel mean curvature vector field. In this section, we extend their result to the class of Einstein submanifolds by proving the following:

Theorem 5.6. *Let* $f\colon (M^n, g) \to \mathbb{Q}_c^N, n \geq 3$*, be an isometric immersion of a connected Einstein manifold with* Ric $= \lambda g$*, flat normal bundle and parallel mean curvature vector field. Then one of the following holds:*

(i) The immersion f *is totally umbilical.*

(ii) $\lambda = 0 = c$ *and*

$$f(M^n) \subset \mathbb{S}^1(r_1) \times \cdots \times \mathbb{S}^1(r_k) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+k}.$$

(iii) $\lambda = 0 < c$ *and*

$$f(M^n) \subset \mathbb{S}^1(r_1) \times \cdots \times \mathbb{S}^1(r_n) \subset \mathbb{S}_c^{2n-1} \subset \mathbb{R}^{2n},$$

where $r_1^2 + \cdots + r_n^2 = 1/c$*.*

(iv) $\lambda = 0 > c$ *and*

$$f(M^n) \subset \mathbb{H}^1(r_1) \times \mathbb{S}^1(r_2) \times \cdots \times \mathbb{S}^1(r_n) \subset \mathbb{H}_c^{2n-1} \subset \mathbb{L}^{2n},$$

where $-r_1^2 + r_2^2 + \cdots + r_n^2 = 1/c$.

(v) $\lambda = c(n-k) > 0$ *and*

$$f(M^n) \subset \mathbb{S}^{m_1}(\rho_1) \times \cdots \times \mathbb{S}^{m_k}(\rho_k) \subset \mathbb{S}_c^{n+k-1} \subset \mathbb{R}^{n+k},$$

where $\rho_i = \sqrt{(m_i - 1)/\lambda}$, $m_i \geq 2$ *for all* $1 \leq i \leq k$.

(vi) $f = j \circ h$, *where* h *is as in (ii), (iii), (iv) or (v) and* j *is a totally umbilical inclusion.*

5.2.2 Auxiliary lemmas

Let $f\colon (M^n, g) \to \mathbb{Q}_c^N$ be an isometric immersion as in Theorem 5.6. In the following we are working on an open subset G_k with $k \geq 2$.

Lemma 5.7. *Around every point* $p \in G_k$ *there is a neighborhood* U *that is a Riemannian product of Riemannian manifolds* $M_1, \ldots, M_k$. *Moreover,* $f|_U$ *is the extrinsic product of totally umbilical isometric immersions* $f_1, \ldots, f_k$.

Proof: We claim that each distribution E_i, $1 \leq i \leq k$, is parallel, that is

$$\nabla_X Y \in E_i \text{ for all } X \in TG_k,\ Y \in E_i \text{ and } 1 \leq i \leq k.$$

First we prove that $\nabla_X Y \in E_i$ for all $X, Y \in E_i$ and $1 \leq i \leq k$. Indeed, from (1.7) we have that

$$g(\nabla_X Y, Z)(\eta_i - \eta_j) = g(X,Y)\nabla_Z^{\perp}\hat{\eta}_i$$

for any $X, Y \in E_i$ and $Z \in E_j$ with $j \neq i$. Thus, we obtain

$$2g(\nabla_X Y, Z)\langle \eta_i - \eta_j, \hat{\eta}_i \rangle = g(X,Y)Z\|\hat{\eta}_i\|^2. \tag{5.4}$$

Using (5.2) we observe that

$$\langle \eta_i - \eta_j, \hat{\eta}_i \rangle = \|\hat{\eta}_i\|^2 - \langle \hat{\eta}_i, \hat{\eta}_j \rangle \neq 0.$$

Then (5.4) implies that $\nabla_X Y \in E_i$ for all $X, Y \in E_i$ and any $1 \leq i \leq k$.

Now, we show that $\nabla_X Y \in E_i$ for all $X \in E_j$ and $Y \in E_i$ with $j \neq i$. To this aim, we consider $Y \in E_i$, $X \in E_j$, $Z \in E_l \subset E_i^{\perp}$ and distinguish the following two cases.

If $l = j$, then by using the previous argument, we have

$$g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = 0. \tag{5.5}$$

If $l \neq j$, then from (1.8) we obtain

$$g(\nabla_X Y, Z)(\eta_l - \eta_i) = g(\nabla_Y X, Z)(\eta_l - \eta_j).$$

Thus, by using Lemma 5.3, we obtain that

$$g(\nabla_X Y, Z) = 0 \text{ for all } X \in E_j, Y \in E_i, Z \in E_l, \tag{5.6}$$

with $l \neq i \neq j$. Therefore, from (5.5) and (5.6), we have that $\nabla_X Y \in E_i$ for all $X \in E_j$ and $Y \in E_i$ with $j \neq i$. This completes the proof of our claim.

Now, de Rham's theorem implies that around every point $p \in G_k$ there is a neighborhood U that is the Riemannian product of the integral manifolds $M_1, \ldots,$ M_k of the distributions $E_1, \ldots, E_k$ respectively, through a point $q \in U$. Moreover, since the second fundamental form of f is adapted, Theorem 1.5 implies that $f|_U$ is an extrinsic product of isometric immersions $f_1, \ldots, f_k$, which due to (1.5) have to be totally umbilical. ∎

Lemma 5.8. *Let $m_i = \dim M_i$. Then the following holds:*

$$(m_i - 1)(c + \|\eta_i\|^2) = \lambda, \; 1 \leq i \leq k. \tag{5.7}$$

Moreover, if $m_i \geq 2$ then the sectional curvature of M_i is

$$K_{M_i} = \frac{\lambda}{m_i - 1}. \tag{5.8}$$

Furthermore, if $m_i \geq 2$ for all $1 \leq i \leq k$, then $\lambda > 0$.

Proof: It follows from Lemma 5.7 and the Gauss equation that the principal normals $\eta_1, \ldots, \eta_k$ of f satisfy

$$\langle \eta_i, \eta_j \rangle = -c, \; 1 \leq i \neq j \leq k. \tag{5.9}$$

Equation (5.7) follows from the fact that M^n is Einstein, (5.1) and (5.9). If $m_i \geq 2$ then (5.7) and the Gauss equation imply (5.8).

Now, suppose that $m_i \geq 2$ for all $1 \leq i \leq k$ and assume to the contrary that $\lambda \leq 0$. Then (5.7) implies that

$$\|\eta_i\|^2 \leq -c \;\text{ for all }\; 1 \leq i \leq k \tag{5.10}$$

and thus $c < 0$. Therefore, from (5.9), (5.10) and the Cauchy-Schwarz inequality we obtain that

$$-c = \langle \eta_i, \eta_j \rangle \leq \|\eta_i\| \|\eta_j\| \leq -c \ \text{ for all } \ i \neq j. \tag{5.11}$$

Hence, $\eta_j = \mu_{ij}\eta_i$ for some $\mu_{ij} > 0$ and $1 \leq i \neq j \leq k$. From (5.9) and (5.10) we have that $\mu_{ij} \geq 1$ for all $1 \leq i \neq j \leq k$. Since $\eta_j = \mu_{ij}\eta_i = \mu_{ij}\mu_{ji}\eta_j$, it follows that $\mu_{ij} \leq 1$. Therefore, $\mu_{ij} = 1$ which is a contradiction. ∎

Lemma 5.9. *Assume that there exists $1 \leq i \leq k$ such that $m_i = 1$. Then U is flat.*

Proof: The proof follows from Lemmas 5.7 and 5.8. ∎

5.2.3 Proof of Theorem 5.6

Let $f\colon (M^n, g) \to \mathbb{Q}_c^N$, $n \geq 3$, be an isometric immersion of a connected Einstein manifold with $\mathrm{Ric} = \lambda g$, flat normal bundle and parallel mean curvature vector field. If $c \neq 0$, we always view $\mathbb{Q}_c^N$ as an umbilical hypersurface of the Euclidean space $\mathbb{R}^{N+1}$ or the Lorentzian space $\mathbb{L}^{N+1}$ according to the sign of c. In the following we are working on an open subset G_k.

If $k = 1$, then f is a totally umbilical immersion. In the sequel we assume that $k \geq 2$ and let $p \in G_k$. Then, according to Lemma 5.7 we have that there exists a neighborhood U of p that is a Riemannian product of Riemannian manifolds $M_1, \ldots, M_k$ and $f|_U$ is an extrinsic product of totally umbilical isometric immersions $f_1, \ldots, f_k$.

We distinguish two cases.

If there exists $1 \leq i \leq k$ such that $m_i = 1$ then the result follows from Lemma 5.9 and Theorem 1 of [21].

We now assume that $m_i \geq 2$ for all $1 \leq i \leq k$. If $c = 0$, then each f_i is an umbilical isometric immersion into $\mathbb{R}^{m_i+1}$, where

$$\mathbb{R}^N = \mathbb{R}^m \times \Pi_{i=1}^k \mathbb{R}^{m_i+1}.$$

Therefore, bearing in mind (5.8) we obtain that

$$f(U) \subset \mathbb{S}^{m_1}(\rho_1) \times \cdots \times \mathbb{S}^{m_k}(\rho_k) \subset \mathbb{R}^N.$$

If $c > 0$, then each f_i is an umbilical isometric immersion into $\mathbb{S}^{m_i+1}(r_i) \subset \mathbb{R}^{m_i+2}$, where

$$\mathbb{R}^{N+1} = \mathbb{R}^m \times \Pi_{i=1}^k \mathbb{R}^{m_i+2}$$

Thus, by using (5.8) we obtain that

$$f(U) \subset \mathbb{S}^{m_1}(\rho_1) \times \cdots \times \mathbb{S}^{m_k}(\rho_k) \subset \mathbb{S}_c^N.$$

Finally, if $c < 0$ then f is the extrinsic product of umbilical isometric immersions $f_1, \ldots, f_k$ of either hyperbolic, elliptic or parabolic type.

If $f_1, \ldots, f_k$ is of hyperbolic type, then f_1 is an umbilical isometric immersion into $\mathbb{H}^{m_1+1}(r_1) \subset \mathbb{L}^{m_1+2}$ and each f_i is an umbilical isometric immersion into

$$\mathbb{S}^{m_i+1}(r_i) \subset \mathbb{R}^{m_i+2}, \;\; 2 \leq i \leq k,$$

where

$$\mathbb{L}^{N+1} = \mathbb{L}^{m_1+2} \times \Pi_{i=2}^{k} \mathbb{R}^{m_i+2} \times \mathbb{R}^{m_{k+1}}.$$

Using (5.8) we obtain that

$$f(U) \subset \mathbb{S}^{m_1}(\rho_1) \times \cdots \times \mathbb{S}^{m_k}(\rho_k) \subset \mathbb{H}_c^N.$$

Clearly, $f(U)$ is contained in a totally umbilical submanifold of $\mathbb{H}_c^N$ of positive sectional curvature.

If $f_1, \ldots, f_k$ is of elliptic type then each f_i is an umbilical isometric immersion into $\mathbb{S}^{m_i+1}(r_i) \subset \mathbb{R}^{m_i+2}$, where

$$\mathbb{L}^{N+1} = \Pi_{i=1}^{k} \mathbb{R}^{m_i+2} \times \mathbb{L}^{m}.$$

Bearing in mind (5.8) we obtain that

$$f(U) \subset \mathbb{S}^{m_1}(\rho_1) \times \cdots \times \mathbb{S}^{m_k}(\rho_k) \subset \mathbb{H}_c^N.$$

Clearly, $f(U)$ is contained in a flat totally umbilical submanifold of $\mathbb{H}_c^N$.

If $f_1, \ldots, f_k$ is of parabolic type then there exists $s \leq k$ such that each f_i is an umbilical isometric immersion into $\mathbb{R}^{m_i+1}$ for $1 \leq i \leq s$ and each f_j is an umbilical isometric immersion into

$$\mathbb{S}^{m_j+1}(r_j) \subset \mathbb{R}^{m_j+2} \;\text{ for }\; s+1 \leq j \leq k \;\text{ if }\; s < k,$$

where

$$\mathbb{L}^{N+1} = \mathbb{L}^{l} \times \Pi_{i=s+1}^{k} \mathbb{R}^{m_i+2} \times \mathbb{R}^{m_{k+1}}.$$

Thus, from (5.8) we obtain that

$$f(U) \subset \mathrm{i}\big(\Pi_{i=1}^{s} \mathbb{S}^{m_i}(\rho_i)\big) \times \Pi_{i=s+1}^{k} \mathbb{S}^{m_i}(\rho_i) \subset \mathbb{H}_c^N.$$

Again in this case $f(U)$ is contained in a flat totally umbilical submanifold of $\mathbb{H}_c^N$.

Finally, since M^n is connected and the above different type of submanifolds cannot be smoothly attached we have that $f(M^n)$ is an open subset of one of the above and this completes the proof. ∎

ΠΕΡΙΛΗΨΗ

Η παρούσα διατριβή χωρίζεται σε δύο μέρη:

Στο πρώτο μέρος, (Κεφάλαια 2, 3 και 4) αποδεικνύουμε φράγματα, που εξαρτώνται από τους αριθμούς Betti, για ολοκληρωτικές καμπυλότητες συμπαγών υποπολυπτυγμάτων του Ευκλειδείου χώρου, με χαμηλή συνδιάσταση. Ως συνέπεια,

(i) Υπολογίζουμε την ομολογία των σχεδόν σύμμορφα ισόπεδων (almost conformally flat) υπερεπιφανειών σε πλήρη και απλά συνεκτικά πολυπτύγματα Riemann, με σταθερή καμπυλότητα τομής (χώροι μορφής).

(ii) Αποδεικνύουμε μια αναγκαία συνθήκη έτσι ώστε ένα συμπαγές πολύπτυγμα Riemann να επιδέχεται ελαχιστική ισομετρική εμβάπτιση ως υπερεπιφάνεια στη σφαίρα.

(iii) Επεκτείνουμε ένα αποτέλεσμα των Shiohama και Xu [58] για συμπαγείς υπερεπιφάνειες σε χώρους μορφής.

(iv) Δίνουμε τοπολογικούς περιορισμούς για δ-pinched εμβαπτίσεις.

(v) Δίνουμε εσωτερικούς περιορισμούς για συμπαγή ελαχιστικά υποπολυπτύγματα της σφαίρας με pinched δεύτερη θεμελιώδη μορφή.

Στο δεύτερο μέρος (Κεφάλαιο 5), ασχολούμαστε με το πρόβλημα της ταξινόμησης των υποπολυπτυγμάτων Einstein με ισόπεδη κάθετη δέσμη σε χώρους μορφής. Συγκεκριμένα, αποδεικνύουμε ότι τέτοια υποπολυπτύγματα είναι (τουλάχιστον τοπικά) holonomic. Ως εφαρμογή, κάτω από την υπόθεση ότι ο δείκτης μηδενοκατανομής (relative nullity) της εμβάπτισης είναι θετική σταθερά, συμπεραίνουμε ότι το υποπολύπτυγμα έχει τη δομή γενικευμένου κύλινδρου επί ενός υποπολυπτύγματος με ισόπεδη κάθετη δέσμη. Τέλος, δίνουμε ταξινόμηση των ανωτέρω υποπολυπτυγμάτων υπό την επιπλέον συνθήκη ότι το διανυσματικό πεδίο μέσης καμπυλότητας είναι παράλληλο ως προς την κάθετη συνοχή.

ABSTRACT

The present thesis is divided into two parts:

In the first part, (Chapters 2, 3 and 4) we provide integral curvature bounds in terms of the Betti numbers for compact submanifolds of the Euclidean space with low codimension. As a consequence,

(i) We determine the homology of almost conformally flat hypersurfaces in space forms.

(ii) We provide a necessary condition for a compact Riemannian manifold to admit an isometric minimal immersion as a hypersurface in the round sphere.

(iii) We extend a result due to Shiohama and Xu [58] for compact hypersurfaces in any space form.

(iv) We obtain topological obstructions for δ-pinched immersions.

(v) We obtain intrinsic obstructions for compact minimal submanifolds in spheres with pinched second fundamental form.

In the second part (Chapter 5), we deal with the classification problem of Einstein submanifolds with flat normal bundle in space forms. In particular, we prove that such submanifolds are (at least locally) holonomic. As an application, when assuming that the index of relative nullity of the immersion is a positive constant we conclude that the submanifold has the structure of a generalized cylinder over a submanifold with flat normal bundle. Moreover, we provide a complete classification for such submanifolds, under the additional hypothesis that the mean curvature vector field is also parallel in the normal connection.

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