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Renormalization of four-fermion theories in a mean-field expansion

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The generating functional of a $(\bar\psi\psi)^2$ four-fermion field theory in four dimensions is expressed in terms of collective boson variables and then expanded in a Laplace expansion. All divergences are absorbed in a renormalized Yukawa-type coupling and a renormalized fermion mass. The collective boson mass and self-couplings are fixed to all orders in terms of the Yukawa coupling. The renormalized theory is formally equivalent to the Yukawa theory expanded the same way.

I. INTRODUCTION

Spinor models with quartic self-interaction are not renormalizable in the perturbation expansion with respect to the four-fermion coupling. This is a result of the fact that this parameter has dimensions $(\text{mass})^{-2}$ and thus each higher order of an expansion in it stays dimensionally correct by increasing powers of the momentum in numerators, resulting in ever more divergent integrals.

It has recently been suggested that particular four-fermion theories, within a certain approximation of the Hartree type, are formally equivalent to renormalizable theories and thus renormalizable themselves in this approximation.[1-3] Formal equivalence means that all the renormalized Green's functions are exactly the same functions of the renormalized parameters and the momenta, but the definitions of the renormalized parameters in terms of the bare parameters and the cutoff differ.

Approximations of the above type can be formulated as bound-state mean-field expansions,[3] i.e., Laplace expansions of the functional integral describing the generating functional of the theory. The ordinary loop expansion is also a Laplace expansion of the generating functional. What characterizes our mean-field expansion is that in this expansion the functional integral is written first in terms of a different dummy variable from the fundamental field, a bound-state mean field, and then expanded. The guide toward the choice of a particular mean field is the form of the interaction Lagrangian.

In this paper we study in detail the scalar-scalar four-fermion theory in such a bound-state mean-field expansion.[4] This is the simplest four-Fermi model. Consequently, the results we obtain are not complicated by indices associated with the Lorentz group or internal-symmetry groups. To derive the expansion we rewrite the original Lagrangian

$$L=\bar\psi(i\not\partial)\psi+\frac{g_0^{\,2}}{2\mu_0^{\,2}}(\bar\psi\psi)^2$$

in terms of an auxiliary field σ,

$$L=\bar\psi(i\not\partial+g_0\sigma)\psi-\frac{\mu_0^{\,2}}{2}\sigma^2\,.$$

Even though σ enters the Lagrangian as an auxiliary field, it acquires a kinetic energy term in the renormalized effective action arising from the structure of the vacuum polarization diagrams. Thus the resulting theory will have properties of interactions of fundamental scalars with fermions.

We shall demonstrate renormalizability and formal equivalence to the Yukawa theory for all orders of the σ-mean-field expansion. In this particular model if we were to have N fermion fields the mean-field expansion would be identical to the $1/N$ expansion.[5] The use of N fermions provides us from the beginning with a small parameter $1/N$ in the case that N is large. However, as we shall show the $1/N$ expansion has a Landau ghost which can be bypassed through further expansion of our theory. Our methods work for $N=1$ as well as N large and for the N-small case, the ghost problem is naturally avoided.

The cubic and quartic induced σ self-couplings required by renormalization are not arbitrary, but are given as functions of the renormalized Yukawa coupling and fermion mass. This is demonstrated by deriving differential equations of the Callan-Symanzik type for these couplings. Similarly the σ mass is fixed. As the major result of this paper, we show that the theory is renormalizable in terms of only two parameters, a dimensionless coupling constant and a fermion mass.

The paper is organized as follows: In Sec. II we develop the expansion scheme. In Sec. III we examine the lowest order of the theory. In Sec. IV we formally renormalize the theory by showing that the divergences can be absorbed by the physical parameters of the theory. In Sec. V we derive equations fixing the boson self-couplings. In Sec. VI we demonstrate the equivalence with the Yukawa theory. In Sec. VII we summarize our conclusions.

II. THE MEAN-FIELD EXPANSION

We consider the scalar-scalar four-fermion Lagrangian

$$L=\bar{\psi}(i\partial\!\!\!/)\psi+\frac{\lambda_0}{2}(\bar{\psi}\psi)^2\,. \tag{2.1}$$

The Fermi coupling λ_0 has dimensions of $(\text{mass})^{-2}$. Consequently it is convenient to write the coupling as

$$\lambda_0=\frac{g_0^{\,2}}{\mu_0^{\,2}}\,,$$

where g_0 is a dimensionless bare coupling and $\mu_0^{\,2}$ is a bare mass parameter. Then Eq. (2.1) takes the form

$$L=\bar{\psi}(i\partial\!\!\!/)\psi+\frac{g_0^{\,2}}{2\mu_0^{\,2}}(\bar{\psi}\psi)^2\,. \tag{2.2}$$

The connected generating functional is given by the functional integral

$$e^{iW(\eta,\bar{\eta})}=N\int(d\psi)(d\bar{\psi})\exp\left[i\int d^4x\left(\bar{\psi}(i\partial\!\!\!/)\psi+\frac{g_0^{\,2}}{2\mu_0^{\,2}}[(\bar{\psi}\psi)^2+\bar{\psi}\eta+\eta\psi]\right)\right], \tag{2.3}$$

where η, $\bar{\eta}$ are anticommuting c-number sources. Next we observe that the Gaussian integral over the boson variable σ,

$$\int(d\sigma)\exp\left[-i\int d^4x\,\frac{\mu_0^{\,2}}{2}\left(\sigma-\frac{g_0\bar{\psi}\psi}{\mu_0^{\,2}}\right)^2\right]\,,$$

is just a constant. Inserting the above integral in Eq. (2.3) will change only the normalization constant N. The resulting functional is an integral over both boson and fermion variables:

$$e^{iW(\eta,\bar{\eta},J)}=N'\int(d\psi)(d\bar{\psi})(d\sigma)\exp\left\{i\int d^4x\left[\bar{\psi}(i\partial+g_0\sigma)\psi-\frac{\mu_0^{\,2}}{2}\sigma^2+\bar{\eta}\psi+\bar{\psi}\eta+J\sigma\right]\right\}. \tag{2.4}$$

Here J is an external source coupled to σ. The effective interaction Lagrangian of this model, except for the absence of a kinetic term, is of the Yukawa type with the fermions entering quadratically. Nevertheless, it is easy to see directly that this effective Lagrangian leads to the same field equation as the original Lagrangian (2.2) with sources added. The integral over the Fermi fields of (2.4) can be evaluated exactly to obtain

$$e^{iW(\eta,\bar{\eta},J)}=N'\int(d\sigma)\exp\left\{i\int d^4x\left[-\bar{\eta}S\eta-i\,\mathrm{tr}\ln(iS^{-1})-\frac{\mu_0^{\,2}}{2}\sigma^2+J\sigma\right]\right\}. \tag{2.5}$$

Here

$$S^{-1}(x,y)\equiv(i\gamma^\mu\partial_\mu+g_0\sigma)\,\delta^{(4)}(x,y)\,. \tag{2.6}$$

The corresponding Euclidean space functional is

$$e^{W}=N'\int(d\sigma)\exp\left\{-\int\left[+\bar{\eta}S\eta-\mathrm{tr}\ln S^{-1}+\frac{\mu_0^{\,2}}{2}\sigma^2-J\sigma\right]\right\}\,. \tag{2.7}$$

This expression suggests the modified form

$$e^{W_\epsilon/\epsilon}=N\int(d\sigma)\,e^{-F(\sigma)/\epsilon}\equiv\langle|\rangle^\epsilon\,, \tag{2.8a}$$

where

$$F(\sigma)=\int\left(\bar{\eta}S\eta-\mathrm{tr}\ln S^{-1}+\frac{\mu_0^{\,2}}{2}\sigma^2-J\sigma\right). \tag{2.8b}$$

We have introduced the parameter ϵ as a bookkeeping device to serve as an intermediate expansion parameter. At the end of computations ϵ must be set equal to one to regain the original theory. In the case where N fermion fields participate in the interaction we could rescale the fields so that the effective Lagrangian is multiplied by N. Then it is possible to make the identification $\epsilon = 1/N$ so for large N we would possess a candidate for a small parameter to begin with. Thus, there is no need to send $\epsilon \to 1$. On the other hand, if $N=1$ there is no obvious initial small parameter. Then our only hope of making the ordering obtained by introducing ϵ by hand meaningful in the limit $\epsilon \to 1$ is if in the explicit iterative solution obtained ϵ is always multiplied by a small parameter. We shall show that this is the case for the example under consideration. The mean field σ_0 is defined by the mean-field conditions[3]

$$\frac{\delta F}{\delta \sigma}\bigg|_{\sigma_0} = 0 \tag{2.9}$$

and

$$\frac{\delta^2 F}{\delta\sigma(x)\delta\sigma(y)}\bigg|_{\sigma_0} \equiv A(x,y) > 0\,. \tag{2.10}$$

If we introduce

$$B(x,y,z) \equiv \frac{\delta^3 F}{\delta\sigma(x)\delta\sigma(y)\delta\sigma(z)}\bigg|_{\sigma_0},$$

$$C(x,y,z,w) \equiv \frac{\delta^4 F}{\delta\sigma(x)\delta\sigma(y)\delta\sigma(z)\delta\sigma(w)}\bigg|_{\sigma_0},$$

then we may expand (2.8) around σ_0:

$$\begin{aligned} e^{W_\epsilon/\epsilon} \sim e^{-F(\sigma_0)/\epsilon} e^{-\mathrm{tr}\ln A/2} \Big\{ 1 &- \frac{\epsilon}{8} \int\!\!\int\!\!\int\!\!\int C(x,y,z,w) A^{-1}(x,y) A^{-1}(z,w) \\ &+ \frac{\epsilon}{24} \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int B(x,y,z)B(a,b,c)[2A^{-1}(x,a)A^{-1}(y,b)A^{-1}(z,c) \\ &+ 3A^{-1}(x,y)A^{-1}(z,a)A^{-1}(b,c)] + O(\epsilon^2) \Big\}\,. \end{aligned} \tag{2.11}$$

If (2.8) were an ordinary integral this formula would define an asymptotic expansion for $\epsilon \to 0+$.

Expression (2.11) has been considered in detail in Ref. 3, and consequently we only outline its most essential features here. The lowest-order Fermi propagator $S(x,y)$ as given by Eq. (2.6) is graphically represented by a solid line from point x to point y, while $\psi(y)$ is represented by a cross at the point y and $\bar\psi(y)$ by a barred cross at the point y. With these conventions $B(x,y,z)$ is represented by a triangle connecting the three points x, y, and z, plus all possible graphs made by removing one side of the triangle and placing a cross and a barred cross at the bare ends. Associated with each of the points x, y, and z is a factor of g_0 so that $B(x,y,z)$ behaves as g_0^3. Similarly $C(x,y,z,w)$ is constructed by connecting the four points x, y, z, and w together in all possible ways with four-Fermi propagators, and then systematically removing one propagator at a time and placing a cross and a bared cross at the exposed ends. The sum of all these graphs forms $C(x,y,z,w)$ which is proportional to g_0^4. In general

$$\frac{\delta^n F}{\delta\sigma(x_1)\cdots\delta\sigma(x_n)}\bigg|_{\sigma_0}$$

is represented by the sum of all possible n-sided closed polygons and all possible $(n-1)$-sided open polygonal paths with crosses and barred crosses at their end points and is proportional to g_0^n.

The inverse of the second derivative of F plays a prominent role in the expansion (2.11). Because of this, we graphically denote $A^{-1}(x,y)$ as a wiggly line from x to y. Using this rule it is easy to see that the terms of order ϵ or higher in the curly brackets of Eq. (2.11) can be graphically represented by the appropriate product of polygons or amputated polygons generated by taking the derivative of F in which all points are connected by wiggly lines. In fact, only the three types of vertices shown in Fig. 1 are generated by this procedure. When the Fermi sources are turned off, only the Yukawa-type vertices of Fig. 1(a) remain. We will show in the next section that this resemblance is even analytically accurate since A^{-1} will be identified with the lowest-order propagator of a scalar particle.

With these conventions we may graphically construct W_ϵ as given by Eq. (2.8). It is found that the graphs of order $\epsilon^n (n \geq 2)$ that contribute to W_ϵ are all the topologically possible combinations formed out of the basic vertices of Fig. 1 that obey a simple rule: The number of independent momentum loop integrations is arbitrary, but n-in-

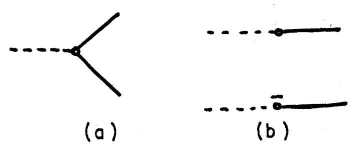

FIG. 1. Basic vertices.

dependent loops have at least one wiggly line in them.

The vacuum functional W_ϵ is primarily of interest to us because by differentiation of W_ϵ with respect to the external sources $\bar{\eta}$, η and J the appropriate number of times any Green's function can be produced. It is straightforward but tedious to verify that differentiation of W_ϵ creates vertices of only the type given in Fig. 1. Consequently, the graphical rules of mean-field expansion with the sources off are very simple. To find any Green's function of order n draw and evaluate all topologically possible graphs containing only the Yukawa-type vertex of Fig. 1(a) and obeying the rule: Exactly n of the independent momentum loop integrals contain at least one wiggly line. The numerical weighting factors of each graph are determined by explicit reference to Eq. (2.11).

III. THE LOWEST-ORDER THEORY

The lowest-order connected functional according to formula (2.11) and (2.8) is

$$W_0 = -F(\sigma_0;\ \eta,\bar{\eta},J)\,, \tag{3.1}$$

where the mean field σ_0 is defined by Eq. (2.9). Because of this definition, it is straightforward to verify that σ_0 is, in fact, the lowest-order vacuum expectation of the quantum σ field in our approximation. Thus, if we make the definition $\sigma^\epsilon(x) \equiv \delta W_\epsilon/\delta J(x)$ it follows that

$$\sigma^0(x) = \frac{\delta W_0}{\delta J(x)} = -\frac{\delta F(\sigma_0;\eta,\bar{\eta},J)}{\delta J(x)} = \sigma_0(x)\,. \tag{3.2a}$$

Similarly it is convenient to introduce the definitions

$$\psi_\epsilon \equiv \frac{\delta W_\epsilon}{\delta \bar{\eta}(x)}$$

and

$$\bar{\psi}_\epsilon \equiv \frac{\delta W_\epsilon}{\delta \eta(x)}\,.$$

Then in lowest order

$$\psi_0(x) = \frac{\delta W_0}{\delta \bar{\eta}(x)} = -\int d^4y\, S_0(x,y)\eta(y) \tag{3.2b}$$

and

$$\bar{\psi}_0(x) = \frac{\delta W_0}{\delta \eta(x)} = -\int d^4y\, \bar{\eta}(y)\, S_0(y,x)\,. \tag{3.2c}$$

Having made these observations, we will hereafter suppress the ϵ label on the classical fields. The zero subscript will be used for lowest order and the context will make it clear if ϵ is not one. Here we have defined

$$S(x,y)\big|_{\sigma=\sigma_0} \equiv S_0(x,y)\,. \tag{3.2d}$$

In terms of these quantities Eq. (2.9) applied to (2.7) through (2.11) determines σ_0 as

$$\mu_0^2\sigma_0(x) = g_0\bar{\psi}_0(x)\psi_0(x) + g_0 \operatorname{tr} S_0(x,x) + J(x)\,. \tag{3.2e}$$

The reader should recognize this as a result obtainable by a factorization of the vacuum expectation of a field equation obtained from the Lagrangian associated with Eq. (2.4).

The positivity condition (2.10) is also easily explicitly computed. We find that it is

$$\begin{aligned} A(x,y) = g_0^2[&\bar{\psi}_0(y)S_0(y,x)\psi_0(x) \\ &+ \bar{\psi}_0(x)S_0(x,y)\psi_0(y)] \\ &+ g_0^2 \operatorname{tr}[S_0(x,y)S_0(y,x)] + \mu_0^2\,\delta(x-y) > 0\,. \end{aligned} \tag{3.3a}$$

Although this condition does have a very positive look to it, it is in fact divergent so in order to determine if (3.3a) is valid we must evaluate it within our explicit renormalization scheme. We leave this verification for later in this section.

In general the integrability condition as given by (3.3a) is a very complicated condition since it must be valid with all the sources on. In the special case where the sources are constant (or zero), $A(x,y) = A(x-y)$ and the condition that A is a positive matrix can be written in the form

$$\int d^4y\, A(x-y)V(y) = \lambda_v V(x)\,,$$

where λ_v is a strictly positive eigenvalue of A associated with the eigenvector $V(y)$. Fourier transforming this relation shows that

$$A(p^2)\,V(p^2) = \lambda_v V(p^2)\,.$$

If $A(p^2) = A(0) + p^2B(p^2)$ with $B(p^2)$ regular at $p^2=0$ we must have $V(p^2) \propto \delta(p^2)$ and

$$A(0) = \lambda_v > 0\,. \tag{3.3b}$$

As we shall see, this condition is the condition that the lowest-order effective potential be at a minimum.

In order to study our results in terms of the effective action, we rewrite W_0 as

$$W_0(\eta,\bar\eta,J)=\int\left(-\psi_0 S_0^{-1}\psi_0+\operatorname{tr}\ln S_0^{-1}-\frac{\mu_0^2}{2}\sigma_0^2\right)+\int J\sigma_0 .$$

The effective action functional of σ is defined, as usual, by a Legendre transformation

$$\Gamma_\epsilon(\sigma)=W_\epsilon(J)-\int J\sigma ,$$

and consequently is to lowest order

$$\Gamma_0(\sigma_0)=\int\left(-\bar\psi_0 S_0^{-1}\psi_0+\operatorname{tr}\ln S_0^{-1}-\frac{\mu_0^2}{2}\sigma_0^2\right).$$

The effective action is the generating functional of all one-particle-irreducible (1PI) graphs. The graphical interpretation of the bare effective action and the bare mean-field functional to this order is very simple. Adding a constant term $-\operatorname{tr}\ln(i\not\partial+g_0v)$, with v the value of σ_0 in the absence of sources, and translating the classical field $\sigma_0=s_0+v$ will not change the action. Thus it can be written in the form

$$\begin{aligned}\Gamma_0(\sigma_0)&=\int\left[\bar\psi_0 S_0^{-1}\psi_0-\frac{\mu_0^2}{2}(s_0+v)^2+\operatorname{tr}\ln\left(1+\frac{g_0s_0}{i\not\partial+g_0v}\right)\right]\\&=\int\left[-\bar\psi_0 S_0^{-1}\psi_0-\frac{\mu_0^2}{2}(s_0^2+2s_0v)-\sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{\nu}\operatorname{tr}\left(\frac{g_0s_0}{i\not\partial+g_0v}\right)^\nu\right].\end{aligned}$$

The infinite series representing the functional logarithm corresponds to the infinite set of diagrams shown in Fig. 2.

It is of interest to note that the mean-field conditions (2.9) and (2.10) coincide with the condition that we are at a minimum of the σ effective potential to lowest order. From above the effective potential to lowest order is just

$$V(\sigma_0)=\frac{\mu_0^2}{2}\sigma_0^2-\operatorname{tr}\ln S_0^{-1}.$$

The mean-field conditions (3.2e) and (3.3) with the sources off and at zero momentum can be written in the forms

$$\frac{\partial V}{\partial\sigma_0}=0$$

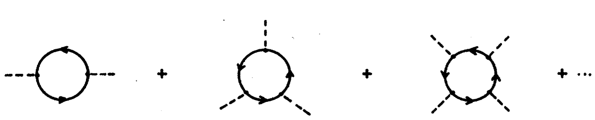

FIG. 2. Infinite series for the functional logarithm.

and

$$\frac{\partial^2V}{\partial\sigma_0^2}>0 .$$

In making the above comparison, we have used the mean-field condition (3.2e) with the sources off which is the gap equation

$$\mu_0^2\sigma_0=g_0\operatorname{tr}S_0$$

and have evaluated $A(p^2)$ at zero momentum as directed to do by our previous analysis. The expression obtained is

$$A(0)=g_0^2\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\left(\frac{1}{(\gamma p+g_0\sigma)^2}\right)+\mu_0^2 .$$

One should keep in mind that for higher orders the mean-field conditions are not obviously the same as demanding a minimum of the effective potential to that order. We have made no attempt (nor is it necessary to do so within our calculational procedure) to confirm the identity. If they should differ (3.3) is preferred.

The fermion inverse propagator[6] to this order is

$$-\frac{\delta\Gamma_0}{\delta\psi_0\delta\bar\psi_0}=S_0^{-1}=\not p+g_0\sigma_0 ; \tag{3.4}$$

the nonvanishing of σ_0 and hence the expectation value of the σ field (with the sources off) breaks the discrete symmetry $\psi\to\gamma_5\psi$ of the original Lagrangian. Of course, no Goldstone boson is associated with such discrete symmetry breaking. Since the standard form of the Fermi propagator is taken to be $S_0^{-1}(p)=\not p-m$ the above form describes a free fermion with mass

$$m=-g_0\sigma_0 . \tag{3.5}$$

The lowest-order inverse σ propagator[6] is defined as usual (see Fig. 3):

$$\begin{aligned}\frac{\delta\Gamma_0}{\delta\sigma_0\delta\sigma_0}&=\Delta_0^{-1}(p^2)\\&=-\mu_0^2-g_0^2\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p)S_0(k) .\end{aligned}$$

Note that $\Delta_0^{-1}(p^2)=-A(p^2)$ so the positivity condition is

$$\Delta_0^{-1}(0)<0 . \tag{3.6}$$

In Minkowski space the above σ propagator is

$$\Delta_0^{-1}(p^2)=-\mu_0^2+ig_0^2\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p)S_0(k) . \tag{3.7}$$

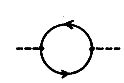

FIG. 3. Self-energy.

The fermion bubble

$$\Pi(p^2) \equiv i\,\mathrm{tr}\int \frac{d^4k}{(2\pi)^4} S_0(k+p)S_0(k)$$

$$= 4i\int \frac{d^4k}{(2\pi)^4} \frac{k^2+k\cdot p+m^2}{[(k+p)^2-m^2](k^2-m^2)}$$

$$= 4i\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx\, \frac{k^2-p^2x(1-x)+m^2}{[k^2+p^2x(1-x)-m^2]^2}$$

is quadratically divergent.[7] Subtracting at zero we obtain

$$\Delta_0^{-1}(p^2) = -\mu_0^2 + g_0^2\,\Pi(0) + p^2 g_0^2\left(\frac{\partial\Pi}{\partial p^2}\right)_0 + g_0^2\,\mathrm{sub}_0^2\,\Pi(p^2)\,.$$

The symbol sub_0^2 stands for

$$\mathrm{sub}_0^2\,\Pi(p^2) = \Pi(p^2) - \Pi(0) - p^2\left(\frac{\partial\Pi}{\partial p^2}\right)_0 .$$

We introduce a lowest-order renormalized mass μ^2 and a renormalization factor Z_σ by the definitions

$$\Delta_0^{-1}(0) = -\frac{\mu^2}{Z_\sigma} \tag{3.8a}$$

and

$$\left(\frac{\partial\Delta_\sigma^{-1}(p^2)}{\partial p^2}\right)_{p^2=0} = \frac{1}{Z_\sigma}\,. \tag{3.8b}$$

Note that the positivity condition (3.6) is simply the condition that μ^2/Z_σ is positive. With these definitions the following form for the propagator is obtained:

$$\Delta_0^{-1}(p^2) = -\frac{\mu^2}{Z_\sigma} + \frac{p^2}{Z_\sigma} + g_0^2\,\mathrm{sub}_0^2\,\Pi(p^2)\,.$$

To this lowest order a renormalized coupling can be defined by

$$g^2 = g_0^2 Z_\sigma, \tag{3.9}$$

so the renormalized propagator $\overline{\Delta}_0(p^2)$ can be written as

$$\overline{\Delta}_0^{-1}(p^2) = p^2 - \mu^2 + g^2\,\mathrm{sub}_0^2\,\Pi(p^2) \tag{3.10}$$

using the definition

$$\Delta_0^{-1}(p^2) = Z_\sigma^{-1}\,\overline{\Delta}_0^{-1}(p^2)\,. \tag{3.11}$$

This result naturally leads to the definition of a renormalized σ field to lowest order g

$$\sigma = Z_\sigma^{1/2}\,\overline{\sigma}\,. \tag{3.12}$$

The quantity

$$Z_\sigma^{-1} = \left(\frac{\partial\Delta_\sigma^{-1}}{\partial p^2}\right)_{p^2=0} = g_0^2\,\frac{\partial\Pi}{\partial p^2}\bigg|_{p^2=0}$$

is logarithmically divergent. The renormalized propagator, however, is a finite function of p^2, μ^2, m, and g^2. The renormalized coupling does not depend on the bare coupling except implicitly through the mass m:

$$\frac{1}{g^2} = \frac{Z_\sigma^{-1}}{g_0^2} = \left(\frac{\partial\Pi}{\partial_p^2}\right)_0 = \frac{2}{3}\int \frac{(d^4k)_E}{(2\pi)^4}\,\frac{3k_E^2-m^2}{(k_E^2+m^2)^3}\,.$$

Unfortunately $1/g^2$ is logarithmically divergent and hence is not a fixed number in our theory. We must instead identify in the conventional way $g_0^2 Z_\sigma$ as being a quantity needing renormalization and hence to be replaced by an arbitrary finite number. We shall choose the renormalized coupling g^2 to be a small number and the expansion parameter of the theory.[8] If we were in $4-\epsilon$ dimensions the analogous dimensionless coupling would be a fixed number in terms of the fermion mass m due to its depending only implicitly (through m) on g_0.

At this stage it might appear to the reader that the mass parameter of Eq. (3.8) is entirely free because it is the result of a renormalization of a quadratically divergent integral. Because of the internal consistency conditions imposed on the theory through (3.2e), this mass is in fact fixed. To show this we examine (3.2e) with $\overline{\eta}=\eta=0$ and $J(x)=J$ where J is a space-time-independent source. In Minkowski space we have

$$\mu_0^2\,\sigma_0 = g_0 i\,\mathrm{tr}\,S_0(x,x) + J\,. \tag{3.13a}$$

Using (3.5) we can rewrite (3.13) in the form

$$\mu_0^2 m = -m g_0^2 f(m) - g_0 J\,, \tag{3.13b}$$

where

$$f(m) \equiv \frac{i\,\mathrm{tr}\,S_0(x,x)}{m}\,. \tag{3.14}$$

If $\mathrm{tr}\,S_0(x,x)$ is explicitly evaluated we find it to be quadratically divergent since it is the Fermi mass to this order. Accordingly $f(m)$ is not defined except through a subtraction procedure. The remaining finite part of $f(m)$, which is all that is of interest, is then determined in terms of J and the lowest-order renormalized mass m, and μ_0^2, by (3.13b). To use (3.13b) to fix the boson mass we need the relationship[4,12]

$$\frac{d^nF(\sigma_0(J);J)}{d^n\sigma_0(J)} = \int d^4x_1 \cdots \times \int d^4x_n\, \frac{\delta^n F(\sigma_0(x;J(x))}{\prod_{i=1}^{n}\delta\sigma_0(x_i;J(x_i))}\bigg|_{\sigma_0(x;J(x))=\sigma_0(J)} \tag{3.15}$$

which is valid for an arbitrary differentiable function with J a space-time–independent source. Applying (3.16) with $F=J$ to (3.14) we find

$$\mu_0^2=-g_0^2 f(m)-mg_0^2 f'(m)-\Delta_0^{-1}(0). \qquad (3.16)$$

We may now apply the internal consistency condition (3.13b) to this equation to eliminate $f(m)$ and obtain with $J=0$

$$\Delta_0^{-1}(0)=mg_0^2 f'(m). \qquad (3.17)$$

Using (3.8a) and (3.9) the above is expressed entirely in renormalized quantities (to this order) as

$$f'(m)=-\frac{u^2}{mg^2}. \qquad (3.18)$$

Of course $f'(m)$ is logarithmically divergent so (3.17) is regarded as its value after an appropriate subtraction procedure.

So far we have not made any progress in restricting the number of parameters of our theory. To this end we need to be more explicit. Using (3.13) in (3.7) we find, using the usual Feynman parametrization, that

$$\Delta_0^{-1}(p^2)=\frac{4i}{(2\pi)^4}\int_0^1 dx\int d^4k\,\frac{2m^2+p^2(x-1)-k\cdot p}{[k^2-m^2+p^2x(1-x)]^2}. \qquad (3.19)$$

In all that we have done to now and all that follows we do not refer to cutoffs or need them. At this stage we interject a minimal assumption which is easily justified in any common rational cutoff procedure. We assume that the last term in the numerator in (3.18) vanishes by oddness. It is then easily seen that after a change in variables for the x integration again by oddness

$$\Delta_0^{-1}(4m^2)=0.$$

Consequently the σ mass to this order lies at threshold:

$$\mu_\sigma^2=4m^2.$$

Of course, higher-order corrections should change this relation.

The mass parameter μ^2 that has been introduced is related to the physical mass μ_σ^2. It is

$$\mu^2=4m^2+g^2\,\mathrm{sub}_0^2\,\Pi(4m^2).$$

Evaluating the subtracted bubble we find

$$\mu^2=4m^2\left(1+\frac{g^2}{12\pi^2}\right). \qquad (3.20)$$

Since we take $g^2>0$ and hence have $Z_\sigma^0>0$, Eq. (3.20) with (3.8) guarantees that the integrability condition (3.6) or equivalently (3.3b) is met.

It is straightforward to explicitly evaluate $\mathrm{sub}_0^2\,\Pi(p^2)$. Doing this calculation and inserting the results into (3.9), we find the expression for the σ propagator in Euclidean space as follows:

$$\frac{-\bar{\Delta}_0^{-1}(p^2)}{4m^2}=x^2+1+\frac{g^2}{2\pi^2}-\frac{g^2}{2\pi^2}\left[(x^2+1)\ln\left(\frac{1+\dfrac{x}{(x^2+1)^{1/2}}}{1-\dfrac{x}{(x^2+1)^{1/2}}}\right)+\frac{x^3}{3(x^2+1)^{1/2}}-x\frac{(2+3x^2)}{(x^2+1)^{1/2}}\right]. \qquad (3.21)$$

Here we have made the definition $x^2=p^2/4m^2$.

Even though we have demonstrated that the integrability condition (3.3b) is satisfied and hence that we are at a minimum of the effective potential the propagator $\Delta_0(p^2)$ is not problem-free. This can be seen by noting that if g^2 is small $\Delta_0^{-1}(p^2)$ as given by (3.21) has a zero at Euclidean four-momentum,

$$\frac{p^2}{4m^2}\sim\tfrac{1}{2}e^{\pi^2/g^2}.$$

More accurate calculations show that for the relatively large value $g^2=0.9$ this zero occurs at $p^2\sim 4m^2\times 10^5$. Thus, for large spacelike momentum the bound-state propagator has a pole.

This pathology is, in fact, in no way unique to our model but occurs in all four-dimensional theories involving fermion propagation. It was first observed by Landau in quantum electrodynamics and is called the Landau ghost. If in electrodynamics the photon Green's function equation $D_F^{\lambda\nu}(q)$ is worked out one obtains to second order[9] in e_0^2 the result

$$[q^2 g_{\mu\lambda}-e_0^2\,\Pi_{\mu\lambda}(q)]D_{F\nu}^{\lambda}=-g_{\mu\nu}.$$

Here $\Pi_{\mu\nu}(q)$ is the usual fermion loop. After renormalization it can be seen that the photon propagator also has a pole at Euclidean four-momentum $p^2/4m^2\sim e^{3\pi/\alpha}$ where α is the fine-structure constant.[10]

In our problem for small g^2, as well as in electrodynamics, this pole occurs at such large values of momentum as to be beyond any conceivable physical relevance.[10] The spectral representation for the photon Green's function does not demonstrate this "ghost" and presumably as higher and higher orders of perturbation theory are calculated the "ghost" moves to higher and higher values of spacelike momentum.

It might be argued that even though this pathology is acceptable in QED, it is in fact not evident that it is acceptable in our resumed four-Fermi model. In particular, a striking difference of our model with the coupling-constant perturbation theory of QED is the lack of symmetry between the bound-state propagator and the fermion propagator. In lowest order of our expansion scheme the bound-state propagator contains momentum-dependent correction terms involving g^2 (g^2 will be identified as our small parameter after $\epsilon \to 1$ since $N=1$) while the fermion propagator has no such correction terms. This situation remedies itself in a logical way after renormalization. We may expand all Green's functions in a power series in the small parameter g^2 and set $\epsilon=1$. It is easily seen from derivations later in this paper that to calculate any Green's function to order $(g^2)^n$ we need only calculate to the ϵ^n term of our expansion, set $\epsilon=1$, expand in g^2, and discard terms of order g^{2n+2} or higher. We shall show later in this paper that after going through this procedure we will then obtain a perturbation theory that looks no different from a renormalized Yukawa theory in a coupling-constant expansion. The advantage we will have gained from the four-Fermi theory is that parameters that are normally free in the Yukawa theory will be fixed here. After having expanded all the Green's functions in powers of g^2 they may be resummed in any way desired and in particular, in a way which treats the σ propagation and the ψ propagation symmetrically in g^2. The point of this discussion has been to indicate that the Landau ghost problem is no worse (or better) than it is in any conventional theory involving fermions in four dimensions.

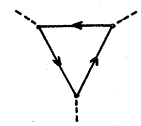

FIG. 4. 3σ vertex.

We shall complete the examination of the lowest order by looking at the 3σ and 4σ vertex functions. The 3σ vertex is just the fermion triangle (Fig. 4)

$$\frac{\delta\Delta_0^{-1}}{\delta\sigma_0}=\Gamma_0^{(3\sigma)}(p_1,p_2)=-ig_0^3\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p_1)S_0(p+p_1+p_2)S_0(k)+(xT)\,. \tag{3.22}$$

Here (xT) stands for the various structurally identical cross terms. The fermion triangle is logarithmically divergent. One subtraction can remove the divergence. Introducing a dimensionless renormalized cubic coupling by

$$\Gamma_0^{(3\sigma)}(0,0)=\frac{m\lambda'}{Z_\sigma^{3/2}}$$

we obtain

$$\Gamma_0^{(3\sigma)}(p_1,p_2)=\frac{m\lambda'}{Z_\sigma^{3/2}}-ig_0^3\,\mathrm{sub}_0^1\left\{\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p_1)S_0(k+p_1+p_2)S_0(k)+(xT)\right\}.$$

Thus, we have

$$\overline{\Gamma}_0^{(3\sigma)}(p_1,p_2)=m\lambda'-ig^3\,\mathrm{sub}_0^1\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p_1)S_0(k+p_1+p_2)S_0(k)+(xT)\,, \tag{3.23}$$

where

$$\Gamma_0^{(3\sigma)}(p_1,p_2)\equiv Z_\sigma^{-3/2}\,\overline{\Gamma}_0^{(3\sigma)}(p_1,p_2)\,. \tag{3.24}$$

Unlike the usual case with divergent renormalizations λ' is not a free parameter but is entirely fixed by the internal consistency conditions of this model. To demonstrate this we note that according to (3.15) and (3.22)

$$\Gamma_0^{(3\sigma)}(0,0)=\frac{d}{d\sigma_0}\,\Delta_0^{-1}(0)=-g_0\frac{d}{dm}\,\Delta_0^{-1}(0)\,.$$

Using (3.16) we have

$$\Gamma_0^{(3\sigma)}(0,0)=g_0^3\frac{d}{dm}\,(f(m)+mf'(m))=g_0^3\,(-6mg(m)-2m^2g'(m))\,. \tag{3.25}$$

Here we have isolated the divergent part of $f(m)$ by making the definition

$$g(m)\equiv -f'(m)/2m=-u^2/2g^2m^2\,. \tag{3.26}$$

Direct calculation shows that

$$g'(m) = -1/8\pi^2 m \tag{3.27a}$$

and

$$g''(m) = 1/8\pi^2 m^2 . \tag{3.27b}$$

Thus, combining (3.26) and (3.27a) with (3.24) we conclude that

$$\Gamma^{(3\sigma)}(0,0) = g\,3u^2/m + g^3 m/4\pi^2 .$$

Substitution of (3.20) allows this to be displayed as a function of the two parameters m and g,

$$\bar{\Gamma}^{(3\sigma)}(0,0) = 12gm + g^3 m^5/4\pi^2 . \tag{3.28}$$

Thus, to this order in our ϵ expansion

$$\lambda' = 12g + 5g^3/4\pi^2 . \tag{3.29}$$

Similarly the 4σ-vertex function is (Fig. 5)

$$\Gamma_0^{(4\sigma)}(p_1 p_2 p_3) = \frac{\delta \Delta_0^{-1}}{\delta\sigma\delta\sigma} = i g_0^4 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} S_0(k+p_1) S_0(k+p_1+p_2) S_0(k+p_1+p_2+p_3) S_0(k) + (xT) . \tag{3.30}$$

The fermion quadrangle is logarithmically divergent. Introducing a dimensionless renormalized quartic coupling by

$$\Gamma_0^{(4\sigma)}(0,0,0) = -\frac{\lambda}{Z_\sigma^2} , \tag{3.31}$$

we obtain

$$\Gamma_0^{(4\sigma)}(p_1 p_2 p_3) = Z_\sigma^{-2} \bar{\Gamma}^{(4\sigma)}(p_1 p_2 p_3) , \tag{3.32}$$

where the renormalized 4σ vertex is

$$\bar{\Gamma}_0^{(4\sigma)}(p_1 p_2 p_3) = -\lambda + i g^4 \operatorname{sub}_0^1 \operatorname{tr} \int \frac{d^4k}{(2\pi)^4} S_0(k+p_1) S_0(k+p_1+p_2) S_0(k+p_1+p_2+p_3) S_0(k) + (xT) . \tag{3.33}$$

Just as it was possible to explicitly calculate λ', it is also straightforward to calculate λ. Using (3.15), (3.30), and (3.22) we find

$$\Gamma_0^{(4\sigma)}(0,0,0) = \frac{d^2}{d\sigma_0^2} \Delta_0^{-1}(0)$$
$$= -g_0 \frac{d}{dm} \Gamma^{(3\sigma)}(0,0) ,$$

which with (3.25) becomes

$$\Gamma^{(4\sigma)}(0,0,0) = g_0^4 [6g(m) + 10mg'(m) + 2m^2 g''(m)] .$$

Using (3.32), (2.26), and (3.27) this is

$$\bar{\Gamma}^{(4\sigma)}(0,0,0) = -\frac{3g^2u^2}{m^2} - \frac{g^4}{\pi^2} ,$$

which through application of (3.20) becomes

$$\bar{\Gamma}^{(4\sigma)}(0,0,0) = -12g^2 - \frac{2g^4}{\pi^2} . \tag{3.34}$$

From this we find

$$\lambda = 12g^2 . \tag{3.35}$$

Thus we have shown (not surprisingly) that because of renormalization cubic and quartic renormalized σ self-couplings appear in the theory, although bare couplings of these types are absent in the original Lagrangian. What is exciting is that, because of the fact that all propagation is described through the fermion propagator (3.4) in a self-consistent manner regulated by (3.2e), all these renormalized couplings are completely fixed in lowest order by the renormalized parameter g. Later in this paper we shall show to all orders that this coupling constant and the Fermi mass are the only free parameters of the theory.

All 1PI vertex functions except those that we already examined are superficially finite. Thus,

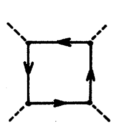

FIG. 5. 4σ vertex.

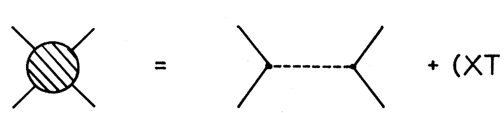

FIG. 6. Lowest-order scattering.

all lowest-order Green's functions are finite functions of the momenta and the renormalized parameters. For example, four-Fermi scattering is, to this order

$$\frac{\delta S_0}{\delta\eta\delta\bar{\eta}} = g_0{}^2 S_0 S_0 \Delta_0 S_0 S_0 + (xT)$$
$$= g^2 S_0 S_0 \overline{\Delta}_0 S_0 S_0 + (xT)\,,$$

and thus is saturated by the exchange of a σ bound state (see Fig. 6). Note that no remnant of the original explicit four-fermion contact interaction appears. This property is essential for the renormalizability of our theory as demonstrated in the next section.

IV. RENORMALIZATION

With our knowledge of how to construct any graph in the mean-field perturbation theory out of the lowest-order propagators and our explicit knowledge of the behavior of these propagators we can now study the renormalization properties through Weinberg's theorem (see Ref. 9). The large-momentum behavior of the lowest-order σ propagator as can be seen from (3.15) is

$$\Delta_0(p^2) \sim O(1/p^2) \quad (p^2 \to \infty)\,.$$

The fermion propagator is free and has the usual behavior

$$S_0(p) \sim \frac{\gamma \cdot p}{p^2} \quad (p^2 \to \infty)\,.$$

From these and the fact that only the vertex of Fig. 1(a) contributes when the sources are off, we can deduce the degree of superficial ultraviolet divergence of any graph using conventional power counting. The superficial degree of divergence associated with a graph having B external σ lines and F external fermion lines is easily seen to be

$$D = 4 - B - \tfrac{3}{2}F\,. \tag{4.1}$$

According to (4.1) vacuum graphs will have maximal $D=4$. These divergences are irrelevant because the vacuum bubbles are always divided out of any Green's function. Graphs with one external σ line have $D=3$, but they are always absorbed in the mass renormalization and will not be explicitly discussed further.

The remaining superficially divergent graphs of our expansion are (Fig. 7)

(a) graphs with two external σ lines ($D=2$),
(b) graphs with two external ψ lines ($D=1$),
(c) graphs with two external ψ lines and one external σ line ($D=0$),
(d) graphs with three external σ lines ($D=1$).
(e) graphs with four external σ lines ($D=0$).

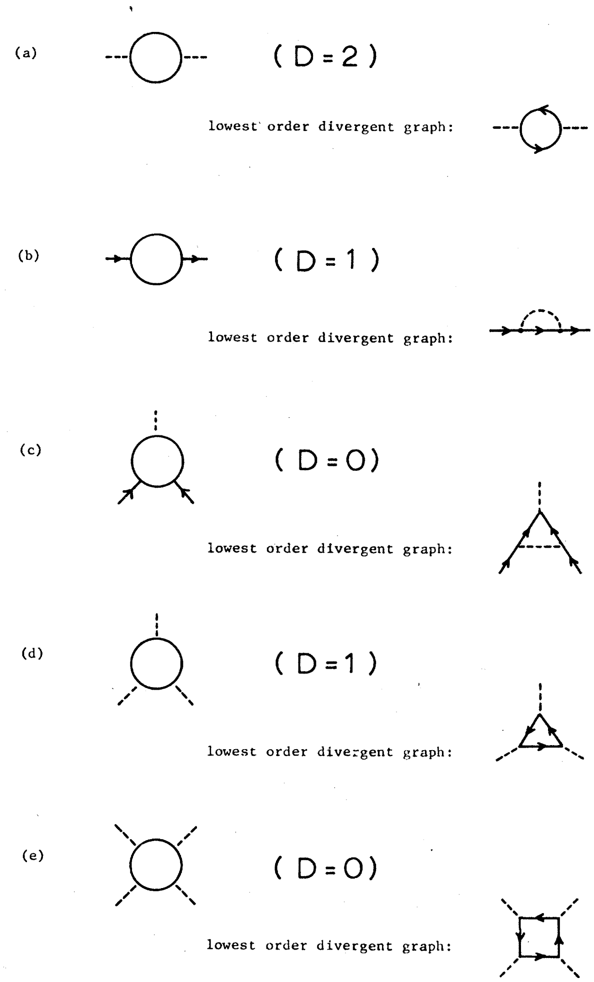

FIG. 7. The graphs on the left-hand side represent the classes of divergences which occur, and the graphs on the right-hand side represent the lowest-order divergent graphs in each class.

Graphs having two external meson lines have a maximal D of 2. For large internal momentum the leading behavior of the lowest-order inverse σ propagator is

$$\int \frac{d^4k\, k^2}{(k^2)^2} \sim \Lambda^2\,.$$

Thus, as we have already seen, this graph requires two subtractions in order to rid it of the quadratic divergence.

Graphs having two external fermion lines are linearly divergent behaving like

$$\int \frac{d^4k\, \not{k}}{(k^2)^2} \sim \gamma \cdot \Lambda$$

and thus require two subtractions.

The vertex graphs having two external fermion lines and one σ line have $D=0$. They are logarithmically divergent:

$$\int \frac{d^4k}{(\not{k})^2 k^2} \sim \ln\Lambda .$$

Graphs with three external σ lines have $D=1$, but their true superficial divergence is logarithmic. Having dimensions of mass and vanishing if the discrete γ_5 symmetry is preserved, they must be proportional to the bare fermion mass $m_0 = -g_0\sigma$. The other dimensional parameter in the theory, the renormalized σ mass μ^2, is always determined in terms of m and the couplings, as we shall show later. Consequently, the graphs with three external σ lines behave as

$$\int \frac{d^4k}{(k^2)^2} \sim \ln\Lambda .$$

No other graphs have superficial divergence.

Note that the above arguments have in effect nearly demonstrated that the theory is renormalizable since it is clear that there are a finite number of (superficially) divergent objects with a limited degree of (superficial) divergence. To finish the renormalization program we only need to observe how the renormalization constants of our model group together to form the renormalized parameters. All renormalized Green's functions of the theory must be finite functions of these renormalized parameters. In order to demonstrate how this works we shall examine the Schwinger-Dyson equations for the superficially divergent Green's functions. It is straightforward to show as well that the superficially finite Green's functions are finite functions of the same renormalized parameters.

We introduce the exact Green's functions associated with (2.8) in exact analog to the lowest-order Green's functions we have been discussing. Thus in Minkowski space using

$$W_\epsilon = -i\epsilon \ln \langle | \rangle^\epsilon ,$$

we define all the superficially divergent all-order Green's functions as follows:

$$\sigma(x) \equiv \frac{\delta W_\epsilon}{\delta J(x)} , \tag{4.2a}$$

$$\psi(x) = \frac{\delta W_\epsilon}{\delta \bar{\eta}(x)} , \tag{4.2b}$$

$$\bar{\psi}(x) \equiv \frac{\delta W_\epsilon}{\delta \eta(x)} , \tag{4.2c}$$

$$S(x,y) \equiv \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\bar{\eta}(y)} W_\epsilon , \tag{4.2d}$$

$$\Delta(x,y) = \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} W_\epsilon , \tag{4.2e}$$

$$\Gamma_\sigma(z,x,y) \equiv \frac{\delta}{\delta g_0\sigma(z)} S^{-1}(x,y) , \tag{4.2f}$$

$$\Gamma^{(3\sigma)}(z,x,y) \equiv \frac{\delta\Delta^{-1}(x,y)}{\delta\sigma(z)} , \tag{4.2g}$$

$$\Gamma^{(4\sigma)}(w,z,x,y) \equiv \frac{\delta^2}{\delta\sigma(w)\delta\sigma(z)} \Delta^{-1}(x,y) . \tag{4.2h}$$

The next step in our renormalization program is to write down a set of suitable renormalization conditions for the superficially divergent functions at some arbitrary point in momentum space. For convenience we choose to write the renormalization conditions at zero momentum. We emphasize that we are defining the all-order parameters in the following. When it is necessary to use the lowest-order parameters again either it will be pointed out or it will be clear from the context.

We make the following definitions:

$$S^{-1}(0) \equiv -\frac{m}{Z_2} , \tag{4.3a}$$

$$\left(\frac{\partial S^{-1}}{\partial \not{p}}\right)_0 \equiv \frac{1}{Z_2} , \tag{4.3b}$$

$$\Delta^{-1}(0) \equiv -\frac{\mu^2}{Z_\sigma} , \tag{4.3c}$$

$$\left(\frac{\partial\Delta^{-1}(p^2)}{\partial p^2}\right)_0 \equiv \frac{1}{Z_\sigma} , \tag{4.3d}$$

$$\Gamma_\sigma(0,0) \equiv \frac{1}{Z_1} , \tag{4.3e}$$

$$\Gamma^{(3\sigma)}(0,0) \equiv \frac{m\lambda'}{Z_\sigma^{3/2}} , \tag{4.3f}$$

$$\Gamma^{(4\sigma)}(0,0,0) \equiv \frac{-\lambda}{Z_\sigma^{2}} . \tag{4.3g}$$

When we examine the Green's function equations of this theory it will be apparent that the following are the only grouping of the parameters of (4.3) which occur in the renormalized theory:

(a) a renormalized fermion mass m,
(b) a renormalized σ mass μ^2,
(c) a dimensionless renormalized Yukawa-type coupling $g^2 \equiv g_0{}^2 Z_\sigma (Z_2/Z_1)^2$,
(d) a dimensionless renormalized 3σ coupling λ',
(e) a dimensionless renormalized 4σ coupling λ.

We shall show that the theory is renormalizable in terms of these parameters (m, μ^2, g^2, λ', and λ). Thus, all the divergences can be absorbed in them to all orders in our mean-field expansion. In addition we shall show in the next section that μ^2, λ', and λ are dependent parameters. To every order in our expansion $\lambda' = \lambda'(g^2)$, $\mu^2 = \mu^2(m^2, g^2)$, and $\lambda = \lambda(g^2)$. Thus, the theory will be shown to be characterized by two parameters, a fermion mass m

and a dimensionless Yukawa coupling g^2.

The Schwinger-Dyson equations of this model appropriate for our expansion scheme can be derived in a straightforward way by differentiation of (2.8) with respect to the external sources J, η, and $\bar{\eta}$. The following equations are obtained and can be used to generate the all-order Green's functions[3] with $\eta=\bar{\eta}=0$:

$$\left[i\not{\partial}+g_0\sigma(x)+\epsilon g_0-\frac{i\delta}{\delta J(x)}\right]\psi(x)+\eta=0\,, \tag{4.4a}$$

$$\mu_0{}^2\sigma(x)=J(x)+g_0 i\,\mathrm{tr}S(x,x)\,.^{11} \tag{4.4b}$$

In (4.4b) we have set

$$\eta=\bar{\eta}=0\,.$$

From these we obtain

$$S^{-1}(p)=\not{p}+g_0\sigma+ig_0{}^2\epsilon\int\frac{d^4k}{(2\pi)^4}S(k+p)\Delta(k)\Gamma_\sigma(k+p,k)\,, \tag{4.5}$$

$$\Delta^{-1}(p^2)=-\mu_0{}^2+ig_0{}^2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}S(k+p)\Gamma_\sigma(k+p,k)S(k)\,, \tag{4.6}$$

$$\Gamma_\sigma(k,k+p)=1+ig_0{}^2\epsilon\frac{\delta}{\delta g_0\sigma}\int S\Delta_\sigma\Gamma_\sigma\,, \tag{4.7}$$

$$\Gamma^{(3\sigma)}=g_0{}^3 i\frac{\delta}{\delta g_0\sigma}\mathrm{tr}(S\Gamma_\sigma S)\,, \tag{4.8}$$

$$\Gamma^{(4\sigma)}=ig_0{}^4\frac{\delta^2}{\delta(g_0\sigma)^2}\mathrm{tr}(S\Gamma_\sigma S)\,. \tag{4.9}$$

In all of these it is understood that the sources have been turned off after taking the variational derivatives. We emphasize that the above are equations for the all-order Green's functions for arbitrary ϵ.

Now we begin our subtraction procedure. The number of necessary subtractions is dictated by the degree of superficial divergence. Further, the momentum-space analysis is done with the sources off.

The inverse fermion propagator is of the form

$$S^{-1}(p)=\not{p}+g_0\sigma+\Sigma(p)\,,$$

where

$$\Sigma(p)\equiv ig_0{}^2\epsilon\int\frac{d^4k}{(2\pi)^4}S(k+p)\Delta(k^2)\Gamma_\sigma(k+p,k)\,.$$

The fermion self-energy $\Sigma(p)$ is in general of the form $\Sigma(p)=\not{p}A(p^2)+B(p^2)$. Subtracting at zero we obtain

$$S^{-1}(p)=\not{p}+g_0\sigma+B(0)+\not{p}A(0)+\not{p}(A(p^2)-A(0))+B(p^2)-B(0)\,.$$

Using the renormalization conditions we are led to

$$\begin{aligned}S^{-1}(p)&=\not{p}\left(\frac{\partial S^{-1}}{\partial\not{p}}\right)_0-\frac{m}{Z_2}+\not{p}(A(p^2)-A(0))+B(p^2)-B(0)\\&=(\not{p}-m)\frac{1}{Z_2}+\not{p}(A(p^2)-A(0))+B(p^2)-B(0)\,.\end{aligned}$$

Defining

$$\mathrm{sub}_0^2\Sigma(p)\equiv\not{p}\,\mathrm{sub}_0^1A(p^2)+\mathrm{sub}_0^1B(p^2)\,,$$

we obtain the renormalized fermion propagator

$$\begin{aligned}&S^{-1}=Z_2{}^{-1}\bar{S}^{-1}\,,\\&\bar{S}^{-1}(p)\equiv\not{p}-m+ig_0{}^2Z_2\epsilon\,\mathrm{sub}_0^2\int\frac{d^4k}{(2\pi)^4}S(k+p)\Delta_\sigma(k^2)\times\Gamma_\sigma(k,k+p)\,.\end{aligned} \tag{4.10}$$

The inverse σ propagator is subtracted in the same fashion. It is of the form

$$\Delta^{-1}(p^2)=-\mu_0{}^2+\Pi(p^2)\,,$$

where

$$\Pi(p^2)\equiv ig_0{}^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}(S(k)\Gamma_\sigma(k,k+p)S(k+p))\,.$$

Subtracting twice and using the appropriate renormalization conditions we obtain

$$\begin{aligned}\Delta^{-1}(p^2)&=-\mu_0{}^2+\Pi(0)+p^2\left(\frac{\partial\Pi}{\partial p^2}\right)_0+\mathrm{sub}_0^2\,\Pi(p^2)\\&=-\frac{\mu^2}{Z_\sigma}+p^2\left(\frac{\partial\Pi}{\partial p^2}\right)_0+\mathrm{sub}_0^2\,\Pi(p^2)\,,\end{aligned}$$

since

$$\frac{1}{Z_\sigma}\equiv\left(\frac{\partial\Delta^{-1}}{\partial p^2}\right)_0=\left(\frac{\partial\Pi}{\partial p^2}\right)_0.$$

Defining $\bar{\Delta}$ by

$$\Delta^{-1}(p^2)=Z_\sigma{}^{-1}\bar{\Delta}^{-1}$$

leads to

$$\bar{\Delta}^{-1}(p^2)=p^2-\mu^2+iZ_\sigma g_0{}^2\times\mathrm{sub}_0^2\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}(S(k)\Gamma_\sigma(k,k+p)S(k+p))\,. \tag{4.11}$$

The renormalized fermion vertex is defined in a similar way:

$$\begin{aligned}\Gamma_\sigma&=\Gamma_\sigma(0,0)+\mathrm{sub}_0{}^1\Gamma_\sigma\\&=\frac{1}{Z_1}+ig_0{}^2\epsilon\,\mathrm{sub}_0^1\frac{\delta}{\delta g_0\sigma}(S\Delta_\sigma\Gamma_\sigma)=Z_1{}^{-1}\bar{\Gamma}_\sigma\end{aligned}$$

or

$$\bar{\Gamma}_\sigma\equiv1+ig_0{}^2Z_1\epsilon\,\mathrm{sub}_0^1\frac{\delta}{\delta g_0\sigma}\int S\Delta_\sigma\Gamma_\sigma\,. \tag{4.12}$$

The 3σ and 4σ vertices having logarithmic superficial divergence must be subtracted once. The renormalized vertices are

$$\overline{\Gamma}^{(3\sigma)} = Z_\sigma^{3/2}\Gamma^{(3\sigma)} = m\lambda' + g_0^3 Z_\sigma^{3/2}\,\mathrm{sub}_0^1\left(\frac{\delta}{\delta g_0\sigma}\mathrm{tr}(S\Gamma_\sigma S)\right), \tag{4.13}$$

$$\overline{\Gamma}^{(4\sigma)} = Z_\sigma^{2}\Gamma^{(4\sigma)} = -\lambda + ig_0^4 Z_\sigma^{2}\,\mathrm{sub}_0^1\left(\frac{\delta}{\delta (g_0\sigma)^2}\mathrm{tr}(S\Gamma_\sigma S)\right). \tag{4.14}$$

The renormalization of the σ propagator dictates the definition of a renormalized σ field,

$$\overline{\sigma} = Z_\sigma^{-1/2}\sigma. \tag{4.15}$$

Our next step will be to indicate that (4.10), (4.11), (4.12), (4.13), and (4.14) depend on the Z's only through the renormalized parameters defined previously, that is, they are finite functions of the momenta and the renormalized parameters to every order in our expansion.

Before we do that let us consider the fermion vertex

$$\Gamma_\sigma = 1 + ig_0^2\epsilon\frac{\delta}{\delta g_0\sigma}\int S\Delta_\sigma\Gamma_\sigma. \tag{4.16}$$

It is convenient to rewrite (4.16) in the form

$$\Gamma_\sigma(\xi) = 1(\xi')[\delta(\xi'-\xi) - \epsilon g_0^2\Delta(\xi'\xi'')SI(\xi''\xi)]. \tag{4.17}$$

Here we have explicitly displayed the coordinate associated with σ propagation, but have suppressed the coordinate and spin indices associated with the Fermi field in an extended matrix notation. The only information that is relevant to us at this point is that $I(\xi''\xi)$ (which is a function of four coordinates and two Fermi indices) has the momentum behavior of a Green's function having two external Fermi lines and two external σ-meson lines. Consequently, by our preceding analysis $I(\xi'',\xi)$ is superficially finite and its only divergences come from the divergences of the other Green's functions from which it is constructed. Comparing (4.16) and (4.17) we must have

$$g_0^2 S\Delta I = \frac{g_0^2}{i}\frac{\delta}{\delta g_0\sigma}(S\Delta\Gamma_\sigma). \tag{4.18}$$

Reexpressing everything in (4.18) in terms of renormalized quantities, we find

$$g^2\overline{S}\overline{\Delta}\left(\frac{Z_1^2}{Z_2}I\right) = \frac{g^2}{i}\frac{\delta}{\delta g\overline{\sigma}}(\overline{S}\overline{\Delta}\overline{\Gamma}_\sigma). \tag{4.19}$$

Thus, if we define $\overline{I} \equiv (Z_1^2/Z_2)I$, which clearly is expressible in terms of renormalized objects, we conclude that

$$\delta(\xi'-\xi) - \epsilon g_0^2\Delta(\xi'\xi'')SI(\xi'\xi'') = \delta(\xi'-\xi) - \epsilon g^2\overline{\Delta}(\xi'\xi'')\overline{S}\overline{I}(\xi'\xi''). \tag{4.20}$$

Thus, the operator of (4.17) is completely expressible in renormalized (but not finite since it carries the vertex divergence) quantities.

In order to symmetrize our expressions and partially alleviate the overlapping-divergence problem, we replace the bare vertex $1(\xi)$ everywhere in our Green's function equations by using (4.17) in the form

$$1 = \Gamma_\sigma(1-\epsilon g_0^2\Delta SI)^{-1}, \tag{4.21}$$

which in terms of renormalized Green's functions becomes

$$1 = Z_1^{-1}\overline{\Gamma}_\sigma(1-\epsilon g^2\overline{\Delta}\overline{S}\overline{I}) \equiv Z_1^{-1}\overline{\Gamma}_\sigma\overline{B}. \tag{4.22}$$

Now we are ready to examine the renormalized functions one by one with the appropriate number of subtractions to make them finite. The fermion propagator is

$$\overline{S}^{-1} = \not{p} - m + ig_0^2 Z_2\epsilon\,\mathrm{sub}_0^2\int Z_2 Z_\sigma Z_1^{-1}Z_1^{-1}(\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Delta}\overline{\Gamma}_\sigma)$$

or

$$\overline{S}^{-1} = \not{p} - m + ig^2\epsilon\,\mathrm{sub}_0^2\int\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Delta}\overline{\Gamma}_\sigma. \tag{4.23}$$

The σ propagator is

$$\overline{\Delta}^{-1} = p^2 - \mu^2 + ig_0^2 Z_0 Z_1^{-2}Z_2^2\,\mathrm{sub}_0^2\int\mathrm{tr}(\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Gamma}_\sigma\overline{S})$$

or

$$\overline{\Delta}^{-1} = p^2 - \mu^2 + ig^2\,\mathrm{sub}_0^2\int\mathrm{tr}(\overline{\Gamma}_\sigma\overline{B}\overline{S}\,\overline{\Gamma}_\sigma\overline{S}). \tag{4.24}$$

The fermion vertex is

$$\overline{\Gamma}_\sigma = 1 + ig_0^2 Z_2 Z_\sigma^{1/2}\epsilon\,\mathrm{sub}_0^1\frac{\delta}{\delta g_0\overline{\sigma}}\overline{S}\overline{\Delta}\overline{\Gamma}_\sigma$$

or

$$\overline{\Gamma}_\sigma = 1 + ig^2\epsilon\,\mathrm{sub}_0^1\frac{\delta}{\delta g\overline{\sigma}}\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Delta}\overline{\Gamma}_\sigma. \tag{4.25}$$

The 3σ vertex is

$$\overline{\Gamma}^{(3\sigma)} = m\lambda' + iZ_\sigma^{3/2}g_0^3\,\mathrm{sub}_0^1\frac{\delta}{\delta g_0\overline{\sigma}}\mathrm{tr}(\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Gamma}_\sigma\overline{S}) \times Z_2^2 Z_1^{-2}Z_\sigma^{1/2}$$

or

$$\overline{\Gamma}^{(3\sigma)} = m\lambda' + ig^3\,\mathrm{sub}_0^1\frac{\delta}{\delta g\overline{\sigma}}\mathrm{tr}(\overline{\Gamma}_\sigma\overline{B}\overline{S}\overline{\Gamma}_\sigma\overline{S}). \tag{4.26}$$

The 4σ vertex is

$$\overline{\Gamma}^{(4\sigma)} = -\lambda + iZ^2 g_0{}^4 \,\mathrm{sub}_0^1 \frac{\delta}{\delta g_0 \overline{\sigma}} \mathrm{tr}(\overline{\Gamma}_\sigma \overline{BS}\overline{\Gamma}_\sigma \overline{S}\overline{\Gamma}_\sigma \overline{S}) \times Z_2{}^3 Z_1{}^{-3} Z_\sigma{}^{-1/2}$$

or

$$\overline{\Gamma}^{(4\sigma)} = -\lambda + ig^4 \,\mathrm{sub}_0^1 \frac{\delta}{\delta g \overline{\sigma}} \mathrm{tr}(\overline{\Gamma}_\sigma \overline{BS}\overline{\Gamma}_\sigma \overline{S}\overline{\Gamma}_\sigma \overline{S}) . \quad (4.27)$$

We have thus succeeded in expressing the renormalized functions only in terms of renormalized quantities. The renormalized functions thus appear to be finite functions of the momenta and the renormalized parameters. However, we have only partially confronted the problem of overlapping divergences. We have not written down the renormalized Green's functions in a form which is manifestly free of overlaps. One elegant way to begin is to differentiate the divergent Green's functions[8] rather than subtract and apply (4.17) to generate Green's function equations which are (partially) free of overlaps. In fact, the Green's functions we display are adequate for calculational purposes and serve our point of displaying the parametrization of the theory; we will not discuss the details of this procedure here, but defer full explicit removal of overlaps to a paper more specifically aimed at analyzing the details of the Schwinger-Dyson equations.[14]

An important observation that we can make here is that the parameter ϵ in the preceding renormalized Green's function equations is always multiplied by the renormalized coupling g^2. Thus, if we let $\epsilon \to 1$ we can keep an effective small expansion parameter by letting $g^2 \to 0$. This is in accord with our earlier statements.

Finally, we note that with $\epsilon = 1$, and expanding so g^2 is small, the renormalized Green's functions look similar to the equations we would get for a conventional (kinetic terms for the σ field and mass terms for the fermion in the Lagrangian) $g_0 \overline{\psi}\psi\sigma$ interaction expanded and renormalized in a coupling-constant expansion. The esoteric aspects of the mean-field expansion caused by the reordering of terms in the usual expansion are now gone. What remains, which is of interest, are the relations among parameters which result from the way the theory was generated. We have already examined these in first order, and will show how the all-order relations work in the next section. Section VI will further develop the relationship to more conventional theories.

V. THE RENORMALIZED PARAMETERS

In Sec. III we observed that the parameters g and m completely characterized the lowest-order renormalized theory. We proved this remarkable fact by differentiation with respect to the lowest-order mass m or equivalently the lowest-order vacuum expectation σ_0. In this section we will build on this observation and examine the all-order theory through the Callan-Symanzik equations, demonstrating that the entire theory is dependent only on g and m. We will show how to explicitly calculate λ, λ', and μ^2 in terms of these two parameters. For clarity and simplicity in presentation, we will assume for most of what follows that the quantities we will deal with depend only on g and m. At appropriate places we will indicate how modification of our equations and induction using the lowest-order results confirms this assumption.

An ordinary differentiation of a 1PI diagram with respect to a bare mass corresponds to a mass insertion. On the other hand, in our model higher σ-vertex functions with one of the external momenta set equal to zero are also Green's functions with a mass insertion. This suggests that we can derive a set of Callan-Symanzik equations by differentiating the unrenormalized 1PI functions with respect to the constant σ field. We recall that it was pointed out in the preceding section that we are primarily interested in expanding the renormalized Green's functions in a power series in g^2. Consequently, we lose nothing by setting $\epsilon = 1$ for this discussion.

Let us begin with the fermion propagator. The following is true:

$$\sigma \frac{dS^{-1}(p)}{d\sigma} = g_0 \sigma \Gamma_\sigma(p, -p) . \quad (5.1)$$

Inserting the renormalized propagators, we obtain

$$\sigma \frac{dZ_2{}^{-1}}{d\sigma} \overline{S}^{-1}(p) + Z_2{}^{-1} \sigma \frac{d\overline{S}^{-1}(p)}{d\sigma} = Z_1{}^{-1} Z_\sigma{}^{1/2} g_0 \overline{\sigma}\, \overline{\Gamma}_\sigma(p,-p) .$$

Introducing the "anomalous dimension" of the fermion field

$$\gamma_\psi \equiv \tfrac{1}{2}\sigma \frac{\partial \ln Z_2}{\partial \sigma} , \quad (5.2)$$

we are led to

$$\left(-2\gamma_\psi + \sigma \frac{d}{d\sigma}\right) \overline{S}^{-1}(p) = g \overline{\sigma}\, \overline{\Gamma}_\sigma(p,-p) . \quad (5.3)$$

γ_ψ is a superficially finite quantity. Since the theory is renormalizable γ_ψ will be a finite function of the renormalized parameters and since it is dimensionless it will be a function only of the dimensionless coupling g^2.

Considering (5.3) at zero momentum we have

$$\left[-2\gamma_\psi(g^2)+\sigma\frac{d}{d\sigma}\right]m=-g\bar{\sigma}. \qquad (5.4)$$

The derivative $\sigma\, dm/d\sigma$ must be of the form $\sigma\, dm/d\sigma=\alpha(g^2)m$. Note that α is finite. Thus, we can obtain the solution

$$m=-g\bar{\sigma}(\alpha-2\gamma_\psi)^{-1}.$$

To lowest order this leads to the familiar result

$$m=-g\bar{\sigma}. \qquad (5.5)$$

Before we proceed to derive similar equations for the σ vertices, we introduce

$$\beta(g^2)\equiv\sigma\frac{\partial g^2}{\partial\sigma}\bigg|_{g_0^2} \qquad (5.6)$$

and

$$\gamma_\sigma(g^2)\equiv\tfrac{1}{2}\sigma\frac{\partial\ln Z_\sigma}{\partial\sigma}\bigg|_{g_0^2}. \qquad (5.7)$$

$\beta(g^2)$ and $\gamma_\sigma(g^2)$ are finite functions of the dimensionless coupling. To lowest order

$$\beta(g^2)=\sigma\frac{\partial g^2}{\partial\sigma}=g_0^2\sigma\frac{\partial Z_\sigma}{\partial\sigma}=2g^2\gamma_\sigma(g^2).$$

Since to this order $\gamma_\sigma(g^2)=g^2/8\pi^2$ we have $\beta(g^2)=g^4/4\pi^2$.

Differentiating the 3σ vertex we obtain

$$\sigma\frac{d\Gamma^{(3\sigma)}(p_1,p_2)}{d\sigma}=\sigma\Gamma^{(4\sigma)}(p_1,p_2,0),$$

which at zero momentum is according to our definitions

$$\sigma\frac{d}{d\sigma}(m\lambda' Z_\sigma^{-3/2})=\frac{-\sigma\lambda}{Z_\sigma^2}.$$

Carrying out the differentiation we find

$$\left(\alpha(g^2)-3\gamma_\sigma(g^2)+\sigma\frac{d}{d\sigma}\right)\lambda'=\frac{\lambda}{g}(\alpha-2\gamma_\psi). \qquad (5.8)$$

Expanding the Callan-Symanzik operator in terms of independent renormalized parameters g and m (5.8) can be rewritten as

$$\left(\alpha-3\gamma_\sigma+\alpha m\frac{\partial}{\partial m}+\beta(g^2)\frac{\partial}{\partial g^2}\right)\lambda'=\frac{\lambda}{g}(\alpha-2\gamma_\psi).$$

Since λ' is dimensionless, it cannot depend explicitly on the mass m, so we have the simplified equation

$$\left(\alpha-3\gamma_\sigma+\beta\frac{\partial}{\partial g^2}\right)\lambda'=\frac{\lambda}{g}(\alpha-2\gamma_\psi). \qquad (5.9)$$

To lowest order this equation becomes

$$\left(1-\frac{3g^2}{8\pi^2}+\frac{g^4}{4\pi^2}\frac{\partial}{\partial g^2}\right)\lambda'=\frac{\lambda}{g}. \qquad (5.10)$$

Next we derive an equation for the 4σ coupling using

$$\sigma\frac{d\Gamma^{(4\sigma)}(0,0,0)}{d\sigma}=\sigma\Gamma^{(5\sigma)}(0,0,0,0).$$

Repeating the previous procedure at zero momentum, we obtain

$$\left(-4\gamma_\sigma+\sigma\frac{d}{d\sigma}\right)\lambda=-Z_\sigma^{3/2}\bar{\sigma}\,\Gamma^{(5\sigma)}(0)=-\bar{\sigma}\,\bar{\Gamma}^{(5\sigma)}(0). \qquad (5.11)$$

Expanding the differential operator results in

$$\left(-4\gamma_\sigma+\beta(g^2)\frac{\partial}{\partial g^2}\right)\lambda=\frac{m}{g}(\alpha-2\gamma_\psi)\bar{\Gamma}^{(5\sigma)}(0). \qquad (5.11')$$

To lowest order this equation is

$$\left(\frac{-g^2}{2\pi^2}+\frac{g^4}{4\pi^2}\frac{\partial}{\partial g^2}\right)\lambda=\frac{m}{g}\left(\frac{-3}{\pi^2 m}\right)g^5$$
$$=\frac{-3g^4}{\pi^2}.$$

We have used the fact that to lowest order the results of Sec. III easily lead to

$$\Gamma^{(5\sigma)}(0,0,0,0)=-\frac{3g_0^5}{\pi^2 m}.$$

Consequently, our equation becomes

$$\left(-1+\frac{g^2}{2}\frac{\partial}{\partial g^2}\right)\lambda=-6g^2. \qquad (5.12)$$

An ansatz $\lambda=c_1g^2+c_2g^4$ can be inserted. It satisfies (5.12) to order g^2 for $c_1=12$ and c_2 = anything. Thus we obtain the solution

$$\lambda=12g^2+O(g^4). \qquad (5.13)$$

Going back to Eq. (5.10) for the cubic coupling, we obtain

$$\left(1-\frac{3g^2}{8\pi^2}+\frac{g^4}{4\pi^2}\frac{\partial}{\partial g^2}\right)\lambda'=12g+c_2g^3. \qquad (5.14)$$

A solution to the above can be immediately obtained by inserting an ansatz $\lambda'=c_3g+c_4g^3$ to yield

$$\lambda'=12g+O(g^3). \qquad (5.15)$$

Thus we have succeeded in obtaining the renormalized cubic and quartic σ self-couplings as functions of g^2, the renormalized Yukawa-type coupling in exactly the same form as our explicit calculation of Sec. III. It is clear that Eqs. (5.11) and (5.9) will enable us to calculate these quantities to any order of g^2 in our expansion. This is easily seen by an induction argument. All the quantities in (5.11) except λ are explicitly finite. If we know them in one order as functions of g and m we can calculate them in the next higher order by using our Green's function expansion techniques and they will be functions of only g and m. Consequently, we can calculate λ to the next higher order as a function only of g and m. Then using (5.8) the same type of argument can be applied to calculate λ' as a function of g and m to one higher order. Moreover, we can now see how to construct the arguments that these functions depend only on g and m and not λ or λ'. For example, if at some order λ becomes an independent function of another parameter ρ then one could argue that (5.11) should have a term in it of the form $\beta_\rho\partial/\partial\rho$. However, since β_ρ and all other quantities are finite and constructed from lower-order functions of only g and m, ρ cannot be an independent parameter. Thus, since the lowest-order Green's functions have been explicitly constructed and found to be functions of only g and m, we have all we need for an induction argument except the demonstration that μ is not an independent mass. We demonstrated explicitly that this is true to lowest order. Our next step will be to find an equation for μ^2 similar to the above equations for the couplings so that μ^2 can be calculated to any order.

The original theory had one mass parameter μ_0^2. A bare fermion mass appears, however, due to the nonvanishing expectation value of σ. These two parameters are not independent (in the absence of sources), but are related through the gap equation

$$\mu_0^2\sigma = g_0 \operatorname{tr} S .$$

Differentiating this equation with respect to the constant σ field we obtain

$$\mu_0^2 + \sigma\frac{d\mu_0^2}{d\sigma} = -g_0^2 \operatorname{tr}(S\Gamma_\sigma S) .$$

Since we know that

$$\Delta^{-1}(0) = -\mu_0^2 - g_0^2 \operatorname{tr}(S\Gamma_\sigma S) ,$$

we are led to the equation

$$\sigma\frac{d\mu_0^2}{d\sigma} = \Delta^{-1}(0) , \tag{5.16}$$

which can then be written according to (4.3c) as

$$Z_\sigma\sigma\frac{d\mu_0^2}{d\sigma} = -\mu^2 .$$

Introducing the superficially finite function

$$\xi(\mu^2, m, g^2) \equiv Z_\sigma\sigma\frac{d\mu_0^2}{d\sigma} , \tag{5.17}$$

we get the equation

$$\xi(\mu^2, m, g^2) = -\mu^2 . \tag{5.18}$$

Equation (5.17) can be analyzed as follows:

$$\begin{aligned}\xi &= Z_\sigma\sigma\frac{d\mu_0^2}{d\sigma}\\ &= Z_\sigma\sigma\frac{d}{d\sigma}\left[\mu^2 Z_\sigma^{-1} - g_0^2 \operatorname{tr}(S\Gamma_\sigma S)\right]\\ &= \sigma\frac{d\mu^2}{d\sigma} - 2\gamma_\sigma\mu^2 + Z_\sigma\sigma\Gamma^{(3\sigma)}_{(0)}\\ &= \sigma\frac{d\mu^2}{d\sigma} - 2\gamma_\sigma\mu^2 - \frac{m^2}{g}\lambda'(\alpha - 2\gamma_\psi) .\end{aligned}$$

Our Eq. (5.18) then takes the form

$$\left(1 - 2\gamma_\sigma + \sigma\frac{d}{d\sigma}\right)\mu^2 = \frac{m^2\lambda'}{g}(\alpha - 2\gamma_\psi) .$$

Analyzing the mass operator we obtain

$$\left(1 - 2\gamma_\sigma + \alpha m\frac{\partial}{\partial m} + \beta(g^2)\frac{\partial}{\partial g^2}\right)\mu^2 = \frac{m^2\lambda'}{g}(\alpha - 2\gamma_\psi) . \tag{5.19}$$

To lowest order this equation becomes

$$\left(1 - \frac{g^2}{4\pi^2} + m\frac{\partial}{\partial m} + \frac{g^4}{4\pi^2}\frac{\partial}{\partial g^2}\right)\mu^2 = 12m^2 ,$$

and it is automatically satisfied for

$$\mu^2 = 4m^2\left(1 + \frac{g^2}{12\pi^2}\right) ,$$

which, as was established in Sec. III, is the correct lowest-order result.

It is clear now that (5.19) to every order can yield $\mu^2 = \mu^2(m^2, g^2)$. We can apply our construction arguments to support this, and Eq. (5.19) gives us the required information to finish the induction proof. We have succeeded in renormalizing the four-fermion theory in terms of a dimensionless coupling g^2 and a fermion mass m. The other renormalized parameters that were introduced are determined order by order in terms of m and g^2.

VI. EQUIVALENCE WITH THE YUKAWA THEORY

Let us consider the theory described by the Lagrangian

$$\begin{aligned}L(\psi,\bar\psi,\sigma) = {}&\psi(i\partial\!\!\!/ + g_0\sigma)\psi + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \frac{\mu_0^2}{2}\sigma^2\\ &- \frac{\lambda_0}{4!}\sigma^4 + \bar\eta\psi + \bar\psi\eta + J\sigma .\end{aligned} \tag{6.1}$$

The generating function after the integration of the fermion variables takes the form, in Euclidean space,

$$\begin{aligned}e^W = \int(d\sigma)\exp\Bigg[-\int\Bigg(&\bar\eta S\eta - \operatorname{tr}\ln S^{-1} + \tfrac{1}{2}\sigma(-\partial^2 + \mu_0^2)\sigma\\ &+ \frac{\lambda_0}{4!}\sigma^4 + J\sigma\Bigg)\Bigg] .\end{aligned} \tag{6.2}$$

Here S is given as before by (2.6).

The mean-field expansion can be defined in the same way as in Sec. II. An ϵ is introduced and the generating function becomes

$$e^{W_\epsilon/\epsilon}=N\int(d\sigma)\exp[-F(\sigma)/\epsilon]=\langle|\rangle^\epsilon .$$

The mean-field conditions are again

$$\frac{\delta F}{\delta\sigma}\Big|_{\sigma_0}=0,\quad \frac{\delta^2 F}{\delta\sigma\delta\sigma}\Big|_{\sigma_0}>0$$

and the expanded functional is given by (2.11). Note that even though (6.1) is a common renormalizable field theory in a coupling-constant expansion or a loop expansion, we have no *a priori* information about its behavior in our mean-field expansion, and the questions of renormalizability and parameterization must be reexamined from the beginning here.

The lowest-order connected functional in Euclidean space is

$$\begin{aligned}-W_0\{\sigma_0,\eta,\bar\eta,J\}&=F\{\sigma_0,\eta,\bar\eta,J\}\\&=\int\Big[\bar\eta S_0\eta-\operatorname{tr}\ln S_0^{-1}+\tfrac{1}{2}\sigma_0(-\partial^2+\mu_0{}^2)\sigma_0\\&\qquad+\frac{\lambda_0}{4!}\sigma_0{}^4-J\sigma_0\Big].\end{aligned}\tag{6.3}$$

This form is identical to that given by (2.8b) except for the σ^4 term and the kinetic terms for σ. Despite these nontrivial differences we shall show that for a special value of the renormalized parameters the theories are identical.

The mean-field condition (we now return to Minkowski space) takes the form of a gap equation for constant source J,

$$\mu_0{}^2\sigma_0=g_0 i\operatorname{tr}S_0-\frac{\lambda_0}{3!}\sigma_0{}^3+J.\tag{6.4}$$

The fermion propagator to lowest order describes a free fermion of mass $m=-g_0\sigma_0$. The inverse σ propagator is

$$\begin{aligned}\left(\frac{\delta W_0}{\delta J\delta J}\right)^{-1}&=\Delta^{-1}(p^2)\\&=p^2-\mu_0{}^2+ig_0{}^2\operatorname{tr}\int\frac{d^4k}{(2\pi)^4}S_0(k+p)S_0(k)\\&\quad-\frac{\lambda_0}{2}\sigma_0{}^2.\end{aligned}$$

The structure of this is nearly identical to (3.7) so we proceed in the identical manner. Defining

$$\Pi(p^2)=i\int\frac{d^4k}{(2\pi)^4}\operatorname{tr}[S(k+p)S(k)]$$

and subtracting twice, we obtain

$$\begin{aligned}\Delta^{-1}(p^2)&=p^2\left[1+g_0{}^2\left(\frac{\partial\Pi}{\partial p2}\right)_0\right]-\mu_0{}^2-\frac{\lambda_0}{2}\sigma_0{}^2\\&\quad+\Pi(0)g_0{}^2+g_0{}^2\operatorname{sub}_0^2\Pi(p^2).\end{aligned}$$

Introducing a renormalized σ-mass parameter μ^2 and a renormalization factor Z_σ, as in Eqs. (3.8), we are led to the following form for the renormalized σ propagator:

$$\bar\Delta^{-1}(p^2)=p^2-\mu^2+g^2\operatorname{sub}_0^2\Pi(p^2)=Z_\sigma\Delta^{-1}(p^2).\tag{6.5}$$

The renormalized coupling appearing in (6.5) is

$$g^2\equiv g_0{}^2Z_\sigma .\tag{6.6}$$

For this model the renormalization equations defining the mass and the couplings are different from the $(\bar\psi\psi)^2$ model and are

$$\frac{\mu^2}{Z_\sigma}=\mu_0{}^2+\frac{\lambda_0\sigma_0}{2}-g_0{}^2\Pi(0)$$

and

$$\frac{1}{Z_\sigma}=1+g_0{}^2\left(\frac{\partial\Pi}{\partial p^2}\right)_0 .$$

Application of these with (6.6) leads to

$$\frac{1}{g^2}=\frac{1}{g_0{}^2}+\left(\frac{\partial\Pi}{\partial p^2}\right)_0 .\tag{6.7}$$

We now want to carry out the lowest-order constant source analysis for this theory in the same manner as was done for the $(\bar\psi\psi)^2$ theory. Equation (6.4) can be rewritten using the definition (3.14) to obtain

$$\mu_0{}^2\sigma_0=g_0 m f(m)-\frac{\lambda_0}{3!}\sigma_0{}^3+J.\tag{6.8}$$

Turning off the source and using the definition of m and g this becomes

$$\mu_0{}^2=-g_0^2 f(m)-\lambda_0 Z_\sigma\frac{m^2}{3!g^2}.\tag{6.9}$$

Note that, except for the last term, (6.9) is the same as (3.13b) with $J=0$. The extra term in (6.9) clearly reflects that in general Yukawa-type models have more parameters than the pure Fermi models.

By differentiating (6.8) with respect to the space-time independent field σ_0 we obtain the expression for the lowest-order σ propagator at zero momentum,

$$\Delta_0^{-1}(0)=\mu_0{}^2+g_0^2(f(m)+mf'(m))+\frac{\lambda_0\sigma_0{}^2}{2}.\tag{6.10}$$

Using (6.9) $\mu_0{}^2$ can be eliminated to obtain

$$\bar\Delta^{-1}(0)=mg^2f'(m)+\lambda_0 Z_\sigma^2\frac{m^2}{3g^2}=-\mu^2.\tag{6.11}$$

We can solve this equation for $f'(m)$ and find

$$f'(m) = -\frac{\mu^2}{mg^2} - \frac{\lambda_0 Z_\sigma{}^2 m}{3g^4}. \tag{6.12}$$

Except for the last term this is identical to Eq. (3.17). The first term of (6.12) is identical to the Fermi model case and the second term occurs because of the σ^m term in the Lagrangian. Note that in this model, because of the explicit appearance of this bare quartic meson interaction, we can obtain no information about the location of the physical σ pole in terms of the Fermi mass alone.

The 3σ- and 4σ-vertex functions are logarithmically divergent and so are subtracted once at zero. The renormalized 3σ and 4σ couplings are defined as

$$\Gamma^{(3\sigma)}(0,0) \equiv \frac{m\lambda'}{Z_\sigma^{3/2}} \equiv Z_\sigma^{-3/2}\,\overline{\Gamma}^{(3\sigma)}$$

and

$$\Gamma^{(4\sigma)}(0,0,0) \equiv \frac{-\lambda}{Z_\sigma{}^2} \equiv Z_\sigma^{-2}\,\overline{\Gamma}^{(4\sigma)},$$

where

$$\lambda \equiv \lambda_0 \frac{Z_\sigma{}^2}{Z_\lambda}.$$

To lowest order we easily find that

$$\overline{\Gamma}^{(3\sigma)}(p_1 p_2) = m\lambda' - ig^3\,\mathrm{sub}_0^1\mathrm{tr}\int \frac{d^4k}{(2\pi)^4} S_0(k)S_0(k+p_1) \times S_0(k+p_1+p_2) + (xT) \tag{6.13}$$

and

$$\overline{\Gamma}^{(4\sigma)}(p_1p_2p_3) = -\lambda + ig^4\,\mathrm{sub}_0^1\mathrm{tr}\int \frac{d^4k}{(2\pi)^4} S_0(k+p_1) \times S_0(k+p_1+p_2) \times S_0(k+p_1+p_2+p_3). \tag{6.14}$$

By differentiating (6.10) we can generate explicit expressions for λ and λ' as in Sec. III. We find

$$\lambda' = -\frac{g^3}{m}[2f'(m) + mf''(m)] - \frac{\lambda_0 Z_\sigma{}^2}{g} \tag{6.15}$$

and

$$\lambda = -g^4[3f''(m) + mf'''(m)] - \lambda_0 Z_\sigma{}^2. \tag{6.16}$$

Except for the last terms these have the same structure as the corresponding expressions for the $(\overline{\psi}\psi)^2$ model. Using Eq. (6.12) for $f'(m)$ we easily verify from (6.15) and (6.16) that

$$\lambda' = \frac{\lambda}{g} + O(g^3). \tag{6.17a}$$

Using (6.16) and (6.11) we find

$$\mu^2 = \frac{\lambda m^2}{3g^2}\left(1 + \frac{g^4}{\lambda\pi^2}\right). \tag{6.17b}$$

So far we have just discussed this theory in lowest order. We can examine higher orders by working out the expansion of (6.2) through Eq. (2.11). We note that all terms in the expansion (2.11) involving five of more derivatives of F with respect to $\sigma(x)$ are the same for both the $(\overline{\psi}\psi)^2$ model and the Yukawa model. The functions A, B, C are the σ propagator and 3- and 4-vertex, respectively, and are, of course, divergent. Writing these objects in the form (6.5), (6.13), and (6.14) for the Yukawa model we see they are exactly the same functions of p^2, m, μ^2, λ', λ, and g^2 as Eqs. (3.10), (3.22), and (3.33). This observation remains valid even when the Fermi sources are on. We can further verify that all the functional derivatives with respect to the three sources of S_0, Δ_0, $\Gamma_0^{3\sigma}$, and $\Gamma_0^{4\sigma}$ are the same functions of these parameters in both models. These properties along with the expansion (2.11) guarantee that to any order all Green's functions of both theories are identical functions of the above parameters. The real difference between the two theories is the values that the renormalized parameters can take. We have already seen that these values are very restricted for the $(\overline{\psi}\psi)^2$ model with only two parameters being free. We will show shortly that there is a choice of parameters for the Yukawa model so that it touches the $(\overline{\psi}\psi)^2$ theory.

Before we proceed with this identification we can display some all-order results for the Yukawa-type model since its parameterization has been identified. We proceed just as in Sec. IV. For example, the renormalized cubic coupling is related to the quartic coupling since we started with no bare cubic coupling. It satisfies the exact equation

$$\left(\alpha - 3\gamma_\sigma + \beta_g(g^2,\lambda)\frac{\partial}{\partial g^2} + \beta_\lambda(g^2,\lambda)\frac{\partial}{\partial\lambda}\right)\lambda' = \frac{\lambda}{g}(\alpha - 2\gamma_\psi). \tag{6.18}$$

To lowest order, since

$$\beta_\lambda(g^2,\lambda) \equiv \sigma\frac{\partial\lambda}{\partial\sigma}\bigg|_{g_0{}^2,\lambda_0} = 4\gamma_\sigma\lambda - \overline{\sigma}\,\overline{\Gamma}^{(5\sigma)}(0) = \frac{g^2\lambda}{2\pi^2} - \frac{3g^4}{\pi^2},$$

(6.18) becomes

$$\left[1 - \frac{3g^2}{8\pi^2} + \frac{g^4}{4\pi^2}\frac{\partial}{\partial g^2} + \frac{g^2}{2\pi^2}(\lambda - 6g^2)\frac{\partial}{\partial\lambda}\right]\lambda' = \frac{\lambda}{g}.$$

Thus, we again obtain the previous solution $\lambda' = \lambda/g + O(g^3)$. It can be shown that the mass μ^2 satisfies an analogous equation which we use shortly. Since the 4σ coupling is an independent parameter its Callan-Symanzik equation must be an identity. This equation is

$$\left(-4\gamma_\sigma+\beta_g(g^2,\lambda)\frac{\partial}{\partial g^2}+\beta_\lambda(\lambda,g^2)\frac{\partial}{\partial\lambda}\right)\lambda = \frac{m}{g}(\alpha-2\gamma_\psi)\Gamma^{(5\sigma)}(0),$$

which becomes to lowest order

$$-4\gamma_\sigma\lambda+\beta_\lambda(\lambda,g^2)=-\frac{3}{\pi^2}g^4.$$

Since $\gamma_\sigma=g^2/8\pi^2$ and $\beta_\lambda=(g^2/2\pi^2)(\lambda-6g^2)$ this indeed is an identity.

Now we return to demonstrating the equivalence of the two theories for some set of values of the parameters. We have already shown that the lowest-order Green's functions are identical functions of the same renormalized parameters. If we impose $\lambda=12g^2$ (6.17a) becomes (3.26) and (6.17b) becomes (3.20). We can see this directly in the Callan-Symanzik equation for the σ mass with this condition,

$$\left(1-2\gamma_\sigma+\alpha m\frac{\partial}{\partial m}+\beta_g\frac{\partial}{\partial g^2}+\beta_\lambda\frac{\partial}{\partial\lambda}\right)\mu^2=\frac{m^2\lambda'}{g}(\alpha-2\gamma_\psi).$$

This is identical with (3.20) since the $\beta_\lambda\partial/\partial\lambda$ term will not contribute. Thus, the mass $\mu^2=\mu^2(m,g^2)$ becomes the same for both theories. Hence, we conclude that, imposing $\lambda=12g^2$, the lowest order of the Yukawa theory is equivalent to the lowest order of the four-fermion theory. This is enough to guarantee their equivalence to all orders, since all higher-order Green's functions are constructed by iterating the lowest-order functional and its derivatives and the renormalized parameters are constructed by a similar iteration procedure of the renormalization-group equations. The equivalence of the two theories holds, of course, only if we start with our mean-field expansion and then expand in the renormalized coupling, but does not hold in perturbation theory with respect to the bare couplings.

It is profitable to look at this equivalence in another way. Equations (6.9), (6.11), (6.15), and (6.16) would all be identical to the analogous equations in the four-Fermi model except for their last terms. If we require $\lambda_0 Z_\sigma^2=0$ then both models are identical (to all orders). This means that for nonzero λ_0 the models become identical in the limit

$$Z_\sigma=0. \tag{6.19}$$

This is the well-known composite particle limit. We have thus demonstrated that in this limit the operator $\lambda_0\sigma^4$ is irrelevant as it has no effect on quantum field theory.[12,13] It can be argued that an equation of the form (6.19) is valid when Z_σ is the all-order Z.[14]

Finally, we note that if we assumed that our Lagrangian (6.1) had terms of the form $\sum_{n=5}^{\infty}C_n\sigma^n$ added to it, it is not normally renormalizable even in the mean-field approximation.[15] However, this does not affect an analysis of the lowest order and if we proceed through this, we can argue that in the limit $Z_\sigma\to 0$ all these extra operators become irrelevant and the theory collapses to the results we have already obtained. We can then iterate this lowest-order result using (2.11). Since all the lowest-order Green's functions are identical, the higher-order Green's functions will be identical before renormalization but after $Z_\sigma\to 0$. In particular, the quantities needing renormalization of this all-order theory will be the same as those of the $(\bar\psi\psi)^2$ theory. It follows then that the renormalization-group equations are the same and since the lowest-order Green's functions are set, iteration fixes this new theory to be identical to the $(\bar\psi\psi)^2$ theory. Unfortunately, we have essentially defined our way into this result and we probably have gained no real insight into the general behavior of the more complicated nonrenormalizable theory.

VII. CONCLUSIONS

(a) We have developed an expansion scheme for the four-fermion theory. The vacuum functional is expressed as an integral over collective boson variables while the fermions appear only through the sources. A Laplace expansion of the functional integral is written down.

(b) The theory is renormalizable in four dimensions to all orders in this expansion, i.e., all divergences can be absorbed in a finite number of renormalized parameters. After renormalization the theory is reexpanded in the then identifiable small parameter g^2. Cubic and quartic boson self-interactions are induced.

(c) The boson self-couplings are determinable to any order in terms of the renormalized Yukawa-type coupling and the fermion mass. The bound-state mass is also determined in terms of the fermion mass and the coupling. Thus, there are only two free parameters.

(d) The renormalized vacuum functional describes fermions interacting with a fundamental boson in contrast with the bare functional that described fermions interacting with a composite boson. This happens because a kinetic term for the collective boson was created from the vacuum polarization graphs.

(e) The Yukawa theory can be expanded the same way. The lowest-order renormalized connected functional of the Yukawa theory becomes exactly the same function of the renormalized quantities as the vacuum functional of the four-fermion theory if we impose the condition 1/3! (renormalized quartic boson coupling) = 2 (renormalized Yukawa coupling)^2 to lowest order. All renormalized 1PI functions are the same. Note that previous statements[1,2] about required restrictions on the bare parameters of the Yukawa model in order to obtain this equivalence are "inoperative."

(f) Equivalence of the vacuum functional and its derivatives to lowest order is enough to guarantee equivalence to all orders since the mean-field expansion and the various Callan-Symanzik equations used to set renormalized parameters are based on the iteration of the lowest-order functional.

(g) This equivalence occurs at $Z_\sigma = 0$ and at this point the quartic and higher-order σ self-interactions terms in the Lagrangian become irrelevant.

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[1]N. J. Snyderman and G. S. Guralnik, in *Quark Confinement and Field Theory*, edited by D. R. Stump and D. H. Weingarten (Wiley, New York, 1977), p. 33; N. J. Snyderman, Ph. D. thesis, Brown University, 1976 (unpublished).

[2]T. Eguchi, Phys. Rev. D 14, 2765 (1976); T. Eguchi, in *Quark Confinement and Field Theory*, edited by D. R. Stump and D. H. Weingarten (Wiley, New York, 1977), p. 13.

[3]C. Bender, F. Cooper, and G. S. Guralnik, Ann. Phys. (N.Y.) 109, 165 (1977).

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[5]Typical early but far from exhaustive references of the $1/N$ work are L. Dolan and R. Jackiw, Phys. Rev. D 9, 3320 (1974); H. J. Schnitzer, *ibid.* 10, 1800 (1974); 10, 2042 (1974); S. Coleman, R. Jackiw, and H. D. Politzer, *ibid.* 10, 2491 (1974).

[6]We define the connected Green's functions in the following way: $S(x,y) \equiv i\langle(\psi(x)\bar{\psi}(y))_+\rangle$ and $\Delta(x,y) \equiv -i\langle(\sigma(x)\sigma(y))_+\rangle + i\langle\sigma(x)\rangle\langle\sigma(y)\rangle$.

[7]Here and in what follows we have assumed $N=1$. It is straightforward to insert the N's in front of every trace, but since we are not interested in identifying N as the expansion parameter, it is irrelevant for our examination.

[8]The reader should be consciously aware of what he might regard as peculiarities in our renormalization procedure. We do not use explicit Lagrangian counterterm renormalization procedures in our discussions of mean-field theories, but instead regard divergent Green's functions as defined by either subtracting (or differentiating) in momentum space until they are finite and then integrating or adding back in the requisite number of powers of momentum multiplied by finite (but undefined except through identities of the theory) constants. This method is obviously fully equivalent to the Lagrangian counterterm method, but to our own taste is somewhat more elegant for dealing with mean-field problems as well as more conventional coupling-constant renormalization problems. Furthermore, we do not need to use an explicit regularization scheme since the results obtained through the above procedure make no reference to divergent quantities. As a very nice alternative example of this type of procedure, see M. Baker and Choonkyu Lee, Phys. Rev. D 15, 2201 (1977).

[9]See J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 287, Eq. (19.6).

[10]See Julian Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1973), Vol. II, p. 40.

[11]With $\epsilon \neq 1$ this equation has additional nonvanishing terms when n and $\bar{n}$ are not zero. We have, however, lost no generality and just from these equations can construct all Green's functions for $\bar{n}=n=0$. A detailed analysis of this is contained in Ref. 3.

[12]K. Tamvakis and G. S. Guralnik, Brown University Report No. HET-366, 1978 (unpublished).

[13]David Campbell, Fred Cooper, Gerald S. Guralnik, and Neal Snyderman, Phys. Rev. D (to be published).

[14]F. Cooper, G. S. Guralnik, R. Haymaker, and K. Tamvakis (unpublished).

[15]Interesting equivalences of nonrenormalizable ϕ^6 interaction to a ϕ^4 interaction in a $1/N$ expansion in lowest order have been observed by the Brandeis group. These are very different Lagrangians and limits from ours, but also owe their behavior to the simple structure of the lowest-order effective action. See for example H. Schnitzer, Nucl. Phys. B109, 293 (1978); P. K. Townsend, *ibid.* B118, 199 (1977).