

Hawking radiation from a $(4 + n)$ -dimensional rotating black hole on the brane

C.M. Harris^a, P. Kanti^{b,*}

^a *Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, United Kingdom*

^b *Department of Mathematical Sciences, University of Durham, Science Site, South Road, Durham DH1 3LE, United Kingdom*

Received 1 June 2005; received in revised form 23 September 2005; accepted 12 October 2005

Available online 24 October 2005

Editor: G.F. Giudice

Abstract

We study the emission of Hawking radiation in the form of scalar fields from a $(4 + n)$ -dimensional, rotating black hole on the brane. We perform a numerical analysis to solve both the radial and angular parts of the scalar field equation, and derive exact results for the Hawking radiation energy emission rate. We find that, in 5 dimensions, as the angular momentum increases, the emission rate is suppressed in the low-energy regime but significantly enhanced in the intermediate and high-energy regimes. For higher values of n , the Hawking radiation emission rate on the brane is significantly enhanced, with the angular momentum, over the whole energy regime. We also investigate the energy amplification due to the effect of super-radiance and demonstrate that, in the presence of extra dimensions, this effect is again significantly enhanced.

© 2005 Elsevier B.V. All rights reserved.

During the past few years, great interest has been drawn to the concept of trans-Planckian collisions which can be realized in the framework of theories with large extra dimensions [1], under the assumption that the colliding particles have a center-of-mass energy s larger than the $(4 + n)$ -dimensional, fundamental Planck scale M_* . During such high-energy collisions, it is natural to expect that the products will no longer be ordinary particles but heavy objects, arising in the context of a fundamental theory of interactions including gravity [2].

Small, $(4 + n)$ -dimensional black holes are one of the possible products of such collisions [2–4]. As long as their mass M_{BH} is larger than a few times the fundamental Planck scale M_* , these black holes can be treated as classical, with all the laws of black hole physics applying to them. One of the most important characteristics would then be their decay in time, through the emission of Hawking radiation [5], a characteristic that will also be the most striking observable effect of their existence. The produced black holes, after shedding any additional quantum numbers inherited from the colliding par-

ticles, will settle down to Kerr-like, rotating black holes, and a *spin-down phase* will commence [3], in which the black hole angular momentum will be gradually lost through the emission of Hawking radiation and super-radiance. A *Schwarzschild phase* describing a non-rotating black hole will follow next with the emission of Hawking radiation resulting in the decrease of the black hole mass.

The emission of Hawking radiation during the Schwarzschild phase has been studied both analytically [6,7] and numerically [8] (see [9], for a review and related works), however the same is not true for the spin-down phase which has been, apart from a few exceptions [10,11], largely ignored. In this short Letter, we present, for the first time in the literature, exact numerical results for the Hawking radiation energy emission rate coming from a $(4 + n)$ -dimensional, rotating black hole. We focus our attention on the emission of scalar field radiation on the brane where, according to the assumptions of the model [1], the observer is situated. After formulating the problem, we numerically solve both the angular and radial scalar field equations to determine the exact angular eigenvalues and radial wavefunction, respectively, for arbitrarily large black hole angular momentum and energy of the emitted particles. Then, we derive the absorption coefficient and subsequently the differential energy

* Corresponding author.

E-mail address: panagiota.kanti@durham.ac.uk (P. Kanti).

emission rate for Hawking radiation and the energy spectrum amplification due to the super-radiance effect.

The line-element describing a $(4+n)$ -dimensional, rotating, uncharged black hole was found by Myers and Perry [12]. A black hole created by the collision of particles moving in a $(4+n)$ -dimensional spacetime can have up to $(n+3)/2$ angular momentum parameters. However, in the context of the theory with large extra dimensions [1], the colliding partons are restricted to propagate on an infinitely-thin 3-brane and therefore they have a non-zero impact parameter only on a 2-dimensional plane along our brane; thus, it is reasonable to assume that they will acquire only one non-zero angular parameter about an axis in the brane. The 4-dimensional induced spacetime on the brane, in which the emitted particles propagate, takes the form [9]

$$ds^2 = \left(1 - \frac{\mu}{\Sigma r^{n-1}}\right) dt^2 + \frac{2a\mu \sin^2 \theta}{\Sigma r^{n-1}} dt d\varphi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{a^2 \mu \sin^2 \theta}{\Sigma r^{n-1}}\right) \sin^2 \theta d\varphi^2, \quad (1)$$

where

$$\Delta = r^2 + a^2 - \frac{\mu}{r^{n-1}} \quad \text{and} \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (2)$$

The parameters μ and a are related to the mass and angular momentum, respectively, of the black hole through the definitions [12]

$$M_{\text{BH}} = \frac{(n+2)A_{n+2}}{16\pi G} \mu, \quad J = \frac{2}{n+2} M_{\text{BH}} a. \quad (3)$$

In the above, G is the $(4+n)$ -dimensional Newton's constant, and A_{n+2} the area of a $(n+2)$ -dimensional unit sphere given by: $A_{n+2} = 2\pi^{(n+3)/2} / \Gamma[(n+3)/2]$. Note that the induced line-element on the brane has an explicit dependence on the number n of extra dimensions.

The black hole horizon is given by solving $\Delta(r) = 0$. Unlike the 4D Kerr black hole for which there is an inner and outer solution for r_{h} , for $n \geq 1$ there is only one solution of this equation. In addition, in the $n = 0$ and $n = 1$ cases, there is a maximum possible value of a , otherwise there are no solutions of $\Delta = 0$ and thus no horizon to shield the singularity at $r = 0$. On the other hand, for $n > 1$ there is no fundamental upper bound on a and a horizon r_{h} always exists. For general n , the horizon radius is given by $r_{\text{h}}^{n+1} = \mu / (1 + a_*^2)$, where we have defined $a_* = a / r_{\text{h}}$. An upper bound can nevertheless be imposed on the angular momentum parameter of the black hole by demanding the creation of the black hole itself from the collision of the two particles. The maximum value of the impact parameter between the two particles that can lead to the creation of a black hole was found to be [13]

$$b_{\text{max}} = 2 \left[1 + \left(\frac{n+2}{2} \right)^2 \right]^{-\frac{1}{n+1}} \mu^{\frac{1}{n+1}}. \quad (4)$$

If we, then, write $J = b M_{\text{BH}} / 2$ [11] for the angular momentum of the black hole, and use the second of Eq. (3), we obtain [13]

$$a_*^{\text{max}} = \frac{n+2}{2}. \quad (5)$$

By using the Newman–Penrose formalism, the equation for the propagation of a field with spin 0, 1/2 and 1, in the gravitational background induced on the brane, can be found [9]. Here, we will focus on the emission of scalar fields leaving the analysis for non-zero spin fields for a subsequent work [14]. By using the field factorization

$$\phi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} R(r) T_{\ell}^m(\theta), \quad (6)$$

where $T_{\ell}^m(\theta)$ are the so-called spheroidal harmonics [15], we obtain a set of decoupled radial and angular equations,

$$\begin{aligned} \frac{d}{dr} \left(\Delta \frac{dR}{dr} \right) + \left(\frac{K^2}{\Delta} - \Lambda_{\ell}^m \right) R &= 0, \\ \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dT_{\ell}^m(\theta)}{d\theta} \right) &+ \left(-\frac{m^2}{\sin^2 \theta} + a^2 \omega^2 \cos^2 \theta + E_{\ell}^m \right) T_{\ell}^m(\theta) = 0, \end{aligned} \quad (7)$$

respectively, where we have defined

$$\begin{aligned} K &= (r^2 + a^2)\omega - am, \\ \Lambda_{\ell}^m &= E_{\ell}^m + a^2 \omega^2 - 2am\omega. \end{aligned} \quad (9)$$

The Hawking temperature of the $(4+n)$ -dimensional, rotating black hole is found to be

$$T_{\text{H}} = \frac{(n+1) + (n-1)a_*^2}{4\pi(1+a_*^2)r_{\text{h}}}, \quad (10)$$

and leads to the emission of scalar Hawking radiation on the brane, with the corresponding differential energy emission rate given by the expression

$$\frac{dE(\omega)}{dt} = \sum_{\ell, m} |\mathcal{A}_{\ell, m}|^2 \frac{\omega}{\exp[(\omega - m\Omega) / T_{\text{H}}] - 1} \frac{d\omega}{2\pi}. \quad (11)$$

In the above, the rotation velocity Ω is defined as

$$\Omega = \frac{a_*}{(1+a_*^2)r_{\text{h}}}, \quad (12)$$

while $|\mathcal{A}_{\ell, m}|^2$ is the absorption probability for a scalar particle propagating in the background induced on the brane (1). Its presence in the expression for the emission rate modifies the blackbody profile of the spectrum due to its explicit dependence on the energy ω of the emitted particle, its angular momentum numbers (ℓ, m) , and the number of extra dimensions n . The exact form of $|\mathcal{A}_{\ell, m}|^2$ can be found by solving the radial equation (7), a task which we have performed by using numerical analysis. The numerical solution obtained for $R(r)$ interpolates between the asymptotic solutions at the horizon of the black hole and infinity. Near the horizon, Eq. (7) leads to the asymptotic solution

$$R_{\text{h}}(r^*) = A_1 e^{ikr^*} + A_2 e^{-ikr^*}, \quad (13)$$

where $A_{1,2}$ are integration constants,

$$k = \omega - \frac{ma}{r_{\text{h}}^2 + a^2}, \quad (14)$$

and r^* is the tortoise coordinate defined by

$$\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta(r)}. \quad (15)$$

A boundary condition must be applied in the near-horizon regime, namely that the solution must contain only incoming modes; this is satisfied if we set $A_1 = 0$. On the other hand, for fixed a_* and large r , the solution at infinity takes the form

$$R_\infty(r) = B_1 \frac{e^{i\omega r}}{r} + B_2 \frac{e^{-i\omega r}}{r}, \quad (16)$$

where $B_{1,2}$ are again integration constants.

However, before the numerical integration of Eq. (7) can take place, the value of the constant Λ_ℓ^m , or equivalently, the angular eigenvalue E_ℓ^m , must be determined. The eigenvalues of the spheroidal harmonics are functions of $a\omega$, and an analytic form can be found only in the limit of small $a\omega$ [16]—the only work in the literature on the Hawking radiation coming from a $(4+n)$ -dimensional, rotating black hole [11] employs this approximate expression for E_ℓ^m , and the corresponding analysis is valid only in the limit of low energy and low black-hole angular momentum. The exact value of E_ℓ^m , for arbitrarily large values of $a\omega$ can be obtained by using a continuation method [13,17]. This is a generalization of perturbation theory which can be applied for arbitrarily large changes in the initial Hamiltonian for which the eigenvalues are known. Here, we briefly describe this technique for the case $s = 0$. Eq. (8) can be alternatively written as

$$(\mathcal{H}_0 + \mathcal{H}_1)T_\ell^m(\theta, a\omega) = -E_\ell^m(a\omega)T_\ell^m(\theta, a\omega), \quad (17)$$

where \mathcal{H}_0 stands for the first two terms of the differential operator in Eq. (8), and $\mathcal{H}_1 = a^2\omega^2 \cos^2\theta$. For $a\omega = 0$, \mathcal{H}_1 vanishes, and $T_\ell^m(\theta, 0)$ reduce to the usual spherical harmonics $S_\ell^m(\theta)$, with $E_\ell^m(0) = \ell(\ell + 1)$. To employ the continuation method, we write the $T_\ell^m(\theta, a\omega)$ functions in the basis of the θ -parts of the spherical harmonics:

$$T_\ell^m(\theta, a\omega) = \sum_{\ell'} B_{\ell\ell'}^m(a\omega) S_{\ell'}^m(\theta). \quad (18)$$

By differentiating Eq. (17) and applying the same techniques as in perturbation theory, we obtain the equation

$$\frac{dE_\ell^m}{d(a\omega)} = - \sum_{\alpha, \beta} B_{\ell\alpha}^m B_{\ell\beta}^m \langle \alpha | \beta \rangle, \quad (19)$$

where $\langle \alpha | \beta \rangle \equiv \langle \alpha m | d\mathcal{H}_1/d(a\omega) | \beta m \rangle$. The coefficients $B_{\ell\alpha}^m$ satisfy themselves a similar differential equation, i.e.,

$$\frac{dB_{\ell\ell'}}{d(a\omega)} = - \sum_{\alpha, \beta, \gamma \neq \ell} \frac{B_{\gamma\alpha} B_{\ell\beta}}{E_\ell^m - E_\gamma^m} \langle \alpha | \beta \rangle B_{\gamma\ell'}, \quad (20)$$

with initial condition $B_{\ell\ell'}^m(0) = \delta_{\ell\ell'}$. By integrating Eqs. (19) and (20), it is possible to obtain the eigenvalues of the spheroidal functions for any ℓ and m and for arbitrarily large values of $a\omega$. This integration was performed numerically by using a Runge–Kutta method. Fig. 1 shows the angular eigenvalues E_ℓ^m for scalar fields as a function of $a\omega$, for some indicative angular momentum modes.

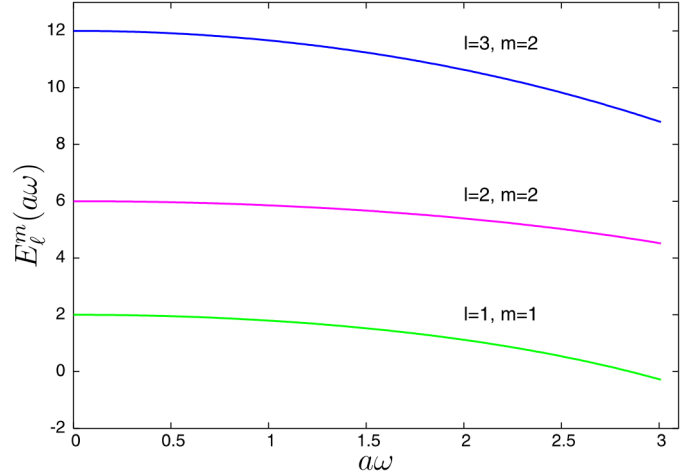


Fig. 1. The angular eigenvalues E_ℓ^m , for $s = 0$, as a function of $a\omega$ for three angular momentum modes: $\ell = m = 1$, $\ell = m = 2$, and $\ell = 3, m = 2$.

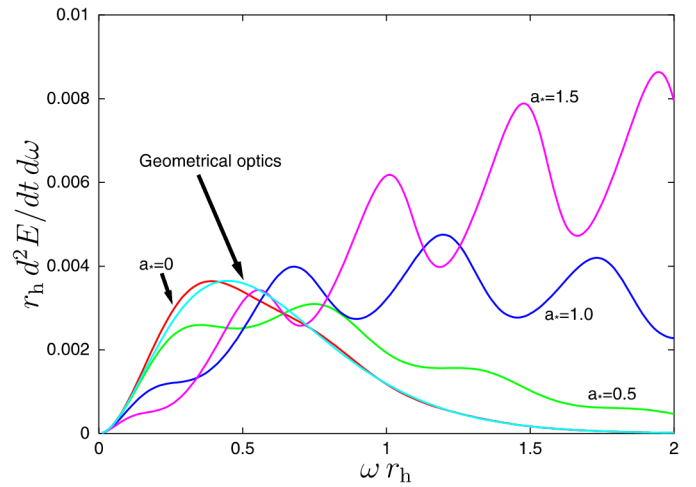


Fig. 2. Power spectra for scalar emission on the brane from rotating black holes, for $n = 1$ and various values of a_* .

Having derived the eigenvalues E_ℓ^m , we can now proceed to integrate Eq. (7) and derive the solution for the radial function $R(r)$. The integration starts at the horizon of the black hole and proceeds towards infinity. Comparing our numerical results with the asymptotic solution at infinity (16), we determine the integration constants B_1 and B_2 . The absorption probability for scalar fields can then be derived from the relation

$$|\mathcal{A}_{\ell,m}|^2 = 1 - |\mathcal{R}_{\ell,m}|^2 = 1 - \left| \frac{B_1}{B_2} \right|^2, \quad (21)$$

where $\mathcal{R}_{\ell,m}$ is the reflection coefficient given by the ratio of the outgoing and ingoing amplitudes at infinity.

The value of the absorption probability $|\mathcal{A}_{\ell,m}|^2$ is then inserted into Eq. (11) to determine the differential energy emission rate per unit time and frequency by the black hole on the brane. This rate is shown in Fig. 2, for the case of a 5-dimensional black hole ($n = 1$). For convenience, we assume that the black hole horizon value remains fixed, and set $r_h = 1$. Fig. 2 shows the emission spectrum on the brane for various values of the angular momentum parameter a_* , up to the value

$a_*^{\max} = 1.5$ defined by the black-hole-creation constraint¹ (5). The dimensionless parameter ωr_h on the horizontal axis adequately covers the low, intermediate and high-energy regimes. The different curves in the figure allow us to compare the energy emission rates for black holes with the same horizon radius and different angular momentum. When r_h is kept fixed, as in this case, the temperature of the black hole decreases sharply with a_* , leading to the observed suppression of the energy emission rate in the low-energy regime—this is in agreement with the behaviour found by analytical methods in the low ω and low a limit [11]. As the energy increases further, however, the absorption probability is significantly enhanced; this enhancement gradually overcomes the decrease in the black hole temperature leading to the observed increase in the energy emission rate, as a_* increases, both in the intermediate and high-energy regimes. This in turn leads to a significant enhancement of the total emissivity of a 5-dimensional rotating black hole (that is, energy emitted per unit time over the whole frequency band) compared to that of a non-rotating black hole of the same dimensionality.

The numerical results produced above allowed the statement made in [3], according to which the emission of Hawking radiation is dominated by modes with $\ell = m$, to be tested. We have found that Fig. 2 looks the same at the $\sim 90\%$ level if only the $\ell = m$ modes are included in the sum of Eq. (11), a result that confirms this statement.

Keeping the black hole horizon value r_h fixed during our analysis was a convenient choice from a calculational point of view. However, as a varies, this leads to the comparison of energy emission rates for black holes with different masses. From a phenomenological point of view, fixing the mass parameter μ of the black hole, instead, makes more sense. In Fig. 3, we present the energy emission spectrum on the brane for a 6-dimensional black hole, i.e., for $n = 2$ and $\mu = 1M_*^{-3}$. The angular momentum parameter now varies from zero to the maximum value—derived from Eqs. (4)–(5)—of $a^{\max} = 1.17M_*^{-1}$, where M_* is the fundamental Planck scale. For fixed μ , the black hole temperature again decreases,² as a increases; however, the increase in this case is only a mild one, which allows for the enhancement of the absorption probability to dominate the spectrum. This leads to a significant enhancement of the energy emission rate on the brane in all energy regimes, and thus to a clear enhancement of the total emissivity of a rotating black

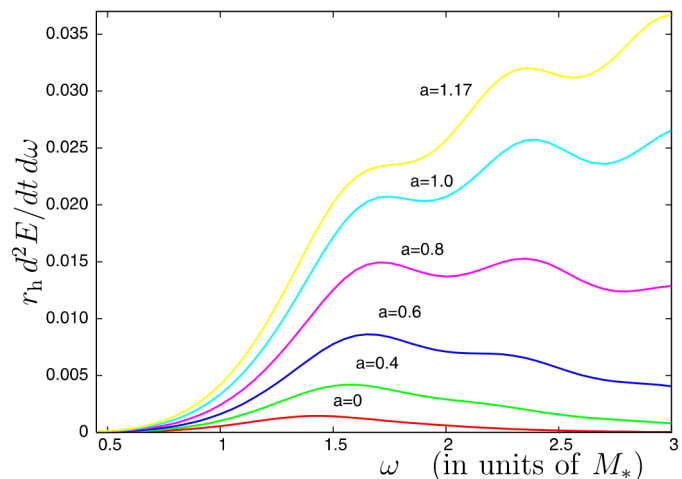


Fig. 3. Power spectra for scalar emission on the brane from rotating black holes, for $n = 2$, $\mu = 1M_*^{-3}$ and various values of a in units of M_*^{-1} .

hole compared to the one of a non-rotating black hole with the same mass.

An interesting effect which takes place during the propagation of a bosonic field in the background of a rotating black hole is super-radiance [19], that is, the amplification of the amplitude of the incident wave. This becomes manifest when the reflection probability becomes larger than unity, or equivalently when the absorption probability becomes negative. In the context of a higher-dimensional model, the only studies in the literature are an analytic approach [10] which confirmed super-radiance for bulk scalars incident on five-dimensional black holes, and a sole numerical result for $n = 6$ and $\ell = m = 1$ [20]. In the context of our numerical analysis, we have investigated this effect for various values of the angular momentum of the black hole and number of extra dimensions. In Fig. 4, we compare the energy amplification due to super-radiant scattering of scalar fields for a 4-dimensional, maximally-rotating black hole with $a_* = 1$ (equivalent to $a = M_{\text{BH}}$), and a 6-dimensional black hole with again $a_* = 1$. The vertical axis is the absorption probability (in fact $-\hat{\mathcal{A}}_{\ell,m}^2$) expressed as a percentage so that it gives the percentage energy amplification of the incident wave. Fig. 4(a) shows excellent agreement with the results produced for $n = 0$ and $a_* = 1$ in [21]. Comparing the vertical axes of Fig. 4(a) and (b), we clearly see that, in the presence of extra dimensions, the peak amplification is more significant. For small values of a_* , it is found that the $\ell = m = 1$ mode provides the greatest amplification as in the 4-dimensional case. However, for larger values of a_* or n , the maximum amplification can occur in modes with larger values of ℓ : for $n = 6$, and $a_* = 4.0$, the maximum energy amplification is found to be around 9% and occurs in the $\ell = m = 7$ mode.

In summary, in this work we have studied the emission of Hawking radiation in the form of scalar fields from a $(4 + n)$ -dimensional, rotating black hole on the brane. We have performed a numerical analysis to solve both the radial and angular parts of the scalar field equation on the brane, and derived exact results for the Hawking radiation energy emission rate. We found that in the case of a 5-dimensional black hole,

¹ In terms of fixed mass parameter μ , the maximum value of a is $0.83\sqrt{\mu}$. Had the angular momentum parameter been allowed to increase indefinitely, the critical value for the existence of the horizon, i.e., $a = \sqrt{\mu}$, would have been reached. For this value, both the black hole horizon and the temperature vanish, the latter leading to the suspension of the emission of Hawking radiation. The non-vanishing value of the area of the black hole induced on the brane, given by $A_{\text{H}}^{(4)} = 4\pi(r_h^2 + a^2)$, as opposed to the vanishing one of the $(4+n)$ -dimensional black hole, given by $A_{\text{H}}^{(4+n)} = \Omega_{2+n}r_h^n(r_h^2 + a^2)$ [with Ω_{2+n} the solid angle of the $(2+n)$ -dimensional space], would lead to a substantial suppression of the emission of energy in the bulk compared to the one on the brane during these last stages, in addition to the suppression found in the non-rotating case [8,18].

² To be exact, the temperature decreases up to the point $a = 1.1M_*^{-1}$, and then starts increasing up to $a^{\max} = 1.17M_*^{-1}$; however, the increase in its value is only 0.5%, which leads to no observable effect.

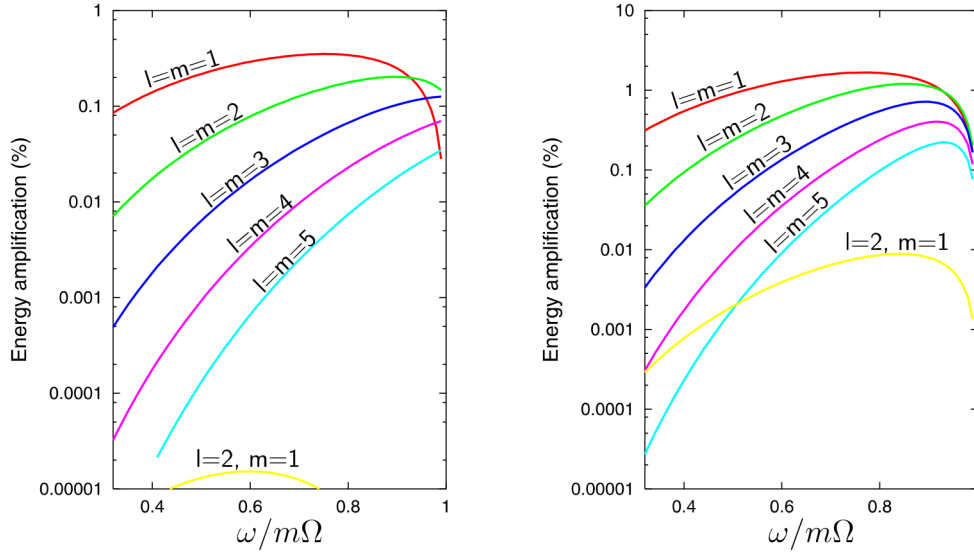


Fig. 4. Left: super-radiant scattering of scalars by a maximally rotating ($a_* = 1$) 4-dimensional black hole; right: super-radiant scattering of scalars by an induced-on-the-brane 6-dimensional black hole with $a_* = 1$.

the emission rate is suppressed in the low-energy regime, as the angular momentum increases—in agreement with previous approximate results—but is significantly enhanced in the intermediate and high-energy regimes, that were until now unexplored. We have extended our analysis to black holes of higher-dimensionality and, as an illustrative example, we have presented the spectrum of a 6-dimensional black hole: in this case, the energy emission rate on the brane is enhanced with the angular momentum over the whole energy band, a behaviour that persists for all higher values of n . We have also investigated the amplification of the incident wave due to the effect of superradiance, and showed that this effect is also significantly enhanced in the presence of extra dimensions.

Acknowledgements

We would like to thank B. Webber and E. Winstanley for useful discussions. During most of this work, C.M.H. was funded by the UK PPARC Research Studentship PPA/S/S/2000/03001. The work of P.K. was funded by the UK PPARC Research Grant PPA/A/S/2002/00350.

References

- [1] N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B 429 (1998) 263.
- [2] T. Banks, W. Fischler, hep-th/9906038.
- [3] S.B. Giddings, S. Thomas, Phys. Rev. D 65 (2002) 056010.
- [4] S. Dimopoulos, G. Landsberg, Phys. Rev. Lett. 87 (2001) 161602.
- [5] S.W. Hawking, Commun. Math. Phys. 43 (1975) 199.
- [6] P. Kanti, J. March-Russell, Phys. Rev. D 66 (2002) 024023; P. Kanti, J. March-Russell, Phys. Rev. D 67 (2003) 104019.
- [7] V.P. Frolov, D. Stojkovic, Phys. Rev. D 66 (2002) 084002; V.P. Frolov, D. Stojkovic, Phys. Rev. Lett. 89 (2002) 151302.
- [8] C.M. Harris, P. Kanti, JHEP 0310 (2003) 014.
- [9] P. Kanti, Int. J. Mod. Phys. A 19 (2004) 4899.
- [10] V.P. Frolov, D. Stojkovic, Phys. Rev. D 67 (2003) 084004.
- [11] D. Ida, K.Y. Oda, S.C. Park, Phys. Rev. D 67 (2003) 064025; D. Ida, K.Y. Oda, S.C. Park, Phys. Rev. D 69 (2004) 049901, Erratum.
- [12] R.C. Myers, M.J. Perry, Ann. Phys. 172 (1986) 304.
- [13] C.M. Harris, hep-ph/0502005.
- [14] M. Casals, P. Kanti, E. Winstanley, in preparation.
- [15] C. Flammer, Spheroidal Wave Functions, Stanford Univ. Press, Stanford, 1957.
- [16] E. Seidel, Class. Quantum Grav. 6 (1989) 1057.
- [17] E. Wasserstrom, J. Comput. Phys. 9 (1972) 53.
- [18] R. Emparan, G.T. Horowitz, R.C. Myers, Phys. Rev. Lett. 85 (2000) 499.
- [19] Y.B. Zel'dovich, JETP Lett. 14 (1971) 180.
- [20] D. Ida, K.Y. Oda, S.C. Park, hep-ph/0501210.
- [21] W. Press, S.A. Teukolsky, Nature 238 (1972) 211.