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OSCILLATION TESTS FOR DELAY EQUATIONS

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ABSTRACT. This paper is concerned with the oscillatory behavior of first-order delay differential equations of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq T, \tag{1}$$

where $p, \tau \in C([T,\infty), \mathbf{R}^+)$, $\mathbf{R}^+ = [0,\infty)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq T$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let the numbers k and L be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\, ds.$$

It is proved that, when $L < 1$ and $0 < k \leq 1/e$, all solutions of Equation (1) oscillate if the condition

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2},$$

where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$, is satisfied.

1. Introduction. Consider the linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq T, \tag{1}$$

where p and τ are continuous functions defined on $[T,\infty)$, $p(t) > 0$, $\tau(t) < t$ for $t \geq T$, $\tau(t)$ is nondecreasing and $\lim_{t\to\infty} \tau(t) = \infty$.

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By a solution of Equation (1) we understand a continuously differentiable function defined on $[\tau(T_1), \infty)$ for some $T_1 \geq T$ and such that (1) is satisfied for $t \geq T_1$. Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

The first systematic study for the oscillation of all solutions of Equation (1) was made by Myshkis. In 1950 [**20**] he proved that every solution of Equation (1) oscillates if

$$\text{(C1)} \qquad \limsup_{t\to\infty}[t-\tau(t)] < \infty, \quad \liminf_{t\to\infty}[t-\tau(t)] \cdot \liminf_{t\to\infty} p(t) > \frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [**16**] proved that the same conclusion holds if

$$\text{(C2)} \qquad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 1.$$

In 1979, Ladas [**15**] and, in 1982, Koplatadze and Chanturija [**11**] improved (C1) to

$$\text{(C3)} \qquad \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e}.$$

Concerning the constant $1/e$ in (C3), it is to be pointed out that, if the inequality

$$\int_{\tau(t)}^{t} p(s)\,ds \leq \frac{1}{e}$$

holds eventually, then, according to a result in [**11**], (1) has a nonoscillatory solution.

In 1982, Ladas, Sficas and Stavroulakis [**17**] and, in 1984, Fukagai and Kusano [**9**] established oscillation criteria (of the type of the conditions (C2) and (C3)) for Equation (1) with oscillating coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C2) and (C3) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

Before the work of Erbe and Zhang [**8**] not much was known about the class of linear delay differential equations for which neither (C2) nor (C3) was satisfied. As far as we know, only the papers [**4, 9, 10**] contained results that could be applied also in some cases that were not covered by the above mentioned results. In 1988, Erbe and Zhang [**8**] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Equation (1). Their result, when formulated in terms of the numbers k and L defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds \quad \text{and} \quad L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds,$$

says that all the solutions of Equation (1) are oscillatory if $0 < k \le 1/e$ and

$$\text{(C4)} \qquad L > 1 - \frac{k^2}{4}.$$

Since then, several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991, Jian Chao [**2**] derived the condition

$$\text{(C5)} \qquad L > 1 - \frac{k^2}{2(1-k)},$$

while, in 1992, Yu, Wang, Zhang and Qian [**21**] obtained the condition

$$\text{(C6)} \qquad L > 1 - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2}.$$

In 1990, Elbert and Stavroulakis [**6**] and, in 1991, Kwong [**14**], using different techniques, improved (C4) in the case where $0 < k \le 1/e$, to the conditions

$$\text{(C7)} \qquad L > 1 - \left(1 - \frac{1}{\sqrt{\lambda_1}}\right)^2$$

and

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C8}$$

respectively, where λ_1 is the smaller root of the equation

$$\lambda = e^{k\lambda}. \tag{2}$$

Following this historical (and chronological) review, we also mention that, in the case where

$$\int_{\tau(t)}^{t} p(s)\,ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds = \frac{1}{e},$$

this problem has been studied in 1993 by Elbert and Stavroulakis [**7**] and in 1995 by Kozakiewicz [**13**], Li [**19**] and by Domshlak and Stavroulakis [**5**].

The purpose of this paper is to combine the methods previously used in [**14**] and [**21**] to show that the conditions (C2) and (C4)–(C8) may be weakened to

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{C9}$$

where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$.

It is to be noted that, as $k \to 0$, then all conditions (C4)–(C8) and also our condition (C9) reduce to the condition (C2). However, the improvement is clear as $k \to 1/e$. For illustrative purposes, we give the values of the lower bound in these conditions when $k = 1/e$: (C2): 1.000000, (C4): 0.966166, (C5): 0.892951, (C6): 0.863457, (C7): 0.845182, (C8): 0.735759, (C9): 0.599216.

We see that our condition (C9) essentially improves all the known results in the literature.

2. Main results. In what follows we will denote by k and L the lower and upper limits of the average $\int_{\tau(t)}^{t} p(s)\,ds$ as $t \to \infty$, respectively, i.e.,

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds$$

and

$$L = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds.$$

Set

$$w(t) = \frac{x(\tau(t))}{x(t)}.$$

We begin with the preliminary analysis of asymptotic behavior of the function $w(t)$ for a possible nonoscillatory solution $x(t)$ of Equation (1) in the case that $k \le 1/e$. For this purpose, assume that (1) has a solution $x(t)$ which is positive for all large t. Dividing first Equation (1) by $x(t)$ and then integrating it from $\tau(t)$ to t leads to the integral equality

$$w(t) = \exp \int_{\tau(t)}^{t} p(s)w(s)\,ds \tag{3}$$

which holds for all sufficiently large t, say for $t \ge T_1$, where both $x(t)$ and $x(\tau(t))$ are positive on $[T_1, \infty)$.

Lemma 1. *Suppose that $k > 0$ and Equation* (1) *has an eventually positive solution $x(t)$. Then $k \le 1/e$ and*

$$\lambda_1 \le \liminf_{t\to\infty} w(t) \le \lambda_2,$$

where λ_1 and λ_2 are the roots of the equation $\lambda = e^{k\lambda}$.

Proof. Let $\alpha = \liminf_{t\to\infty} w(t)$. From (3), we have

$$w(t) = \exp \int_{\tau(t)}^{t} p(s)w(s)\,ds$$

for sufficiently large t. This obviously implies that

$$\alpha \ge \exp k\alpha,$$

which is impossible if $k > 1/e$, since a simple calculus argument shows that, in this case, $\lambda < e^{k\lambda}$ for all λ. This implies that (1) has no eventually positive solution if $k > 1/e$. On the other hand, if $0 < k \le 1/e$, then $\lambda = e^{k\lambda}$ has roots $\lambda_1 \le \lambda_2$, with equality $\lambda_1 = \lambda_2 = e$ if and only if $k = 1/e$, and $\alpha \ge e^{k\alpha}$ if and only if $\lambda_1 \le \alpha \le \lambda_2$. □

The next lemma is taken from [**21**] and it gives an upper bound for the function $w(t)$ as $t \to \infty$.

Lemma 2. *Let* $0 < k \le 1/e$ *and* $x(t)$ *be an eventually positive solution of Equation* (1). *Then*

$$\limsup_{t\to\infty} w(t) \le \frac{2}{1-k-\sqrt{1-2k-k^2}}. \tag{4}$$

Theorem 1. *Let* $0 < k \le 1/e$, *and let* $x(t)$ *be an eventually positive solution of Equation* (1). *Then*

$$L \le \frac{1+\ln\lambda_1}{\lambda_1} - M, \tag{5}$$

where λ_1 *is the smaller root of the equation* $\lambda = e^{k\lambda}$ *and*

$$M = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}. \tag{6}$$

Proof. Let θ be any number in $(1/\lambda_1, 1)$. From Lemma 1 and the definition of M, there is a $T_1 > T$ such that

$$\frac{x(\tau(t))}{x(t)} > \theta\lambda_1, \quad t \ge T_1, \tag{7}$$

and

$$\frac{x(t)}{x(\tau(t))} > \theta M, \quad t \ge T_1. \tag{8}$$

Now let $t \ge T_1$. Since the function $g(s) = x(\tau(t))/x(s)$ is continuous, $g(\tau(t)) = 1 < \theta\lambda_1$ and $g(t) > \theta\lambda_1$, there is a $t^*(t) \in (\tau(t), t)$ such that

$$\frac{x(\tau(t))}{x(t^*(t))} = \theta\lambda_1.$$

Dividing (1) by $x(t)$, integrating from $\tau(t)$ to $t^*(t)$, and taking into account (7) yields

$$\int_{\tau(t)}^{t^*(t)} p(s)\,ds \le -\frac{1}{\theta\lambda_1}\int_{\tau(t)}^{t^*(t)} \frac{x'(s)}{x(s)}\,ds = \frac{\ln(\theta\lambda_1)}{\theta\lambda_1}. \tag{9}$$

Integrating (1) over $[t^*(t), t]$ and using (8) and the fact that $x(\tau(s)) \ge x(\tau(t))$ if $s \le t$ yields

$$\begin{aligned}\int_{t^*(t)}^{t} p(s)\,ds &\le \frac{x(t^*(t))}{x(\tau(t))} - \frac{x(t)}{x(\tau(t))}\\ &= \frac{1}{\theta\lambda_1} - \frac{x(t)}{x(\tau(t))}\\ &\le \frac{1}{\theta\lambda_1} - \theta M.\end{aligned} \tag{10}$$

Adding (10) and (9) yields

$$\int_{\tau(t)}^{t} p(s)\,ds \le \frac{1+\ln(\theta\lambda_1)}{\theta\lambda_1} - \theta M.$$

Letting $t \to \infty$ yields

$$L \le \frac{1+\ln(\theta\lambda_1)}{\theta\lambda_1} - \theta M.$$

Letting $\theta \to 1$ completes the proof. □

This theorem, in view of Lemma 2, implies the following

Corollary 1. *Consider the differential equation* (1) *and assume that when* $L < 1$ *and* $0 < k \le 1/e$ *the following condition holds*

$$L > \frac{\ln\lambda_1 + 1}{\lambda_1} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2} \tag{C9}$$

where λ_1 *is the smaller root of the equation*

$$\lambda = e^{k\lambda}.$$

Then all solutions of Equation (1) *oscillate.*

Example. Consider the delay differential equation

$$(11)\qquad x'(t)+\frac{0.6}{\alpha\pi+\sqrt{2}}(2\alpha+\cos t)x\left(t-\frac{\pi}{2}\right)=0,$$

where $\alpha=(\sqrt{2}(0.6e+1))/(\pi(0.6e-1))$. Then

$$\liminf_{t\to\infty}\int_{t-\pi/2}^{t}0.6(2\alpha+\cos u)/(\alpha\pi+\sqrt{2})\,du=\frac{1}{e}$$

and

$$\limsup_{t\to\infty}\int_{t-\pi/2}^{t}0.6(2\alpha+\cos u)/(\alpha\pi+\sqrt{2})\,du=0.6.$$

Thus, according to Corollary 1, all solutions of Equation (11) are oscillatory. We remark that none of the results mentioned in the introduction can be applied to this equation.

3. Extensions. It is easy to see that the conclusions of Lemmas 1 and 2 remain valid if we replace Equation (1) by the differential inequality

$$(12)\qquad x'(t)+p(t)x(\tau(t))\le 0,\quad t\ge T.$$

It is also clear that if $x(t)$ is a solution of (12) then $-x(t)$ is a solution of the differential inequality

$$(13)\qquad x'(t)+p(t)x(\tau(t))\ge 0.$$

Thus, we conclude the following

Corollary 2. *Assume that the conditions of Corollary* 1 *are satisfied. Then Equation* (12) *has no eventually positive solutions and Equation* (13) *has no eventually negative solutions.*

Our results can be extended to advanced differential equations and inequalities of the form

$$(1')\qquad x'(t)-p(t)(\tau(t))=0,$$

$$(12)'\qquad x'(t)-p(t)x(\tau(t))\ge 0,$$

and

$$(13)' \qquad x'(t) - p(t)x(\tau(t)) \le 0,$$

where $\tau(t) > t$ for $t \ge T$. Since the proofs are very similar we omit them and formulate only the corresponding results.

Corollary 3. *Assume that the conditions of Corollary* 1 *are satisfied with*

$$k = \liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds \quad \textit{and} \quad L = \limsup_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds.$$

Then Equation $(12)'$ *has no eventually positive solutions, Equation* $(13)'$ *has no eventually negative solutions, and Equation* $(1)'$ *has oscillatory solutions only.*

We can also apply our results to equations with positive and negative coefficients of the form, cf. [**21**],

$$x'(t) + p(t)x(t-\tau) - q(t)x(t-\sigma) = 0,$$

where

$$p, q \in C([T,\infty), \mathbf{R}^+) \quad \text{and} \quad \tau, \sigma \in \mathbf{R}^+,$$

to neutral differential equations of the form, cf. [**3**],

$$\frac{d}{dt}[x(t) + p(t)x(t-\tau)] + q(t)x(t-\sigma) = 0,$$

where

$$p \in C([T,\infty), \mathbf{R}), \quad q \in C([T,\infty), \mathbf{R}^+) \quad \text{and} \quad \tau, \sigma \in \mathbf{R}^+,$$

and also to higher-order equations and essentially improve the existing results in the literature.

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