Miskolc Mathematical Notes
Vol. 9 (2008), No. 2, pp. 73–80

HU e-ISSN 1787-2413

# MULTIPLE SOLUTIONS FOR IMPULSIVE HIGHER ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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*Received 17 May, 2006*

*Abstract.* The Leggett-Williams fixed point theorem is applied to obtain triple positive solutions for impulsive $n$th order functional differential equations.

2000 *Mathematics Subject Classification:* 34A37, 34K25

*Keywords:* impulsive functional differential equations, multiple solutions, concave functional

## 1. INTRODUCTION

In this paper, we are concerned with the existence of three nonnegative solutions for initial value problems for $n$th order functional differential equations with impulsive effects

$$y^{(n)}(t) = f(t, y_t), \quad t \in J = [0,T],\ t \neq t_k,\ k = 1,\dots,m, \tag{1.1}$$

$$\Delta y^{(i-1)}|_{t=t_k} = I_k^i(y(t_k^-)), \quad i = 1,\dots,n,\ k = 1,\dots,m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \tag{1.3}$$

$$y^{(i-1)}(0) = \eta_i, \quad i = 2,\dots,n, \tag{1.4}$$

where $f: J \times D \to \mathbb{R}$ is a given function, $D = \{\psi: [-r,0] \to \mathbb{R}_+ \mid \psi$ is continuous everywhere except for a finite number of points $s$ at which $\psi(s)$ and the right limit $\psi(s^+)$ exist and $\psi(s^-) = \psi(s)\}$, $\phi \in D$, $0 < r < \infty$, $\eta_i \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $I_k^i \in C(\mathbb{R}, \mathbb{R}_+)$ $(i = 1,\dots,n,\ k = 1,2,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $\mathbb{R}$ is a real Banach space with the norm $|\cdot|$.

For any function $y$ defined on $[-r,T]$ and any $t \in J$, we denote by $y_t$ the function defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0].$$

Impulsive differential equations can be used to model and describe real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. That is why, in recent years, they have become an object of investigation. We refer to the monographs of Bainov and Simeonov [5], Benchohra, Henderson and Ntouyas [6], Lakshmikantham et al [14], and Samoilenko and Perestyuk [16] where numerous properties of their solutions are studied, and a detailed bibliography is given.

Recently extensions to functional differential equations with impulsive effects have been done by Benchohra et al. [7], Dong [9], and Franco et al. [11].

The existence of multiple solutions for boundary value problems for impulsive differential equations was studied by Agarwal and O'Regan [1], Erbe et al. [10], and Guo and Liu [13]. Notice that when the impulses are absent (i. e., $I_k = 0$, $k = 1,\dots,m$) the existence of three solutions and multiple solutions for ordinary differential equations was studied in [2–4].

The main results here make use of the Leggett–Williams fixed point theorem [15] in obtaining multiple positive solutions for the problem (1.1)–(1.4). This paper can be viewed as a generalization of the recent paper [8].

## 2. Preliminaries

In this section, we provide notations, definitions, and preliminary facts which are used throughout the paper.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y\colon J \to \mathbb{R}$ which are Lebesgue integrable, with

$$\|y\|_{L^1} = \int_0^T |y(t)|dt.$$

Let $(E, \|\cdot\|)$ be a Banach space and $C$ be a cone in $E$. By a concave nonnegative continuous functional $\psi$ on $C$ we mean a continuous mapping $\psi\colon C \to [0,\infty]$ with

$$\psi(\lambda x + (1-\lambda)y) \geq \lambda\psi(x) + (1-\lambda)\psi(y) \quad \text{for all } x, y \in C \text{ and } \lambda \in [0,1].$$

For $k = 0,\dots,m$, let $J_k = (t_k, t_{k+1}]$ and for $y\colon[-r,T] \to \mathbb{R}$, define $y_k$ to be the restriction of $y$ to $J_k$. In order to define the solution of (1.1)–(1.4), we shall consider the space

$$\Omega = \big\{y\colon[-r,T] \to \mathbb{R} \mid y_k \in C(J_k,\mathbb{R}),\ k = 0,\dots,m, \text{ and there exists } y(t_k^+),\ k = 1,\dots,m\big\},$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k} : k = 0,\dots,m\}.$$

**Definition 1.** A map $f\colon J \times D \to \mathbb{R}$ is said to be $L^1$-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in D$;
(ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \le h_q(t) \quad \text{for all } \|u\| \le q \text{ and for almost all } t \in J.$$

Our consideration is based on the following fixed point theorem by Leggett and Williams [15] (see also Guo and Lakshmikantham [12]).

**Theorem 2.1.** *Let $E$ be a Banach space, $C \subset E$ a cone of $E$ and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$. Suppose a concave nonnegative continuous functional $\psi$ exists on the cone $C$ with $\psi(y) \le \|y\|$ for $y \in \overline{C}_R$, and let $N : \overline{C}_R \to \overline{C}_R$ be a completely continuous operator. Assume there are numbers $\varrho$, $L$ and $K$ with $0 < \varrho < L < K \le R$ such that*

(A1) *$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \varnothing$ and $\psi(N(y)) > L$ for all $y \in C(\psi, L, K)$;*
(A2) *$\|N(y)\| < \varrho$ for all $y \in \overline{C}_\varrho$;*
(A3) *$\psi(N(y)) > L$ for all $y \in C(\psi, L, R)$ with $\|N(y)\| > K$, where $C(\psi, L, K) = \{y \in C : \psi(y) \ge L$ and $\|y\| \le K\}$.*

*Then $N$ has at least three fixed points $y_1$, $y_2$, $y_3$ in $\overline{C}_R$. Furthermore, we have*

$$y_1 \in C_\varrho, \quad y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$

*and*

$$y_3 \in \overline{C}_R \setminus \{C(\psi, L, R) \cup \overline{C}_\varrho\}.$$

## 3. Triple solutions of impulsive functional differential equations

In what follows we will assume that $f$ is an $L^1$-Carathéodory function. Let us start by defining what we mean by a solution of problem (1.1)–(1.4).

**Definition 2.** A function $y \in \Omega$ such that $y \in \Omega \Cap AC^{(n-1)}((t_k, t_{k+1}), \mathbb{R})$, for $k = 0, \dots, m$, is said to be a solution of (1.1)–(1.4) if $y$ satisfies the equation $y^{(n-1)}(t) = f(t, y_t)$ a. e. on $J \setminus \{t_1, \dots, t_m\}$, the conditions $\Delta y^{(i-1)}|_{t=t_k} = I_k^i(y(t_k^-))$, $i = 1, \dots, n$, $k = 1, \dots, m$, $y(t) = \phi(t)$, $t \in [-r, 0]$, and $y^{(i-1)}(0) = \eta_i$, $i = 2, \dots, n$.

Here, $AC^{(n-1)}((t_k, t_{k+1}), \mathbb{R}) = \{y : (t_k, t_{k+1}) \to \mathbb{R} \mid y^{(i)}$ is absolutely continuous, $i = 0, \dots, n-1\}$.

In our application of Theorem 2.1 to obtain triple positive solutions of (1.1)–(1.4), we will draw upon the following list of hypotheses.

(H1) There exist constants $c_k^i$, $i = 1, \dots, n$, $k = 1, \dots, m$, such that

$$I_k^i(y) \le c_k^i, \quad i = 1, \dots, n, \; k = 1, \dots, m, \text{ for each } y \in \mathbb{R};$$

(H2) There exists a function $h \in L^1(J, \mathbb{R}_+)$ and $\varrho > 0$ and $0 < M < 1$ such that

$$f(t,u) \le M h(t) \quad \text{for a. e. } t \in J \text{ and each } u \in D$$

and

$$\|\phi\| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + M \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)ds + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i < \varrho;$$

(H3) There exist $L > \varrho$ and $M \leq M_1 < 1$ and an interval $[a,b] \subset (0,T)$ such that, for each $y \in \Omega$ with $\min_{t\in[a,b]} y(t) \geq L$,

$$\begin{aligned} &\min_{t\in[a,b]} \Big( \phi(0) + \sum_{i=2}^{n} \eta_i \frac{t^{i-1}}{(i-1)!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,y_s)ds \\ &\qquad + \sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \Big) \\ &\geq M_1 \Big( \phi(0) + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s,y_s)ds \\ &\qquad + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) \Big) \\ &> L; \end{aligned}$$

(H4) There exist $R > L$ and $M_1 \leq M_2 < 1$ such that

$$\|\phi\| + \sum_{i=2}^{n} \eta_i \frac{T^{i-1}}{(i-1)!} + M_2 \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)ds + \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i \leq R.$$

**Theorem 3.1.** *Suppose that hypotheses (H1)–(H4) are satisfied. Then problem (1.1)–(1.4) has three nonnegative solutions $y_1$, $y_2$, and $y_3$ with*

$$\|y_1\|_\Omega < \varrho, \quad y_2(t) > L \quad \text{for } t \in [0,T],$$

*and*

$$\|y_3\|_\Omega > \varrho \quad \text{with} \quad \min_{t\in[a,b]} y_3(t) < L.$$

*Proof.* In making application of Theorem 2.1, we naturally transform problem (1.1)–(1.4) into a fixed point problem. Define an operator $N\colon \Omega \to \Omega$ by

$$N(y)(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0], \\ \phi(0) + \displaystyle\sum_{i=2}^{n} \eta_i \frac{t^{i-1}}{(i-1)!} + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,y_s)\,ds & \\ \qquad + \displaystyle\sum_{0<t_k<t} \sum_{i=1}^{n} \frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-)) & \text{if } t \in [0,T]. \end{cases}$$

It is fairly standard that $N$ is completely continuous. Rather than repeat the usual complete continuity arguments, we refer the reader to [7, 8] for details. We content ourselves in verifying that the other hypotheses of Theorem 2.1 are satisfied. Let

$$C = \{y \in \Omega : y(t) \geq 0 \text{ for } t \in [-r,T]\}$$

be a cone in $\Omega$. Since $f$ and $I_k^i$ are all positive functions, it is immediate that $N(C) \subset C$, and moreover, $N|_{\overline{C}_R}$ is completely continuous. In fact, if $y \in \overline{C}_R$, then by (H4), $\|N(y)\|_\Omega \leq R$. In particular, $N\colon \overline{C}_R \to \overline{C}_R$ is completely continuous.

Next, let $\psi\colon C \to [0,\infty)$ be defined by

$$\psi(y) = \min_{t\in[a,b]} y(t).$$

Then, it is clear that $\psi$ is a nonnegative concave continuous functional and

$$\psi(y) \leq \|y\|_\Omega \quad \text{for } y \in \overline{C}_R.$$

Now it remains to show that the hypotheses (A1) and (A2) of Theorem 2.1 are satisfied. First, for $y \in \overline{C}_\varrho$, we have from (H1) and (H2),

$$\begin{aligned} |N(y)(t)| \leq& |\phi(0)| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{(T-s)^{n-1}}{(n-1)!} f(s,y_s)\,ds \\ &+ \sum_{0<t_k<t} \sum_{i=1}^{n} + \frac{(T-t_k)^{i-1}}{(i-1)!} I_k(y(t_k^-)) \\ \leq& \|\phi\| + \sum_{i=2}^{n} |\eta_i| \frac{T^{i-1}}{(i-1)!} + M\int_0^T \frac{(T-s)^{n-1}}{(n-1)!} h(s)\,ds \\ &+ \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(T-t_k)^{i-1}}{(i-1)!} c_k^i \\ <& \varrho. \end{aligned}$$

As a consequence, (A2) of Theorem 2.1 holds. Next, let $K$ be such that $\frac{L}{M} \leq K \leq R$ and let $z(t) = \frac{1}{2}(L+K)$ for $t \in [-r,T]$. By the definition of $C(\psi,L,K)$, the function

$z$ belongs to $C(\psi, L, K)$. Moreover, $\psi(z) = \frac{1}{2}(L+K) > L$, so that

$$\{y \in C(\psi, L, K) : \psi(y) > L\} \neq \varnothing.$$

So, if we choose $y \in C(\psi, L, K)$, then from (H3), we have

$$\begin{aligned}
\psi(N(y)) = \min_{t\in[a,b]}\Bigg(&\phi(0) + \sum_{i=2}^{n}\eta_i \frac{t^{i-1}}{(i-1)!} + \sum_{0<t_k<t}\sum_{i=1}^{n}\frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k)) \\
&+ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s)ds\Bigg) \\
\geq M_1\Bigg(&\phi(0) + \sum_{i=2}^{n}\eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s, y_s)ds \\
&+ \sum_{k=1}^{m}\sum_{i=1}^{n}\frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-))\Bigg) \\
> L.&
\end{aligned}$$

As a consequence, (A1) of Theorem 2.1 is fulfilled.

Finally, to see that (A3) of Theorem 2.1 holds, let $y \in C(\psi, L, R)$ with $\|N(y)\|_\Omega > K$. Again from (H3), we have

$$\begin{aligned}
\psi(N(y)) = \min_{t\in[a,b]}\Bigg(&\phi(0) + \sum_{i=2}^{n}\eta_i \frac{t^{i-1}}{(i-1)!} + \sum_{0<t_k<t}\sum_{i=1}^{n}\frac{(t-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k)\Bigg) \\
&+ \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s, y_s)ds) \\
\geq M_1\Bigg(&\phi(0) + \sum_{i=2}^{n}\eta_i \frac{T^{i-1}}{(i-1)!} + \int_0^T \frac{T^{n-1}}{(n-1)!} f(s, y_s)ds \\
&+ \sum_{k=1}^{m}\sum_{i=1}^{n}\frac{(T-t_k)^{i-1}}{(i-1)!} I_k^i(y(t_k^-))\Bigg) \\
\geq M_1&||N(y)||_\Omega > M_1 K \geq L.
\end{aligned}$$

Thus, Theorem 2.1 implies that $N$ has at least three fixed points $y_1$, $y_2$, and $y_3$ which are positive solutions of problem (1.1)–(1.4). Furthermore, we have $y_1 \in C_\varrho$,

$$y_2 \in \{y \in C(\psi, L, R) : \psi(y) > L\},$$

and $y_3 \in C_R \setminus (C(\psi, L, R) \cup C_\varrho)$.The proof is complete. □

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