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A NOTE ON A NONLINEAR *m*-POINT BOUNDARY VALUE PROBLEM FOR *p*-LAPLACIAN DIFFERENTIAL INCLUSIONS

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ABSTRACT. In this note a selection theorem due to Bressan and Colombo for lower semi-continuous multi-valued operators with nonempty closed decomposable values combined with Schaefer's fixed point theorem is used to investigate the existence of positive solutions for *m*-point boundary value problems for one dimensional *p*-Laplacian differential inclusions.

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1. INTRODUCTION

This note is concerned with the existence of positive solutions for the following class of boundary value problems for *m*-point one dimensional *p*-Laplacian differential inclusions

$$(\varphi(y'))' \in F(t,y), \qquad \text{a. e. } t \in J := [0,1]; \tag{1.1}$$

$$y'(0) = \sum_{i=1}^{m-2} b_i y'(\xi_i), \qquad y(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i), \tag{1.2}$$

where $\varphi : \mathbb{R}_+^* \to \mathbb{R}_+$ defined by $\varphi(v) := |v|^{p-2}v$, $p > 1$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R}_+)$ is a multi-valued map, $\mathcal{P}(\mathbb{R}_+)$ is the family of all subsets of $\mathbb{R}_+$, $\xi_i \in (0,1)$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < 1$, and a_i, b_i, $i = 1, \ldots, m-2$, are positive and satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, $\sum_{i=1}^{m-2} b_i < 1$. The study of multi-point boundary value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [15,16]. Then Gupta [10, 11] studied three-point boundary value problems for nonlinear ordinary differential equations, the m-point boundary value problem was studied by Gupta *et al*, [12, 14], Ma [18]. Very recently, in a series of papers by Benchohra and Ntouyas (see [2–5]) some extensions to multi-point differential inclusions have been proposed with the aid of fixed point arguments in the cases when the right-hand side is convex as well as nonconvex valued. Some existence results for one dimensional p-Laplacian differential equations are given in the papers by Bai and Fang [1] and Jian and Guo [17]. Our goal here is to give the existence of at least one positive solution for m-point boundary value problems for one dimensional p-Laplacian differential inclusions. Our approach relies on Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo [6] for lower semi-continuous operators with nonempty closed and decomposable values.

2. Preliminaries

In this Section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper.
$C([0,1],\mathbb{R})$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : 0 \le t \le 1\}.$$

$AC([0,1],\mathbb{R})$ is the space of all absolutely continuous functions y from $[0,1]$ into $\mathbb{R}$. $L^1(J,\mathbb{R})$ denotes the Banach space of functions $y : J \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^1 |y(t)|dt.$$

Let $\mathcal{A}$ be a subset of $[0,1]\times\mathbb{R}$. $\mathcal{A}$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{N}\times D$ where $\mathcal{N}$ is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $\mathbb{R}$. A subset $\mathcal{I}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in \mathcal{I}$ and $\mathcal{N} \subset [0,1]$ measurable the function $u\chi_{\mathcal{N}} + v\chi_{[0,1]-\mathcal{N}} \in \mathcal{I}$, where $\chi_{[0,1]}$ stands for the characteristic function of $[0,1]$.

Let E be a Banach space, X a nonempty closed subset of E and $G : X \to \mathcal{P}(E)$ a multi-valued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multi-valued maps we refer to the books by Deimling [7], Górniewicz [9] and Hu and Papageorgiou [19].

Definition 1. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,b],\mathbb{R}))$ be a multi-valued operator. We say N has the property (BC) if

(1) N is lower semi-continuous (l.s.c.);
(2) N has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator

$$\mathcal{F} : C([0,1],\mathbb{R}^+) \to \mathcal{P}(L^1([0,1],\mathbb{R}^+))$$

by letting

$$\mathcal{F}(y) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, y(t)) \text{ for a. e. } t \in [0,1]\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2. Let $F : J \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ be a multi-valued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 1 ([6]). *Assume that Y is a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multi-valued operator which has the property (BC). Then N has a continuous selection, i.e. there exists a continuous function (single-valued) $g : Y \to L^1(J,\mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Let us introduce the following hypotheses which are assumed hereafter:

(H1) $F : [0,1] \times \mathbb{R}^+ \to \mathcal{P}(\mathbb{R}^+)$ is a nonempty compact valued multi-valued map such that:
  (a) $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
  (b) $y \mapsto F(t,y)$ is lower semi-continuous for a. e. $t \in [0,1]$;

(H2) There exists a function $h \in L^1([0,1],\mathbb{R}_+)$ such that

$$\|F(t,y)\| := \sup\{|v| : v \in F(t,y)\} \le h(t) \text{ for a. e. } t \in [0,1] \text{ and for } y \in \mathbb{R}.$$

3. Main result

Let us start by defining what we mean by a solution of problem (1.1)–(1.2).

Definition 3. A function $y \in C^1((0,1),\mathbb{R})$ with $\varphi(y') \in AC((0,1),\mathbb{R})$ is said to be a solution of (1.1), (1.2) if there exists $v(t) \in L^1(J,\mathbb{R})$ such that y satisfies the equation $(\varphi(y'))' = v(t)$ a. e. on J and the condition (1.2).

Theorem 2. *Suppose that hypotheses (H1), (H2) are satisfied. Then the m-point BVP* (1.1), (1.2) *has at least one positive solution.*

Proof. (H1) and (H2) imply by Lemma 2.2 in Frigon [8] that F is of the lower semi-continuous type. Then from Theorem 1 there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C([0,1],\mathbb{R})$. Consider the following problem

$$(\varphi(y'))' = f(y(t)), \quad \text{a. e. } t \in J, \tag{3.1}$$

$$y'(0) = \sum_{i=1}^{m-2} b_i y'(\xi_i), \qquad y(1) = \sum_{i=1}^{m-2} a_i y'(\xi_i). \tag{3.2}$$

Clearly, if y is a solution of problem (3.1), (3.2), then y is a solution to problem (1.1), (1.2).

Transform the problem (3.1), (3.2) into a fixed point problem. Consider the operator $N : C([0,1],\mathbb{R}^+) \to C([0,1],\mathbb{R})$ defined by:

$$\begin{aligned} N(y)(t) = &-\int_0^t \psi\left(\int_0^s f(y(\tau))d\tau\right)ds - tB\sum_{i=1}^{m-2} b_i\psi\left(\int_0^{\xi_i} f(y(\tau))d\tau\right) \\ &+ A\left\{\int_0^1 \psi\left(\int_0^s f(y(\tau))d\tau\right)ds - \sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\psi\left(\int_0^s f(y(\tau))d\tau\right)ds \right.\\ &\left. + B\sum_{i=1}^{m-2} b_i\psi\left(\int_0^{\xi_i} f(y(\tau))d\tau\right)\left(1-\sum_{i=1}^{m-2} a_i\xi_i\right)\right\} \end{aligned}$$

where ψ is the inverse of the function φ defined by $\psi(w) := |w|^{q-2}w$, with $q = \frac{p}{p-1} > 1$ and

$$A = \left(1-\sum_{i=1}^{m-2} a_i\right)^{-1}, \qquad B = \left(1-\sum_{i=1}^{m-2} b_i\right)^{-1}.$$

The fixed points of the operator N are solutions to problem (3.1), (3.2) (see [1]). It is clear that $N(y)(t) \geq 0$ on J for any $y \in C([0,1],\mathbb{R}^+)$. We shall first show that N is completely continuous. The proof will be given in three steps.

Step 1: *N is continuous.* Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,1],\mathbb{R})$. Set

$$L(y)(t) := \int_0^t |f(y(s))|ds.$$

Then

$$|L(y_n)(s) - L(y(s))| \leq \int_0^t |f(y_n(s)) - f(y(s))|ds \leq \int_0^1 |f(y_n(s)) - f(y(s))|ds.$$

Since f is a continuous function, it follows that

$$\|L(y_n) - L(y)\|_\infty \leq \|f(y_n(.)) - f(y(.))\|_{L^1} \to 0 \quad \text{as } n \to \infty.$$

Then

$$
\begin{aligned}
|N(y_n(t)) - N(y(t))| \leq & \int_0^1 |\psi(L(y_n(t)) - \psi(L(y(t)))|\, ds \\
& + tB \sum_{i=1}^{m-2} b_i\, |\psi(L(y_n(\xi_i))) - \psi(L(y_n(\xi_i)))| \\
& + A \int_0^1 |\psi(L(y_n(s))) - \psi(L(y_n(s)))|\, ds \\
& + A \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} |\psi(L(y_n(s))) - \psi(L(y(s)))|\, ds \\
& + AB \sum_{i=1}^{m-2} b_i\, |\psi(L(y_n(\xi_i)) - \psi(L(y(\xi_i))| \left(1 - \sum_{i=1}^{m-2} a_i \xi_i\right).
\end{aligned}
$$

Since ψ is a continuous function, then

$$
\begin{aligned}
\|N(y_n) - N(y)\|_\infty \leq & \|\psi(L(y_n)) - \psi(L(y))\|_\infty + B \sum_{i=1}^{m-2} b_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \\
& + A\|\psi(L(y_n)) - \psi(L(y))\|_\infty + A \sum_{i=1}^{m-2} a_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \\
& + AB \sum_{i=1}^{m-2} b_i \|\psi(L(y_n)) - \psi(L(y))\|_\infty \left(1 - \sum_{i=1}^{m-2} a_i \xi_i\right) \ as\ n \to \infty.
\end{aligned}
$$

Step 2: *N maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$.* Indeed, it is enough to show that, for any $q > 0$, there exists a positive constant ℓ such that, for each $y \in B_q = \{y \in C([0, 1], \mathbb{R}) : \|y\|_\infty \leq q\}$, we have $\|N(y)\|_\infty \leq \ell$. From $(H2)$, we have

$$
\left| \int_0^1 f(y(s)) ds \right| \leq \|h\|_{L^1} := q^*,
$$

and

$$|N(y)(t)| \leq \int_0^1 |\psi(L(y(s))|ds + tB\sum_{i=1}^{m-2} b_i|\psi(L(y(\xi_i)))|$$
$$+ A\int_0^1 |\psi(L(y(s)))|ds + A\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} |\psi(L(y(s)))|ds$$
$$+ AB\sum_{i=1}^{m-2} b_i|\psi(L(y(\xi_i)))|\left(1 - \sum_{i=1}^{m-2} a_i\xi_i\right).$$

Then

$$||N(y)||_\infty \leq \sup_{w\in[-q^*,q*]} |\psi(w)| + B\sum_{i=1}^{m-2} b_i \sup_{w\in[-q^*,q^*]} |\psi(w)|$$
$$+ A \sup_{w\in[-q^*,q^*]} |\psi(w)| + A\sum_{i=1}^{m-2} a_i \sup_{w\in[-q^*,q^*]} |\psi(w)|$$
$$+ AB\sum_{i=1}^{m-2} b_i \sup_{w\in[-q^*,q^*]} |\psi(w)|\left(1 - \sum_{i=1}^{m-2} a_i\xi_i\right) := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $l_1, l_2 \in [0,1]$, $l_1 < l_2$ and B_q be a bounded set of $C([0,1],\mathbb{R})$ as in Step 2. Let $y \in B_q$, then

$$|N(y)(l_2) - N(y)(l_1)| \leq (l_2 - l_1) \sup_{w\in[-q^*,q^*]} |\psi(w)|$$
$$+ B(l_2 - l_1)\sum_{i=1}^{m-2} b_i \left|\psi\left(\int_0^{\xi_i} f(y(\tau))d\tau\right)\right|.$$

As $l_2 \to l_1$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Step 4: *The set*

$$\mathcal{E}(N) := \{y \in C([0,1],\mathbb{R}) : y = \lambda N(y), \ \ \textit{for some} \ \ 0 < \lambda < 1\}$$

is bounded.

The reasoning used in the proof of Step 2 shows that the set $\mathcal{E}(N)$ is bounded.

Set $X := C([0,1],\mathbb{R})$. As a consequence of Schaefer's fixed point theorem [20, p. 29] we deduce that N has a fixed point y which is a solution to problem (3.1), (3.2), and hence, a solution to problem (1.1), (1.2).

□

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