Forbidden states and the three-body bound state collapse

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The appearance of bound states with large binding energies of several hundred MeV in the three-body system, known as bound state collapse, is investigated. For this purpose three classes of two-body potentials are employed; local potentials equivalent to nonlocal interactions possessing a continuum bound state, in addition to the usual negative-energy bound state; local potentials with a strong attractive well sustaining a forbidden state; and supersymmetric transformation potentials. It is first shown that local potentials equivalent to the above nonlocal ones have a strong attractive well in the interior region which supports, in addition to the physical deuteron state, a second bound state (usually called a pseudobound state) with a large binding energy, which is responsible for the bound state collapse in the three-body (and in general to the *N*-body) system. Second, it is shown that local potentials with a forbidden state also generate a three-body bound state collapse, implying that the role played by the forbidden state is similar to the one played by the pseudobound state. Finally, it is shown that the removal of the forbidden state via supersymmetric transformations also results in the disappearance of the collapse. Thus one can safely argue that the presence of unphysical bound states with large binding energies in the two-body system is responsible for the bound state collapse in the three-body system.

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I. INTRODUCTION

Three-body bound state collapse (BSC) [1–5], i.e., the appearance of bound states with large binding energies in the three-body system, has been the subject of several studies in the past. It was found, long ago, that the rank-1 nonlocal separable potential of Tabakin [1] generates a large binding energy for the three-body system [2]. This came as a surprise, as this potential predicts the two-body data, fairly well and, there was no apparent reason why such an unphysical bound state with large binding energy should appear in the three-body system.

Subsequent studies with rank-2 separable potentials [3,4] showed that BSC could be related to two-body continuum bound states (CBS's), i.e., to the existence of an *S*-matrix pole on the real positive-energy axis. This was the case for separable potentials, while for purely local potentials or superpositions of local and nonlocal potentials the collapse was not observed. Some aspects of BSC for the Tabakin potential were also studied by Rupp *et al.* [5]. In that work a rank-1 potential, similar to the Tabakin potential, was constructed and used to calculate the three-body binding energy as a function of the nonlocal parameter β, i.e., the inverse of the nonlocality range of the potential. Several radial *S*-wave functions for different values of β were constructed, and compared with the deuteron wave function of the Graz-II potential [6]. This comparison revealed that, in contrast to the deuteron wave function of the Graz-II potential, these bound state wave functions have a node at short distances which moves outward with increasing β, while the corresponding three-body binding energies increase eventually leading to a collapse of the three-body system. This was an indication that BSC is related to the range of the nonlocality, at least for this kind of potential.

Almost a decade later, Delfino *et al.* [7,8] were able to show that for rank-1 separable potentials with a one-term form factor of Yamaguchi type, the BSC could essentially be linked to the Thomas effect [9], i.e., to a drastic increase of the three-particle binding energy as the range of the two-particle potential tends to zero. Thus these authors were able to establish an equivalence between the Thomas effect and the phenomenon of collapse by means of the range of the potential. In contrast, when the form factor is a sum of Yamaguchi terms and the nonlocality parameter could not be taken as a measure of the range of the potential, Delfino *et al.* [8] noted that, if this type of potentials supports a CBS, in addition to a physical dueteron state, the CBS wave function is identical to the wave function of a negative-energy bound state, the so called pseudobound state, which is responsible for the collapse. An important suggestion of that work was that one should expect similar results for all potentials which support at least another bound state in addition to the physical deuteron state. In the resonating group model, this bound state is usually called a Pauli-forbidden state (PFS) [10].

In the presence of a PFS the physical deuteron state becomes an excited bound state and its wave function has a node. The relation between that node and the three-body binding energy was investigated by Nakaichi-Maeda [11] who employed the Kukulin nucleon-nucleon (*NN*) potential [12]. This interaction, in addition to the physical deuteron state, also sustains a PFS which, for the triplet channel, is

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~58 MeV, for the singlet is ~440 MeV, and for the couple channel system is even deeper, ~525 MeV [13]. Thus, for this potential, the physical deuteron state is an excited bound state, and the corresponding wave function has a node. By varying the magnitude of the inner amplitude of the wave function and the position of the node, Nakaichi-Maeda confirmed that the BSC is connected to the nodal behavior of the deuteron wave function.

The purpose of this paper is threefold. First, we take the findings of Delfino *et al.* [7,8] further, and show that they can be generalized to any kind of two-particle potential which has a bound state with large binding energy, in addition to the physical deuteron state. By looking at the nodal behavior of the wave functions we will also confirm, in a more rigorous way, the results of Nakaichi-Maeda [11]. Second, by constructing a local potential sustaining at least one forbidden state, we will show that it can cause a BSC in a three-body system. Finally, we will remove the forbidden state via supersymmetric (SUSY) transformations [14,15], and show that this results in the disappearance of the BSC.

The paper is organized as follows: In Sec. II, we present a short description of nonlocal potentials of rank 1 sustaining a CBS, we construct their phase-equivalent local interactions, and use them to obtain the trinucleon binding energy. The relevance of forbidden states of purely local interactions to the BSC are discussed in Sec. III. In Sec. IV, we briefly present the supersymmetric transformations used to remove the forbidden states. Finally, in Sec. V, we discuss our results and draw our conclusions.

II. NONLOCAL INTERACTIONS

A. Short review

For convenience, let us briefly recall the relevant formulas describing a two-particle system in an S state. In our investigations we shall use the nonlocal potentials of Table I of Ref. [8] which are rank-1 separable interactions,

$$V(p,q)=\lambda g(p)g(q), \tag{1}$$

with a form factor consisting of a sum of Yamaguchi terms:

$$g(k)=\frac{\alpha_1}{k^2+\beta_1^2}+\frac{\alpha_2}{k^2+\beta_2^2}. \tag{2}$$

The corresponding two particle t matrix, at a given energy $E=k^2$ ($\hbar^2/2\mu=1$), is given by

$$t(p,q;k^2)=g(p)\frac{\lambda}{D(k^2)}g(q), \tag{3}$$

with

$$D(k^2)=1-\frac{2\lambda}{\pi}\int_0^\infty\frac{g^2(p)p^2dp}{k^2-p^2+i\epsilon}. \tag{4}$$

If the system has a bound state at $E_b=-\gamma^2$ then, from Eq. (4), we obtain

TABLE I. The parameters for the nonlocal potentials [Eqs. (1) and (2)], together with the collapse momentum p_c and the three-body binding energy B_3 for $\alpha_2=-1.0$ and $\beta_1=1.4$ (fm^{-1}).

Pot.	α_1	β_2 (fm^{-1})	λ (fm^{-3})	p_c (fm^{-1})	B_3 (MeV)
1	0.05	8.47	2491.4	1.307	902.3
2	0.1	5.73	1096.1	1.213	644.8
3	0.13	4.92	831.3	1.167	578.5
4	0.15	4.53	729.3	1.149	539.4
5	0.2	3.82	573.9	1.097	486.5

$$D(\gamma^2)=1+\lambda\sum_{i,j=1}^{2}\frac{\alpha_i\alpha_j}{(\gamma+\beta_i)(\gamma+\beta_j)(\beta_i+\beta_j)}. \tag{5}$$

The corresponding bound state wave function is given by

$$\Phi_b(p)=-N\frac{g(p)}{(\gamma^2+p^2)}, \tag{6}$$

where N is the normalization constant.

The potentials of Table I support a CBS at a positive energy $E_c=p_c^2$ if

$$g(p_c)=0 \quad \text{and} \quad D(p_c^2)=0. \tag{7}$$

From Eqs. (2) and (4) and conditions (7), for a CBS wave function one obtains

$$\Phi_c(p)=-N\frac{(\alpha_1+\alpha_2)}{(p^2+\beta_1^2)(p^2+\beta_2^2)}. \tag{8}$$

Two important aspects concerning this wave function were already noted in Ref. [8], namely, that it does not depend directly on the energy p_c^2, and that at large distances it decays exponentially as $\exp(-\beta_i r)$ and not as $\exp(-p_c r)$. It was further noted that it reduces to a bound state [Eq. (6)] if one identifies the binding energy γ^2 with β_i^2 and the form factor with $(p^2+\beta_j^2)^{-1}$, where $\beta_i(\beta_j)$ is the smaller (larger) of β_1 and β_2. In other words, the CBS wave function has the behavior of a normal negative-energy bound state wave function, and this pseudobound state is responsible for the appearance of an extra bound state in the three-body system.

It is noted that for this type of nonlocal potential the parameter β_i cannot be taken as a measure of the range of the potential. However, the range of the nonlocality can be deduced by constructing equivalent local interactions in coordinate space, which we shall discuss next.

B. Equivalent local interactions

There are many ways to construct local interactions equivalent to nonlocal ones. A particular localization method which is well suited for our investigation is the one based on two linear independent solutions of the Schrödinger equation. The method was outlined in Refs. [16–18] and we refer to these works for more detail. This type of equivalent local potential (ELP) (sometimes called the quantal or Wronskian

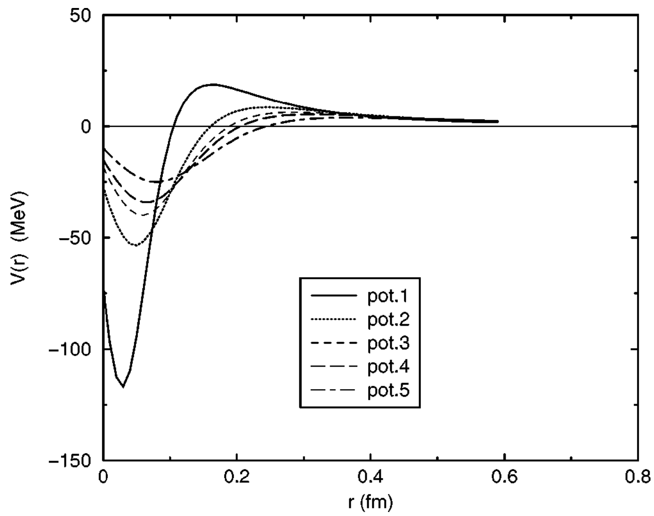

FIG. 1. Local potentials equivalent to the two-body nonlocal interactions of Table I for $k=2.0$ fm^{-1}. The strong, short-ranged, attractive well of potential 1 generates BSC in the three-body system.

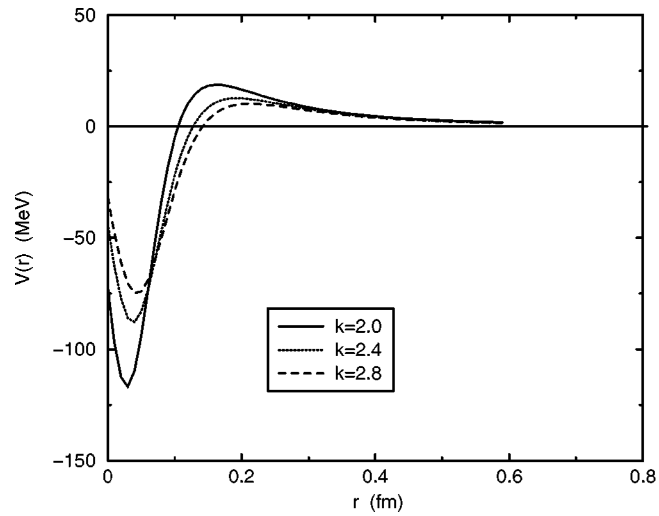

FIG. 2. Local potentials equivalent to potential 1 of Table I for $k=2.0$, 2.4, and 2.8 fm^{-1}.

ELP) depends on energy. Since we are interested in the shape and range of these potentials as well as in their strength for interpretation purposes, it suffices to construct them at some fixed energy. Another way is, of course, to construct ELP's via inverse scattering techniques in order to generate *l*-dependent but energy-independent interactions [19–21]. However, this would lead to unnecessary complications without gaining more insight into what we are trying to do.

The ELP's of the nonlocal interactions of Table I are shown in Fig. 1 for $k=2.0$ fm^{-1}. It is seen that they are similar in shape, and have an attractive strength at short distances which is quite large compared with the strength of a typical *NN* interaction. Another aspect of these potentials should be noted, namely, the existence of a hump, which suggests a repulsion in the interaction region, and thus resonances may also appear [21]. Such a hump is characteristic of local potentials equivalent to nonlocal interactions, which fit the two-nucleon scattering data at high energies [11,17,22]. The striking similarity of the n-α local potentials, equivalent to nonlocal potentials, and of the corresponding two-body bound state wave functions presented in Ref. [21] with those of the present work is worth mentioning. Looking now at the three-body binding energies generated by the nonlocal potentials of Table I, we note that the shorter the range of the potential the larger the three-body binding energy. This is in agreement with the findings of Delfino *et al.* [7] obtained with rank-1 nonlocal potentials with a one-term Yamaguchi form factor. The results of Ref. [7] were shown to be related to the Thomas effect. The present results suggest that this relation is also valid for potentials with a form factor consisting of a sum of Yamaguchi terms, and therefore it is a more general statement.

In Fig. 2 one of the ELP's employed, potential 1, is shown for different momenta. It is seen that the main characteristics of the potential do not change significantly and thus the previous results of Rupp *et al.* [5], obtained with a local potential equivalent to that of Tabakin, are corroborated.

The wave functions of the excited states, i.e., the physical two-body states, are shown in Fig. 3. It is seen that there is a node in the interior region which moves to shorter distances as the energy of the ground state increases. This is not unexpected, as the attractive well of the potential is shifted in the interior region and assumes the characteristics of a δ function (hence the relation to the Thomas effect, see Fig. 2); therefore, the position of the node of the excited state is also shifted closer to zero. These results are in qualitative agreement with those of Nakaichi-Maeda [11]. In other words, the appearance of a bound state with a large binding energy in the three-body system (collapse) is related to the nodal behavior of the physical two-body wave function.

III. LOCAL INTERACTIONS

There are many local potential models which determine the deuteron properties fairly well. Since our main concern is

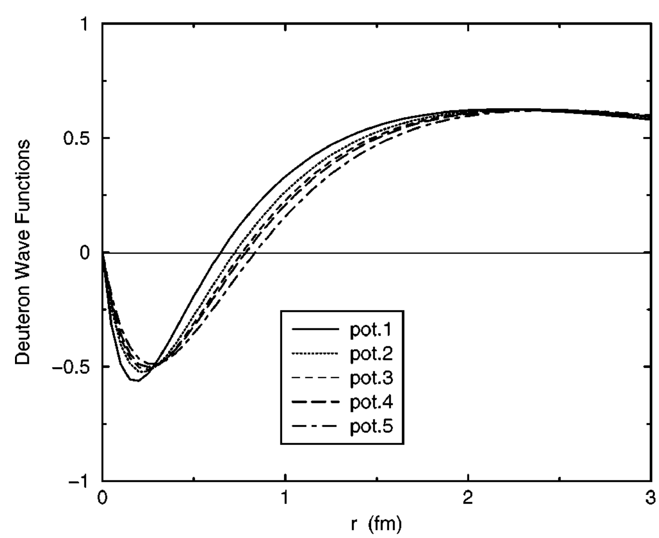

FIG. 3. The short-range behavior of the physical deuteron wave function generated by the potentials of Table I. The position of the node depends on the range of the attractive well.

TABLE II. The set of parameters used in the two-body local potential of Eq. (9) together with the corresponding two- and three-body binding energies B_2 and B_3, respectively.

Pot.	V_0 (MeV)	α (fm^{-2})	V_1 (MeV)	β (fm^{-1})	μ (fm^{-1})	B_2 (MeV)	B_3 (MeV)
1	178.24	0.2	5.0	1.0	0.7	2.225	4.60
						77.513	163.85
2	36.5	0.03	10.0	3.0	0.7	2.225	7.26
						17.486	38.35
3	45.88	0.5	30.0	1.4	0.7	2.225	4.74
4	4.7	0.007	3.0	1.4	0.2	0.074	0.308
						2.822	5.96

three-body bound state collapse, it is sufficient to choose out of this multitude a simple local model which reproduces the binding energy of the deuteron. The aforementioned *NN* potential of Kukulin *et al.* [12] is best suited for our investigations. This potential has a deep attractive well at short distances which results from a six-quark model in the interior region, and generates a PFS with a large binding energy. This implies that the corresponding physical two-body bound state wave function has an inner node which simulates the repulsive core of the traditional *NN* potentials. The form of this potential is

$$V(r)=V_0\exp(-\alpha r^2)+V_1[1-\exp(-\beta r)]\frac{\exp(-\mu r)}{\mu r}. \tag{9}$$

By varying its parameters one can move the forbidden state above or below the physical deuteron state, which we keep fixed at $E_b=2.225$ MeV. Thus the state corresponding to E_b can be an excited or a ground state of the two-body system, and thus it may or may not have a node.

Several sets of parameters were used which give rise to different shapes, ranges, and depths of the potential. Four characteristic examples are presented in Table II together with the resulting two- and three-body binding energies. These potentials are plotted in Fig. 4. Potential 1 is much more attractive, and supports a forbidden state at $E_g=77.513$ MeV. As compared to the other three potentials, potential 2 has a different shape and a much longer range, the forbidden state being at $E_g=17.486$ MeV. Potential 3 sustains only the physical deuteron state, while potential 4 supports, in addition to this an excited bound state at $E_e=0.074$ MeV. It is noted that potentials 3 and 4 are similar to soft-core *NN* potentials, and fit the binding energy but not the scattering phase shifts. Therefore, with these examples we can pinpoint which property of the potential is most important for the collapse.

In order to calculate the three-body binding energy, we utilize the Faddeev formalism. For this we transform the potentials [Eq. (9)] in momentum space, and the potential matrix,

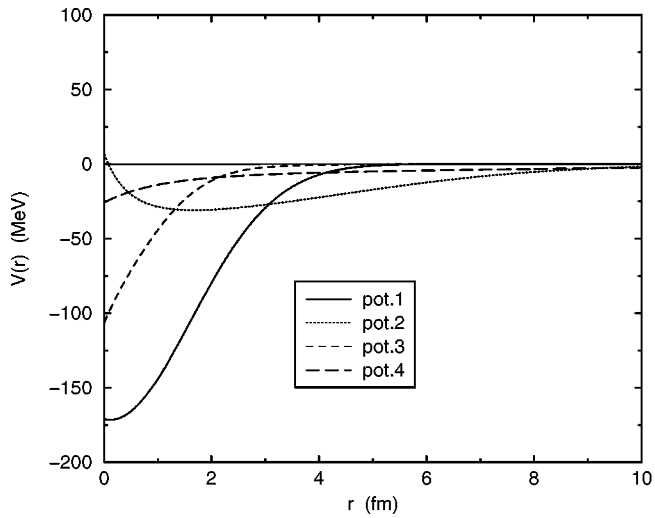

FIG. 4. The local potentials corresponding to Eq. (9). The parameters are those given in Table II.

$$V(p,q)=\frac{V_0}{4}\sqrt{\pi}\alpha\left[\exp\left(-\frac{(p-q)^2}{4\alpha}\right)-\exp\left(-\frac{(p+q)^2}{4\alpha}\right)\right]+\frac{V_1}{4\mu}\ln\left[\frac{[\mu^2+(p+q)^2][(\mu+\beta)^2+(p-q)^2]}{[\mu^2+(p+q)^2][(\mu+\beta)^2+(p+q)^2]}\right], \tag{10}$$

is then used to obtain the two-body t matrix needed in the Faddeev equation for the bound states. For three bosons in an S state, one has [23]

$$\psi(p,q)=\frac{8}{\pi q\sqrt{3}}\int_0^\infty q'\,dq'\int_{|2q-q'|/\sqrt{3}}^{(2q+q')/\sqrt{3}}p'\,dp'\times\frac{t[p,(p'^2+q'^2-q^2)^{1/2};s-q^2]\psi(p',q')}{s-p'^2-q'^2}, \tag{11}$$

where

$$t(p,k;z)=V(p,k)-\frac{2}{\pi}\int_0^\infty\frac{V(p,k')t(k',k;z)k'^2dk'}{k'^2-z}. \tag{12}$$

Due to the variable limits of p', it is impractical to solve Eq. (11) by converting it to a matrix form and then applying the usual eigenanalysis techniques. Furthermore, the form of our potentials (a deep attractive well) requires special attention and care. Thus the method of successive iterations has been employed [24], and the results obtained were reasonably stable.

Here we point out that the two-body ground-state energies for potentials 1 and 2 are larger than the binding energy of the deuteron. In contrast, the two-body ground-state energy for potential 3 is fixed at 2.225 MeV, while that for potential 4 is fixed at 2.282 MeV for reasons which we shall explain

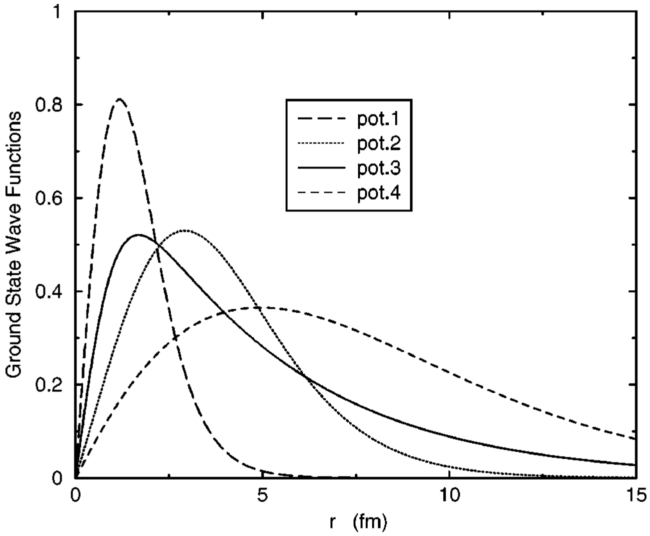

FIG. 5. Ground-state wave functions of the potentials of Table II. Their peak and spread are directly linked to the shape and strength of the attractive well.

later. The corresponding wave functions for the two-body ground states are shown in Fig. 5, and those for the excited states in Fig. 6.

We have searched for three-body binding energies in the region of (−1000, 0) MeV. The results are presented in Table II. For potentials 1, 2, and 4, we located two three-body bound states corresponding to the two two-body bound states. The three-body binding energy is at its maximum value at $E_t = 163.85$ MeV for potential 1, and decreases to $E_t = 38.35$ MeV for potential 2. There is no collapse for potential 3, which supports only the physical deuteron state. Comparing these results with the range of potentials shown in Fig. 4, we see that the three-body ground-state energy is larger when the range of the potential is smaller—an indirect manifestation of the relation between the BSC and the Thomas effect. Looking now at the nodal behavior of the wave functions in Fig. 6, we note that the collapse is more enhanced when the node is in the interior region. This is in line with the results found in Sec. II.

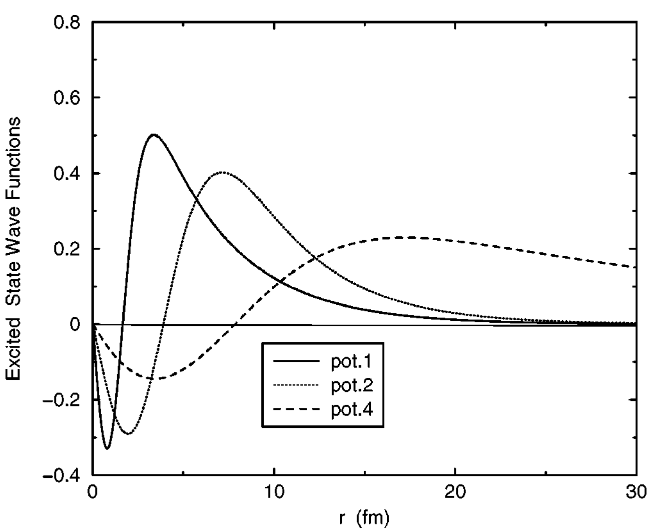

FIG. 6. Excited-state wave functions of the potentials of Table II. Short-range attractive wells shift the node in the interior region.

In order to obtain an excited bound state that lies above the physical deuteron state, we slightly increased, for potential 4, the two-body ground-state energy to $E_b = 2.822$ MeV, and obtained $E_e = 0.074$ MeV. The corresponding three-body binding energies are 0.308 and 5.96 MeV, i.e., no collapse was detected in this case. It seems that there is a contradiction here with the results of the nonlocal potentials, which also showed a collapse for potentials fulfilling condition (7). As already noticed by Delfino *et al.* [7,8], however, for this type of potential the two-body pseudobound state has the same behavior as the physical bound state. Therefore, we can conclude that only forbidden states with large binding energies cause the three-body bound state collapse. This argument will be also supported by removing the forbidden states via supersymmetric transformations.

IV. SUPERSYMMETRIC POTENTIALS

Quite often, in nuclear physics problems, the constructed two-body potential has an attractive well that sustains unphysical bound states which must be removed or projected out from the spectrum before the potential is used in calculations. One way to achieve this is via SUSY transformations, in which one can add or remove a bound state from the spectrum. The method was discussed extensively in Refs. [14,15], and therefore here we shall recall its main features only briefly.

We consider the radial Schrödinger equation

$$H_0\psi_0(r) \equiv \left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V_0(r) \right] \psi_0(r) = E\psi_0(r) \tag{13}$$

for the Hamiltonian H_0, which has in its spectrum an unphysical two-body bound state $E \equiv E_b^{(0)}$. This state can be removed by factorizing H_0 and its supersymmetric partner Hamiltonian H_1,

$$H_0 = A_0^+ A_0^- + \epsilon_0, \quad H_1 = A_0^- A_0^+ + \epsilon_0, \tag{14}$$

where

$$A^- = (A_0^+)^\dagger = -\frac{d}{dr} + \frac{d}{dr}\ln[\psi_0(r,\epsilon_0)]. \tag{15}$$

Here $\psi_0(r,\epsilon_0)$ is the solution of the Schrödinger equation at the factorization energy ϵ_0. The SUSY-1 potential V_1, that corresponds to the partner Hamiltonian H_1, is given by

$$V_1 = V_0 - 2\frac{d^2}{dr^2}\ln[\psi_0(E_0^{(0)})], \tag{16}$$

where $\epsilon_0 \leq E_0^{(0)}$, $E_0^{(0)}$ being the ground state energy of H_0. Whereas V_1 and V_0 have the same spectra (except that V_1 does not sustain a two-body ground state at $E_0^{(0)}$), they are not phase equivalent. To achieve phase equivalence a second supersymmetric transformation is needed to obtain the SUSY-2 potential [15]:

$$V_2 = V_0 - 2\frac{d^2}{dr^2}\ln[\psi_0(E_0^{(0)})\psi_1(E_0^{(0)})]. \tag{17}$$

The latter potential is fully phase equivalent to V_0, except that the ground state of V_0 has been removed from the spectrum. The fundamental difference between V_0 and V_2 is that the latter has a repulsive singular core at short distances,

$$V_2 \sim V_0 + \frac{2(2l+3)}{r^2} \sim \frac{(2+l)(3+l)}{r^2}, \tag{18}$$

instead of an attractive well. It is interesting to note that the difference $\delta_l(0) - \delta_l(\infty)$ may be a multiple of π despite the fact that the ground state is removed, i.e., the Levinson theorem is not applicable to this type of singular potentials, as shown long ago by Swan [25].

One may argue here that, as the resulting potential is shallow and singular, it should be expected that no deep bound state is generated in the three-body system. However, as the three-body bound state collapse was found to be related to the resonance behavior of the Jost function, we endeavored to go through the three-body calculations once more, as the SUSY transformations might generate a new resonance spectrum [21] that could be of relevance.

We applied the above method to potentials 1 and 2 of Sec. III, and checked again for three-body binding energies in the region $(-1000, 0)$ MeV. Only one three-body bound state was found for each potential, namely, at 4.72 MeV for potential 1 and at 7.14 MeV for potential 2, i.e., no collapse was detected. Thus the removal of the forbidden state of the two-body system resulted in the disappearance of the second three-body bound state as well.

V. DISCUSSION AND CONCLUSIONS

The role played by unphysical two-body bound states in the appearance of a collapsed state in the three-body system has been investigated. For this purpose three classes of potentials were employed; local potentials equivalent to nonlocal interactions which produce BSC; local interactions which, in addition to the physical deuteron state, sustain a second bound state with large binding energy; and SUSY transformation potentials. We shall discuss these in turn.

Rank-1 nonlocal separable potentials with a Yamaguchi-type form factor may possess bound states in the continuum, i.e., the S matrix has a pole on the real positive-energy axis. These bound states at positive energies are called pseudobound states, because in all respects they behave similarly to negative-energy states. Since the nonlocal potentials cannot be used directly for interpretation purposes, we resorted to the construction of ELP's which can provide information about how the underlying nonlocality is manifested in configuration space. In the present work, we constructed quantal ELP's to above-rank-1 nonlocal interactions, and showed that they have a strong attractive well that tends to have a δ-function behavior and sustain a deep bound state of a nature similar to the PFS. This implies that bound states in the continuum are mapped, via the localization procedure, onto a positive imaginary energy axis. The three-body ground-state energy then becomes extremely large. In other words, when at a two-body level one has a strong, short-range attractive potential that generates the Thomas effect, then there is a BSC in the three-body system and, in general, in N-body system—a finding which is in agreement with that of Delfino *et al.* [7,8].

We have extended our investigations to include purely local interactions having unphysical two-body bound states, i.e., PFS's with large binding energies, and calculated the corresponding three-body binding energies. We have demonstrated that an increase (decrease) of the two-body binding energy of the unphysical state results in an increase/(decrease) of the three-body binding energy. Thus the BSC of the three-body system is directly connected to the presence of an unphysical two-body state. We wish to mention further that variations of the binding energy of the unphysical two-body state also resulted in variations of the inner amplitude and the position of the node of the physical deuteron state. This is an indirect proof of the results of Nakaichi-Maeda [11]. Of course, one must be careful about the role of the node. The existence of a node in the physical two-body state wave function, generated by a local or nonlocal interaction, is also a manifestation of a strong attraction in the interior region.

The PFS is usually present in two-cluster systems, where antisymmetrization is used to construct the underlying intercluster interaction which is, in general, nonlocal. However, local potentials can also be constructed to have one PFS or more, which in few-cluster systems generate the appearance of a set of unphysical bound states. One such potential is the α-α potential of Buck *et al.* [26] employed in Ref. [27] to study the spectrum of the 3α and 4α systems. It was found in Ref. [27] that the appearance of a set of unphysical bound states disappears once the PFS's are removed from the spectrum via SUSY transformations. In the present work we found a similar result, namely, that the removal of an unphysical bound state from the two-body spectrum also results in the disappearance of BSC in the three-body system. In conclusion, from the above discussion one can safely argue that the presence of unphysical bound states with large binding energies in the two-body system is responsible for BSC in the three-body system and, in general, in the N-body system.

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